São Paulo J. Math. Sci. (2016) 10:180–218
DOI 10.1007/s40863-015-0013-5

Relativization, absolutization, and latticization in Ring and Module Theory

Toma Albu[1]

Published online: 6 October 2015
© Instituto de Matemática e Estatística da Universidade de São Paulo 2015

Abstract In this survey paper we illustrate a general strategy which consists on putting a module-theoretical result into a latticial frame (we call it *latticization*), in order to translate that result to Grothendieck categories (we call it *absolutization*) and module categories equipped with hereditary torsion theories (we call it *relativization*). The renowned Hopkins–Levitzki Theorem and Osofsky–Smith Theorem from Ring and Module Theory, we will discuss in the last two sections of the paper, are among the most relevant illustrations of the power of this strategy.

Keywords Hereditary torsion theory · τ-Artinian module · τ-Noetherian module · Quotient category · Localization · Grothendieck category · The Gabriel–Popescu Theorem · Modular lattice · Upper continuous lattice · Artinian lattice · Noetherian lattice · The Hopkins–Levitzki Theorem · The Osofsky–Smith Theorem · Krull dimension

Mathematics Subject Classification 16-02 · 16D90 · 16S90 · 18E15 · 18E40 · 06C05

1 Introduction

The main purpose of this survey paper is to illustrate a ***general strategy*** which consists on putting a *module-theoretical* result into a *latticial frame* (we call it *latticization*), in

✉ Toma Albu
Toma.Albu@imar.ro

1 Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit 5,
P.O. Box 1 - 764, 010145 Bucharest 1, Romania

order to translate that result to *Grothendieck categories* (we call it *absolutization*) and *module categories* equipped with *hereditary torsion theories* (we call it *relativization*).

More precisely, if $\mathbb{P}$ is a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories, our strategy consists of the following three steps:

I. *Translate/formulate,* if possible, the problem $\mathbb{P}$ into a *latticial setting.*
II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame.*
III. *Back to basics,* i.e., to Grothendieck categories and module categories equipped with hereditary torsion theories.

This approach is very natural and simple, because we ignore the specific context of Grothendieck categories and module categories equipped with hereditary torsion theories, focusing only on those latticial properties which are relevant to our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The renowned *Hopkins–Levitzki Theorem* and *Osofsky–Smith Theorem* from Ring and Module Theory are among the most relevant illustrations of the power of this strategy.

In Sect. 2 we explain what is *Relativization.* First, we briefly present the concept of a hereditary torsion theory, with a great emphasis on the lattice $\mathrm{Sat}_\tau(M_R)$ of all τ-saturated submodules of a right module M_R, where $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on the category Mod-R of all right modules over a unital ring R. Next, we present the Hopkins–Levitzki Theorem, abbreviated H-LT, we call Classical H-LT, saying that any right Artinian ring with identity is right Noetherian, or equivalently, any Artinian right R-module over a right Artinian ring R with identity is Noetherian. Then, we define the concepts of a τ-Artinian and τ-Noetherian module and present the Relative H-LT, which is obtained by replacing in the Classical H-LT all occurrences “Artinian” and “Noetherian” with “τ-Artinian” and “τ-Noetherian”, respectively.

Section 3 discusses *Absolutization,* which consists on putting a module-theoretical property/result into a categorical setting in a Grothendieck category. We briefly recall the definition of a Grothendieck category and the construction of the quotient category Mod-$R/\mathcal{T}$ of Mod-R modulo any of its Serre subcategories $\mathcal{T}$. This quotient category turns out to be a Grothendieck category when the given Serre subcategory $\mathcal{C}$ is closed under arbitrary direct sums, i.e., is a localizing subcategory of Mod-R. We present and explain then the statement of the renowned *Gabriel–Popescu Theorem*, which roughly says that all Grothendieck categories are obtained in this way, up to a category equivalence.

In Sect. 4 we discuss the main features of a third procedure in Module Theory we call *Latticization.* This consists on translating/formulating, if possible, the terms appearing in a problem $\mathbb{P}$ involving subobjects or submodules, to be studied in Grothendieck categories or in module categories with respect to hereditary torsion theories, into a latticial setting, and then, on investigating the obtained problem $\mathbb{P}$ in this latticial frame. The best illustration of this procedure is to place both the Relative H-LT and Absolute H-LT into a latticial frame in order to obtain a general Latticial H-LT, which gives an exhaustive answer to the following natural question: *When an arbitrary Artinian modular lattice L is Noetherian?* Applying the obtained result to the opposite lattice L^o of L we answer immediately the dual question: *When an arbitrary Noetherian modular lattice L is Artinian?*

The aim of Sect. 5 is two-fold: firstly, to provide all the connections between the Classical, Relative, Absolute, and Latticial H-LT discussed in the previous sections, and secondly, to present other aspects of the H-LT including the Faith's Δ-Σ and counter versions of the Relative H-LT, the Dual H-LT, as well as a Krull dimension-like H-LT. Let us mention that the only two module-theoretical proofs available in the literature of the Relative H-LT, due to Miller and Teply [41] and Faith [30], are very long and complicated. We show in a unified manner that this result, as well as the Absolute H-LT, are immediate consequences of the Latticial H-LT, whose proof is very short and simple, illustrating thus the power of our main strategy explained above.

Section 6 is devoted to another famous theorem in Module Theory, the *Osofsky–Smith Theorem,* abbreviated O-ST, saying that a finitely generated (respectively, cyclic) right R-module such that all of its finitely generated (respectively, cyclic) subfactors are CS modules is a finite direct sum of uniform submodules. We present a sketch of the proof of the latticial counterpart of this theorem, and then apply it to derive immediately the Categorical (or Absolute) O-ST and the Relative O-ST. We believe that the reader will be once more convinced of the power of our strategy when extending some important results of Module Theory to Grothendieck categories and to module categories equipped with hereditary torsion theories by passing first through their latticial counterparts.

2 Relativization

The aim of this section is to illustrate through the Relative Hopkins–Levitzki Theorem a general direction in Module Theory which, roughly speaking, deals with the investigation of properties of submodules of a module M_R in the lattice $\mathrm{Sat}_\tau(M_R)$ of all τ-saturated submodules of M_R for a hereditary torsion theory τ on Mod-R. We call this procedure *Relativization.*

2.1 Hereditary torsion theories

The concept of *torsion theory* for Abelian categories has been introduced by S.E. Dickson [27] in 1966. For our purposes, we present it only for module categories in one of the many equivalent ways that can be done.

All rings considered in this paper are associative with unit element, and all modules are unital right modules. If R is a ring, then Mod-R denotes the category of all right R-modules. We often write M_R to emphasize that M is a right R-module, and $\mathcal{L}(M_R)$, or just $\mathcal{L}(M)$, stands for the lattice of all submodules of M_R. The notation $N \leqslant M$ will mean that N is a submodule of M.

A *hereditary torsion theory* on Mod-R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of non-empty subclasses $\mathcal{T}$ and $\mathcal{F}$ of Mod-R such that $\mathcal{T}$ is a *localizing subcategory* of Mod-R in the Gabriel's sense [31] and

$$\mathcal{F} = \{ F_R \mid \mathrm{Hom}_R(T, F) = 0,\ \forall\, T \in \mathcal{T} \}.$$

Thus, any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is uniquely determined by its first component $\mathcal{T}$.

Recall that a *localizing subcategory* of Mod-R is a Serre class of Mod-R which is closed under direct sums. By a *Serre class* (or *Serre subcategory*) of Mod-R we mean a non-empty subclass $\mathcal{T}$ of Mod-R such for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

in Mod-R, one has

$$X \in \mathcal{T} \Longleftrightarrow X' \in \mathcal{T} \ \ \& \ \ X'' \in \mathcal{T}.$$

We say that $\mathcal{T}$ is *closed under direct sums* if for any family $(X_i)_{i\in I}$, I arbitrary set, with $X_i \in \mathcal{T}$, $\forall\, i \in I$, it follows that $\bigoplus_{i\in I} X_i \in \mathcal{T}$.

The prototype of a hereditary torsion theory is the pair $(\mathcal{A}, \mathcal{B})$ in Mod-$\mathbb{Z}$, where $\mathcal{A}$ is the class of all torsion Abelian groups and $\mathcal{B}$ is the class of all torsion-free Abelian groups.

If I is a right ideal of a unital ring R, M is a right R-module, $r \in R$, and $x \in M$, then we denote

$$(I : r) := \{\, a \in R \mid ra \in I \,\} \ \text{ and } \ \mathrm{Ann}_R(x) := \{\, a \in R \mid xa = 0 \,\}.$$

A (right) *Gabriel filter* (or *Gabriel topology*) on R is a non-empty set F of right ideals of R satisfying the following two conditions:

- If $I \in F$ and $r \in R$, then $(I : r) \in F$;
- If I and J are right ideals of R such that $J \in F$ and $(I : r) \in F$ for all $r \in J$, then $I \in F$.

Each Gabriel filter F on R defines two classes of right R-modules

$$\mathcal{T}_F := \{\, M_R \mid \mathrm{Ann}_R(x) \in F,\ \forall\, x \in M \,\}$$

and

$$\mathcal{F}_F := \{\, M_R \mid \mathrm{Ann}_R(x) \notin F,\ \forall\, x \in M,\ x \neq 0 \,\},$$

and the pair $(\mathcal{T}_F, \mathcal{F}_F)$ is a hereditary torsion theory on Mod-R. Conversely, to any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R we can associate the Gabriel filter

$$F_\tau := \{\, I \leqslant R_R \mid R/I \in \mathcal{T} \,\}.$$

It is well-known that the assignment $F \longmapsto (\mathcal{T}_F, \mathcal{F}_F)$ establishes a bijective correspondence between the set of all (right) Gabriel filters on R and the class of all hereditary torsion theories on Mod-R, with inverse correspondence given by $\tau \longmapsto F_\tau$ (see, e.g., [50, Chapter VI, Theorem 5.1]). In particular, the class of all hereditary torsion theories on Mod-R is actually a set.

Throughout this section $\tau = (\mathcal{T},\ \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R. For any module M_R we denote

$$\tau(M) := \sum_{N \leqslant M,\, N \in \mathcal{T}} N.$$

Since $\mathcal{T}$ is a localizing subcategory of Mod-R, we have $\tau(M) \in \mathcal{T}$, and we call it the *τ-torsion submodule* of M. Note that, as for Abelian groups, we have

$$(M \in \mathcal{T} \iff \tau(M) = M) \quad \text{and} \quad (M \in \mathcal{F} \iff \tau(M) = 0).$$

The members of $\mathcal{T}$ are called *τ-torsion modules*, while the members of $\mathcal{F}$ are called *τ-torsion-free modules*.

For further basic torsion-theoretic notions and results the reader is referred to [32] and/or [50].

2.2 The lattice $\mathrm{Sat}_{\tau(M)}$

For any M_R and any $N \leqslant M$ we denote

$$\mathrm{Sat}_\tau(M) := \{\, N \mid N \leqslant M,\ M/N \in \mathcal{F}\,\},$$

and call

$$\overline{N} := \bigcap \{\, C \mid N \leqslant C \leqslant M,\ M/C \in \mathcal{F}\,\}$$

the *τ-saturation* of N in M. We say that N is *τ-saturated* if $N = \overline{N}$. Note that $\overline{N}/N = \tau(M/N)$ and

$$\mathrm{Sat}_\tau(M) = \{\, N \mid N \leqslant M,\ N = \overline{N}\,\},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all τ-saturated submodules of M, which explains the notation. Clearly, $\mathrm{Sat}_\tau(M)$ is a non-empty subset of the partially ordered set $\mathcal{L}(M)$ of all submodules of M ordered by inclusion $\subseteq$.

For any family $(N_i)_{i \in I}$ of elements of $\mathrm{Sat}_\tau(M)$ we set

$$\bigvee_{i \in I} N_i := \overline{\sum_{i \in I} N_i} \qquad \text{and} \qquad \bigwedge_{i \in I} N_i := \bigcap_{i \in I} N_i.$$

By [50, Chapter IX, Proposition 4.1], the set $\mathrm{Sat}_\tau(M)$ of all τ-saturated submodules of M is an upper continuous modular lattice with respect to the inclusion $\subseteq$ and the operations $\bigvee$ and $\bigwedge$ defined above, and with least element $\tau(M)$ and greatest element M.

Note that though $\mathrm{Sat}_\tau(M)$ is a subset of the lattice $\mathcal{L}(M)$ of all submodules of M, it is not a sublattice, because the sum of two τ-saturated submodules of M is not necessarily τ-saturated.

Definition A module M_R is said to be *τ-Noetherian* (resp. *τ-Artinian*) if $\mathrm{Sat}_\tau(M)$ is a Noetherian (resp. Artinian) poset. The ring R is said to be right *τ-Noetherian* (resp. *τ-Artinian*) if the module R_R is *τ-Noetherian* (resp. *τ-Artinian*). □

Recall that a partially ordered set, shortly poset, $(P, \leqslant)$ is called *Noetherian* (resp. *Artinian*) if it satisfies the ACC (resp. DCC), i.e., if there is no strictly ascending (resp. descending) chain $x_1 < x_2 < \ldots$ (resp. $x_1 > x_2 > \ldots$) in P.

We end this subsection with some basic properties of the lattice $\mathrm{Sat}_\tau(M)$ that will be used later.

Lemma 2.1 ([7, Lemmas 3.4.2 and 3.4.4]) *The following statements hold for a module M_R and submodules $P \subseteq N$ of M_R.*

(1) *The mapping*

$$\alpha : \mathrm{Sat}_\tau(N/P) \longrightarrow \mathrm{Sat}_\tau(\overline{N}/\overline{P}),\ \ X/P \mapsto \overline{X}/\overline{P},$$

is a lattice isomorphism.

(2) $\mathrm{Sat}_\tau(N) \simeq \mathrm{Sat}_\tau(\overline{N})$.

(3) *If $N \in \mathcal{T}$, then* $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/N)$.

(4) *If $M/N \in \mathcal{T}$, then* $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(N)$.

(5) *If $N, P \in \mathrm{Sat}_\tau(M)$, then the assignment $X \mapsto X/P$ defines a lattice isomorphism from the interval $[P, N]$ of the lattice* $\mathrm{Sat}_\tau(M)$ *onto the lattice* $\mathrm{Sat}_\tau(N/P)$. □

2.3 The Classical Hopkins–Levitzki Theorem

One of the most lovely results in Ring Theory is the *Hopkins–Levitzki Theorem*, abbreviated H-LT. This theorem, saying that any right Artinian ring with identity is right Noetherian, has been proved independently in 1939 by *Charles Hopkins* [36] (1902–1939) for left ideals and by *Jacob Levitzki* [39](1904–1956) for right ideals. More details about the history of this theorem may be found in [4].

