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Noncompact global attractors for scalar reaction–diffusion equations

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Abstract This paper considers a class of nondissipative reaction–diffusion equations. We are particularly interested in globally well-posed equations exhibiting blow-up in infinite time. These are known as slowly nondissipative equations. We review the recently developed theory for this class of problems, where a characterization for the associated noncompact global attractor is obtained. In addition, we derive an extension for the permutation realization result that holds for dissipative equations. The outlined results are then illustrated with an example. A brief discussion on the similarities with the dissipative case closes the paper.

Keywords Global attractors · Slowly non-dissipative equations · Heteroclinic orbits · Blow-up solutions

1 Introduction

In this survey we consider the following scalar reaction–diffusion equation with Neumann boundary conditions

Dedicated to Prof. Orlando Lopes on the occasion of his 70th birthday.

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$$\begin{cases} u_t = u_{xx} + f(x, u, u_x), \quad x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{cases} \tag{1}$$

where $f \in C^2$. It is known, from standard semigroup theory (see, for instance, [1,12, 15]), that Eq. (1) with initial condition

$$u(0, x) = u_0(x) \tag{2}$$

defines a (local) solution semigroup with u_0 in the appropriate state space.

A well established theory considers the case where f is dissipative. Sufficient conditions for f to be dissipative are the following:

$$f(x, u, 0) \cdot u < 0, \tag{3}$$

for $|u|$ large enough, and moreover,

$$|f(x, u, p)| \leq \left(1 + |p|^{\gamma}\right), \tag{4}$$

with $c > 0$ and $0 \leq \gamma < 2$, uniformly for x and u in compact sets. Under these conditions, (1), (2) generates a global semigroup on the phase space

$$X = H^2([0, \pi]) \cap \{u_x(0) = u_x(\pi) = 0\} \tag{5}$$

which admits a global attractor $\mathcal{A} \subset X$, that is, a nonempty compact invariant set attracting every bounded subset of X. See for instance [2,12,13].

It is also known that the semigroup associated with (1), (2) possesses a gradient-like structure. This is due to the existence of a Lyapunov function [16,22] for the semigroup. As a consequence, the global attractor is composed of the set of equilibria E for (1) and their heteroclinic connections

$$\mathcal{A} = E \cup \{\text{heteroclinic connections}\}.$$

If all the equilibria in E are hyperbolic, the gradient-like structure of the semigroup along with the dissipativity property implies that the global attractor $\mathcal{A}$ is also given by the union of the unstable manifolds of all the equilibria

$$\mathcal{A} = \cup_{e \in E} W^u(e).$$

The investigation of nondissipative equations of the form (1) exhibiting finite time blow-up is an ongoing topic of research. These are known as *fast nondissipative equations*. The remaining class of dynamical systems associated with Eq. (1) is composed by those which exhibit blow-up only in infinite time. We refer to this class as *slowly nondissipative equations*.

A positive linear growth of the nonlinearity is sufficient to ensure slowly nondissipativity of the equation. That is, by assuming that the nonlinearity in Eq. (1) is of the form

$$f(x,u,u_x) := bu + g(x,u,u_x), \tag{6}$$

where $b > 0$ and $g : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}$ is a bounded C^2 function, we obtain a slowly nondissipative equation. Indeed, the global existence of solutions is a consequence of the boundedness of the function g (see [15]). Moreover, we have the following result, obtained in [3].

Lemma 1.1 *Consider the Eq.* (1) *with initial condition* (2) *and nonlinearity f given by* (6). *If $b > 0$ then there exists at least one solution blowing-up in infinite time.*

In the context of slowly nondissipative equations, the existence of global solutions that are unbounded in t implies that one cannot obtain compactness for the attractor. In this case, the object to be considered is the nonempty minimal set which attracts all bounded sets in the state space. This is referred to as *noncompact global attractor*. Also, we refer to the solutions that blow-up in infinite time as *grow-up solutions*. Estimates for the behavior of grow-up solutions of (1), (6) in terms of the initial conditions were presented in [17].

