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On algebras of polynomial codimension growth

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Abstract Let A be an associative algebra over a field F of characteristic zero and let $c_n(A), n = 1, 2, \ldots$, be the sequence of codimensions of A. It is well-known that $c_n(A), n = 1, 2, \ldots$, cannot have intermediate growth, i.e., either is polynomially bounded or grows exponentially. Here we present some results on algebras whose sequence of codimensions is polynomially bounded.

Keywords Polynomial identities · Codimensions · Codimension growth

Mathematics Subject Classification 16R10 · 16P90

1 Codimensions and algebras with 1

Let A be an associative algebra over a field F of characteristic zero, $F\langle X\rangle$ the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over F and $\mathrm{Id}(A) \subseteq F\langle X\rangle$ the T-ideal of polynomial identities of A. Recall that a polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity for A, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. Then

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$$

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is a T-ideal of $F\langle X\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$. An effective way of studying such an ideal is that of determining some numerical invariants allowing to give a quantitative description.

A very useful numerical invariant that can be attached to $\mathrm{Id}(A)$ is given by the sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots,$ of A. Recall that such a sequence is defined as follows:

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}, n = 1, 2, \ldots,$$

where P_n denotes the space of multilinear polynomials in $x_1, \ldots, x_n$, for $n \geq 1$. In general $c_n(A)$ is bounded from above by $n!$, but in case A is a PI-algebra, i.e., satisfies a non-trivial polynomial identity, a celebrated theorem of Regev asserts that $c_n(A)$, $n = 1, 2, \ldots,$ is exponentially bounded [24], i.e., there exist constants $\alpha, a > 0$ such that $c_n(A) \leq \alpha a^n$ for all n.

Later in [10,11] Kemer showed that, given any PI-algebra A over a field of characteristic zero, $c_n(A)$, $n = 1, 2, \ldots$, cannot have intermediate growth, i.e., either is polynomially bounded or grows exponentially. Moreover, if $c_n(A)$ is polynomially bounded then it was proved in [1] that

$$c_n(A) = qn^k + O(n^{k-1}) \approx qn^k,\ n \to \infty, q \in \mathbb{Q}.$$

For general PI-algebras the exponential rate of growth of the sequence of codimensions was explicitly computed in [8,9]. The authors proved that for any associative algebra A, satisfying an ordinary identity, the limit

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer. It is called the PI-exponent of A. In case A is an algebra with 1, $\mathrm{Id}(A)$ is completely determined by its multilinear proper polynomials (see for instance [2]).

Recall that $f(x_1, \ldots, x_n) \in P_n$ is a proper polynomial if it is a linear combination of products of (long) Lie commutators $[x_{i_1}, \ldots, x_{i_k}]$. We denote by Γ_n the subspace of P_n of proper polynomials in $x_1, \ldots, x_n$; we put also $\Gamma_0 = \mathrm{span}\{1\}$. Then, the sequence of proper codimensions is defined as $c_n^p(A) = \dim \frac{\Gamma_n}{\Gamma_n \cap Id(A)}$, $n = 0, 1, 2, \ldots$. For a unitary algebra A, the relation between ordinary codimensions and proper codimensions (see for instance [3]), is given by the formula

$$c_n(A) = \sum_{i=0}^{n} \binom{n}{i} c_i^p(A),\ n = 1, 2, \ldots. \tag{1}$$

In particular, if A is a unitary algebra whose sequence of codimensions is polynomially bounded, then $c_n(A) = qn^k + \cdots$ is a polynomial with rational coefficients [1,6]. In [3] it was proved that in case $k > 1$ the leading coefficient q is a rational number satisfying the inequality

$$\frac{1}{k!} \leq q \leq \sum_{j=2}^{k} \frac{(-1)^j}{j!} \to \frac{1}{e}, k \to \infty,$$

where $e = 2.71\ldots$. In the non-unitary case, for any $q \in \mathbb{Q}$ there exists an algebra A such that $c_n(A) \approx qn^k$ for a suitable k. For k odd the lower bound was improved in [6]. The authors proved that if $c_n(A) \approx qn^k$, for some odd integer $k > 1$ and rational number q, then $q \geq \frac{k-1}{k!}$. Moreover, they proved that for any k the upper and the lower bound of q are actually reached.