An equivalent form of the H-LT, referred in the sequel also as the *Classical H-LT*, is the following.

Theorem 2.2 (Classical H-LT) *If R be a right Artinian ring with identity, then any Artinian right module is Noetherian.* □

2.4 The Relative H-LT

The next result is due to Albu and Năstăsescu [14, Théorème 4.7] for commutative unital rings, conjectured for non commutative rings by Albu and Năstăsescu [14, Problème 4.8], and proved for arbitrary unital rings by Miller and Teply [41, Theorem 1.4].

Theorem 2.3 (Relative H-LT) *Let R be a ring with identity, and let τ be a hereditary torsion theory on* Mod-*R. If R is a right τ-Artinian ring, then every τ-Artinian right R-module is τ-Noetherian.* □

The importance of the Relative H-LT in investigating the structure of some relevant classes of modules, including injectives as well as projectives, is revealed in [15] and [30], where the main body of both these monographs deals with this topic.

2.5 Relativization

The Relative H-LT nicely illustrates a general direction in Module Theory, namely the so called *Relativization.* Roughly speaking, this topic deals with the following matter:

Given a property $\mathbb{P}$ *in the lattice* $\mathcal{L}(M_R)$ *investigate the property* $\mathbb{P}$ *in the lattice* $\mathrm{Sat}_\tau(M_R)$.

Since more than forty years module theorists were dealing with the following problem:

Having a theorem $\mathbb{T}$ *on modules, is its relativization* τ-$\mathbb{T}$ *true?*

Notice that the module-theoretical proofs available in the literature of the Relative H-LT, namely the original one in 1979 due to Miller and Teply [41, Theorem 1.4] and another one in 1982 due to Faith [30, Theorem 7.1 and Corollary 7.2], are very long and complicated, so, the relativization of a result on modules is not always a simple job, and as this will become immediately clear, sometimes it may be even impossible.

Indeed, consider the following nice result of Lenagan [38, Theorem 3.2]:

$$\mathbb{T}: \textit{ If } R \textit{ has right Krull dimension then its prime radical } \mathrm{N}(R) \textit{ is nilpotent}.$$

The relativization of $\mathbb{T}$ is the following:

$$\tau\text{-}\mathbb{T}: \textit{ If } R \textit{ has right } \tau\textit{-Krull dimension then its } \tau\textit{-prime radical } \mathrm{N}_\tau(R) \textit{ is } \tau\textit{-nilpotent}.$$

Recall that $\mathrm{N}_\tau(R)$ is the intersection of all τ-saturated two-sided prime ideals of R, and a right ideal I of R is said to be τ-nilpotent if $I^n \in \mathcal{T}$ for some positive integer n. For the concept of Krull dimension of rings, modules, and posets, see Sect. 5.

The truth of the relativization τ-$\mathbb{T}$ of $\mathbb{T}$ has been asked by Albu and Smith [16, Problem 4.3]. Surprisingly, the answer is "no" in general, even if R is (left and right) Noetherian, by [11, Example 3.1].

However, τ-$\mathbb{T}$ is true for any ring R and any *ideal invariant* hereditary torsion theory τ, including any commutative ring R and any τ (see [11, Section 6]).

3 Absolutization

By taking as pattern the *Absolute Hopkins–Levitzki Theorem,* we shall discuss in this section another general direction in Module Theory called *Absolutization.*

The reader is referred to [15,31], and [50] for the concepts, constructions, and facts on Grothendieck categories presented in this section.

3.1 The Absolute Hopkins–Levitzki Theorem

The next result is due to Năstăsescu, who actually gave two different short nice proofs: [43, Corollaire 1.3], based on the Loewy length, and [44, Corollaire 2], based on the length of a composition series.

Theorem 3.1 (ABSOLUTE H-LT) *Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator. Then any Artinian object of $\mathcal{G}$ is Noetherian.* □

Recall that a *Grothendieck category* is an Abelian category $\mathcal{G}$ with exact direct limits (or, equivalently, satisfying the axiom AB5 of Grothendieck) and having a generator G (this means that for every object X of $\mathcal{G}$ there exist a set I and an epimorphism $G^{(I)} \twoheadrightarrow X$). Also, recall that an object $X \in \mathcal{G}$ is said to be *Noetherian* (respectively, *Artinian*) if the lattice $\mathcal{L}(X)$ of all subobjects of X is Noetherian (respectively, Artinian).

3.2 Quotient categories

Clearly, for any ring R with identity element, the category Mod-R is a Grothendieck category. A procedure to construct new Grothendieck categories is by taking the *quotient category* Mod-$R/\mathcal{T}$ of Mod-R modulo any of its localizing subcategories $\mathcal{T}$. This construction is quite complicated and goes back to Serre's "langage modulo $\mathcal{T}$" (1953), Grothendieck (1957), and Gabriel (1962) [31].

We are now going to present the construction of the quotient category. We shall perform it starting with Mod-R, but it can be done "mutatis mutandis" for any locally small Abelian category instead of Mod-R. Recall that a category $\mathcal{C}$ is said to be *locally small* if the class Sub(X) of all subobjects of X is a set for each $X \in \mathcal{C}$, and in this case, it is actually a poset.

Let $\mathcal{T}$ be an arbitrary *Serre subcategory* of Mod-R. We shall construct a new category called the *quotient category* of Mod-R *modulo* $\mathcal{T}$ and denoted by Mod-$R/\mathcal{T}$. This category is expected to have similar properties with that of a quotient module; so, Mod-$R/\mathcal{T}$ should be an Abelian category equipped with a covariant exact functor

$$T : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

such that T "kills" each $X \in \mathcal{T}$ (this means that $T(X) = 0,\ \forall\, X \in \mathcal{T}$), and moreover, T should be universal with these properties.

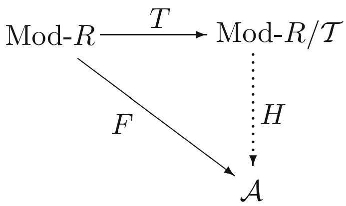
More precisely, we want to construct for the given Serre subcategory $\mathcal{T}$ a pair (Mod-$R/\mathcal{T}$, T), where Mod-$R/\mathcal{T}$ is an Abelian category and

$$T : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

is a covariant exact functor, such that $T(X) = 0,\ \forall\, X \in \mathcal{T}$, and such that, for any Abelian category $\mathcal{A}$ and for any exact covariant functor

$$F : \text{Mod-}R \longrightarrow \mathcal{A}$$

with $F(X) = 0,\ \forall X \in \mathcal{T}$, there exists a unique functor H making commutative the diagram:



The construction of the *quotient category* Mod-$R/\mathcal{T}$ of Mod-R *modulo* $\mathcal{T}$ is the following.

The objects of the category Mod-$R/\mathcal{T}$ are the same as those of Mod-R, i.e.,

$$\mathrm{Obj}\,(\text{Mod-}R/\mathcal{T}) \;:=\; \mathrm{Obj}\,(\text{Mod-}R),$$

while the morphisms in this category are more complicatedly to be defined.

For any $M,\ N \in$ Mod-R denote

$$I_{M,N} \;:=\; \{\,(M', N')\,|\,M' \leqslant M,\ N' \leqslant N,\ M/M' \in \mathcal{T},\ N' \in \mathcal{T}\,\},$$

and define the following order relation in $I_{M,N}$:

$$(M', N') \;\preceq\; (M'', N'') \iff M'' \leqslant M' \text{ and } N' \leqslant N''.$$

Clearly, $I_{M,N}$ is a directed set.

Define now for $M,\ N \in$ Mod-R

$$\mathrm{Hom}_{\text{Mod-}R/\mathcal{T}}\,(M, N) \;:=\; \varinjlim_{(M',N')\in I_{M,N}} \mathrm{Hom}_R\,(M', N/N').$$

Theorem 3.2 *Let $\mathcal{T}$ be a Serre subcategory of* Mod-R*. Then, the construction above defines an Abelian category* Mod-$R/\mathcal{T}$*, and the assignment*

$$\begin{array}{rcl} T:\ \text{Mod-}R & \longrightarrow & \text{Mod-}R/\mathcal{T} \\ X & \longmapsto & T(X) = X \\ (X \xrightarrow{f} Y) & \longmapsto & (T(f): X \longrightarrow Y) = \textit{the image of } f \textit{ in} \\ & & \textit{the inductive limit} \end{array}$$

is an exact functor. Moreover, the pair (Mod-$R/\mathcal{T},\ T$) *has the above described universal property.*

Proof See [31, Chapitre III] or [42, Corollario 25.10, Teorema 25.13]. □

3.3 The Gabriel–Popescu Theorem

Next, we are interested in knowing when the Abelian quotient category Mod-$R/\mathcal{T}$ of Mod-R modulo a Serre subcategory $\mathcal{T}$ is a Grothendieck category. It can be shown (see, e.g., [31] or [42]) that the Serre subcategory $\mathcal{T}$ of Mod-R is a localizing subcategory of Mod-R if and only if the canonical functor

$$T : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

has a *right adjoint*

$$S : \text{Mod-}R/\mathcal{T} \longrightarrow \text{Mod-}R.$$

This means that for every $X \in$ Mod-R and $Y \in$ Mod-$R/\mathcal{T}$ there exists a "functorial" isomorphism, i.e., *natural* in both first and second argument,

$$\text{Hom}_{\text{Mod-}R/\mathcal{T}}(TX, Y) \xrightarrow{\sim} \text{Hom}_R(X, SY),$$

and in this case Mod-$R/\mathcal{T}$ is a Grothendieck category.

Thus, we have a procedure to construct new Grothendieck categories starting with Mod-R by taking quotient categories of Mod-R modulo arbitrary localizing subcategories of Mod-R.

Roughly speaking, the renowned semicentennial *Gabriel–Popescu Theorem*, states that in this way we obtain **all** the Grothendieck categories. More precisely,

Theorem 3.3 (Gabriel–Popescu Theorem) *Let $\mathcal{G}$ be an arbitrary Grothendieck category, and consider an arbitrary generator U of $\mathcal{G}$. Denote by R the ring* $\text{End}_{\mathcal{G}}(U)$ *of endomorphisms of U. Then there exists a localizing subcategory $\mathcal{T}$ of* Mod-R *such that*

$$\mathcal{G} \simeq \text{Mod-}R/\mathcal{T}. \qquad \square$$

For an error-free and detailed proof we strongly recommend to the interested reader the approach in [42, pp. 130-138 and Osservazione 25.16].

Notice that the ring R and the localizing subcategory $\mathcal{T}$ of Mod-R in Theorem 3.3 can be obtained in the following (non canonical) way. Let U be any generator of the Grothendieck category $\mathcal{G}$, and let R_U be the ring $\text{End}_{\mathcal{G}}(U)$ of endomorphisms of U. If

$$S_U : \mathcal{G} \longrightarrow \text{Mod-}R_U$$

is the functor $\text{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and

$$\text{Ker}(T_U) := \{\, M \in \text{Mod-}R_U \mid T_U(M) = 0 \,\}$$

is a localizing subcategory of Mod-R_U. Take now as R any such R_U and as $\mathcal{T}$ such a $\text{Ker}(T_U)$.

3.4 Absolutization

The Absolute H-LT illustrates another general direction in Module Theory, namely the so called *Absolutization.* Roughly speaking, this topic deals with the following matter:

> *Given a property* $\mathbb{P}$ *on modules, investigate the property* $\mathbb{P}$ *on objects of a Grothendieck category.*

As for relativization, the following problem naturally arises:

> *Having a theorem* $\mathbb{T}$ *on modules, is its absolutization* abs-$\mathbb{T}$ *true?*

For example, the absolutization of the H-LT is true by Theorem 3.1, but the absolutization of the property that any non-zero module has a simple factor module is not true. Indeed, let R be an infinite direct product of copies of a field, and let $\mathcal{A}$ be the localizing subcategory of Mod-R consisting of all semi-Artinian R-modules. Then, the quotient category Mod-$R/\mathcal{A}$ has no simple object (see, e.g., [1, Remarks 1.4(1)]).

We shall discuss now the interplay *Relativization* $\longleftrightarrow$ *Absolutization.* Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Then, one can form the quotient category Mod-$R/\mathcal{T}$, and denote by

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

the canonical functor from the category Mod-R to its quotient category Mod-$R/\mathcal{T}$.

Proposition 3.4 ([15, Proposition 7.10]) *With the notation above, for every module* M_R *there exists a lattice isomorphism*

$$\text{Sat}_\tau(M) \overset{\sim}{\longrightarrow} \mathcal{L}(T_\tau(M)).$$

In particular, M is a τ-Noetherian (respectively, τ-Artinian) module if and only if $T_\tau(M)$ is a Noetherian (respectively, Artinian) object of Mod-$R/\mathcal{T}$. □

We may also think that *Absolutization* is a technique to pass from τ-relative results in Mod-R to absolute properties in the quotient category Mod-$R/\mathcal{T}$ via the canonical functor T_τ : Mod-R $\longrightarrow$ Mod-$R/\mathcal{T}$. This technique is, in a certain sense, opposite to *Relativization*, meaning that absolute results in a Grothendieck category $\mathcal{G}$ can be translated, via the Gabriel–Popescu Theorem, into τ-relative results in Mod-R as follows.

Let U be any generator of the Grothendieck category $\mathcal{G}$, and let R_U be the ring $\text{End}_{\mathcal{G}}(U)$ of endomorphims of U. As we have mentioned just after Theorem 3.3, if $S_U : \mathcal{G} \longrightarrow \text{Mod-}R_U$ is the functor $\text{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U and $\text{Ker}\,(T_U) := \{\, M \in \text{Mod-}R_U \mid T_U(M) = 0 \,\}$ is a localizing subcategory of Mod-R_U. Let now τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory Ker (T_U) of Mod-R_U. Many properties of an object $X \in \mathcal{G}$ can now be translated into relative τ_U-properties of the right R_U-module $S_U(X)$; e.g., $X \in \mathcal{G}$ is an Artinian (respectively, Noetherian) object if and only if $S_U(X)$ is a τ_U-Artinian (respectively, τ_U-Noetherian) right R_U-module.

As mentioned before, the two module-theoretical proofs available in the literature of the Relative H-LT due to Miller and Teply [41] and Faith [30], are very long and complicated. On the contrary, the two categorical proofs of the Absolute H-LT due to Năstăsescu [43,44] are short and simple. We shall prove in Sect. 5 that *Relative H-LT* $\Longleftrightarrow$ *Absolute H-LT*; this means exactly that any of this theorems can be deduced from the other one. In this way we can obtain two short categorical proofs of the Relative H-LT.

