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ARTICLE

The transformation problem: a mathematical approach of its solution within Marx's original framework

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Abstract

This study demonstrates that the transformation problem concerns whether a contradiction exists between Marx's concept of labor value and that of production price. In a narrow sense, this is just a mathematical problem. It can be boiled down to the construction of a mathematical model that can simulate the conversion from value to production price to conclude whether the constructed model meets certain constraints (so-called "the double invariance"). However, models of most researchers so far met only one of the two constraints, some of them couldn't satisfy any one of them. So, they tried to explain this problem by modifying Marx's conditions or conclusions, and hence doubted or even denied Marx's conclusions. This was actually an act of self-deception. In contrast to related attempts that tampered with Marx's transformation conditions or conclusions, this study is based on the solution of the transformation problem within Marx's original framework. It extends Bortkiewicz (Third volume of Capital, Sweezy, 1907) and Samuelson (Am Econ Rev 47:884–912, 1957, Proc Natl Acad Sci 67:423–425, 1970, J Econ Lit 9:399–431, 1971) to overcome their shortcomings. This study provides the model of static transformation and that of dynamic transformation and illustrates them with examples of mathematical simulation: the static transformation is the result of the dynamic transformation and its end point. Both the general dynamic transformation model and the general static transformation model satisfy two constraints of Marx simultaneously. So, this study offers a new mathematical approach to the transformation problem. The author obtained a solution to the transformation problem in 2000 (a static version in line with Marx's original intention), continued to conduct research, and published several research results. However, so far, the author's publication was mainly in Chinese and Japanese and did not receive due attention in international academia. As I received the invitation to the Marx special feature of from this journal (*This plan was not realized. Some of contributed papers to this plan were published in this journal's special feature "Promenade in the history of economic thought" (vol. 18–1). [Associate Editor]), I endeavored to describe my research in this topic deeper.

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1 The transformation problem and the basic principles for the construction of its model

The so-called transformation problem concerns whether a contradiction exists between the concept of labor value and that of production price in Marx's theoretical system.

Scholarly controversies over this problem have ranged very wide. Yet, in a narrow sense, the true nature of the transformation problem is a mathematical issue. It can be boiled down to the construction of a mathematical model that can simulate the conversion from value to production price, and then to conclude whether this model meets certain constraints (so-called "the double invariance": (1) total production price = total value; (2) total average profit = total surplus value).

Hitherto, researchers of Marxian economics have constructed a large number of mathematical models to deal with this problem. Nevertheless, prior to Zhang (2000), most of their models satisfied only one of the two constraints—some satisfied none of two.[1] They had to modify the constraints, or conclusion of Marx, to get out of the impasse. As a result, the research on this problem became complicated whilst moving farther and farther away from Marx's original intention.

We would not say that the efforts of researchers were totally wasted. The research has gradually approached to the correct path of the solution. Of the many literatures, the most important contributions were Bortkiewicz (1907), Samuelson (1957, 1970, 1971), and Zhang (2000). In the next section, we will follow them in chronological order to clarify the process to approach to the correct path.

From the perspective of mathematics, the construction of models of the transformation problem needs to observe a number of basic principles. Put specifically, the following questions need to be categorically answered:

1) Whether or not to adhere to Marx's original framework for studying the transformation problem? A choice to be made, first of all.
2) How should mathematical conditions be set, as prerequisites for building a model of the transformation problem, without losing generality?
3) How to distinguish between endogenous variables and exogenous variables?
4) How to establish the mathematical relation between value and production prices?
5) How to construct a system of equations that reflects the relationship between value and production price?
6) How to deal with the relationship between the above equations and the constraint conditions of the transformation problem?

[1] See more details in Zhang (2004, 2019).

These questions will be answered below in Sect. 3 of the paper.

In Sect. 4, we will discuss several issues related to the transformation problem. In particular, we provide a model of inverse transformation. We will also mathematically give the static transformation model (the relation between value and production price in a certain year) and the dynamic transformation model (simulating the process of transformation from value to production price over several years[2]).

In Sect. 5, we provide a scheme to unify static transformation and dynamic transformation from a mathematical point of view. Section 6 gives a brief conclusion.

2 Explorations and misconceptions in the construction of transformation models

Regarding the conversion of value to production price, Marx provided a transitional calculation in the third volume of *Capital*.[3] Marx's starting point was the value system:

$$c_i + v_i + m_i = w_i \quad (i = 1, 2, \dots, n) \tag{1}$$

Here, c_i, v_i, m_i, and w_i represent constant capital, variable capital, surplus value, and total value of the i^{th} department under the value system, respectively.

Marx's production price system is derived as

$$(1 + r)\left(c_i + v_i\right) = P_i \quad (i = 1, 2, \dots, n) \tag{2}$$

Here, P_i represents the total production price of the ith department, and r is the average profit rate. Marx wrote production price system as follows:

$$r = \frac{\sum_{i=1}^{n} m_i}{\sum_{i=1}^{n} \left(c_i + v_i\right)} \tag{3}$$

In this way, the calculation of r in formula (1) becomes extremely simple. Table 1 presents a specific example of this calculation. Because the cost price part $(c_i + v_i)$ in formula (2) is not converted into production price, this way of calculation is incomplete. Therefore, Zhang (2004) called (2) the "half- transformation formula".

