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ORIGINAL ARTICLE

Korovkin type approximation theorems via power series method

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Abstract In this paper we consider power series method which is also member of the class of all continuous summability methods. The power series method includes Abel method as well as Borel method. We investigate, using the power series method, Korovkin type approximation theorems for the sequence of positive linear operators defined on $C[a, b]$ and $L_q[a, b]$, $1 \leq q < \infty$, respectively. We also study some quantitative estimates for L_q approximation and give the rate of convergence of these operators.

Keywords Power series method · Korovkin type theorem · Quantitative estimate · Second-order modulus of smoothness

Mathematics Subject Classification 41A25 · 41A36

1 Introduction

The classical Korovkin type theorems provide conditions for whether a given sequence of positive linear operators converges to the identity operator in the space of continuous functions on a compact interval [1,9]. If the sequence of positive linear operators does not converge to the identity operator then it might be usefull to use some summability methods [14,18]. The main purpose of using summability theory has always been to make a non-convergent sequence to converge. This was the motivation behind Fejer's famous theorem showing Cesàro method being effective in making the Fourier series

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of continuous periodic functions to converge [4]. In this paper we investigate the approximation properties of positive linear operators by means of power series method which is also member of the class of all continuous summability methods. The method includes Abel method as well as the Borel method. The results presented in this paper are motivated by those of [11] and [18].

In Sect. 2, we prove some Korovkin type approximation theorems for a sequence of positive linear operators defined on $C[a, b]$ via power series method and also give the rate of convergence of these operators. Section 3 is devoted to a Korovkin type result for a sequence of positive linear operators acting from $L_q[a, b]$, $1 \leq q < \infty$, into itself and some quantitative estimates for L_q approximation.

First of all, we recall some basic definitions and notations used in the paper. Let (p_k) be a real sequence with $p_0 > 0$ and $p_k \geq 0$ $(k \in N)$, and such that the corresponding power series $p(t) := \sum_{k=0}^{\infty} p_k t^k$ has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} x_k p_k t^k = L$$

exists then we say that $x = (x_k)$ is convergent in the sense of power series method [10,15]. Note that the power series method is regular if and only if

$$\lim_{t \to R^-} \frac{p_k t^k}{p(t)} = 0, \quad for\ each\ k \in N \tag{1}$$

holds [5]. We assume throughout the paper that the methods fulfill condition (1).

2 Approximation properties on C[a,b] via power series method

We denote the space of all bounded and continuous real valued functions on the interval [a,b] by $B[a, b]$ and $C[a, b]$, respectively. Note that $B[a, b]$ and $C[a, b]$ are Banach spaces with norm

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Let T: C[a,b]$\to$ $B[a, b]$ be a linear operator. Then T is called positive if $Tf \geq 0$ whenever $f \geq 0$. If T is a positive linear operator then $f \leq g$ implies that $Tf \leq Tg$ and $|f| \leq g$ implies $|Tf| \leq Tg$. In this section we assume that (T_k) is a sequence of positive linear operators from $C[a, b]$ into $B[a, b]$ such that

$$\sup_{0<t<R} \frac{1}{p(t)} \sum_{k=0}^{\infty} \|T_k(1)\| p_k t^k < \infty. \tag{2}$$

Also $V_t\{(.); x\}$ given by

$$V_t\{(f(y); x)\} := \frac{1}{p(t)} \sum_{k=0}^{\infty} T_k(f(y); x) p_k t^k$$

is well-defined operator from $C[a, b]$ into $B[a, b]$ as we can see from the following inequality

$$\|\{V_t(.); x\}\| \leq \sup_{0<t<R} \frac{1}{p(t)} \sum_{k=0}^{\infty} \|T_k(1)\| p_k t^k < \infty. \tag{3}$$

Observe that $V_t\{(.); x\}$ is also linear positive operator. Throughout the paper, we also use the following test functions $f_i(x) = x^i$, $i = 0, 1, 2$.