However, some module theorists are not so comfortable with categorical proofs of module-theoretical theorems. Moreover, as we shall see in Sect. 6, statements like "*basically the same proof for modules works in the categorical setting*" may lead sometimes to wrong statements and results.

There exists an alternative for those people, namely the *latticial setting*. Indeed, if τ is a hereditary torsion theory on Mod-R and M_R is any module then $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice, and if $\mathcal{G}$ is a Grothendieck category and X is any object of $\mathcal{G}$ then $\mathcal{L}(X)$ is also an upper continuous modular lattice. Therefore, a strong reason to study chain conditions in such lattices exists, and this will be amply performed in the next section.

4 Latticization

In this section we shall discuss a third general direction in Module Theory called *Latticization.* This consists on putting a module-theoretical result into a *latticial frame* in order to translate that result to Grothendieck categories and module categories equipped with hereditary torsion theories. We shall illustrate this procedure through the latticial counterpart of the Classical H-LT.

4.1 A latticial strategy

Let $\mathbb{P}$ be a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories. Our *main strategy* in this direction since more than thirty years, we call *latticization,* consists of the following three steps:

I. *Translate/formulate,* if possible, the problem $\mathbb{P}$ into a *latticial setting.*
II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame.*
III. *Back to basics,* i.e., to Grothendieck categories and module categories equipped with hereditary torsion theories.

The advantage to deal in such a way is, in our opinion, that this is the most *natural* and *simple* approach as well, because we ignore the specific context of Grothendieck categories and module categories equipped with hereditary torsion theories, focusing only on those latticial properties which are relevant in our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The best illustration of this approach is, as we shall see in Sect. 5 that both the *Relative H-LT* and the *Absolute H-LT* are, in a unified manner, immediate consequences of the so called *Latticial H-LT*, which will be amply discussed in this section.

4.2 Lattice background (I)

Throughout this section L always denotes a lattice and L^0 its *dual* or *opposite* lattice. For a lattice L, or more generally, for a partially ordered poset $(P, \leqslant)$ (briefly, a poset) and elements $a, b \in P$ such that $a \leqslant b$ we set

$$b/a := [a, b] = \{ x \in P \mid a \leqslant x \leqslant b \}.$$

A *subfactor* of P is any interval b/a with $a \leqslant b$. For any $a \in P$, we also set

$$[a) := \{ x \in P \mid a \leqslant x \} \text{ and } (a] := \{ x \in P \mid x \leqslant a \}.$$

Recall that a lattice L is called *modular* if

$$a \wedge (b \vee c) = b \vee (a \wedge c), \ \forall\, a,\ b,\ c \in L \ \text{ with } b \leqslant a.$$

A lattice L is said to be *upper continuous* if it is complete and

$$a \wedge \Big(\bigvee_{c \in C} c \Big) = \bigvee_{c \in C} (a \wedge c)$$

for every $a \in L$ and every chain (or, equivalently, directed subset) $C \subseteq L$.

An element e of a lattice L with a least element 0 is called *essential* (in L) if $e \wedge a \neq 0,\ \forall\, a \in L,\ a \neq 0$. Dually, an element s of a lattice L with a greatest element 1 is called *superfluous* or *small* (in L) if $s \vee b \neq 1,\ \forall\, b \in L,\ b \neq 1$, i.e., if s is an essential element in L^0.

Let $a \leqslant b$ be elements of a lattice L. We say that the sublattice b/a of L is *simple* in case $a \neq b$ and $b/a = \{a, b\}$, i.e., the interval b/a has exactly two elements. An element a of a lattice L with a least element 0 is said to be an *atom* of L if the interval $a/0$ is simple. As in [47], a lattice L with a greatest element 1 is called *semi-atomic* (respectively, *semi-Artinian*) if 1 is a join of atoms of L (respectively, if for every $x \in L,\ x \neq 1$, the sublattice $1/x$ of L contains an atom). The *socle* $\mathrm{Soc}(L)$ of a complete lattice L is the join of all atoms of L. If L has no atoms, then $\mathrm{Soc}\,(L) = 0$.

Notice that if M is a right R-module, then a submodule N of M is an atom in the lattice $\mathcal{L}(M)$ of all submodules of M if and only if N is a simple submodule of M. Moreover, the lattice $\mathcal{L}(M)$ is semi-atomic if and only if M is a semisimple module.

If $a \leqslant b$ are elements of L, by a *composition series* for b/a, if it exists, we mean a finite chain

$$a = c_0 < c_1 < \cdots < c_n = b,$$

for some positive integer n and elements c_i such that c_i/c_{i-1} $(1 \leqslant i \leqslant n)$ is simple, i.e., the chain above has no refinement, except by introducing repetitions of the existing elements c_i. The integer n is called the *length* of the series.

Given a non-zero lattice L with least element 0 and greatest element 1, we say that the lattice L *has a composition series* (or *has finite length*) in case $1/0$ has a composition series, and in this case any two composition series of L have the same length, called the *length* of L and denoted by $\ell(L)$. A modular lattice is of finite length if and only if L is both Noetherian and Artinian. Note that L has a composition series if and only if the opposite lattice L^o has a composition series.

A set S of non-zero elements of a lattice L with least element 0 is said to be *independent* if for every finite subset F of S and for each $s \in S \setminus F$, one has $s \wedge (\bigvee_{x \in F} x) = 0$. We say that L has finite *Goldie (*or *uniform) dimension* if L contains no infinite independent subset. L is said to have finite *dual Goldie dimension* if its dual lattice L^0 has finite Goldie dimension. The lattice L is called *QFD* (acronym for *Q*uotient *F*inite *D*imensional) if the quotient interval $[a) = \{x \in L \mid x \geqslant a\}$ of L has finite Goldie dimension for all $a \in L$. A thorough investigation of the Goldie dimension of arbitrary modular lattices may be found in [7].

We denote by $\mathcal{L}$ (respectively, $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_{0,1}$) the class of all lattices (respectively, lattices with least element 0, lattices with greatest element 1, lattices with least element 0 and greatest element 1), and L will always designate a member of $\mathcal{L}$. In addition, we shall denote by $\mathcal{M}$ the class of all modular lattices. The notation $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_{0,1}$ have similar meanings.

For all undefined notation and terminology on lattices, as well as for more results on them, the reader is referred to [24,25,35], and/or [50].

4.3 When an arbitrary Artinian modular lattice is Noetherian?

The Classical/Relative/Absolute H-LT asks when a particular Artinian lattice $\mathcal{L}(M_R)$/$\mathrm{Sat}_\tau(M_R)$/$\mathcal{L}(X)$ is Noetherian. Our contention is that the natural setting for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices. Therefore we shall present in this subsection the Latticial H-LT which gives an exhaustive answer to the following more general question:

When an arbitrary Artinian modular lattice with 0 is Noetherian?

To do that, the following property that a lattice L with least element 0 may have ("$\mathcal{E}$" for Essential) plays a decisive role:

($\mathcal{E}$) *For all $a \leqslant b$ in L there exists $c \in L$ such that $b \wedge c = a$ and $b \vee c$ is an essential element of* $[a)$.

Any QFD lattice $L \in \mathcal{M}_0$, in particular, any Noetherian lattice and any upper continuous modular lattice satisfy ($\mathcal{E}$). Observe that the set $\mathbb{N}$ of all natural numbers ordered by the usual divisibility is an Artinian modular lattice (even distributive) which does not satisfy ($\mathcal{E}$).

In order to characterize Artinian lattices which are Noetherian, we introduce the following condition ("$\mathcal{BL}$" for Bounded Length) for a lattice L with least element 0:

($\mathcal{BL}$) *There exists a positive integer n such that for all $x < y$ in L with $y/0$ having a composition series there exists $c_{xy} \in L$ such that $c_{xy} \leqslant y$, $c_{xy} \nleqslant x$, and $\ell(c_{xy}/0) \leqslant n$.*

The next result is the *Hopkins–Levitzki Theorem* for an arbitrary modular lattice with least element, that will be used in the next section to provide very short proofs of the *absolute* (or *categorical*) and *relative* counterparts of the *Classical Hopkins–Levitzki Theorem.*

Theorem 4.1 (LATTICIAL H-LT [17, Theorem 1.9]) *Let $L \in \mathcal{M}_0$ be an Artinian lattice. Then L is Noetherian if and only if L satisfies both conditions ($\mathcal{E}$) and ($\mathcal{BL}$).* □

Other results which answer the question when an Artinian upper continuous modular lattice is Noetherian involve the concept of *Loewy length* of a lattice, and may be found in [2].

4.4 Lattice generation

We now exhibit a natural situation, involving the concept of "lattice generation", where the condition ($\mathcal{BL}$) occurs. In order to define it, first recall some definitions and facts on module generation. If R is a unital ring and M, U are two unital right R-modules, then M is said to be *U-generated* if there exist a set I and an epimorphism $U^{(I)} \twoheadrightarrow M$. The fact that M is U-generated can be equivalently expressed as follows: for every proper submodule N of M there exists a submodule P of M which is not contained in N, such that P is isomorphic to a quotient of the module U. Further, M is said to be *strongly U-generated* if every submodule of M is U-generated. These concepts can be naturally extended to arbitrary lattices as follows:

Definitions We say that a lattice $L \in \mathcal{L}_1$ is *generated* by a lattice $G \in \mathcal{L}_1$ (or is *G-generated*) if for every $a \neq 1$ in L there exist $c \in L$ and $g \in G$ such that $c \nleqslant a$ and $(c\,] \simeq 1/g$. A lattice $L \in \mathcal{L}$ is called *strongly generated* by a lattice $G \in \mathcal{L}_1$ (or *strongly G-generated*) if for every $b \in L$, the interval $(b\,]$ is G-generated, i.e., for all $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $(c\,] \simeq 1/g$. □

Recall that we have denoted by $\mathcal{L}$ (respectively, $\mathcal{L}_0$, $\mathcal{L}_1$) the class of all lattices (respectively, lattices with least element 0, lattices with greatest element 1). Of course, as in [3], the above definitions can be obviously further extended from lattices to posets.

Proposition 4.2 ([7, Proposition 4.3.7]) *Let $L \in \mathcal{M}_0$ be such that L is strongly generated by an Artinian lattice $G \in \mathcal{M}_1$. Then L satisfies the condition ($\mathcal{BL}$).* □

Combining Theorem 4.1 and Proposition 4.2 we obtain at once:

Theorem 4.3 ([7, Theorem 4.3.8]) *Let $L \in \mathcal{M}_0$ be an Artinian lattice which is strongly generated by an Artinian lattice $G \in \mathcal{M}_1$. Then L is Noetherian if and only if L satisfies the condition ($\mathcal{E}$). In particular, if additionally L is upper continuous, then L is Noetherian.* □

4.5 When an arbitrary Noetherian modular lattice is Artinian?

The results of the previous subsection can be now easily dualized by asking when a Noetherian lattice is Artinian. From now on, L will denote a modular lattice with a greatest element, i.e., $L \in \mathcal{M}_1$.

The dual properties of $(\mathcal{E})$ and $(\mathcal{BL})$ for a lattice $L \in \mathcal{M}_1$ are the following:

$(\mathcal{E}^o)$ *For all $a \leqslant b$ in L there exists $c \in L$ such that $a \vee c = b$ and $a \wedge c$ is a superfluous element of* $(\, b\,]$.

and

$(\mathcal{BL}^o)$ *There exists a positive integer n such that for all $x < y$ in L with $1/x$ having a composition series there exists c_{xy} in L such that $x \leqslant c_{xy},\ y \nleqslant c_{xy}$, and $\ell(1/c_{xy}) \leqslant n$.*

Examples of modular lattices that satisfy or not the condition $(\mathcal{E}^o)$ can be easily obtained by taking the opposites of the lattices discussed in the previous subsection; e.g., the opposite lattice $\mathbb{N}^o$ of the lattice $\mathbb{N}$ of all natural numbers ordered by the usual divisibility is Noetherian and does not satisfy $(\mathcal{E}^o)$.

Because the opposite of a modular lattice (respectively, lattice of finite length) is also a modular lattice (respectively, lattice of finite length) and

$$L \in \mathcal{M}_1 \text{ satisfies } (\mathcal{E}^o) \text{ (respectively, } (\mathcal{BL}^o))$$
$$\iff L^o \in \mathcal{M}_0 \text{ satisfies } (\mathcal{E}) \text{ (respectively, } (\mathcal{BL})),$$

the next result follows immediately from Theorem 4.1.

Theorem 4.4 (Latticial Dual H-LT [17, Theorem 1.11]) *Let $L \in \mathcal{M}_1$ be a Noetherian lattice. Then L is Artinian if and only if L satisfies both conditions $(\mathcal{E}^o)$ and $(\mathcal{BL}^o)$.* □

4.6 Lattice cogeneration

Recall that if R is a unital ring and $M,\ U$ are unital right R-modules, then M is said to be *U-cogenerated* if there exist a set I and a monomorphism $M \rightarrowtail U^I$. The fact that M is U-cogenerated can be equivalently expressed as follows: for any non-zero submodule N of M there exist a submodule P of M and a submodule U' of U such that $N \nsubseteq P$ and $M/P \simeq U'$. Further, we say that a module M is *strongly U-cogenerated* in case any quotient module of M is U-cogenerated. These concepts can be naturally extended to arbitrary lattices as follows:

Definitions A lattice $L \in \mathcal{L}_0$ is said to be *cogenerated* by a lattice $C \in \mathcal{L}_0$ or *C-cogenerated* if for any $x \neq 0$ in L there exist $z \in L$ and $c \in C$ with $x \nleqslant z$ and $[z) \simeq c/0$. A lattice $L \in \mathcal{L}$ is called *strongly cogenerated* by a lattice $C \in \mathcal{L}_0$ or *strongly C-cogenerated* if for any $y \in L$, the interval $[y)$ is C-cogenerated, that is, for any $y < x$ in L there exist $z \in L$ and $c \in C$ such that $y \leqslant z,\ x \nleqslant z$, and $[z) \simeq c/0$. □

Of course, the above definitions can be obviously further extended from lattices to posets. Observe that C-cogeneration is dual to G-generation:

$$L \text{ is } C\text{-cogenerated} \iff L^o \text{ is } C^o\text{-generated}.$$

The dual statements of Proposition 4.2 and Theorem 4.3 can now be easily obtained:

Proposition 4.5 *Let $L \in \mathcal{M}_1$ be such that L is strongly cogenerated by a Noetherian lattice $G \in \mathcal{M}_0$. Then L satisfies the condition $(\mathcal{BL}^o)$.* □

Theorem 4.6 *Let $L \in \mathcal{M}_1$ be a Noetherian lattice which is strongly cogenerated by a Noetherian lattice $G \in \mathcal{M}_0$. Then L is Artinian if and only if L satisfies the condition $(\mathcal{E}^o)$.* □

Theorems 4.1 and 4.4 have the following immediate consequence:

Corollary 4.7 ([17, Corollary 1.12]) *The following statements are equivalent for a lattice $L \in \mathcal{M}_{0,1}$.*

(1) *L has a composition series.*
(2) *L is Artinian, satisfies $(\mathcal{BL})$, and $1/a$ has finite Goldie dimension for all $a \in L$.*
(3) *L is Artinian, satisfies $(\mathcal{BL})$, and is upper continuous.*
(4) *L is Artinian and satisfies both $(\mathcal{BL})$ and $(\mathcal{E})$.*
(5) *L is Noetherian, satisfies $(\mathcal{BL}^o)$, and $a/0$ has finite dual Goldie dimension for all $a \in L$.*
(6) *L is Noetherian, satisfies $(\mathcal{BL}^o)$, and is lower continuous.*
(7) *L is Noetherian and satisfies both $(\mathcal{BL}^o)$ and $(\mathcal{E}^o)$.* □

5 The Hopkins–Levitzki Theorem

The aim of this section is twofold: firstly, to establish all the connections between the Classical, Relative, Absolute, and Latticial H-LT discussed in the previous sections, and secondly, to present other aspects of the H-LT including the *Faith's* Δ-Σ and *counter* versions of the Relative H-LT, the *Dual H-LT,* as well as a *Krull dimension-like H-LT.* In particular, we show in a unified manner that both the Relative H-LT and Absolute H-LT are immediate consequences of the Latticial H-LT.