We start by exhibiting PI-algebras realizing the smallest and the largest value of q (see for instance [6]). Let

$$U_k = U_k(F) = \left\{ \alpha E + \sum_{1 \leq i < j \leq k} \alpha_{ij} e_{ij} \mid \alpha, \alpha_{ij} \in F \right\},$$

where $E = E_{k \times k}$ denotes the identity $k \times k$ matrix and the e_{ij}'s are the usual matrix units.

In what follows Lie commutators are left-normed, i.e., $[x_1, x_2, \ldots, x_k] = [[\cdots[[x_1, x_2], x_3], \ldots], x_k]$. The next theorem shows that the algebra U_k has the largest possible polynomial growth of degree $k-1$, namely $c_n(U_k) \approx qn^{k-1}$, where $q = \sum_{j=2}^{k-1} \frac{(-1)^j}{j!}$.

Theorem 1.1 [6, Theorem 3.1] *Let F be an infinite field. Then:*

1. *A basis of the identities of U_k is given by all products of commutators of total degree k*

$$[x_1, \ldots, x_{a_1}][x_{a_1+1}, \ldots, x_{a_2}] \cdots [x_{a_{r-1}+1}, \ldots, x_{a_r}] \tag{2}$$

   with $a_r = k$ in case k is even, and by the polynomials in (2) plus the polynomial of degree $k+1$

$$[x_1, x_2] \cdots [x_k, x_{k+1}]$$

   in case k is odd.

2. $$c_n(U_k) = \sum_{j=0}^{k-1} \frac{n!}{(n-j)!} \theta_j \approx \theta_{k-1} n^{k-1}, \quad n \to \infty,$$

   where $\theta_i = \sum_{j=0}^{i} \frac{(-1)^j}{j!}$, for $i \in \mathbb{N}$.

The importance of U_k is shown in the following.

Theorem 1.2 *Let A be a unitary algebra over an infinite field F such that $c_n(A) \approx qn^k$, $n \to \infty$. Then $Id(A) \supseteq Id(U_{k+1})$.*

We now turn to the problem of constructing algebras with 1 realizing the minimal possible value for q.

For $k \geq 2$ let

$$N_k = \text{span}\{E, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq U_k,$$

where $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in U_k$ denote the diagonal just above the main diagonal of U_k.

Let also G_{2k} denote the Grassmann algebra with 1 on a $2k$-dimensional vector space over F. Recall that

$$G_{2k} = \langle 1, e_1, \ldots, e_{2k} \mid e_i e_j = -e_j e_i \rangle.$$

Theorem 1.3 [6, Theorem 3.4] *Let $k \geq 3$ and let F be an infinite field. Then*

1. *A basis of the identities of N_k is given by the polynomials*

$$[x_1, \ldots, x_k], \; [x_1, x_2][x_3, x_4].$$

2.

$$c_n(N_k) = 1 + \sum_{j=2}^{k-1} (j-1)\binom{n}{j} \approx \frac{k-2}{(k-1)!} n^{k-1}, \quad n \to \infty.$$

Theorem 1.4 [6, Theorem 3.5] *Let F be an infinite field. Then*

1. *A basis of the identities of G_{2k} is given by the polynomials*

$$[x_1, x_2, x_3], \; [x_1, x_2] \cdots [x_{2k+1}, x_{2k+2}].$$

2.

$$c_n(G_{2k}) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}, \quad n \to \infty.$$

Notice that the smallest value of q is realized by N_{k+1} in case k is odd and by G_k in case k is even.

Recall that if $\mathcal{V}$ is a variety of algebras then $c_n(\mathcal{V}) = c_n(A)$, where $\mathcal{V} = \text{var}(A)$ and the growth of $\mathcal{V}$ is the growth of the codimensions of $\mathcal{V}$. We have the following.