The next theorem is a particular case of Theorem 1 of [12]. For the completeness, we give the proof of our theorem by using an alternative way.

Theorem 1 *Let $\{T_k\}$ be a sequence of positive linear operators from $C[a, b]$ into $B[a, b]$ such that (2) holds. Then for any function $f \in C[a, b]$ we have*

$$\lim_{t \to R^-} \|V_t\{(f(y); x)\} - f(x)\| = 0 \tag{4}$$

if and only if

$$\lim_{t \to R^-} \|V_t\{(f_i(y); x)\} - f_i(x)\| = 0, \quad i = 0, 1, 2. \tag{5}$$

Proof It is obvious that (4) implies (5). In order to show that (5) implies (4), let $\{T_k\}$ be a sequence of positive linear operators from $C[a, b]$ into $B[a, b]$ and let $f \in C[a, b]$. Since f is continuous on $[a, b]$, for every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y, x \in [a, b]$ satisfying $|y - x| < \delta$. Note that

$$|f(y) - f(x)| \leq |f(y)| + |f(x)| < \frac{2H}{\delta^2}(y - x)^2$$

for all $y, x \in [a, b]$ satisfying $|y - x| \geq \delta$ where $H := \|f\|$. Hence, as in the classical case [9], for any $y, x \in [a, b]$ we have

$$|f(y) - f(x)| < \varepsilon + \frac{2H}{\delta^2}(y - x)^2. \tag{6}$$

On the other hand from (6) one can get

$$|V_t\{(f(y); x)\} - f(x)| \leq V_t\{(|f(y) - f(x)|; x)\} + H|V_t\{(f_0(y); x)\} - f_0(x)| \tag{7}$$

for all $t \in (0, R)$. Using (6), linearity and positivity of the operators $V_t\{(.); x\}$, we get

$$\begin{aligned} V_t\{(|f(y) - f(x)|; x)\} &\leq \varepsilon V_t\{(f_0(y); x)\} + \frac{2H}{\delta^2} V_t\{((y-x)^2; x)\} \\ &\leq \varepsilon (V_t\{(f_0(y); x)\} - f_0(x)) + \varepsilon \\ &\quad + \frac{2H}{\delta^2} \Big\{ (V_t\{(f_2(y); x)\} - f_2(x)) \\ &\quad - 2x(V_t\{(f_1(y); x)\} - f_1(x)) \\ &\quad + x^2 (V_t\{(f_0(y); x)\} - f_0(x)) \Big\}. \end{aligned} \tag{8}$$

It follows from (7) and the last inequality, for all $t \in (0, R)$, that

$$\begin{aligned} |V_t\{(f(y); x)\} - f(x)| &\leq \left(\varepsilon + H + \frac{2H}{\delta^2}\sigma^2\right) |V_t\{(f_0(y); x)\} - f_0(x)| \\ &\quad + \frac{4\sigma H}{\delta^2} |V_t\{(f_1(y); x)\} - f_1(x)| + \frac{2H}{\delta^2} |V_t\{(f_2(y); x)\} - f_2(x)| + \varepsilon \end{aligned}$$

where $\sigma = \max\{|a|, |b|\}$. Then we have

$$\begin{aligned} \|V_t\{(f(y); x)\} - f(x)\| &\leq K\{\|V_t\{(f_2(y); x)\} - f_2(x)\| + \|V_t\{(f_1(y); x)\} - f_1(x)\| \\ &\quad + \|V_t\{(f_0(y); x)\} - f_0(x)\|\} + \varepsilon \end{aligned} \tag{9}$$

where $K = \max\{\varepsilon + H + \frac{2H}{\delta^2}\sigma^2, \frac{4\sigma H}{\delta^2}, \frac{2H}{\delta^2}\}$. Hence it follows from (5) and (9) that