In the last row of Table 1, we see that the sum of production price is equal to the sum of value, and the sum of average profit is equal to the sum of surplus value. Whether these two equivalences can be established is the condition for judging

[2] It should be noted that the exposition in this paper is limited to mathematical simulations. It does not involve the discussion of the specific historical process of the formation of production prices in any particular country.

[3] Marx did not directly give this method, which is summed up based on Marx's relevant calculations, see Marx (1966), pp. 163–164. Bortkiewicz (1907) and others have cited it.

Table 1 Marx's half-transformation: an example

Dept	a	b	c	d	e	f	g	h	p
	Constant capital	Variable capital	Surplus value rate	Surplus value	Profit rate	Value	Average profit rate (%)	Average profit	Production price
			$c=d/b$ (%)	$d=bc$	$e=d/(a+b)$ (%)	$f=a+b+d$		$h=g(a+b)$	$p=a+b+h$
I	80	20	100	20	20	120	20	20	120
II	90	10	100	10	10	110	20	20	120
III	70	30	100	30	30	130	20	20	120
Total	240	60		60		360		60	360

Source: Marx (1966), pp. 163–164

whether Marx's concept of the transformation from value to production price is sustainable or not.

However, though Marx was conscious of the incompleteness of his transitional transformation (because the cost part is not converted into production price), he did not provide any further solution.

The mathematical refinement of Marx's transformation method—i. e. the attempt to convert the cost price part into production prices—was initiated by Ladislaus von Bortkiewicz in 1907.

Bortkiewicz (1907) constructed a transformation model based on Marx's simple reproduction theory. However, he erroneously expanded Marx's two departments into three which resulted in incorrect balances. These errors reflected in his failure of the conversion of value into the production price.

One of Bortkiewicz's main contributions was the establishment of a link between value and production price through the deviation rate of production price from value. Following the symbols in the translation by Sweezy (1949),[4] we assume that the means of production, workers' consumption goods, and capitalists' consumption goods are produced in Department I, II, and III respectively.

Bortkiwicz believed that, under the condition of simple reproduction, following equilibrium relations must hold[5]:

$$\left.\begin{aligned} c_1 + v_1 + s_1 &= c_1 + c_2 + c_3 \\ c_2 + v_2 + s_2 &= v_1 + v_2 + v_3 \\ c_3 + v_3 + s_3 &= s_1 + s_2 + s_3 \end{aligned}\right\} \tag{4}$$

Suppose that the relationship between the price and the value of the products is (on average) x for Department I, y for Department II, and z for Department III. Furthermore, let ρ be the profit rate that is common to all departments; it is also called the average profit rate.

Then, the following equations should hold:

$$\left.\begin{aligned} (1+\rho)\left(c_1x + v_1y\right) &= \left(c_1 + c_2 + c_3\right)x \\ (1+\rho)\left(c_2x + v_2y\right) &= \left(v_1 + v_2 + v_3\right)y \\ (1+\rho)\left(c_3x + v_3y\right) &= \left(s_1 + s_2 + s_3\right)z \end{aligned}\right\} \tag{5}$$

Currently, the number of unknowns is four, while there are only three equations. There are two ways to solve this problem: adding an equation or reducing an unknown. Hence, Bortkiwicz considered that, if we were to choose an appropriate price unit in such a way that the total price and total value become equal, we must set

[4] See Sweezy (1949), pp. 199–221.

[5] Because the 3 Departments of Bortkiewicz are misconceived, this balanced relationship cannot be established. For specific analysis, see Zhang (2004), p. 54, and Zhang (2018).

$$Cx + Vy + Sz = C + V + S \tag{6}$$

where

$$C = c_1 + c_2 + c_3, \quad V = v_1 + v_2 + v_3, \quad S = s_1 + s_2 + s_3.$$

If, on the other hand, the price unit and the value unit are to be regarded as identical, then we have to consider in which of the three departments the good which serves as the value and price unit (numeraire) is produced. If gold is the good in question, then Department III is involved and in place of (6), we get $z = 1$. If we follow this procedure, the number of unknowns is reduced to three (x, y, and ρ).[6]

Therefore, the new equations are as follows:

$$\left.\begin{aligned} (1+\rho)\left(c_1x + v_1y\right) &= \left(c_1 + c_2 + c_3\right)x \\ (1+\rho)\left(c_2x + v_2y\right) &= \left(v_1 + v_2 + v_3\right)y \\ (1+\rho)\left(c_3x + v_3y\right) &= s_1 + s_2 + s_3 \end{aligned}\right\} \tag{7}$$

Then, he defined $\sigma = 1 + \rho; f_i = \frac{c_i}{v_i}, g_i = c_i + v_i + s_i (i = 1, 2, 3)$, from which he obtained the general solution of the equation system as

$$x = \frac{\sigma f_1 y}{g_1 - \sigma} \quad y = \frac{g_3}{g_2 + (f_3 - f_2)\sigma}.$$

However, when Bortkiewicz used data to test the model, he found that not all data can satisfy the two constraints ("double invariance"). Therefore, Bortkiewicz concluded that Marx's transformation is valid only under certain special conditions and cannot be generalized. In fact, he denied the validity of the transformation problem.