5.1 Latticial H-LT $\Longrightarrow$ Relative H-LT

As mentioned above, the module-theoretical proofs available in the literature of the Relative H-LT are very long and complicated. We present below a very short proof of it based on the Latticial H-LT.

So, let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian, and let M_R be a τ-Artinian module. The Relative H-LT states that M_R is a τ-Noetherian module.

Set $G := \mathrm{Sat}_\tau(R_R)$ and $L := \mathrm{Sat}_\tau(M_R)$. Then G and L are Artinian upper continuous modular lattices. We have to prove that M_R is a τ-Noetherian module, i.e.,

L is a Noetherian lattice. By Theorem 4.3, it is sufficient to check that L is strongly G-generated, i.e., for every $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$.

Since $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/\tau(M))$ by Lemma 2.1(3), we may assume, without loss of generality, that $M \in \mathcal{F}$. Let $a = A < B = b$ in $L = \mathrm{Sat}_\tau(M_R)$. Then, there exists $x \in B \setminus A$. Set $C := \overline{xR}$ and $I := \mathrm{Ann}_R(x)$. We have $R/I \simeq xR \leqslant M \in \mathcal{F}$, so $R/I \in \mathcal{F}$, i.e., $I \in \mathrm{Sat}_\tau(R_R) = G$. By Lemma 2.1, we deduce that

$$[\,I,\ R\,] \simeq \mathrm{Sat}_\tau(R/I\,) \simeq \mathrm{Sat}_\tau(xR\,) \simeq \mathrm{Sat}_\tau(\overline{xR}\,) = \mathrm{Sat}_\tau(C) = [\,0,\ C\,],$$

where the intervals $[\,I,\ R\,]$ and $[\,0,\ C\,]$ are considered in the lattices G and L, respectively. Then, if we denote $c = C$ and $g = I$, we have $c \in L$, $g \in G$, $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$, which shows that L is strongly G-generated, as desired.

5.2 Latticial H-LT $\Longrightarrow$ Absolute H-LT

We show how the Absolute H-LT is an immediate consequence of the Latticial H-LT. Let $\mathcal{G}$ be a Grothendieck category, and let $U,\ X$ be Artinian objects of $\mathcal{G}$ such that X is strongly U-generated (this means that each subobject of X is U-generated). We are going to prove that X is Noetherian.

Set $G := \mathcal{L}(U)$ and $L := \mathcal{L}(X)$. Then G and L are both Artinian upper continuous modular lattice. We have to prove that L is a Noetherian lattice. By Theorem 4.3, it is sufficient to check that L is strongly G-generated. To do that, let $a = A < B = b$ in $L = \mathcal{L}(X)$. Because B is U-generated by hypothesis, there exists a morphism $\alpha : U \longrightarrow B$ in $\mathcal{G}$ such that $\mathrm{Im}\,(\alpha) \nleqslant A$. But $\mathrm{Im}\,(\alpha) \simeq U/\mathrm{Ker}\,(\alpha)$, so, if we set $c := \mathrm{Im}\,(\alpha)$ and $k := \mathrm{Ker}\,(\alpha)$, then we have $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/k$, which shows exactly that the lattice L is strongly G-generated.

In particular if U is an Artinian generator of $\mathcal{G}$, then any Artinian object $X \in \mathcal{G}$ is Noetherian, which is exactly the Absolute H-LT.

5.3 Absolute H-LT $\Longrightarrow$ Relative H-LT

We are going to show how the Relative H-LT can be deduced from the Absolute H-LT. Let $\tau = (\mathcal{T},\ \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian ring, and let M_R be a τ-Artinian module. We pass from Mod-R to the Grothendieck category Mod-$R/\mathcal{T}$ with the aid of the canonical functor T_τ : Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. Since R_R is a generator of Mod-R and T_τ is an exact functor that commutes with direct sums we deduce that $T_\tau(R)$ is a generator of Mod-$R/\mathcal{T}$, which is Artinian by Proposition 3.4. Now, again by Proposition 3.4, $T_\tau(M)$ is an Artinian object of Mod-$R/\mathcal{T}$, so, it is also Noetherian by the Absolute H-LT, i.e., M is τ-Noetherian, and we are done.

5.4 Relative H-LT $\Longrightarrow$ Absolute H-LT

We prove that the Absolute H-LT is a consequence of the Relative H-LT. Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator U. Set $R_U := \mathrm{End}_{\mathcal{G}}(U)$, and let

$$S_U = \mathrm{Hom}_{\mathcal{G}}(U, -) : \mathcal{G} \longrightarrow \text{Mod-}R_U \quad \text{and} \quad T_U : \text{Mod-}R_U \longrightarrow \mathcal{G}$$

be the pair of functors from the Gabriel–Popescu Theorem setting, described in Sect. 3 just after Theorem 3.3. Then $T_U \circ S_U \simeq 1_{\mathcal{G}}$ and

$$\mathrm{Ker}(T_U) := \{ M \in \text{Mod-}R_U \mid T_U(M) = 0\}$$

is a localizing subcategory of Mod-R_U. Let τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory $\mathcal{T}_U := \mathrm{Ker}(T_U)$ of Mod-R_U. By the Gabriel–Popescu Theorem we have

$$\mathcal{G} \simeq \text{Mod-}R_U/\mathcal{T}_U \quad \text{and} \quad U \simeq (T_U \circ S_U)(U) = T_U(S_U(U)) = T_U(R_U).$$

Since U is an Artinian object of $\mathcal{G}$, so is also $T_U(R_U)$, which implies, by Proposition 3.4, that R_U is a τ_U-Artinian ring.

Now, let $X \in \mathcal{G}$ be an Artinian object of $\mathcal{G}$. Then, there exists a right R_U-module M such that $X \simeq T_U(M)$, so $T_U(M)$ is an Artinian object of $\mathcal{G}$, i.e., M is a τ_U-Artinian module. By the Relative H-LT, M is τ_U-Noetherian, so, again by Proposition 3.4, $X \simeq T_U(M)$ is a Noetherian object of $\mathcal{G}$, as desired.

5.5 Faith's Δ-Σ version of the Relative H-LT

Recall that an injective module Q_R is said to be Σ-*injective* if any direct sum of copies of Q is injective. This concept is related with the concept of a τ-Noetherian module as follows.

Let Q_R be an injective module, and denote

$$\mathcal{T}_Q := \{ M_R \mid \mathrm{Hom}_R(M, Q) = 0\}.$$

Then $\mathcal{T}_Q$ is a localizing subcategory of Mod-R, and let τ_Q be the hereditary torsion theory on Mod-R (uniquely) determined by $\mathcal{T}_Q$. Note that for any hereditary torsion theory τ on Mod-R there exists an injective module Q_R such that $\tau = \tau_Q$.

A renowned theorem of Faith (1966) says that an injective module Q_R is Σ-injective if and only if R_R is τ_Q-Noetherian, or equivalently, if R satisfies the ACC on annihilators of subsets of Q. In order to uniformize the notation, Faith [30] introduced the concept of a Δ-*injective module* as being an injective module Q such that R_R is τ_Q-Artinian, or equivalently, R satisfies the DCC on annihilators of subsets of Q. Thus, the Relative H-LT is equivalent with the following *Faith's* Δ-Σ version of it.

Theorem 5.1 ([30, p. 3]) *Any* Δ-*injective module is* Σ-*injective.* □

5.6 Faith's counter version of the Relative H-LT

Let M_R be a module, and let $S := \mathrm{End}_R(M)$. Then M becomes a left S-module, and the module ${}_SM$ is called the *counter module* of M_R. We say that M_R is *counter-Noetherian* (respectively, *counter-Artinian*) if ${}_SM$ is a Noetherian (respectively, Artinian) module.

The next result is an equivalent version, in terms of counter modules, of the Relative H-LT.

Theorem 5.2 ([30, Theorem 7.1]) *If Q_R is an injective module which is counter-Noetherian, then Q_R is counter-Artinian.* □

5.7 Absolute H-LT $\Longrightarrow$ Classical H-LT

Grothendieck categories having an Artinian generator are very special in view of the following surprising result of Năstăsescu.

Theorem 5.3 ([46, Théorème 3.3]) *A Grothendieck category $\mathcal{G}$ has an Artinian generator if and only if $\mathcal{G} \simeq$ Mod-A, with A a right Artinian ring with identity.* □

A heavy artillery has been used in the original proof of Theorem 5.3, namely: the Gabriel–Popescu Theorem, the Relative H-LT, as well as structure theorems for Δ-injective and Δ^*-projective modules. The Σ^*-*projective* and Δ^*-*projective* modules, introduced and investigated in [45,46], are in a certain sense dual to the notions of Σ-injective and Δ-injective modules.

A more general result whose original proof is direct, without involving the many facts listed above, is the following.

Theorem 5.4 ([22, Theorem 2.2]) *Let $\mathcal{G}$ be a Grothendieck category having a (finitely generated) generator U such that $\mathrm{End}_{\mathcal{G}}(U)$ is a right perfect ring. Then $\mathcal{G}$ has a (finitely generated) projective generator.* □

Observe now that if $\mathcal{G}$ has an Artinian generator U, then, by the Absolute H-LT, U is also Noetherian, so, an object of finite length. Then $S = \mathrm{End}_{\mathcal{G}}(U)$ is a semiprimary ring, in particular it is right perfect. Now, by Theorem 5.4, $\mathcal{G}$ has a finitely generated projective generator, say P. If $A = \mathrm{End}_{\mathcal{G}}(P)$ then A is a right Artinian ring, and $\mathcal{G} \simeq$ Mod-A, which shows how Theorem 5.3 is an immediate consequence of Theorem 5.4.

Clearly Relative H-LT $\Longrightarrow$ Classical H-LT by taking as τ the hereditary torsion theory (0, Mod-R) on Mod-R, and Absolute H-LT $\Longrightarrow$ Classical H-LT by taking as $\mathcal{G}$ the category Mod-R.

We conclude that the following implications between the various aspects of the H-LT discussed so far hold:

Latticial H-LT $\Longrightarrow$ Relative H-LT $\Longleftrightarrow$ Absolute H-LT $\Longleftrightarrow$ Classical H-LT

Faith's Δ-Σ Theorem $\Longleftrightarrow$ Relative H-LT $\Longleftrightarrow$ Faith's Counter Theorem

5.8 The Absolute and Relative Dual H-LT

Remember that the Absolute H-LT states that if $\mathcal{G}$ is a Grothendieck category with an Artinian generator, then any Artinian object of $\mathcal{G}$ is necessarily Noetherian, so it is natural to ask whether its dual holds:

Problem (ABSOLUTE DUAL H-LT) If $\mathcal{G}$ is a Grothendieck category having a Noetherian cogenerator, then does it follow that any Noetherian object of $\mathcal{G}$ is Artinian? □

Notice that the Absolute Dual H-LT fails even for a module category Mod-R (see [1]). However the Absolute Dual H-LT holds for large classes of Grothendieck categories, namely for the so called commutative Grothendieck categories, introduced and investigated in [13]. A Grothendieck category $\mathcal{C}$ is said to be *commutative* if there exists a commutative ring A with identity such that $\mathcal{G} \simeq$ Mod-$A/\mathcal{T}$ for some localizing subcategory $\mathcal{T}$ of Mod-A. These are exactly those Grothendieck categories $\mathcal{G}$ having at least a generator U with a commutative ring of endomorphisms.

Recall that an object G of a Grothendieck category $\mathcal{G}$ is a *generator* of $\mathcal{G}$ if every object X of $\mathcal{G}$ is an epimorphic image $G^{(I)} \twoheadrightarrow X$ of a direct sum of copies of G for some set I. Dually, an object $C \in \mathcal{G}$ is said to be a *cogenerator* of $\mathcal{G}$ if every object X of $\mathcal{G}$ can be embedded $X \rightarrowtail C^I$ into a direct product of copies of C for some set I.

Theorem 5.5 ([1, Theorem 3.2]) *The following assertions are equivalent for a commutative Grothendieck category $\mathcal{G}$.*

(1) *$\mathcal{G}$ has a Noetherian cogenerator.*
(2) *$\mathcal{G}$ has an Artinian generator.*
(3) *$\mathcal{G} \simeq$ Mod-A for some commutative Artinian ring with identity.* □

An immediate consequence of Theorem 5.5 is the following.

Theorem 5.6 (ABSOLUTE DUAL HL-T) *If $\mathcal{G}$ is any commutative Grothendieck category having a Noetherian cogenerator, then every Noetherian object of $\mathcal{G}$ is Artinian.* □

If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R, then a module C_R is said to be a *τ-cogenerator* of Mod-R if $C \in \mathcal{F}$ and every module in $\mathcal{F}$ is cogenerated by C. The next result is the relative version of the Absolute Dual H-LT.