$$\lim_{t \to R-} \|V_t\{(f(y); x)\} - f(x)\| < \varepsilon$$

which concludes the proof, since ε is arbitrary. □

Example 1 We now exhibit an example of a sequence of positive linear operators satisfying the conditions of Theorem 1 but that does not satisfy the conditions of the classical Korovkin theorem. Let $p_k = 1$, in this case R = 1 and $p(t) = \frac{1}{1-t}$, $t \in (-1, 1)$. Thus, the power series method corresponds to the Abel method. Consider the sequence (T_k) defined by $T_k : C[0, 1] \to B[0, 1]$, $T_k(f; x) = (1 + \alpha_k)B_k(f; x)$ where (B_k) is the sequence of Bernstein polynomials. Take $(\alpha_k) = ((-1)^k)$. Observe that (α_k) is Abel convergent to zero, but it is not convergent. Then one can see that the sequence (T_k) satisfies our Theorem 1, but it does not satisfy the classical Korovkin theorem.

We now study the rate of convergence of the sequence of positive linear operators examined in Theorem 1 by means of the modulus of continuity.

The modulus of continuity, denoted by $\omega(f, \delta)$, is defined by

$$\omega(f, \delta) = \sup_{|y-x| \leq \delta} |f(y) - f(x)|.$$

It is known that for any $\delta > 0$ and each $x, y \in [a, b]$, $f \in C[a, b]$

$$|f(y) - f(x)| \leq \left(\left[\!\left[\frac{|y-x|}{\delta}\right]\!\right] + 1\right)\omega(f, \delta)$$

where $[\![\lambda]\!]$ denotes the integer part of λ.

Theorem 2 *Let $\{T_k\}$ be a sequence of positive linear operators from $C[a, b]$ into $B[a, b]$ such that (2) holds. If*

(i) $\lim_{t\to R^-} \|V_t\{(f_0(y); x)\} - f_0(x)\| = 0$,

(ii) $\lim_{t\to R^-} \omega(f, \alpha(t)) = 0$,

then for all $f \in C[a, b]$ we have

$$\lim_{t\to R^-} \|V_t\{(f(y); x)\} - f(x)\| = 0$$

where $\alpha(t) = \sqrt{\|V_t\{((y-x)^2; x)\}\|}$.

Proof Using the linearity and positivity of $V_t\{(.); x\}$ and also for every $x, y \in [a, b]$ taking into account $\left[\!\left[\frac{|y-x|}{\delta}\right]\!\right] \leq \frac{(y-x)^2}{\delta^2}$, for all $t \in (0, R)$ and $\delta > 0$, we have

$$\begin{aligned}
&|V_t\{(f(y); x)\} - f(x)| \leq V_t\{(|f(y) - f(x)|; x)\} + H|V_t\{(f_0(y); x)\} - f_0(x)| \\
&\quad \leq V_t\left\{\left(\left(1 + \left[\!\left[\frac{|y-x|}{\delta}\right]\!\right]\right)\omega(f, \delta); x\right)\right\} + H|V_t\{(f_0(y); x)\} - f_0(x)| \\
&\quad \leq \omega(f, \delta)V_t\left\{\left(1 + \frac{(y-x)^2}{\delta^2}; x\right)\right\} + H|V_t\{(f_0(y); x)\} - f_0(x)| \\
&\quad \leq \omega(f, \delta)V_t\{(f_0(y); x)\} + \frac{\omega(f, \delta)}{\delta^2}V_t\{((y-x)^2; x)\} \\
&\qquad + H|V_t\{(f_0(y); x)\} - f_0(x)|.
\end{aligned} \tag{10}$$