Definition 1.1 A variety $\mathcal{V}$ is minimal of polynomial growth n^k if $c_n(\mathcal{V}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n(\mathcal{U}) \approx q'n^t$ with $t < k$.

Theorem 1.5 [14,15] *The algebras N_k and G_{2t} generate minimal varieties of polynomial growth, for any $k \geq 3$ and for any $t \geq 1$.*

2 Characterizing algebras of polynomial codimension growth

Much effort has been put into the study of varieties $\mathcal{V}$ of polynomial growth, i.e., such that $c_n(\mathcal{V})$ is polynomially bounded [5,12,17–21].

A classification of varieties of polynomial growth was started in [4,6]. More precisely the authors gave a complete list of finite dimensional algebras generating varieties of at most linear growth and, in the unitary case, of at most cubic growth. A celebrated theorem of Kemer [11] characterizes the varieties of polynomial growth as follows. Let G be the infinite dimensional Grassmann algebra over F and UT_2 the algebra of 2×2 upper triangular matrices over F. Then $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $G, UT_2 \notin \mathrm{var}(A)$. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially. The sequence of codimensions for the algebras G and UT_2 are well known: in [13] it was shown that $c_n(G) = 2^{n-1}$. Also, it follows from [22] that $c_n(UT_2) = 2^{n-1}(n-2)+2$. Therefore, these algebras generate the only two minimal varieties of exponent 2, in the sense that any of their proper subvarieties has exponent ≤ 1, that it has polynomial growth.

In [14,15] the author classified all the subvarieties of $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$, by giving a complete list of finite dimensional algebras generating them.

We start by giving the classification of the subvarieties of $\mathrm{var}(G)$. By [13], $\mathrm{Id}(G) = \langle [x_1, x_2, x_3] \rangle_T$; hence $G_{2k} \in \mathrm{var}(G)$, for any $k \geq 1$ (see Theorem 1.4).

We recall the following definition.

Definition 2.1 Let A and B be algebras. We say that A is PI-equivalent to B and we write $A \sim_{PI} B$ when A and B satisfy the same identities, that is $\mathrm{Id}(A) = \mathrm{Id}(B)$.

Theorem 2.1 [14] *Let $A \in var(G)$. Then either $A \sim_{PI} G$ or $A \sim_{PI} G_{2k} \oplus N$ or $A \sim_{PI} N$ or $A \sim_{PI} C \oplus N$, where N is a nilpotent algebra, C is a commutative non-nilpotent algebra and $k \geq 1$.*

Notice that the previous theorem allows us to classify all codimension sequences of the algebras lying in the variety generated by G. We can also classify all algebras generating minimal varieties inside $\mathrm{var}(G)$.

Corollary 2.1 *Let $A \in var(G)$ be such that $var(A) \subsetneq var(G)$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n(A) = 0$ or $c_n(A) = 1$ or $c_n(A) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}$, $k = 1, 2, \ldots$.*

Corollary 2.2 *An algebra $A \in var(G)$ generates a minimal variety of polynomial growth if and only if $A \sim_{PI} G_{2k}$, for some $k \geq 1$.*

Before giving the classification of the subvarieties of $\mathrm{var}(UT_2)$, we need to introduce a family of algebras without unit inside $\mathrm{var}(UT_2)$.

Let $UT_k = UT_k(F)$ be the algebra of $k \times k$ upper triangular matrices over F.

For $k \geq 2$ let

$$A_k = \mathrm{span}\{e_{11}, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq UT_k,$$

where $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$ and let A_k^* be the the subalgebra of UT_k obtained by flipping A_k along its secondary diagonal.

Lemma 2.1 [14, Lemma 3.1] *If* $k \geq 3$, *then*

1. $Id(A_k) = \langle [x_1, x_2][x_3, x_4], [x_1, x_2]x_3 \ldots x_{k+1} \rangle_T$.
2. $c_n(A_k) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l-1)+1 \approx qn^{k-1}$, *where* $q \in \mathbb{Q}$ *is a non-zero constant.*

Hence $Id(A_k^*) = \langle [x_1, x_2][x_3, x_4], x_3 \ldots x_{k+1}[x_1, x_2] \rangle_T$ *and* $c_n(A_k^*) = c_n(A_k)$.