Theorem 5.7 (RELATIVE DUAL HL-T) *Let R be a commutative ring with identity, and let τ be a hereditary torsion theory on* Mod-R *such that* Mod-R *has a τ-Noetherian τ-cogenerator. Then every τ-Noetherian R-module is τ-Artinian.* □

5.9 Krull dimension

The idea of *dimension* is fundamental in many parts of Mathematics. Very intuitively, each kind of dimension *"takes the measure"* of the involved concepts from Mathematics in the form of *numerical invariants, cardinal invariants,* or *ordinal invariants.* Usually, it measures the deviation of a certain system from some ideal situation, or how likely or unlikely a certain object is to enjoy a certain property, or the progress in some inductive procedure. The most important dimensions encountering in Algebra, and in particular in Ring and Module Theory, are *Krull dimension, Goldie dimension, Gabriel dimension, (co)homological dimension,* and *Gelfand-Kirillov dimension.*

The *Krull dimension* of a poset P (also called *deviation* of P and denoted by $\operatorname{dev}(P)$) is an ordinal number denoted by $k(P)$, which may or may not exist, and is defined recursively as follows:

- $k(P) = -1 \iff P$ is the zero poset 0 consisting of a single element, where -1 is the predecessor of the ordinal number 0.
- $k(P) = 0 \iff P \neq 0$ and P is Artinian.
- Let $\alpha \geqslant 1$ be an ordinal number, and assume that we have already defined which posets have Krull dimension β for any ordinal $\beta < \alpha$. Then we define what it means for a poset P to have Krull dimension α: $k(P) = \alpha$ if and only if we have not defined $k(P) = \beta$ for some $\beta < \alpha$, and for any descending chain

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant x_{n+1} \geqslant \ldots$$

  of elements of P, $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geqslant n_0$, $k(x_n/x_{n+1}) < \alpha$, i.e., $k(x_n/x_{n+1})$ has previously been defined and it is an ordinal $< \alpha$.
- If no ordinal α exists such that $k(P) = \alpha$, we say that P does not have Krull dimension.

An alternative more compact equivalent definition of the Krull dimension of a poset is that involving the concept of an *Artinian poset relative* to a class of posets. If $\mathcal{X}$ is an arbitrary non-empty subclass of the class $\mathcal{P}$ of all posets, a poset P is said to be $\mathcal{X}$-*Artinian* if for every descending chain $x_1 \geqslant x_2 \geqslant \ldots$ in P, $\exists\, k \in \mathbb{N}$ such that $x_i/x_{i+1} \in \mathcal{X}$, $\forall\, i \geqslant k$. The notion of an $\mathcal{X}$-*Noetherian* poset is defined similarly.

For every ordinal $\alpha \geqslant 0$, we denote by $\mathcal{P}_\alpha$ the class of all posets having Krull dimension $< \alpha$. Then, it is easily seen that a poset P has Krull dimension an ordinal $\alpha \geqslant 0$ if and only if $P \notin \mathcal{P}_\alpha$ and P is $\mathcal{P}_\alpha$-Artinian. So, roughly speaking, the Krull dimension of a poset P measures how close P is to being Artinian.

5.10 The definition of the dual Krull dimension of a poset

The *dual Krull dimension* of a poset P (also called *codeviation* of P and denoted by $\operatorname{codev}(P)$), denoted by $k^o(P)$, is defined as being (if it exists!) the Krull dimension $k(P^o)$ of the opposite poset P^o of P. If α is an ordinal, then the notation $k(P)\leqslant\alpha$ (respectively, $k^o(P) \leqslant \alpha$) will be used to indicate that P has Krull dimension (respectively, dual Krull dimension) and it is less than or equal to α.

The existence of the dual Krull dimension $k^o(P)$ of a poset P is equivalent with the existence of the Krull dimension $k(P)$ of P in view of the following nice result of Lemonnier [37, Théorème 5, Corollaire 6]:

Theorem 5.8 *A poset P does not have Krull dimension if and only if P contains a copy of the (usually) ordered set $\mathbb{D} = \{\, m/2^n \mid m \in \mathbb{Z},\ n \in \mathbb{N}\,\}$ of dyadic real numbers. Consequently, P has Krull dimension if and only if P has dual Krull dimension.* □

Remember that

$$P \text{ is Artinian (respectively, Noetherian)} \iff k(P) \leqslant 0 \ \text{(respectively, } k^o(P) \leqslant 0).$$

So, we immediately deduce from Theorem 5.8 the following fact, which usually is proved in a more complicated way: *any Noetherian poset has Krull dimension.*

The following problem naturally arises:

Problem Let P be a poset with Krull dimension. Then P also has dual Krull dimension. How are the ordinals $k(P)$ and $k^o(P)$ related? □

For other basic facts on the Krull and dual Krull dimension of a module or an arbitrary poset the reader is referred to [33,37], and [40].

5.11 Krull dimension and dual Krull dimension of modules and rings

Recall that for a module M one denotes by $\mathcal{L}(M)$ the lattice of all submodules of M. The following ordinals (if they exist) are defined in terms of the lattice $\mathcal{L}(M)$.

- *Krull dimension* of M: $k(M) := k(\mathcal{L}(M))$.
- *Dual Krull dimension* of M: $k^o(M) := k^o(\mathcal{L}(M))$.
- *Right Krull dimension* of R: $k(R) := k(R_R)$.
- *Right dual Krull dimension* of R: $k^o(R) := k^o(R_R)$.

The problem we presented above for arbitrary posets can be specialized to modules and rings as follows.

Problem Compare the ordinals $k(M)$ and $k^o(M)$ of a given module M_R with Krull dimension. In particular, compare the ordinals $k(R)$ and $k^o(R)$ of a ring R with right Krull dimension. □

Related to the Problem above, the following question has been raised by Albu and Smith in 1991, and also mentioned in [19, Question 1].

$$\textit{If } R \textit{ is any ring with right Krull dimension, is it true that } k^0(R) \leqslant k(R)?$$

Observe that the answer is *yes* for $k(R) = 0$, which is exactly the Classical H-LT. Other cases when the answer is *yes*, according to [19], are when R is one of the following types of rings: a commutative Noetherian ring, or a commutative ring with Krull dimension 1, or a commutative domain with Krull dimension 2, or a valuation domain with Krull dimension, or a right Noetherian right V-ring.

5.12 A Krull dimension-like extension of the Absolute H-LT

If $\mathcal{G}$ is a Grothendieck category and X is an object of $\mathcal{G}$, then recall that the *Krull dimension* of X, denoted by $k(X)$, is defined as $k(X) := k(\mathcal{L}(X))$, where $\mathcal{L}(X)$ is the lattice of all subobjects of X.

The definition of the Krull dimension of an object in a Grothendieck category $\mathcal{G}$ can also be given using a transfinite sequence of Serre subcategories of $\mathcal{G}$ and suitable quotient categories of $\mathcal{G}$ (see [34, Proposition 1.5]). Using this approach, the following extension of the Absolute H-LT has been proved:

Theorem 5.9 ([12, Theorem 3.1]) *Let $\mathcal{G}$ be a Grothendieck category, and let U be a generator of $\mathcal{G}$ such that $k(U) = \alpha + 1$ for some ordinal $\alpha \geqslant -1$. Then, for every object X of $\mathcal{G}$ having Krull dimension and for every ascending chain*

$$X_1 \leqslant X_2 \leqslant \ldots \leqslant X_n \leqslant \ldots$$

of subobjects of X, $\exists\, m \in \mathbb{N}$ such that $k(X_{i+1}/X_i) \leqslant \alpha,\ \forall\, i \geqslant m$. □

Note that for $\alpha = -1$ we obtain exactly the Absolute H-LT, because in this case, $X \in \mathcal{G}$ has Krull dimension if and only if $k(X) \leqslant 0$, i.e., if and only if X is Artinian.

It seems that the above result is really a *categorical property* of Grothendieck categories. As we already stressed before, the natural frame for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices, and therefore we shall present in the next subsection a very general version of Theorem 5.9 for upper continuous modular lattices.

5.13 A Krull dimension-like extension of the Latticial H-LT

In order to present an extension of Theorem 5.9 to lattices, which, on one hand, is interesting in its own right, and, on the other hand, provides another proof of it, avoiding the use of quotient categories, we need first a latticial substitute for the notion of generator of a Grothendieck category, which has been already presented in Sect. 4.

Theorem 5.10 ([18, Theorem 3.16]) *Let L and G be upper continuous modular lattices. Suppose that $k(G) = \alpha + 1$ for some ordinal $\alpha \geqslant -1$ and L is strongly generated by G. If L has Krull dimension, then $k(L) \leqslant \alpha + 1$, and for every ascending chain*

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$$

of elements of L, $\exists\, m \in \mathbb{N}$ such that $k(x_{i+1}/x_i) \leqslant \alpha,\ \forall\, i \geqslant m$. □

Two main ingredients are used in the proof of Theorem 5.10, namely the *Latticial H-LT* and a *localization* technique for modular lattices developed in [17] and [18] analogously with that for Grothendieck categories. In the next subsection we shall briefly discuss this technique.

Problem Does the result of Theorem 5.10 fail when $k(G)$ is a limit ordinal? We suspect that the answer is *yes*, even in the module case. □

5.14 Localization of modular lattices

The terminology and notation below are taken from the localization theory in Grothendieck categories. First, in analogy with the notion of a Serre subcategory of an Abelian category, we present below, as in [17], the notion of a *Serre class of lattices.*

Definition By an *abstract class of lattices* we mean a non-empty subclass $\mathcal{X}$ of the class $\mathcal{M}_{0,1}$ of all modular lattices with 0 and 1, which is closed under lattice isomorphisms (i.e., if $L,\ K \in \mathcal{M}_{0,1},\ K \simeq L$ and $L \in \mathcal{X}$, then $K \in \mathcal{X}$).

We say that a subclass $\mathcal{X}$ of $\mathcal{M}_{0,1}$ is a *Serre class for* $L \in \mathcal{M}_{0,1}$ if $\mathcal{X}$ is an abstract class of lattices, and for all $a \leqslant b \leqslant c$ in L, $c/a \in \mathcal{X}$ if and only if $b/a \in \mathcal{X}$ and $c/b \in \mathcal{X}$. A *Serre class of lattices* is an abstract class of lattices which is a Serre class for all lattices $L \in \mathcal{M}_{0,1}$. □

Let $\mathcal{X}$ be an arbitrary non-empty subclass of $\mathcal{M}_{0,1}$ and let $L \in \mathcal{M}_{0,1}$ be a lattice. Define a relation $\sim_{\mathcal{X}}$ on L by

$$a \sim_{\mathcal{X}} b \iff (a \vee b)/(a \wedge b) \in \mathcal{X}.$$

Then $\sim_{\mathcal{X}}$ is a congruence on L if and only if $\mathcal{X}$ is a Serre class for L. Recall that a *congruence* on a lattice L is an equivalence relation $\sim$ on L such that for all $a,\ b,\ c \in L$, $a \sim b$ implies $a \vee c \sim b \vee c$ and $a \wedge c \sim b \wedge c$. It is well-known that in this case the quotient set $L/\sim$ has a natural lattice structure, and the canonical mapping $L \longrightarrow L/\sim$ is a lattice morphism. If $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}_{0,1}$, then the lattice $L/\sim_{\mathcal{X}}$ is called the *quotient lattice of L by* (or *modulo*) $\mathcal{X}$.

We define now for any non-empty subclass $\mathcal{X}$ of $\mathcal{M}_{0,1}$ and for any lattice L a certain subset $\mathrm{Sat}_{\mathcal{X}}(L)$ of L, called the $\mathcal{X}$-*saturation* of L:

$$\mathrm{Sat}_{\mathcal{X}}(L) := \{\, x \in L \mid x \leqslant y \in L,\ y/x \in \mathcal{X} \implies x = y \,\}.$$

This is the precise analogue of the subset

$$\mathrm{Sat}_{\tau}(M) = \{\, N \leqslant M_R \mid M/N \in \mathcal{F} \,\}$$

of the lattice $\mathcal{L}(M_R)$ of all submodules of a given module M_R, where $\tau = (\mathcal{T},\ \mathcal{F})$ is a hereditary torsion theory on Mod-R.

Definition Let $\mathcal{X}$ be an arbitrary non-empty subclass of $\mathcal{M}_{0,1}$. We say that a lattice $L \in \mathcal{M}_{0,1}$ has an $\mathcal{X}$-*saturation* if there exists a mapping, called the $\mathcal{X}$-*saturation* of L

$$L \longrightarrow \mathrm{Sat}_{\mathcal{X}}(L)\ ,\quad x \longmapsto \overline{x},$$

satisfying the following two conditions:

(1) $x \leqslant \overline{x}$ and $\overline{x}/x \in \mathcal{X}$ for all $x \in L$.
(2) $x \leqslant y$ in $L \implies \overline{x} \leqslant \overline{y}$. □

If $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}_{0,1}$ such that L has an $\mathcal{X}$-saturation $x \longmapsto \overline{x}$, and if we define

$$x \overline{\vee} y := \overline{x \vee y},\ \forall x,\ y \in \mathrm{Sat}_{\mathcal{X}}(L),$$

then the reader can easily check that $\mathrm{Sat}_{\mathcal{X}}(L)$ becomes a modular lattice with respect to $\leqslant,\ \wedge,\ \overline{\vee},\ \overline{0},\ 1$.

By Proposition 3.4, for any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R and any module M_R, the lattice $\mathrm{Sat}_\tau(M)$ is isomorphic to the lattice $\mathcal{L}(T_\tau(M))$ of all subobjects of the object $T_\tau(M)$ in the quotient category Mod-$R/\mathcal{T}$, where

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

is the canonical functor. The same happens also in our latticial frame: if $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}_{0,1}$ such that L has an $\mathcal{X}$-saturation, then

$$L/\sim_{\mathcal{X}} \simeq \mathrm{Sat}_{\mathcal{X}}(L).$$

Consequently, the lattice L is $\mathcal{X}$-Noetherian (respectively, $\mathcal{X}$-Artinian) $\Longleftrightarrow$ the lattice $\mathrm{Sat}_{\mathcal{X}}(L)$ is Noetherian (respectively, Artinian) $\Longleftrightarrow$ the lattice $L/\sim_{\mathcal{X}}$ is Noetherian (respectively, Artinian).

If $\mathcal{X}$ is a Serre class of lattices for a lattice L, one may define as in [17] the relative conditions $(\mathcal{E})_{\mathcal{X}}$ and $(\mathcal{BL})_{\mathcal{X}}$ in order to prove the following Latticial H-LT relative to $\mathcal{X}$.