In [3,4] Ben-Gal addresses slowly nondissipative equations with nonlinearities of the form (6) with $g = g(u)$. The general case $g = g(x,u,u_x)$ is considered by the authors in [18]. In the next two sections we outline the results obtained in these references for the characterization of the noncompact global attractor related to the equation. As in the dissipative case this characterization is combinatoric in nature as it uses a permutation defined on the set of equilibria. In Sect. 4 we extend to the slowly nondissipative case the permutation realization results obtained in the dissipative case, and characterize the corresponding noncompact global attractors. A specific example is considered in Sect. 5 and its noncompact global attractor is described. In the final section we present some comments and discuss the results.

2 Noncompact global attractors

Henceforward we will consider the slowly nondissipative equation

$$\begin{cases} u_t = u_{xx} + bu + g(x,u,u_x), & x \in [0,\pi] \\ u_x(t,0) = u_x(t,\pi) = 0. \end{cases} \tag{7}$$

We let $X = L^2([0,\pi])$ with norm $\|\cdot\|$ and consider the densely defined sectorial operator $A := -\partial_{xx} - bI$. Then the fractional power spaces associated with the operator $A_1 = A + (b+1)I$ are well-defined and given by

$$X^\alpha := D(A_1^\alpha),$$

for each $\alpha \geq 0$, with the graph norm $\|x\|_\alpha := \|A_1^\alpha x\|$, $x \in X^\alpha$.

Despite the nondissipativity, the Eq. (7) has a gradient-like structure. This is due to the existence of the Lyapunov function already mentioned. As a consequence we have the following result, derived in [3].

Lemma 2.1 *Any solution $u(t, x)$ of* (1)*,* (2) *either converges to some bounded equilibrium as t goes to infinity, or is a grow-up solution.*

By assuming that all the equilibria are hyperbolic, it also follows that the noncompact global attractor $\mathcal{A}_f$ for Eq. (7) can be characterized as the union of the unstable manifolds of all the equilibria

$$\mathcal{A}_f = \cup_{e \in E_f^c} W^u(e)$$

where E_f^c denotes the set of (bounded) equilibria of Eq. (7).

By analysing the asymptotic behavior of grow-up solutions it is obtained via Poincaré projection, as used in [14], the existence of some objects at infinity for which these unbounded solutions will limit to. These objects are defined as *equilibria at infinity*. These are given more precisely as follows. Consider an orthonormal basis $\{\varphi_j(\cdot)\}_{j\in\mathbb{N}}$ of $L^2([0, \pi])$ comprised of eigenvalues of the operator A with Neumann boundary conditions. It was obtained in [3] that, given a grow-up solution $u(t, \cdot)$, there exists $j \leq \sqrt{b}$ such that the normalized trajectory $\frac{u(t,\cdot)}{\|u(t,\cdot)\|}$ converges to either $\varphi_j(\cdot)$ or $-\varphi_j(\cdot)$. The projections to infinite norm of the functions $\{\pm\varphi_j(\cdot)\}_{0\leq j\leq\sqrt{b}}$, denoted by $\pm\Phi_j(\cdot)$ for each $0 \leq j \leq \sqrt{b}$, are the equilibria at infinity.

Therefore, the set of equilibria E_f of Eq. (7) is composed by the set of (bounded) equilibria E_f^c and the set of equilibria at infinity $E^\infty := \{\pm\Phi_0, \pm\Phi_1, \ldots, \pm\Phi_{[\sqrt{b}]}\}$, that is,

$$E_f = E_f^c \cup E_f^\infty \tag{8}$$

Under the setting of slowly nondissipative equations, grow-up solutions are interpreted as heteroclinics to equilibria at infinity. That is, if $u(t, \cdot)$ is a grow-up solution in the unstable manifold of $v \in E_f^c$, $W^u(v)$, and $\frac{u(t,\cdot)}{\|u(t,\cdot)\|}$ converges to $\iota\varphi_j(\cdot)$ with $\iota \in \{-1, +1\}$, we say that $u(t, \cdot)$ is a heteroclinic connection from v to the equilibrium at infinity $\iota\Phi_j(\cdot)$.