By [22], $\mathrm{Id}(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle_T$; hence $N_k, A_k, A_k^* \in \mathrm{var}(UT_2)$.
The following theorem allows us to classify all the subvarieties of $\mathrm{var}(UT_2)$.

Theorem 2.2 [14, Theorem 5.4] *If* $A \in var(UT_2)$ *then* A *is PI-equivalent to one of the following algebras:*

$$UT_2, N, N_t \oplus N, N_t \oplus A_k \oplus N, N_t \oplus A_r^* \oplus N, N_t \oplus A_k \oplus A_r^* \oplus N,$$

where N *is a nilpotent algebra and* $k, r, t \geq 2$.

It is worth noticing that the previous theorem allows us to classify all algebras generating minimal varieties inside $\mathrm{var}(UT_2)$.

Corollary 2.3 *Let* $A \in var(UT_2)$. *Then* A *generates a minimal variety of polynomial growth if and only if either* $A \sim_{PI} N_t$ *or* $A \sim_{PI} A_k$ *or* $A \sim_{PI} A_k^*$, *for some* $k \geq 2,\ t > 2$.

The previous theorem allows to classify all codimension sequences of the algebras belonging to the variety generated by UT_2.

Next we show that the algebras $N_k,\ A_k$ and A_k^* play a prominent role in the classification of the varieties of at most linear growth and, in the unitary case, of at most cubic growth.

Theorem 2.3 [6, Theorem 3.6] *Let* A *be an* F*-algebra with* 1. *If* $c_n(A) \approx qn^k$, *for some* $q \geq 1$, $k \leq 3$, *then either* $A \sim_{PI} F$ *or* $A \sim_{PI} N_3$ *or* $A \sim_{PI} N_4$.

Remark If A satisfies the hypotheses of the above theorem then $A \in \mathrm{var}(UT_2)$.

The following corollary follows easily.

Corollary 2.4 *Let* A *be an* F*-algebra with* 1. *If* $c_n(A) \approx qn^k$, *for some* $q \geq 1$, $k \leq 3$, *then either* $c_n(A) = 1$ *or* $c_n(A) = \frac{n(n-1)}{2} + 1$ *or* $c_n(A) = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} + 1$. *Hence either* $q = 1$ *or* $q = \frac{1}{2}$ *or* $q = \frac{1}{3}$.

Notice that if A is an algebra with 1 then A cannot have linear growth of the codimensions.

The following theorem characterizes the varieties of at most linear growth.

Theorem 2.4 [4, Theorem 22] *Let* A *be an* F*-algebra. Then* $c_n(A) \leq kn$ *if and only if* A *is PI-equivalent to either* N *or* $C \oplus N$ *or* $A_2 \oplus N$ *or* $A_2^* \oplus N$ *or* $A_2 \oplus A_2^* \oplus N$ *where* N *is a nilpotent algebra and* C *is a commutative non-nilpotent algebra.*

Notice that the previous theorem allows us to classify all possible linearly bounded codimension sequences.

Corollary 2.5 *Let A be an F-algebra such that $c_n(A) \leq kn$ for all $n \geq 0$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n(A) = 0$ or $c_n(A) = 1$ or $c_n(A) = n$ or $c_n(A) = 2n - 1$.*

3 About minimal varieties

In the previous section we have presented the classification of all minimal subvarieties of $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ and it turned out that there are only a finite number of them. For each such variety, we have exhibited a finite dimensional generating algebra. The relevance of such classification relies in the fact that these were the building blocks that allowed us to give a complete classification of the subvarieties of $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$. In what follows we shall restrict ourselves to varieties generated by algebras with 1.

We shall give the classification, up to PI-equivalence, of the algebras with 1 generating minimal varieties of polynomial growth $\leq n^4$. We start with the following

Theorem 3.1 [7] *Let A be an algebra with 1 such that $c_n(A) \leq qn^3$. Then A generates a minimal variety of polynomial growth if and only if either $A \sim_{PI} N_3$ or $A \sim_{PI} N_4$.*

Let

$$M = \left\{ \begin{pmatrix} a & b & d & e & f \\ 0 & a & c & g & h \\ 0 & 0 & a & c & i \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d, e, f, g, h, i \in F \right\}.$$

In [7] it was proved that M generates a minimal variety of growth n^4.