Theorem 5.11 (Relative Latticial H-LT [17, Theorem 4.9]) *Let $\mathcal{X} \subseteq \mathcal{M}_{0,1}$ be a Serre class for a lattice $L \in \mathcal{M}_{0,1}$ such that L has an $\mathcal{X}$-saturation and L is $\mathcal{X}$-Artinian. Then L is $\mathcal{X}$-Noetherian if and only if L satisfies both conditions $(\mathcal{E})_{\mathcal{X}}$ and $(\mathcal{BL})_{\mathcal{X}}$.* □

Serre classes of lattices which are closed under taking arbitrary joins, we next introduce, are called *localizing classes of lattices* and they play the same role as that of localizing subcategories in the setting of Grothendieck categories. More precisely, we have the following:

Definition Let $\mathcal{X}$ be a non-empty subclass of $\mathcal{M}_{0,1}$ and let L be a complete modular lattice. We say that $\mathcal{X}$ is a *localizing class for L* if $\mathcal{X}$ is a Serre class for L, and for any $x \in L$ and for any family $(x_i)_{i \in I}$ of elements of $1/x$ such that $x_i/x \in \mathcal{X}$ for all $i \in I$, we have $(\bigvee_{i \in I} x_i)/x \in \mathcal{X}$. By a *localizing class of lattices* we mean a Serre class of lattices which is a localizing class for every complete modular lattice. □

Note that if $\mathcal{X}$ is a localizing class for a complete modular lattice L then L has an $\mathcal{X}$-saturation, which is uniquely determined. For more details on localization of modular lattices, the reader is referred to [17,18], and [20].

6 The Osofsky–Smith Theorem

In this section we discuss various aspects of another renowned result of Module Theory.

Theorem 6.1 (Osofsky–Smith Theorem (O- ST) [49, Theorem 1]) *A finitely generated (respectively, cyclic) right R-module such that all of its finitely generated (respectively, cyclic) subfactors are CS modules is a finite direct sum of uniform submodules.* □

Recall that a module M is said to be *CS* (or *extending*) if every submodule of M is essential in a direct summand of M, or, equivalently, any complement submodule of M is a direct summand of M. By subfactor of M one understands any submodule of a factor module of M. Recall that in Module Theory one says that a submodule N of M is a *complement* if there exists a submodule L of M such that $N \cap L = 0$ and N is maximal in the set of all submodules P of M with $P \cap L = 0$, i.e., the element N of the lattice $\mathcal{L}(M)$ of all submodules of M is a pseudo-complement element in this lattice (see the next subsection for the concept of a pseudo-complement element in a lattice). The name CS is an acronym for *C*omplements submodules are direct *S*ummands. More about CS modules can be found in the monograph [29], entirely devoted to them.

Though the Osofsky–Smith Theorem is a module-theoretical result, our contention is that it is a result of a strong latticial nature. In this section a latticial version of this theorem is presented and its applications to Grothendieck categories and module categories equipped with a torsion theory are given.

6.1 Lattice background (II)

In this subsection we shall explain all the latticial concepts that have not been presented in the Lattice background (I) subsection of Sect. 4, but will show up when discussing the Latticial Osofsky–Smith Theorem.

Let L be a lattice with a least element 0. Recall that an element $e \in L$ is called *essential (in L)* if $e \wedge a \neq 0$ for all $0 \neq a \in L$. By $E(L)$ we shall denote the set of all essential elements of L.

We say that L *uniform* if $L \neq \{0\}$ and $x \wedge y \neq 0$ for any non-zero elements $x, y \in L$. An element u of a lattice L is called *uniform* if $u/0$ is a uniform lattice, i.e., if $u \neq 0$ and $a \wedge b \neq 0$ for all non-zero elements a and b in $u/0$, or equivalently, if every non-zero element of $u/0$ belongs to $E\,(u/0)$. Not every lattice contains uniform elements. For example, if R is a non-commutative domain which is not right Ore then the lattice $\mathcal{L}(R_R)$ of all right ideals of R does not contain a uniform element.

For a lattice L with 0 and $a, b, c \in L$, the notation $a = b \dot{\vee} c$ will mean that $a = b \vee c$ and $b \wedge c = 0$, and then we say that a is a *direct join* of b and c. Also, for a non-empty subset S of L, we use the *direct join* notation $a = \dot{\bigvee}_{b \in S} b$ or $a = \dot{\bigvee} S$ if S is an independent subset of L and $a = \bigvee_{b \in S} b$; this is the latticial counterpart of the concept of *internal direct sum* of submodules of a module. Recall that a non-empty subset S of L is called *independent* if $0 \notin S$, and for every $x \in S$, positive integer n, and subset $T = \{t_1, \ldots, t_n\}$ of S with $x \notin T$, one has $x \wedge (t_1 \vee \cdots \vee t_n) = 0$. Clearly a subset S of L is independent if and only if every finite subset of S is independent.

If L is a lattice with least element 0 and greatest element 1, then an element $c \in L$ is a *complement* (in L) if there exists $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$, i.e., $1 = a \dot{\vee} c$; we say in this case that c is a *complement* of a (in L). For example, 1 is a complement of 0 and 0 is a complement of 1. One denotes by $D(L)$ the set of all complements of L, so $\{0, 1\} \subseteq D(L)$. The lattice L is called *indecomposable* if $L \neq \{0\}$ and $D(L) = \{0, 1\}$. An element $a \in L$ is said to be an *indecomposable* element if $a/0$ is an indecomposable lattice.

However, a given element a need not have a complement. The lattice L is called *complemented* if it has least element and greatest element, and every element has a complement. For example, if R is an arbitrary unital ring and M a right unital R-module, then the lattice $\mathcal{L}(M)$ of all submodules of M is complemented (respectively, indecomposable) if and only if the module M is semisimple (respectively, indecomposable).

A lattice L is called *E-complemented* (E for "Essential") provided for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$. Clearly, any complemented lattice is E-complemented. In fact, a lattice L is complemented if and only if L is E-complemented and $E(L) = \{1\}$.

By a *closed element* of a lattice L we mean an element c such that whenever $a \in L$ with $c \leqslant a$ and $c \in E(a/0)$ then $a = c$. We denote by $C(L)$ the set of all closed elements of L. Note that $0 \in C(L)$ and every element of a complemented lattice is closed.

A lattice L is called *essentially closed* if for each $c \in L$ there exists $e \in L$ maximal in the set of elements $f \in L$ such that $c \in E(f/0)$. Let R be any unital ring, and let M be a right R-module. For any submodule A of M, let $\mathcal{S}_A$ denote the set of all submodules B of M such that A is an essential submodule of B (i.e., in the lattice $\mathcal{L}(M)$ of all submodules of M, $A \in E(B/0)$). Clearly, A belongs to $\mathcal{S}_A$ and, by Zorn's Lemma, $\mathcal{S}_A$ has a maximal member. Thus $\mathcal{L}(M)$ is essentially closed, and similarly, with the same arguments, so is any upper continuous modular lattice.

Given an element $a \in L$, an element $b \in L$ is called a *pseudo-complement* of a (in L) provided b is maximal in the set of all elements c in L such that $a \wedge c = 0$. By a *pseudo-complement* of L we mean any element $b \in L$ such that b is a pseudo-complement (in L) of some element $a \in L$. We shall denote by $P(L)$ the set of all pseudo-complements of L. The lattice L is called *pseudo-complemented* if every element a has a pseudo-complement.

As in [7], a lattice L is called *strongly pseudo-complemented* if, for all $a, b \in L$ with $a \wedge b = 0$, there exists a pseudo-complement p of a in L such that $b \leqslant p$. Clearly, strongly pseudo-complemented lattices are pseudo-complemented.

Let R be any unital ring, let M be a right R-module, and let A and B be submodules of M such that $A \cap B = 0$. By Zorn's Lemma, the set of all submodules Q of M such that $B \subseteq Q$ and $A \cap Q = 0$ has a maximal member, say C. Thus, the lattice $\mathcal{L}(M)$ of all submodules of M is strongly pseudo-complemented, and, with the same argument, so is any upper continuous modular lattice.

Notice that by [7, Theorem 1.2.24], a lattice L is strongly pseudo-complemented if and only if L is E-complemented and essentially closed.

We shall illustrate most of the concepts presented above with the following simple example.

Example 6.2 Let F be any field, let V be any infinite dimensional vector space over F, and let H denote the lattice of all subspaces of V. Then the set G of all finite dimensional subspaces of V is a sublattice of H. The lattice H has least element the zero subspace and greatest element V and is complemented. The sublattice G has least element the zero subspace and no greatest element. □

In Example 6.2 every element of H and every element of G is closed. However, if R is any unital ring, U a simple right R-module, and E the injective hull of U, then in the lattice $\mathcal{L}(E)$ of all submodules of E the only closed elements are 0 and E. In case R is the ring $\mathbb{Z}$ of rational integers and W the sublattice of $\mathcal{L}(E)$ consisting of all finitely generated submodules of E, then 0 is the only closed element of W. Thus $C(H) = H$, $C(\mathcal{L}(E)) = \{0, E\}$ and $C(W) = \{0\}$.

In general, a lattice L need not possess essential elements; e.g., in Example 6.2, the only essential element of H is V, and $E(G) = \varnothing$.

By Proposition [7, Proposition 1.2.16], for any modular lattice L with least element we have $P(L) \subseteq C(L)$, but the inclusion may be strict: indeed, in Example 6.2, $C(G) = G$ but $P(G) = \varnothing$. However, this is not the case for the E-complemented lattices (see [7, Corollary 1.2.17]). Also, $P(H) = H$ and H is pseudo-complemented, but $P(G) = \varnothing$ and G is not pseudo-complemented.

As in the renowned *Heine-Borel Theorem* from Real Analysis, an element c of a complete lattice L is called *compact* if for any non-empty subset A of L with $c \leqslant \bigvee_{x \in A} x$ there exists a finite subset $F \subseteq A$, with $c \leqslant \bigvee_{x \in F} x$. One denotes by $K(L)$ the set of all compact elements of L.

A complete lattice L is said to be *compact* if its greatest element 1 is a compact element in L, and *compactly generated* if any element of L is a join of compact elements. Note that any compactly generated lattice is upper continuous (see, e.g., [50, Chapter III, Proposition 5.3]).

Let M_R be a right R-module, and let $N \leqslant M_R$. Then N is a compact (respectively, pseudo-complement, complement) element of the lattice $\mathcal{L}(M_R)$ of all submodules of M_R if and only if N is a finitely generated (respectively, complement, direct summand) submodule of M. In particular, because any module is the sum of all its cyclic submodules, it follows that $\mathcal{L}(M_R)$ is a compactly generated lattice for any module M_R.

6.2 The conditions (C_i) for lattices

Throughout this section L will denote a modular lattice with a least element 0 and a greatest element 1. Recall the following notation:

$P(L) :=$ the set of all *pseudo-complement* elements of L
(P for "*Pseudo*"),
$E(L) :=$ the set of all *essential* elements of L
(E for "*Essential*"),
$C(L) :=$ the set of all *closed* elements of L
(C for "*Closed*"),
$D(L) :=$ the set of all *complement* elements of L
(D for "*Direct summand*"),
$K(L) :=$ the set of all *compact* elements of L
(K for "*Kompakt*").

We present now five conditions (C_i), $i = 1, 2, 3, 11, 12$, introduced in [10] as the latticial counterparts of the well-known corresponding conditions in Module Theory.

Definitions For a lattice L one may consider the following conditions:

(C_1) For every $x \in L$ there exists $d \in D(L)$ such that $x \in E(d/0)$.
(C_2) For every $x \in L$ such that $x/0 \simeq d/0$ for some $d \in D(L)$, one has $x \in D(L)$.
(C_3) For every $d_1, d_2 \in D(L)$ with $d_1 \wedge d_2 = 0$, one has $d_1 \vee d_2 \in D(L)$.
(C_{11}) For every $x \in L$ there exists a pseudo-complement p of x with $p \in D(L)$.
(C_{12}) For every $x \in L$ there exist $d \in D(L)$, $e \in E(d/0)$, and a lattice isomorphism $x/0 \simeq e/0$. □

Definitions A lattice L is called *CC* or *extending* if it satisfies (C_1), *continuous* if it satisfies (C_1) and (C_2), and *quasi-continuous* if it satisfies (C_1) and (C_3). □

6.3 CC lattices

The next result provides the connections between the conditions (C_i), $i = 1, 2, 3, 11, 12$, and characterizes essentially closed CC lattices in terms of closeness; in particular, it explains the term of CC, acronym for *C*losed elements are *C*omplements.

Proposition 6.3 ([10, Proposition 1.10]) *The following statements hold for a lattice $L \in \mathcal{M}_{0,1}$.*

(1) *L is uniform $\Longrightarrow$ L is quasi-continuous $\Longrightarrow$ L is CC.*
(2) *If L is indecomposable, then L is CC $\Longleftrightarrow$ L is uniform.*
(3) *If additionally L is essentially closed, then*

$$L \text{ is CC} \iff C(L) \subseteq D(L) \iff C(L) = D(L).$$

(4) *If additionally L is strongly pseudo-complemented, then*

$$L \text{ is CC} \iff C(L) \subseteq D(L) \iff C(L) = D(L) \iff$$
$$\iff P(L) \subseteq D(L) \iff P(L) = D(L).$$

(5) *L satisfies (C_2) $\Longrightarrow$ L satisfies (C_3).*
(6) *L satisfies (C_1) $\Longrightarrow$ L satisfies (C_{11}).*
(7) *L satisfies (C_{11}) $\Longrightarrow$ L satisfies (C_{12}).* □

As well-known, the conditions (C_i), $i = 1, 2, 3$, for modules are inherited by direct summands. The next result presents the latticial counterpart of this property.

Proposition 6.4 ([10, Proposition 1.15]) *Let $L \in \mathcal{M}_{0,1}$ be a strongly pseudo-complemented lattice (in particular, an upper continuous lattice), and let $d \in D(L)$. If L satisfies (C_i), $i = 1, 2, 3$, then $d/0$ also satisfies (C_i), $i = 1, 2, 3$, in other words, the conditions (C_i), $i = 1, 2, 3$, are inherited by complement intervals.* □

Corollary 6.5 ([7, Corollary 5.1.4]) *Let $L \in \mathcal{M}_{0,1}$ be a strongly pseudo-complemented CC lattice. Then L has finite Goldie dimension if and only if 1 is a finite direct join of uniform elements of L.* □

6.4 CEK lattices

We present now the concept of a *CEK* lattice intervening in the key Lemma 6.7, the chief ingredient used in the proof of the main result of this section.