By (3), for all $t \in (0, R)$, $\|V_t\{(.); x\}\|_{C[a,b]\to B[a,b]} \leq M$. Now letting $\delta = \alpha(t) = \sqrt{\|V_t\{((y-x)^2; x)\}\|}$ and by (10) we get, for all $t \in (0, R)$, that

$$\|V_t\{(f(y); x)\} - f(x)\| \leq \beta\{\omega(f, \alpha(t)) + \|V_t\{(f_0(y); x)\} - f_0(x)\|\}$$

where $\beta = \max\{1 + M, H\}$. Then, by (i), (ii) and (10), we have, for all $f \in C[a, b]$, that

$$\lim_{t\to R^-} \|V_t\{(f(y); x)\} - f(x)\| = 0.$$

□

3 Approximation properties on $L_q[a, b]$ via power series method

In this section, using power series method, we study a Korovkin type approximation theorem for positive linear operators acting on L_q spaces. Some results concerning the Korovkin type theorems for a function in $L_q(-\pi, \pi)$ may be found in [6,16,17]. We also study the quantitative estimates for L_q approximation. Throughout the section we assume $1 \leq q < \infty$.

First, we recall some basic definitions and notations used in this section. Let

$$L_q^2[a, b] = \{f \in L_q[a, b] : f' \textit{is absolutely continuous and } f'' \in L_q[a, b]\}$$

where f' and f'' are respectively the first and second derivatives of f.

For $f \in L_q[a, b]$ and $y > 0$, the K-functional of Peetre [13] is defined by

$$K_{2,q}(f; y) = \inf_{g \in L_q^2[a,b]} \{\|f - g\|_q + y(\|g\|_q + \|g''\|_q)\}.$$

Following [2] and [3], we define the second-order modulus of smoothness to be

$$\omega_{2,q}(f, h) = \sup_{0<y\leq h} \|f(x + y) - 2f(x) + f(x - y)\|_{L_q[a+y,b-y]},$$

where $f \in L_q[a, b]$ and $[a+y, b-y] \subset [a, b]$. By [8] we have the following relation between modulus of smoothness and K-functional of Peetre:

$$\begin{aligned} C^{-1}(\min(1, y^2)\|f\|_q + \omega_{2,q}(f; y)) &\leq K_{2,q}(f; y^2) \\ &\leq C(\min(1, y^2)\|f\|_q + \omega_{2,q}(f; y)) \end{aligned} \tag{11}$$

where $C > 0$ is an absolute constant and independent of f and q.

Let $\{T_k\}$ be a sequence of positive linear operators from $L_q[a, b]$ into $L_q[a, b]$ such that

$$H' := \sup_{0<t<R} \sum_{k=0}^{\infty} p_k \|T_k\|_{L_q \to L_q} t^k < \infty. \tag{12}$$

A generalization of the next theorem has been given in [19]. For the completeness, we give the proof of our theorem by using an alternative way for a particular case $L_q[a, b]$.

Theorem 3 *Let $\{T_k\}$ be a sequence of positive linear operators from $L_q[a, b]$ into $L_q[a, b]$ such that (12) holds. Then for any function $f \in L_q[a, b]$ we have*

$$\lim_{t \to R^-} \|V_t\{(f(y); x)\} - f(x)\|_q = 0 \tag{13}$$

if and only if

$$\lim_{t \to R^-} \|V_t\{(f_i(y); x)\} - f_i(x)\|_q = 0, \quad i = 0, 1, 2.$$

Proof Let $f \in L_q[a, b]$. Given $\varepsilon > 0$, by the Lusin theorem, there exists a continuous function φ on $[a, b]$ such that

$$\|f - \varphi\|_q < \varepsilon.$$

From the above inequality, we get

$$\begin{aligned}\|V_t\{(f(y); x)\} - f(x)\|_q &\le \|V_t\{(f(y) - \varphi(y); x)\}\|_q + \|V_t\{(\varphi(y); x)\} - \varphi(x)\|_q \\ &\quad + \|f(x) - \varphi(x)\|_q \\ &< \varepsilon\left(1 + \frac{1}{p_0}\sum_{k=0}^{\infty} p_k \|T_k\|_{L_q \to L_q} t^k\right) + \|V_t\{(\varphi(y); x)\} \\ &\quad - \varphi(x)\|_q.\end{aligned} \tag{14}$$