A bounded equilibrium $v \in E_f^c$ is *hyperbolic* if $\lambda = 0$ is not an eigenvalue of the linearization of (1) at v,

$$\begin{cases} \lambda w = w_{xx} + f_u(x, v, v_x)w + f_p(x, v, v_x)w_x, \quad x \in [0, \pi] \\ w_x(0) = w_x(\pi) = 0. \end{cases} \tag{9}$$

From here on we make the generic nondegeneracy assumption that all bounded equilibria $v \in E_f^c$ are hyperbolic. For the genericity of this assumption see [5,20]. This implies that $E_f^c = \{v_1, \ldots, v_n\}$ is a finite set. Moreover, in this case the *Morse index* $i(v)$ of $v \in E_f^c$ denotes the number of strictly positive eigenvalues of (9), alias the dimension of the unstable manifold of v, $W^u(v)$. We also assume the condition $b \neq n^2$. This additional assumption is a nondegeneracy condition corresponding to hyperbolicity of the equilibria at infinity, (see [18]).

For a continuous function u defined on $[0, \pi]$ the *zero number* $z(u)$ of u is the number of strict sign changes of u in $[0, \pi]$. Then, two equilibrium points v_1 and v_2 are said to be *adjacent* if there does not exist any other equilibrium v such that $z(v_2 - v) = z(v - v_1) = z(v_2 - v_1)$ with $v(0)$ strictly between $v_1(0)$ and $v_2(0)$, [21]. These two notions are essential for the characterization of global attractors in the dissipative setting. In the slowly nondissipative case, given the relation between the objects $\pm\Phi_j(\cdot)$ and the functions $\pm\varphi_j(\cdot)$, one can trivially extend the definition of zero numbers to the equilibria at infinity. Also, the notion of adjacency of equilibria can be naturally extended to this case. Then the following result, obtained in [18], provides a characterization of the noncompact global attractor for Eq. (7), described through the adjacency criterion.

Theorem 2.1 *The noncompact global $\mathcal{A}_f$ is given by*

$$\mathcal{A}_f = E_f \cup \{\textit{heteroclinic connections}\}.$$

Moreover, given $u, v \in E_f$ there exists a heteroclinic orbit connecting them if, and only if, they are adjacent. Also, the equilibrium with higher Morse index is the source of the connection.

3 Permutation characterization

In the dissipative case, the study of heteroclinic orbit connections in the global attractor of (1) was initiated in [6,7]. The existence of a permutation associated with Eq. (1) describing the dynamics on the global attractor was then established in [8]. This permutation $\sigma_f \in S_n$, where n is the number of equilibria and S_n denotes the symmetric group of permutations of degree n, is obtained from the ordering of the equilibria by their values at $x = 0$ and at $x = \pi$.

The equilibria $v \in E_f$ are solutions of the ODE boundary value problem

$$\begin{cases} u_{xx} + f(x, u, u_x) = 0, & x \in [0, \pi] \\ u_x = 0, \ x = 0, \pi. \end{cases} \tag{10}$$

We use the initial value problem corresponding to the ODE in (10) with Neumann initial conditions $(u(0), u_x(0)) = (u_0, 0)$ to define the meander curve γ_0 in the plane $(u, v) = (u, u_x) \in \mathbb{R}^2$,

$$\gamma_0 := \{(u(\pi), u_x(\pi)) : u_0 \in \mathbb{R}\}. \tag{11}$$

Then the equilibria $v \in E_f$ correspond to the intersections of the curve γ_0 with the curve

$$\gamma_1 := \{(u, v) : v = 0\}$$

in agreement with the Neumann conditions at $x = \pi$. We remark that in the Poincaré spherically extended plane both γ_0, γ_1 are Jordan curves, and the previous hyperbolicity assumption corresponds to transversality of the intersections $\gamma_0 \cap \gamma_1$.

In this view, the permutation σ_f corresponds to the ordering of the points of intersection $\gamma_0 \cap \gamma_1$ first along γ_0 and then along γ_1. In Sect. 5 we show some illustrations. Such permutations defined by the intersection of Jordan curves are called *meander permutations*. Also, a permutation $\sigma \in S_n$ is called *dissipative* if $\sigma(1) = 1$ and $\sigma(n) = n$.