Definitions Let L be a lattice.

(1) An element $a \in L$ is called *essentially compact* if there exists a compact element k of L such that $k \in E(a/0)$, in other words, $E(a/0) \cap K(L) \neq \varnothing$, and $E_k(L)$ will denote the set of all essentially compact elements of L.
(2) L is called CEK (for *C*losed are *E*ssentially *C*ompact) if every closed element of L is essentially compact, i.e., $C(L) \subseteq E_k(L)$. □

The next result provides large classes of CEK lattices.

Proposition 6.6 ([7, Proposition 5.1.5]) *Let L be a non-zero complete modular lattice having the following property:*

$$(\dagger) \quad \textit{For every } 0 \neq x \in L \ \textit{ there exists } 0 \neq k \in K(L) \ \textit{ with } k \leqslant x.$$

In particular, L can be any compactly generated lattice.

Then L has finite Goldie dimension if and only if each element of L is essentially compact, i.e., $L = E_k(L)$. *In particular, any modular lattice with finite Goldie dimension satisfying* ($\dagger$) *is CEK.* □

6.5 Three lemmas

We present below three preparatory facts that will be used to prove the main result of this section. Notice that the first one has a very technical 6-page proof in [5].

Lemma 6.7 ([5, Lemma 2.1]) *Let L be a compact, compactly generated, modular lattice. Assume that all compact intervals b/a of L are CEK, i.e., every $c \in C(b/a)$ is an essentially compact element of b/a. Then $D(L)$ is a Noetherian poset.* □

The next result is the latticial counterpart of a well-known result asserting that a non-zero module M_R satisfying ACC or DCC on direct summands is a finite direct sum of finitely many indecomposable submodules (see, e.g., [23, Proposition 10.14]).

Lemma 6.8 ([5, Lemma 3.1]) *Let* $\{0\} \neq L \in \mathcal{M}_{0,1}$ *and assume that the set $D(L)$ of complement elements of L is either Noetherian or Artinian. Then* 1 *is a direct join of finitely many indecomposable elements of L.* □

Lemma 6.9 ([5, Lemma 3.2]) *Any modular, upper continuous, compact, CC lattice is CEK.* □

6.6 The Latticial Osofsky–Smith Theorem

Theorem 6.10 (Latticial O-ST [5, Theorem 3.4]) *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are CC. Then* 1 *is a finite direct join of uniform elements of L.*

Proof First, observe that the given lattice L being compactly generated, is also upper continuous. Recall that by a subfactor of L we mean any interval b/a of L. By assumption, every compact subfactor of L is CC, so CEK by Lemma 6.9. Using now Lemma 6.7, we deduce that $D(L)$ is a Noetherian poset, so, by Lemma 6.8, $1 = \dot{\bigvee}_{1 \leqslant i \leqslant n} d_i$ is a finite direct join of indecomposable elements d_i of L. Since L is CC, so is also any $d_i/0$ by Proposition 6.4. Finally, every d_i is uniform by Proposition 6.3(1), and we are done. □

Following [28], a right R-module M is said to be *CF* if every closed submodule of M is finitely generated. More generally, we say that a lattice L is *CK* (acronym for *C*losed are *K*ompact) if every closed element of L is compact, i.e., $C(L) \subseteq K(L)$. Clearly, any CK lattice is also CEK, so we deduce at once from Lemmas 6.7 and 6.8 the following result.

Proposition 6.11 *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are CK. Then $D(L)$ is a Noetherian poset, in particular* 1 *is a finite direct join of indecomposable elements of L.* □

We extend now the Latticial O-ST to more general lattices, so that it can be also applied to cyclic modules (which have no latticial counterparts).

Denote by $\mathcal{K}$ the class of all compact lattices and by $\mathcal{U}$ the class of all upper continuous lattices, and let $\mathcal{P}$ be a non-empty subclass of $\mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the following three conditions:

(P_1) If $L \in \mathcal{P}$, $L' \in \mathcal{L}$, and $L \simeq L'$ then $L' \in \mathcal{P}$.
(P_2) If $L \in \mathcal{P}$ then $1/a \in \mathcal{P}$, $\forall\, a \in L$.
(P_3) If $L \in \mathcal{P}$ and $b/a \in \mathcal{P}$ is a subfactor of L, then $\exists\, c \in L$ such that $c/0 \in \mathcal{P}$ and $b = a \vee c$.

Examples of classes $\mathcal{P}$ satisfying the conditions $(P_1) - (P_3)$ above are:

- any $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ such that

$$L \in \mathcal{P} \Longrightarrow (1/a \in \mathcal{P} \;\;\&\;\; a/0 \in \mathcal{P},\ \forall\, a \in L);$$

- the class of all compact, compactly generated, modular lattices;
- the class of all compact, semi-atomic, upper continuous, modular lattices;
- the class of lattices isomorphic to lattices of all submodules of all cyclic right R-modules.

For any lattice L we set $\mathcal{P}(L) := \{\, c \in L \mid c/0 \in \mathcal{P}\,\}$. Observe that $\varnothing \neq \mathcal{P}(L) \subseteq K(L)$ if $L \in \mathcal{U}$.

Theorem 6.12 (Latticial $\mathcal{P}$-O-ST [5, Theorem 3.7]) *Let $\mathcal{P}$ be a non-empty subclass of $\mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above, and let $L \in \mathcal{P}$. Assume that all subfactors of L in $\mathcal{P}$ are CC. Then 1 is a finite direct join of uniform elements of L.* □

Corollary 6.13 ([5, Corollary 3.9]) *Let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1)-(P_3)$ above. Then, the following statements are equivalent for a complete modular lattice L such that any of its elements is a join of elements of $\mathcal{P}(L)$.*

(1) *L is semi-atomic.*
(2) *F is CC and $K(F) \subseteq D(F)$ for every subfactor $F \in \mathcal{P}$ of L.* □

Notice that Corollary 6.13 is a latticial version of the following module-theoretical result:

A right R-module M is semisimple $\iff$ every cyclic subfactor of M is M-injective

(see [29, Corollary 7.14]), which, in turn, is a "modularization" of the well-known *Osofsky's Theorem* [48] saying that a ring R is semisimple if and only if every cyclic right R-module is injective. Because we do not have in hand a good latticial substitute for the notion of an injective module, the result above seems to be the best latticial counterpart of the Osofsky's Theorem.

6.7 The Categorical Osofsky–Smith Theorem

This subsection deals with the *absolutization* of the module-theoretical O-ST. Thus, by applying the Latticial O-ST to the specific case of Grothendieck categories we obtain at once the *Categorical* or *Absolute Osofsky–Smith Theorem.*

Throughout this subsection $\mathcal{G}$ will denote a fixed *Grothendieck category,* and for any object $X \in \mathcal{G}$, $\mathcal{L}(X)$ will denote the upper continuous modular lattice of all subobjects of X. For any subobjects Y and Z of X we denote by $Y \cap Z$ their meet and by $Y + Z$ their join in the lattice $\mathcal{L}(X)$.

Recall that we called an object $X \in \mathcal{G}$ *Artinian* (respectively, *Noetherian*) if the lattice $\mathcal{L}(X)$ of all subobjects of X is Artinian (respectively, Noetherian). More generally, if $\mathbb{P}$ is a property on lattices, an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$, and a subobject Y of X is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. However, for a complement (respectively, compact) subobject of an object $X \in \mathcal{G}$ one uses the well established term of a *direct summand* (respectively, *finitely generated* subobject) of X; for this reason, instead of saying that X is a CC object we will say that X is a CS object (acronym for *C*losed subobjects are direct *S*ummands).

Recall that the category $\mathcal{G}$ is called *locally finitely generated* if it has a family $(U_i)_{i \in I}$ of generators (this means that $\bigoplus_{i \in I} U_i$ is a generator of $\mathcal{G}$) and all U_i's are finitely generated, or equivalently if the lattices $\mathcal{L}(X)$ are compactly generated for all objects X of $\mathcal{G}$. We say that an object $X \in \mathcal{G}$ is *locally finitely generated* if the lattice $\mathcal{L}(X)$ is compactly generated.

If we specialize Theorem 6.10 for the particular case $L = \mathcal{L}(X)$ we obtain immediately the next result.

Theorem 6.14 (Categorical O-ST) *Let $\mathcal{G}$ be a Grothendieck category, and let $X \in \mathcal{G}$ be a finitely generated, locally finitely generated object such that every finitely generated subfactor object of X is CS. Then X is a finite direct sum of uniform objects.* □

An object X of a Grothendieck category $\mathcal{G}$ is called *CF* (acronym for *C*losed are *F*initely generated) if every closed subobject of X is finitely generated, and *completely CF* if every quotient object of X is CF. If we specialize Proposition 6.11 for the particular case $L = \mathcal{L}(X)$ we obtain at once the following result.

Corollary 6.15 *Let X be a finitely generated, locally finitely generated object of a Grothendieck category $\mathcal{G}$ such that every finitely generated subobject of X is completely CF. Then X is a finite direct sum of indecomposable subobjects.* □

Denote by $\mathcal{H}$ the class of all finitely generated objects of $\mathcal{G}$, and let $\mathcal{A}$ be a subclass of $\mathcal{H}$ satisfying the following three conditions:

(A_1) If $X \in \mathcal{A},\ X' \in \mathcal{G},$ and $X \simeq X'$, then $X' \in \mathcal{A}$.
(A_2) If $X \in \mathcal{A}$ then $X/X' \in \mathcal{A},\ \forall\, X' \subseteq X$.
(A_3) If $X \in \mathcal{A}$ and $Z \subseteq Y \subseteq X$ with $Y/Z \in \mathcal{A}$, then $\exists\, U \subseteq X$ such that $U \in \mathcal{A}$ and $Y = Z + U$.

The class $\mathcal{H}$ could be empty (see, e.g., [21, p. 1539] and in this case everything that follows makes no sense.

The next result is an immediate specialization of Theorem 6.12 for the particular case $L = \mathcal{L}(X)$.

Theorem 6.16 (Categorical $\mathcal{A}$-O-ST) *Let $\mathcal{A}$ be a class of finitely generated objects of a Grothendieck category $\mathcal{G}$ satisfying the conditions $(A_1) - (A_3)$ above, and let $X \in \mathcal{A}$. Assume that all subfactors of X in $\mathcal{A}$ are CS. Then X is a finite direct sum of uniform objects of $\mathcal{G}$.* □

We present now a consequence, involving injective objects, of the Categorical O-ST. Recall that for any Grothendieck category one can define as in Mod-R the concepts of an *M-injective object, simple object,* and *semisimple object* (see, e.g., [15, p. 9]).

Proposition 6.17 ([6, Proposition 4.14]) *The following assertions are equivalent for a locally finitely generated object X of a Grothendieck category $\mathcal{G}$.*

(1) *X is semisimple.*
(2) *Every finitely generated subfactor of X is X-injective.* □

We end this subsection by mentioning that some statements/results of [49] and [26] related to the Categorical O-ST saying that "*basically the same proof for modules works in the categorical setting*" are not in order (see [6, p. 2670]). Such statements are very risky and may lead to incorrect results. One reason is that we cannot prove equality between two subobjects of an object in a category as we do for submodules by taking elements of them. Notice that the well-hidden errors in the statements/results occurring in the papers mentioned above on the Categorical O-ST could be spotted only by using our latticial approach of it. So, we do not only correctly absolutize the module-theoretical O-ST but also provide a correct proof of its categorical extension by passing first through its latticial counterpart.

6.8 The Relative Osofsky–Smith Theorem

In this subsection we present the relative version with respect to a hereditary torsion theory of the module-theoretical O-ST [49, Theorem 1]. Its proofs is an easy application of the Latticial O-ST.

Throughout the remainder of this section R denotes a ring with non-zero identity, $\tau = (\mathcal{T}, \mathcal{F})$ a fixed hereditary torsion theory on Mod-R, and $\tau(M)$ the τ-torsion submodule of a right R-module M. For any module M_R we have denoted

$$\mathrm{Sat}_\tau(M) := \{ N \mid N \leqslant M,\ M/N \in \mathcal{F} \}.$$

Let M_R be a module. In Sect. 2 we said that M is τ-Artinian (respectively, τ-Noetherian) if the lattice $\mathrm{Sat}_\tau(M)$ is Artinian (respectively, Noetherian). More generally, if $\mathbb{P}$ is any property on lattices, we say that a module M is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$, and a submodule N of M is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, a module M_R is called *τ-CC* (or *τ-extending*) if the lattice $\mathrm{Sat}_\tau(M)$ is CC (or extending). However, in the sequel we shall use the more appropriate term of a *τ-CS* module (respectively, *τ-direct summand* of a module) instead of that of a *τ-CC* module (respectively, *τ-complement submodule* of a module).

Consider the quotient category Mod-$R/\mathcal{T}$ of Mod-R modulo its localizing subcategory $\mathcal{T}$, and let

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

be the canonical functor. We have seen in Proposition 3.4 that for any M_R, the mapping

$$\mathrm{Sat}_\tau(M) \longrightarrow \mathcal{L}(T_\tau(M)), \quad N \mapsto T_\tau(N),$$

is an isomorphism of lattices, so, for any property $\mathbb{P}$ on lattices, the module M_R is/has τ-$\mathbb{P}$ if and only if the object $T_\tau(M)$ in the quotient Grothendieck category Mod-$R/\mathcal{T}$ is/has $\mathbb{P}$.

We present now intrinsic characterizations, that is, without explicitly referring to the lattice $\mathrm{Sat}_\tau(M)$, of the relative concepts that will appear in the Relative O-ST.