Since the function φ is continuous on $[a, b]$, we have

$$|\varphi(y) - \varphi(x)| < \varepsilon + \frac{2M}{\delta^2}(y - x)^2 \tag{15}$$

where $M := \|\varphi\|_{C[a,b]}$. First of all we consider second term on the right hand of inequality (14). Using the latter inequality and the monotonicity of the operator T_k, we obtain

$$\begin{aligned}&\|V_t\{(\varphi(y); x)\} - \varphi(x)\|_q \le \|V_t\{(|\varphi(y) - \varphi(x)|; x)\}\|_q \\ &\quad + M\|V_t\{(f_0(y); x)\} - f_0(x)\|_q \\ &< \left\|V_t\left\{\left(\varepsilon + \frac{2M}{\delta^2}(y - x)^2; x\right)\right\}\right\|_q + M\|V_t\{(f_0(y); x)\} - f_0(x)\|_q \\ &\le \varepsilon(1 + \|V_t\{(f_0(y); x)\} - f_0(x)\|_q) + \frac{2M}{\delta^2}\|V_t\{((y - x)^2; x)\}\|_q \\ &\quad + M\|V_t\{(f_0(y); x)\} - f_0(x)\|_q \\ &\le \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}\sigma^2\right)\|V_t\{(f_0(y); x)\} - f_0(x)\|_q \\ &\quad + \frac{4M}{\delta^2}\sigma\|V_t\{(f_1(y); x)\} - f_1(x)\|_q \\ &\quad + \frac{2M}{\delta^2}\|V_t\{(f_2(y); x)\} - f_2(x)\|_q\end{aligned} \tag{16}$$

where $\sigma = \max\{|a|, |b|\}$. Hence it follows from (14), (15) and (16) that for all $t \in (0, R)$

$$\begin{aligned}\|V_t\{(f(y); x)\} - f(x)\|_q &< \varepsilon\left(2 + \frac{H'}{p_0}\right) \\ &\quad + \left(\varepsilon + M + \frac{2M}{\delta^2}\sigma^2\right)\|V_t\{(f_0(y); x)\} - f_0(x)\|_q\end{aligned}$$

$$+\frac{4M}{\delta^2}\sigma\|V_t\{(f_1(y);x)\}-f_1(x)\|_q$$
$$+\frac{2M}{\delta^2}\|V_t\{(f_2(y);x)\}-f_2(x)\|_q. \quad (17)$$

Letting $t\to R^-$ in both sides of (17) we get

$$\lim_{t\to R^-}\|V_t\{(f(y);x)\}-f(x)\|_q<\varepsilon$$

which proves sufficiency, since ε is arbitrary. Observe that the necessity is trivial. This completes the proof. □

In order to obtain quantitative estimate and an approximation theorem we use the notation

$$\lambda_{tq}:=\{\max_{i=0,1,2}\|V_t\{f_i(y);x\}-f_i(x)\|_q\}^{\frac{q}{2q+1}}.$$

First of all let us give some lemmas.

Lemma 1 *Let $f\in L_q^{(2)}[a,b]$ and fix $\delta>0$. For $x,y\in[a,b]$, we have*

$$\left|\int_x^y(y-u)f''(u)du\right|\le\delta\int_0^\delta|f''(x+u)|du+\frac{(y-x)^2}{\delta^{\frac{1}{q}}}\|f''\|_q$$

(see, e.g. [16]).

Lemma 2 *Let $\{T_k\}$ be a sequence of positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that (12) holds. Then for $x,y\in[a,b]$, for any function $f\in L_q^{(2)}[a,b]$ and for all t sufficiently close to R from left side, we have*

$$\left\|V_t\left\{\int_x^y(y-u)f''(u)du;x\right\}\right\|_q\le C\|f''\|_q\lambda_{tq}^2$$

where C is a positive constant.