A dissipative permutation $\sigma \in S_n$ is called *Morse* if the index vector $(i_m(\sigma))_{1 \le m \le n}$ defined by

$$
\begin{aligned}
i_1(\sigma) &= 0,\\
i_{m+1}(\sigma) &= i_m(\sigma) + (-1)^{m+1} \operatorname{sign}(\sigma^{-1}(m+1) - \sigma^{-1}(m)), \quad m = 1, \ldots, n-1,
\end{aligned}
$$

satisfies $i_j(\sigma) \ge 0$ for all $1 \le j \le n$.

The Morse indices of the equilibria $v_j \in E_f$, $j = 1, \ldots, n$, are determined by the permutation σ_f, (see [8,9,19]). In fact the Morse indices $i(v_j)$ are given by the index vector just defined,

$$
i(v_j) = i_j(\sigma_f), \quad j = 1, \ldots, n.
$$

Also, for any pair of distinct equilibria $v_j, v_k \in E_f$, the *intersection number* z_{jk} defined by the zero number of their difference, $z_{jk} = z(v_j - v_k)$, is determined by the permutation σ_f. Therefore, by the adjacency criterion, σ_f determines which equilibria are heteroclinically connected.

In [18] the authors extended this result to the context of slowly nondissipative equations (7) by defining a permutation which describes the dynamics on the associated noncompact global attractor. The result was obtained through a suspension technique applied to the nondissipative permutation obtained from the usual ordering of the equilibria.

The concept of k-suspension of a permutation $\sigma \in S_n$ is introduced in [18] and is given as follows.

Definition 3.1 Let k be a positive integer. For any permutation $\sigma \in S_n$ the k-suspension of σ, denoted by σ^k, is defined as the permutation in S_{n+2} satisfying:

1. $\sigma^k(j) = \sigma(j-1) + 1$ for $j \in \{2, \ldots, n+1\}$;
2. and

$$
\sigma^k(1) = 1 \text{ and } \sigma^k(n+2) = n+2, \quad \text{if } k \text{ is odd} \tag{12}
$$
$$
\sigma^k(1) = n+2 \text{ and } \sigma^k(n+2) = 1, \quad \text{if } k \text{ is even.} \tag{13}
$$

Let σ_f be the usual permutation in S_n related to (7), defined by the ordering of the equilibria at $x = 0$ and $x = \pi$. Also let $k = 1 + [\sqrt{b}]$. By successively computing the suspensions

$$
\hat{\sigma}_j := (\hat{\sigma}_{j-1})^{k-j} \quad \text{for } j = 1, \ldots, k,
$$

with $\hat{\sigma}_0 := \sigma_f$, we obtain a permutation $\hat{\sigma}_k \in S_{n+2k}$. Then, the kth suspension $\hat{\sigma}_k$ of σ_f is a dissipative Morse meander permutation. This ensures that $\hat{\sigma}_k$ is realizable by a problem (1) with a dissipative nonlinearity $\hat{f}$, that is, $\hat{\sigma}_k = \sigma_{\hat{f}}$. See [10]. Moreover, $\hat{\sigma}_k$ determines completely the Morse indices and intersection numbers of the equilibria in $E_f = E_f^c \cup E_f^\infty$, see [18] for details. For the equilibria at infinity this follows from the extended interpretation of the nodal properties for such objects.

As a consequence, the suspension $\hat{\sigma}_k$ determines the noncompact global attractor $\mathcal{A}_f$ associated with Eq. (7). By considering $\hat{\sigma}_k$ one can answer the question of whether or not two equilibrium points in E_f, bounded or at infinity, are connected by a heteroclinic orbit.

4 Realization of nondissipative permutations

A simple adjustment of the permutation characterization of Sturm global attractors described in [10] for the dissipative case leads to a characterization of the permutations realizable by slowly nondissipative nonlinearities (6).

As in the dissipative case we assume hyperbolicity of all the equilibria and consider the second order boundary value problem corresponding to the stationary problem,

$$u_{xx} + f(x, u, u_x) = 0, \quad u_x(0) = u_x(\pi). \tag{14}$$

Then the orderings of the stationary solutions $v_1, \ldots, v_n$ of (14) according to their values at $x = 0$ and at $x = \pi$ define the permutation $\sigma \in S_n$ and we say that σ is realizable by a slowly nondissipative nonlinearity (6).