Now we are in a position to classify all minimal varieties generated by algebras with 1 of growth n^4.

Theorem 3.2 *Let A be a unitary algebra such that $c_n(A) \approx qn^4$, for some $q > 0$. Then A generates a minimal variety if and only if $Id(A)$ coincides with one of the following T-ideals*

1. $\langle [x_1, x_2][x_3, x_4], [x_1, x_2, x_3, x_4, x_5] \rangle_T$,
2. $\langle [x_1, x_2, x_3], [x_1, x_2][x_3, x_4][x_5, x_6] \rangle_T$,
3. $\langle [x_2, x_1, x_1, x_1], [x_1, x_3, [x_1, x_2]], St_4 \rangle_T$,
4. $\langle [x_2, x_1, x_1, x_1], St_4, [x_1, x_2]^2 \rangle_T$.

In the first three cases we have that $A \sim_{PI} N_5$ or $A \sim_{PI} G_4$, or $A \sim_{PI} M$, respectively.

For $k \geq 5$, the number of minimal varieties of growth n^k is at least $|F|$, the cardinality of the base field and a classification of all minimal varieties of polynomial growth n^5 is given in [7]. There it is also given a recipe for classifying all minimal varieties of polynomial growth n^k, $k > 5$.

4 Polynomial codimension growth and colengths

An equivalent formulation of Kemers result can be given as follows. The symmetric group S_n acts on the left on the space P_n by permuting the variables: if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in P_n$, $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Since T-ideals are invariant under renaming of the variables, the space $\frac{P_n}{P_n \cap Id(A)}$ becomes an S_n-module. The S_n-character of $P_n(A) = \frac{P_n}{P_n \cap Id(A)}$, denoted by $\chi_n(A)$, is called the n-th cocharacter of A.

By complete reducibility we can write

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition λ and m_λ is the corresponding multiplicity. Also

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

is called the n-th colength of A.

Now Kemer's result can be stated as follows [23]: $c_n(A)$ is polynomially bounded if and only if the sequence of colengths is bounded by a constant i.e., $l_n(A) \leq k$, for some $k \geq 0$ and for all $n \geq 1$. A finer classification depending on the value of the constant k was started in [4,16]. There the authors completely classified, up to PI-equivalence, the algebras A such that $l_n(A) \leq 4$ for n large enough. We state such a result in the following.

Theorem 4.1 [16] *Let A be an F-algebra. Then $l_n(A) = k$, $k \leq 4$, for n large enough if and only if A is PI-equivalent to one of the following algebras:*

N, $C \oplus N$, $A_2 \oplus N$, $A_2^* \oplus N$, $A_2 \oplus A_2^* \oplus N$, $N_3 \oplus N$, $N_3 \oplus A_2 \oplus N$, $N_3 \oplus A_2^* \oplus N$, *where N is a nilpotent algebra and C is a commutative non-nilpotent algebra.*

In conclusion we have the following classification: for any algebra A and n large enough

1. $l_n(A) = 0$ if and only if $A \sim_{PI} N$.
2. $l_n(A) = 1$ if and only if $A \sim_{PI} C \oplus N$.
3. $l_n(A) = 2$ if and only if either $A \sim_{PI} A_2 \oplus N$ or $A \sim_{PI} A_2^* \oplus N$.
4. $l_n(A) = 3$ if and only if either $A \sim_{PI} A_2 \oplus A_2^* \oplus N$ or $A \sim_{PI} N_3 \oplus N$.
5. $l_n(A) = 4$ if and only if either $A \sim_{PI} N_3 \oplus A_2 \oplus N$ or $A \sim_{PI} N_3 \oplus A_2^* \oplus N$,

where N denotes a nilpotent algebra and C a commutative non-nilpotent algebra.

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