Proof Let $f\in L_q^{(2)}[a,b]$ and assume f has been extended outside of $[a,b]$ so that $f''(x)=0$ if $x\notin[a,b]$. From Lemma 1 and monotonicity of the operator T_k, we have

$$\left\|V_t\left\{\int_x^y(y-u)f''(u)du;x\right\}\right\|_q\le\left\|\delta\left(\int_0^\delta|f''(x+u)|du\right)V_t\{f_0(y);x\}\right\|_q$$
$$+\frac{\|f''\|_q}{\delta^{\frac{1}{q}}}\|V_t\{(y-x)^2;x\}\|_q \quad (18)$$

Using the Hölder's inequality and the generalised Minkowski inequality, we get

$$\begin{aligned}
&\left\|\delta\left(\int_0^\delta f''(x+u)du\right)V_t\{f_0(y);x\}\right\|_q \\
&= \left\|\delta\left(\int_0^\delta |f''(x+u)|du\right)\{V_t\{f_0(y);x\}-f_0(x)\}+\delta\int_0^\delta |f''(x+u)|du\right\|_q \\
&\le \delta\left\|\int_0^\delta |f''(x+u)|du\{V_t\{f_0(y);x\}-f_0(x)\}\right\|_q+\delta\left\|\int_0^\delta |f''(x+u)|du\right\|_q \\
&\le \delta^{2-\frac{1}{q}}\|f''\|_q\|V_t\{f_0(y);x\}-f_0(x)\|_q+\delta\int_0^\delta \|f''(x+u)\|_q du \\
&\le \|f''\|_q\{\delta^{2-\frac{1}{q}}\|\{V_t\{f_0(y);x\}-f_0(x)\}\|_q+\delta^2\}. \qquad (19)
\end{aligned}$$

Considering (18), (19) and $\sigma=\max\{|a|,|b|\}$, one can get

$$\begin{aligned}
&\left\|V_t\left\{\int_x^y (y-u)f''(u)du;x\right\}\right\|_q \le \|f''\|_q\left[\delta^2+\frac{1}{\delta^{\frac{1}{q}}}\Big\{\|V_t\{f_2(y);x\}-f_2(x)\|_q\right. \\
&\left.+2\sigma\|V_t\{f_1(y);x\}-f_1(x)\|_q+(\delta^2+\sigma^2)\|V_t\{f_0(y);x\}-f_0(x)\|_q\Big\}\right]. \qquad (20)
\end{aligned}$$

If we choose

$$\delta=\lambda_{tq},$$

then we obtain

$$\left\|V_t\left\{\int_x^y (y-u)f''(u)du;x\right\}\right\|_q \le \|f''\|_q\big\{(\sigma^2+2\sigma+2)\lambda_{tq}^2+\lambda_{tq}^4\big\}\le C\|f''\|_q\lambda_{tq}^2.$$

□

Lemma 3 *Let $\{T_k\}$ be a sequence of positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that (12) holds. Then for any function $f\in L_q^{(2)}[a,b]$ and for all t sufficiently close to R from left side,*

$$\|V_t\{f(y);x\}-f(x)\|_q\le C_q'(\|f\|_q+\|f''\|_q)\{\lambda_{tq}^2\}$$

is satisfied, where C_q' is a positive constant independent of f and t.