For $b > 0$ let $k = 1 + [\sqrt{b}]$. A permutation $\sigma \in S_n$ with odd n is called a *k-nondissipative permutation* if

$$\sigma(1) = 1, \quad \sigma(n) = n \quad \text{if } k \text{ is odd,}$$

and

$$\sigma(1) = n, \quad \sigma(n) = 1 \quad \text{if } k \text{ is even.}$$

The index vector $(i_m(\sigma))_{1 \leq m \leq n}$ of a k-nondissipative permutation is given by

$$\begin{aligned} i_1(\sigma) &= k, \\ i_{m+1}(\sigma) &= i_m(\sigma) + (-1)^{m+1+k} \operatorname{sign}(\sigma^{-1}(m+1) - \sigma^{-1}(m)), \quad m = 1, \ldots, n-1. \end{aligned}$$

Then a k-nondissipative permutation $\sigma \in S_n$ is called *Morse* if $i_m(\sigma) \geq 0$ for all $1 \leq m \leq n$. Notice that we always have $k \geq 1$. We remark that if we let $k = 0$, then we obtain the usual definitions in the dissipative case.

Then we have the following characterization of the permutations realizable by boundary value problems corresponding to slowly nondissipative problems of the form (7).

Theorem 4.1 *A permutation $\sigma \in S_n$ is realizable by a slowly nondissipative nonlinearity of the form* (6) *if and only if n is odd and σ is a k-nondissipative Morse meander permutation with $k = 1 + [\sqrt{b}]$.*

For dissipative realizable permutations, it is known that the index vector determines the Morse indices of the equilibria. The following result, derived in [18], shows that the same statement holds true for k-nondissipative realizable permutations.

Lemma 4.1 *Let $\sigma \in S_n$ be a realizable k-nondissipative permutation and $\{v_1, \ldots, v_n\}$ the set of bounded equilibria of the associated slowly nondissipative equation. Then the Morse indices of the equilibria v_j are respectively given by*

$$i(v_j) = i_j(\sigma), \quad \text{for } j = 1, \ldots, n. \tag{15}$$

5 A planar noncompact global attractor

In this section we illustrate the theory for slowly nondissipative systems with a specific example. As a consequence of Theorem 4.1 it is sufficient to consider a permutation realizable by a slowly nondissipative equation. We choose

$$\sigma = \{5, 2, 3, 4, 1\},$$

or in cycle notation $\sigma = (1\ 5) \in S_5$. For even positive k this σ is a k-nondissipative permutation with index vector $(i_m(\sigma))_{1\leq m\leq 5}$ given by

$$(i_m(\sigma))_{1\leq m\leq 5} = (k, k-1, k-2, k-1, k). \tag{16}$$

Then σ is realizable by a slowly nondissipative nonlinearity of the form (6) with an odd value of $[\sqrt{b}]$. Henceforward we let $1 < b < 4$, for which we have $k = 2$.

For our system, generated by Eqs. (6), (7) with $\sigma_f = \sigma$, the set of bounded equilibria and the equilibria at infinity are given by

$$E_f^c = \{v_1, \ldots, v_5\}, \quad E_f^\infty = \{-\Phi_0, -\Phi_1, \Phi_1, \Phi_0\}.$$

We compute the Morse indices of the equilibria in E_f^c using (15). Then, (16) implies that v_3 is stable, $i(v_3) = 0$. Moreover, $i(v_2) = i(v_4) = 1$ and $i(v_1) = i(v_5) = 2$ which implies that the noncompact global attractor $\mathcal{A}_f$ is planar. The existence of heteroclinic orbit connections between all the equilibria in $E_f = E_f^c \cup E_f^\infty$ follows from Theorem 2.1. The needed adjacency relations are determined by the suspension procedure discussed in Sect. 3. Beginning with $\hat{\sigma}_0 = \sigma$, we have

$$\hat{\sigma}_1 = \{7, 6, 3, 4, 5, 2, 1\},$$
$$\hat{\sigma}_2 = \{1, 8, 7, 4, 5, 6, 3, 2, 9\}.$$

The suspension $\hat{\sigma}_2 = (2\,8)(3\,7) \in S_9$, illustrated in Fig. 1, then provides the adjacency relations between the equilibria in E_f. This second suspension of σ is realizable by a dissipative problem with a set of equilibria