Proposition 6.18 ([6, Proposition 5.3]) *The following assertions hold for a module M_R and $N \leqslant M$.*

(1) *N is τ-essential in $M \Longleftrightarrow (\forall\, P \leqslant M,\ P \cap N \in \mathcal{T} \Longrightarrow P \in \mathcal{T})$.*
(2) *M is τ-uniform $\Longleftrightarrow (\forall\, P,\ K \leqslant M,\ P \cap K \in \mathcal{T} \Longrightarrow P \in \mathcal{T}$ or $K \in \mathcal{T})$.*
(3) *N is a τ-pseudo-complement in $M \Longleftrightarrow \exists\, P \leqslant M$ with $N \cap P \in \mathcal{T}$ and N is maximal among the submodules of M having this property; in this case $N \in \mathrm{Sat}_\tau(M)$ and $N \cap \overline{P} = \tau(M)$.*
(4) *N is τ-closed in $M \Longleftrightarrow \forall\, P \leqslant M$ such that $N \subseteq P$ and N is a τ-essential submodule of P one has $P/N \in \mathcal{T}$. If additionally $N \in \mathrm{Sat}_\tau(M)$, then N is τ-closed in $M \Longleftrightarrow N$ has no proper τ-essential extension in M.*

(5) *N is a τ-direct summand in $M \iff \exists\, P \leqslant M$ with $M/(N+P) \in \mathcal{T}$ & $N \cap P \in \mathcal{T}$.*
(6) *M is τ-complemented $\iff \forall\, N \leqslant M$, $\exists\, P \leqslant M$ with $M/(N+P) \in \mathcal{T}$ & $N \cap P \in \mathcal{T}$.*
(7) *M is τ-compact $\iff \forall\, N \leqslant M$ with $M/N \in \mathcal{T}$, $\exists\, N' \leqslant N$ such that N' is finitely generated and $M/N' \in \mathcal{T}$, in other words, the filter $F(M) := \{N \leqslant M \mid M/N \in \mathcal{T}\}$ has a basis consisting of finitely generated submodules.*
(8) *M is τ-CEK $\iff$ any τ-closed submodule of M is a τ-essential submodule of a τ-compact submodule of M.*
(9) *M is τ-compactly generated $\iff \forall\, N \leqslant M$, $\exists\, I_N$ a set and a family $(C_i)_{i\in I_N}$ of τ-compact submodules of M such that $\sum_{i\in I_N} C_i \subseteq N$ and $N/\big(\sum_{i\in I_N} C_i\big) \in \mathcal{T}$.* □

We say that a finite family $(N_i)_{1\leqslant i\leqslant n}$ of submodules of a module M_R is *τ-independent* if $N_i \notin \mathcal{T}$ for all $1 \leqslant i \leqslant n$, and

$$N_{k+1} \cap \sum_{1\leqslant j\leqslant k} N_j \subseteq \tau(M),\ \forall\, k,\ 1 \leqslant k \leqslant n-1,$$

or, equivalently

$$\overline{N_{k+1}} \cap \overline{\sum_{1\leqslant j\leqslant k} N_j} = \overline{N_{k+1}} \wedge \Big(\bigvee_{1\leqslant j\leqslant k} \overline{N_j} \Big) = \tau(M),$$

in other words, the family $\big(\overline{N_i}\big)_{1\leqslant i\leqslant n}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent. More generally, a family $(N_i)_{i\in I}$ of submodules of M is called *τ-independent* if the family $\big(\overline{N_i}\big)_{i\in I}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent.

Theorem 6.19 (Relative O-ST) [6, Theorem 5.8]) *Let M_R be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then there exists a finite τ-independent family $(U_i)_{1\leqslant i\leqslant n}$ of τ-uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

A more simplified version of the Relative O-ST in case the given module M_R is τ-torsion-free is the following:

Theorem 6.20 ([6, Theorem 5.12]) *Let $M_R \in \mathcal{F}$ be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then, there exists a finite independent family $(U_i)_{1\leqslant i\leqslant n}$ of uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

Since M is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is so, in view of Theorem 6.20 we can of course formulate the Relative O-ST in terms of essentiality and independence in the lattice $\mathcal{L}(M/\tau(M))$ instead of the relative ones in the lattice $\mathcal{L}(M)$:

Theorem 6.21 *Let M_R be a τ-compact, τ-compactly generated module. If all τ-compact subfactors of M are τ-CS, then there exists a finite family $(U_i)_{1\leqslant i\leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1\leqslant i\leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

Recall that for a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R we have denoted in Sect. 2 by

$$F_\tau := \{ I \leqslant R_R \mid R/I \in \mathcal{T} \}$$

the *Gabriel filter* associated with τ. By a *basis* of the Gabriel filter F_τ we mean a subset B of F_τ such that every right ideal in F_τ contains some $J \in B$.

By [9, Proposition 2.12], a Grothendieck category $\mathcal{G}$ has a finitely generated generator if and only if there exists a unital ring A and a hereditary torsion theory $\rho = (\mathcal{H}, \mathcal{E})$ on Mod-A such that $\mathcal{G} \simeq$ Mod-$A/\mathcal{H}$ and the Gabriel filter F_ρ has a basis consisting of finitely generated right ideals of A. Therefore, in case F_τ has a basis consisting of finitely generated right ideals of R, the Grothendieck category Mod-$R/\mathcal{T}$ is locally finitely generated, and so, any module M_R is τ-compactly generated. Thus, the next result is an immediate consequence of Theorem 6.21.

Theorem 6.22 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian), and let M_R be a τ-compact module. If all τ-compact subfactors of M are τ-CS, then there exists a finite family $(U_i)_{1 \leqslant i \leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1 \leqslant i \leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1 \leqslant i \leqslant n} U_i) \in \mathcal{T}$.* □

According to our definitions above of module-theoretical concepts relative to a hereditary torsion theory τ, a module U_R is said to be τ-*simple* if the lattice $\mathrm{Sat}_\tau(U)$ is simple, which means that it has exactly two elements, i.e.,

$$U \notin \mathcal{T} \qquad \text{and} \qquad \mathrm{Sat}_\tau(U) = \{\tau(U), U\}.$$

Recall that U_R is called τ-*cocritical* if it is τ-simple and $U \in \mathcal{F}$.

The τ-*socle* of a module M_R, denoted by $\mathrm{Soc}_\tau(M)$, is defined as the τ-saturation of the sum of all τ-simple (or τ-cocritical) submodules of M, and M is said to be τ-*semisimple* if $M = \mathrm{Soc}_\tau(M)$. By [8, Proposition 6.5], $\mathrm{Soc}_\tau(M)$ is exactly the socle of the lattice $\mathrm{Sat}_\tau(M)$, and so, we have

$$M_R \text{ is a } \tau\text{-semisimple module} \iff \mathrm{Sat}_\tau(M) \text{ is a semi-atomic lattice} \iff$$
$$\iff T_\tau(M) \text{ is a semisimple object of the quotient category Mod-}R/\mathcal{T}.$$

The next result is a relative version of the well-known *Osofsky's Theorem* [48].

Proposition 6.23 ([6, Proposition 5.16]) *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian). Assume that R/I is an injective R-module for each $I \in \mathrm{Sat}_\tau(R)$. Then, any right R-module is τ-semisimple.* □

Acknowledgments The author would like to thank the organizers, especially *Jairo Zacharias Gonçalves* and *Vitor de Oliveira Ferreira* for the invitation to deliver a talk at the Conference "Groups, Rings, and Group Rings 2024", Ubatuba, Brazil, 21-26 July 2014, dedicated to Prof. *César Polcino Milies* on his 75th birthday, as well as for financial support.

References

1. Albu, T.: On commutative Grothendieck categories having a Noetherian cogenerator. Arch. Math. (Basel) **34**, 210–219 (1980)
2. Albu, T.: Certain Artinian lattices are Noetherian. Applications to the Relative Hopkins–Levitzki Theorem. In: Van Oystaeyen, F. (ed.) Methods in Ring Theory, pp. 37–52. D. Reidel Publishing, Dordrecht (1984)
3. Albu, T.: Classes of lattices (co)generated by a lattice and their global (dual) Krull dimension. Discrete Math. **185**, 1–18 (1998)
4. Albu, T.: A seventy year jubilee: The Hopkins–Levitzki Theorem. In: Albu, T., Birkenmeier, G.F., Erdoğan, A., Tercan, A. (eds.) Ring and Module Theory, Trends in Mathematics, pp. 1–26. Birkhäuser, Basel (2010)
5. Albu, T.: The Osofsky–Smith Theorem for modular lattices, and applications (I). Commun. Algebra **39**, 4488–4506 (2011)
6. Albu, T.: The Osofsky–Smith Theorem for modular lattices, and applications (II). Commun. Algebra **42**, 2663–2683 (2014)
7. Albu, T.: Topics in Lattice Theory with Applications to Rings, Modules, and Categories, Lecture Notes, XXIII Brazilian Algebra Meeting, Maringá, Paraná, Brasil (2014)
8. Albu, T., Iosif, M.: Lattice preradicals with applications to Grothendieck categories and torsion theories. J. Algebra **444**, 339–366 (2015). doi:10.1016/j.jalgebra.2015.07.030
9. Albu, T., Iosif, M., Teply, M.L.: Dual Krull dimension and quotient finite dimensionality. J. Algebra **284**, 52–79 (2005)
10. Albu, T., Iosif, M., Tercan, A.: The conditions (C_i) in modular lattices, and applications. J. Algebra Appl. 15 (2016). doi:10.1142/S0219498816500018
11. Albu, T., Krause, G., Teply, M.L.: The nilpotence of the τ-closed prime radical in rings with τ-Krull dimension. J. Algebra **229**, 498–513 (2000)
12. Albu, T., Lenagan, T.H., Smith, P.F.: Ascending chains of submodules in modules having Krull dimension. Rev. Roumaine Math. Pures Appl. **41**, 567–581 (1996)
13. Albu, T., Năstăsescu, C.: Décompositions primaires dans les catégories de Grothendieck commutatives (I). J. Reine Angew. Math. **280**, 172–194 (1976)
14. Albu, T., Năstăsescu, C.: Décompositions primaires dans les catégories de Grothendieck commutatives (II). J. Reine Angew. Math. **282**, 172–185 (1976)
15. Albu, T., Năstăsescu, C.: Relative Finiteness in Module Theory, Monographs and Textbooks in Pure and Applied Mathematics, vol. 84. Marcel Dekker Inc., New York, Basel (1984)
16. Albu, T., Smith, P.F.: Dual relative Krull dimension of modules over commutative rings. In: Facchini, A., Menini, C. (eds.) Abelian Groups and Modules, pp. 1–15. Kluwer Academic Publisher, Dordrecht (1995)
17. Albu, T., Smith, P.F.: Localization of modular lattices, Krull dimension, and the Hopkins–Levitzki Theorem (I). Math. Proc. Cambridge Philos. Soc. **120**, 87–101 (1996)
18. Albu, T., Smith, P.F.: Localization of modular lattices, Krull dimension, and the Hopkins–Levitzki Theorem (II). Commun. Algebra **25**, 1111–1128 (1997)
19. Albu, T., Smith, P.F.: Dual Krull dimension and duality. Rocky Mountain J. Math. **29**, 1153–1165 (1999)
20. Albu, T., Smith, P.F.: Corrigendum and Addendum to Localization of modular lattices, Krull dimension, and the Hopkins–Levitzki Theorem (II). Commun. Algebra **29**, 3677–3682 (2001)
21. Albu, T., Van Den Berg, J.: An indecomposable nonlocally finitely generated Grothendieck category with simple objects. J. Algebra **321**, 1538–1545 (2009)
22. Albu, T., Wisbauer, R.: Generators in Grothendieck categories with right perfect endomorphism rings. Osaka J. Math. **28**, 295–304 (1991)
23. Anderson, F.W., Fuller, K.R.: Rings and Categories of Modules, 2nd edn. Springer, New York (1992)
24. Birkhoff, G.: Lattice Theory, 3rd edn. American Mathematical Society, Providence, RI (1967)

25. Crawley, P., Dilworth, R.P.: Algebraic Theory of Lattices. Prentice-Hall, Englewood Cliffs, New Jersey (1973)
26. Crivei, S., Năstăsescu, C., Torrecillas, B.: On the Osofsky–Smith Theorem. Glasgow Math. J. **52A**, 61–87 (2010)
27. Dickson, S.E.: A torsion theory for Abelian categories. Trans. Amer. Math. Soc. **121**, 223–235 (1966)
28. Dung, N.V.: Modules whose closed submodules are finitely generated. Proc. Edinb. Math. Soc. **34**, 161–166 (1991)
29. Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R.: Extending Modules. Longman Scientific & Technical, Harlow (1990)
30. Faith, C.: Injective modules and injective quotient rings, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, Basel (1982)
31. Gabriel, P.: Des catégories abéliennes. Bull. Soc. Math. France **90**, 323–448 (1962)
32. Golan, J.S.: Torsion Theories. Pitman/Longman, New York (1986)
33. Gordon, R., Robson, J.C.: Krull Dimension. Mem. Amer. Math. Soc. **133** (1973)
34. Gordon, R., Robson, J.C.: The Gabriel dimension of a module. J. Algebra **29**, 459–473 (1974)
35. Grätzer, G.: General Lattice Theory, 2nd edn. Birkhäuser Verlag, Basel Boston Berlin (2003)
36. Hopkins, C.: Rings with minimal condition for left ideals. Ann. Math. **40**, 712–730 (1939)
37. Lemonnier, B.: Déviation des ensembles et groupes abéliens totalement ordonnés. Bull. Sci. Math. **96**, 289–303 (1972)
38. Lenagan, T.H.: The nil radical of a ring with Krull dimension. Bull. London Math. Soc. **5**, 307–311 (1973)
39. Levitzki, J.: On rings which satisfy the minimum condition for the right-hand ideals. Compositio Math. **7**, 214–222 (1939)
40. McConnell, J.C., Robson, J.C.: Noncommutative Noetherian Rings. Wiley, New York (1987)
41. Miller, R.W., Teply, M.L.: The descending chain condition relative to a torsion theory. Pac. J. Math. **83**, 207–220 (1979)
42. Năstăsescu, C.: Teorie della torsione. Quaderni dei Gruppi di Ricerca Matematica CNR, Università di Ferrara (1974)
43. Năstăsescu, C.: Conditions de finitude pour les modules. Rev. Roumaine Math. Pures Appl. **24**, 745–758 (1979)
44. Năstăsescu, C.: Théorème de Hopkins pour les catégories de Grothendieck, In: Ring Theory, Proceedings of the 1980 Antwerp Conference, Lecture Notes in Mathematics **825**, pp. 88-93. Springer, Berlin, Heidelberg, New York (1980)
45. Năstăsescu, C.: Conditions de finitude pour les modules (II). Rev. Roumaine Math. Pures Appl. **25**, 615–630 (1980)
46. Năstăsescu, C.: Δ-Anneaux et modules Σ-injectifs. Applications aux catégories localement artiniens. Commun. Algebra **9**, 1981–1996 (1981)
47. Năstăsescu, C., Van Oystaeyen, F.: Dimensions of Ring Theory. D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo (1987)
48. Osofsky, B.L.: Rings all of whose finitely generated modules whose are injective. Pacif. J. Math. **14**, 645–650 (1964)
49. Osofsky, B., Smith, P.F.: Cyclic modules whose quotients have all complement submodules direct summands. J. Algebra **139**, 342–354 (1991)
50. Stenström, B.: Rings of Quotients. Springer, Berlin, Heidelberg, New York (1975)