Proof Let $f\in L_q^{(2)}[a,b]$ and assume f has been extended outside of $[a,b]$ so that $f''(x)=0$ if $x\notin[a,b]$. For $x,y\in[a,b]$ the following equality

$$f(y) - f(x) = f'(x)(y-x) + \int_x^y (y-u) f''(u) du \tag{21}$$

is well known. Considering (21), Lemma 2 and linearity of operator T_k for all t sufficiently close to R from left side we have

$$\begin{aligned}
\|V_t\{f(y);x\} - f(x)\|_q &\le \|V_t\{f(y) - f(x);x\}\|_q \\
&\quad + \|f\|_\infty \|V_t\{f_0(y);x\} - f_0(x)\|_q \\
&\le \|f'\|_\infty \|V_t\{y-x;x\}\|_q + \left\| V_t\left\{ \int_x^y (y-u) f''(u) du; x \right\} \right\|_q \\
&\quad + \|f\|_\infty \|V_t\{f_0(y);x\} - f_0(x)\|_q \\
&\le \|f'\|_\infty \{\|V_t\{f_1(y);x\} - f_1(x)\|_q + \sigma \|V_t\{f_0(y);x\} - f_0(x)\|_q\} \\
&\quad + C\|f''\|_q \{\lambda_{tq}^2\} \\
&\quad + \|f\|_\infty \|V_t\{f_0(y);x\} - f_0(x)\|_q
\end{aligned}$$

where $\sigma = \max\{|a|, |b|\}$ and $\|.\|_\infty$ denotes essential sup norm on L_∞.

On the other hand if we take $n = 2, k = 1$ and $n = 2, k = 0$ in Theorem 3.1 of [7], we get for any $\varepsilon > 0$ that

$$\|f'\|_\infty \le \varepsilon^{\frac{1}{q}} \{16\varepsilon^{-2} \|f\|_q + \|f''\|_q\}$$

and

$$\|f\|_\infty \le \varepsilon^{\frac{1}{q}} \{16\varepsilon^{-1} \|f\|_q + \varepsilon \|f''\|_q\}.$$

From the above inequalities, we have

$$\|V_t\{f(y);x\} - f(x)\|_q \le C'_q (\|f\|_q + \|f''\|_q) \lambda_{tq}^2.$$

This completes the proof. □

Theorem 4 *Let $\{T_k\}$ be a sequence of positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that (12) holds. Then for all t sufficiently close to R from left side and for any function $f \in L_q[a,b]$ the inequality*

$$\|V_t\{f(y);x\} - f(x)\|_q \le C_q \{\min(1, \lambda_{tq}^2) \|f\|_q + \omega_{2,q}(f; \lambda_{tq})\}$$

holds, where C_q is a positive constant independent of f and t.

Proof Let $f \in L_q[a,b]$ and $g \in L_q^{(2)}[a,b]$. Then for all t sufficiently close to R from left side

$$\begin{aligned}\|V_t\{f(y);x\} - f(x)\|_q &\leq \|f-g\|_q \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k \|T_k\|_{L_q \to L_q} t^k \\ &+ \|V_t\{g(y);x\} - g(x)\|_q + \|f-g\|_q \\ &\leq (1+M)\|f-g\|_q + C_q' \left(\|g\|_q + \|g''\|_q\right) \lambda_{tq}^2\end{aligned} \tag{22}$$

is satisfied , where $M := \frac{1}{p_0} \sup\limits_{0<t<R} \sum_{k=0}^{\infty} p_k \|T_k\|_{L_q \to L_q} t^k$. In inequality (22) taking infimum over $g \in L_q^{(2)}[a,b]$, from the definition of K-functional of Peetre of order two and inequality (11) we get

$$\|V_t\{f(y);x\} - f(x)\|_q \leq C_q\{\min(1,\lambda_{tq}^2)\|f\|_q + \omega_{2,q}(f;\lambda_{tq})\}$$

which completes the proof. □

Using the above rate of convergence, we can give the following.

Corollary 1 *Let $\{T_k\}$ be a sequence of positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that (12) holds and $\lim_{t\to R^-} \lambda_{tq} = 0$. Then for any function $f \in L_q[a,b]$ we have*

$$\lim_{t\to R^-} \|V\{(f(y);x);t\} - f(x)\|_q = 0.$$

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