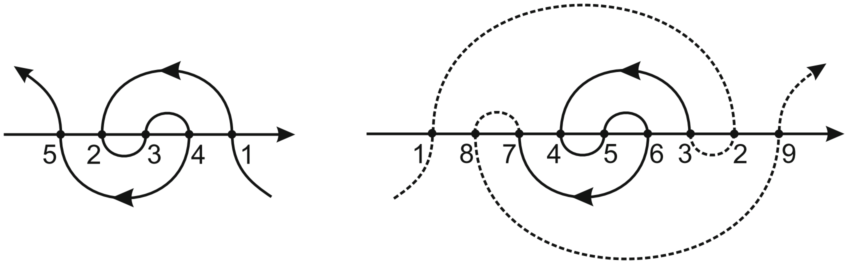

Fig. 1 Canonical representation of meander curves: (*left*) the permutation $\sigma = \{5, 2, 3, 4, 1\}$; (*right*) its second suspension $\hat{\sigma}_2 = \{1, 8, 7, 4, 5, 6, 3, 2, 9\}$

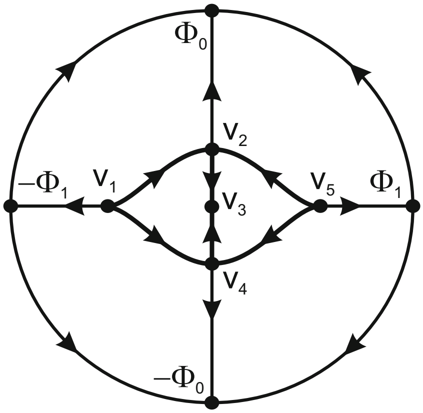

Fig. 2 Noncompact global attractor with permutation $\sigma_f = \{5, 2, 3, 4, 1\}$

$$\{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\}$$

such that $w_{j+2} = v_j$ for $j = 1, \dots, 5$, and w_1, w_2, w_8, w_9, (with very large norms) play the role of $-\Phi_0, -\Phi_1, \Phi_1, \Phi_0$, respectively. Computing the adjacency relations made available by $\hat{\sigma}_2$ then implies the following heteroclinic connections (denoted by $\rightsquigarrow$) between Morse adjacent equilibria, (see for example [9])

$$w_3 \rightsquigarrow w_4, w_6, w_2,\ \ w_7 \rightsquigarrow w_4, w_6, w_8,\ \ w_4 \rightsquigarrow w_5, w_9,\ \ \ w_6 \rightsquigarrow w_5, w_1,$$
$$w_2 \rightsquigarrow w_1, w_9,\ \ w_8 \rightsquigarrow w_1, w_9.$$

The resulting heteroclinic orbit connections in the noncompact global attractor $\mathcal{A}_f$, as illustrated in Fig. 2, are the following

$$v_1 \rightsquigarrow v_2, v_4, -\Phi_1,\ \ v_5 \rightsquigarrow v_2, v_4, \Phi_1\ ,\ \ v_2 \rightsquigarrow v_3, \Phi_0,\ \ \ v_4 \rightsquigarrow v_3, -\Phi_0,$$
$$-\Phi_1 \rightsquigarrow -\Phi_0, \Phi_0,\ \ \Phi_1 \rightsquigarrow -\Phi_0, \Phi_0.$$

By the transitivity of $\rightsquigarrow$ we then obtain the complete heteroclinic orbit connectivity in $\mathcal{A}_f$.

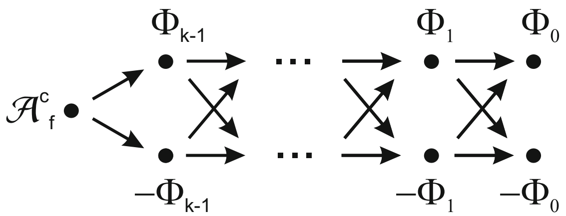

Fig. 3 The connection graph of grow-up solutions in the noncompact global attractor $\mathcal{A}_f$ shows a Chafee–Infante structure of the heteroclinic connections between equilibria at infinity E^∞_f

The noncompact global attractor $\mathcal{A}_f$ contains a compact invariant subset $\mathcal{A}^c_f$ composed of the set of bounded equilibria E^c_f and the heteroclinic orbits connecting them. The information regarding the set of equilibria at infinity, E^∞_f, and the heteroclinic orbits connecting them, essentially derives from the linear equation corresponding to (7) with $g = 0$. In fact, in the general case $k = 1 + [\sqrt{b}]$, we have that $E^\infty_f = \{\pm\Phi_0, \ldots, \pm\Phi_{k-1}\}$, has $2k$ equilibria. Moreover, their heteroclinic connections resemble a k-dimensional Chafee–Infante global attractor (see for example [9]) with the unstable origin replaced by the suspended compact invariant set $\mathcal{A}^c_f$, as illustrated in Fig. 3.

6 Discussion

The results obtained for the slowly nondissipative equation (7) should be compared with the well-established theory of dissipative scalar reaction–diffusion equations. As shown in [18], a k-nondissipative Morse meander permutation can be suspended k times to produce a dissipative Morse meander permutation which ensures that the suspended permutation is realizable by a problem (1) with a dissipative nonlinearity. This realization provides information on the heteroclinic orbit connections between all the equilibria, bounded and at infinity, in the noncompact global attractor of the slowly nondissipative problem (7) realizing the original k-nondissipative permutation. Therefore, the similarities between the characterization results for the global attractors of dissipative and slowly nondissipative scalar reaction–diffusion equations are not surprising.

Taking into account the extended notion of equilibria and the interpretation of grow-up solutions as transfinite heteroclinics, the combinatorial characterizations of both compact and noncompact global attractors in terms of permutations of the equilibria are entirely similar.

The consequences of the permutation characterization of global attractors can also be extended to the noncompact case. It is known from [11] that global attractors in the dissipative case are characterized by the corresponding permutations up to orbit equivalence. More precisely, dissipative equations for which the related permutations coincide, possess globally C^0 orbit equivalent attractors. We recall that two compact global attractors $\mathcal{A}_{f_1}$ and $\mathcal{A}_{f_2}$ are *globally* C_0 *orbit equivalent*, $\mathcal{A}_{f_1} \cong \mathcal{A}_{f_2}$, if there exist a homeomorphism $h : \mathcal{A}_{f_1} \to \mathcal{A}_{f_2}$ which maps f_1-orbits onto f_2-orbits, preserving the time direction of the flows, see [11].

A similar result holds true for the slowly nondissipative equation (7) if we restrict attention to the compact invariant subset $\mathcal{A}^c_f$. Given two nonlinearities f_1 and f_2 of the form (6), i.e.

$$f_1(x,u,u_x) = b_1 u + g_1(x,u,u_x), \;\; f_2(x,u,u_x) = b_2 u + g_2(x,u,u_x),$$

with g_1 and g_2 bounded, the noncompact global attractors $\mathcal{A}_{f_1}$ and $\mathcal{A}_{f_2}$ of (7) have the same connection graphs of grow-up solutions if $[\sqrt{b_1}] = [\sqrt{b_2}]$, (see Fig. 3). Then, if the k-nondissipative permutations σ_{f_1} and σ_{f_2} coincide, the global attractors associated with the dissipative permutations $\hat{\sigma}_k^{f_1}$ and $\hat{\sigma}_k^{f_2}$ are globally C^0 orbit equivalent. This leads to the C^0 orbit equivalence of the compact invariant subsets $\mathcal{A}^c_{f_1}$ and $\mathcal{A}^c_{f_2}$.

More general classes of nonlinearities generating slowly nondissipative systems have not yet been approached. Since the first eigenvalue of the operator $-\partial_x^2$ with Neumann boundary conditions is $\lambda_1 = 0$, the following condition on the nonlinearity $f = f(u)$ produces scalar reaction–diffusion equations with grow-up solutions but without blow-up in finite time,

$$0 < \limsup_{|u|\to\infty} \frac{f(u)}{u} < \infty. \tag{17}$$

It is not difficult to generalize condition (17) to obtain slowly nondissipative equations with more general nonlinearities or boundary conditions. We believe that by analysing the asymptotic behavior of nondissipative equations (1) with more general nonlinearites will provide a larger set of grow-up behaviors and a better understanding of the dynamics for this class of dynamical systems.

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