We can find several defects in the Bortkiewicz model, but the largest consists in his confusion of the roles of constant and variable capital. This error was so subtle that even Morishima (1973) could not notice it.[7] Seton (1957) went further in this mistake and no longer distinguished constant capital from variable capital.

After Bortkiewicz (1907), although many scholars made various attempts on the transformation model, no substantial progress was made untilSamuelson (1957).[8] It was the first that pointed out the abovementioned error in the Bortkiewicz model.

To clarify this we take two departments model. We can represent the assumption of Bortkiewicz (1907), Morishima (1973, 1978), and others, in the style of the input–output Table 2 as follows:

Samuelson (1957) noticed the problem mentioned above and made following corrections:

Based on the new table, Samuelson constructed equations in production price of his two-department model as follows:

[6] See Bortkiwicz (1907), trans. Sweezy (1949), p. 202.

[7] See Zhang (2004), pp. 246–247.

[8] For example, Moszkowska (1929), Winternitz (1948), May (1948), Dobb (1955), Meek (1956, 1973), and so on.

$$\left.\begin{array}{l} p_1K = \left(wL_1 + p_1K_1\right)(1+r) \\ p_2Y = \left(wL_2 + p_1K_2\right)(1+r) \end{array}\right\}.$$

Here, Department I produces homogeneous capital goods K, and Department II produces homogeneous consumer goods Y. p_1,p_2 represent the production prices of the two departments. r is the average profit rate, and w is the wage rate (Table 3).

Samuelson's idea was correct. He expanded this logic into a system of production price equations of n departments. This system of principal equations is also correct. However, in his paper in 1970, he first constructed value equations of n departments before formulating price equations of n departments. He wrote the row vector $\boldsymbol{\pi} = (\boldsymbol{\pi}_j)_{1\times n} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots, \boldsymbol{\pi}_n)$ to denote the commodity value vector,[9] $\mathbf{a}_0 = (\mathbf{a}_{0j})_{1\times n} = (\mathbf{a}_{01}, \mathbf{a}_{02}, \dots, \mathbf{a}_{0n})$ to denote the direct labor consumption vector, and $\mathbf{a} = (\mathbf{a}_{ij})_{n\times n}$ to denote the material (dead labor) consumption coefficient matrix. Therefore, he believed that Marx's value formula in the first volume of *Capital* is written as.[10]

$$\boldsymbol{\pi} = W\mathbf{a}_0 + \boldsymbol{\pi}\mathbf{a} + sW\mathbf{a}_0. \tag{8}$$

Here, s is the rate of surplus value, and W is the wage rate. Furthermore, he denoted $\mathbf{A}_0(0)$ as the vector of all labor consumption coefficients[11]; then,

$$\mathbf{A}_0(0) = \mathbf{a}_0(\mathbf{I}-\mathbf{a})^{-1} = \left(\mathbf{a}_{01}, \mathbf{a}_{02}, \dots, \mathbf{a}_{0n}\right)\begin{bmatrix} 1-\mathbf{a}_{11} & -\mathbf{a}_{12} & \cdots & -\mathbf{a}_{1n} \\ -\mathbf{a}_{21} & 1-\mathbf{a}_{22} & \cdots & -\mathbf{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -\mathbf{a}_{n1} & -\mathbf{a}_{n2} & \cdots & 1-\mathbf{a}_{nn} \end{bmatrix}^{-1}$$

Since $W\mathbf{a}_0(\mathbf{I}-\mathbf{a})^{-1}(1+s) = W\mathbf{A}_0(0)(1+s)$, the following system is obtained by combining (8):

$$\left.\begin{array}{l} \boldsymbol{\pi} = W\mathbf{A}_0(0)(1+s) \\ \boldsymbol{\pi}\mathbf{m} = W \end{array}\right\} \tag{9}$$

[9] In fact, the vector represents the value of one unit of various commodities. This differs from the way Marx expresses it in *Capital*.

[10] The value formula (8) is in a transposition relationship with Marx's manifestation, and he also put constant capital behind variable capital. If we transpose (8) and put the constant capital before the variable capital as Marx does, (8) will become as follows:

Unit value quantity Constant capital Variable capital Surplus value

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{21} & \cdots & \mathbf{a}_{n1} \\ \mathbf{a}_{12} & \mathbf{a}_{22} & \cdots & \mathbf{a}_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_{1n} & \mathbf{a}_{2n} & \cdots & \mathbf{a}_{nn} \end{bmatrix}\begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix} + W_\pi\begin{bmatrix} \mathbf{a}_{01} \\ \mathbf{a}_{02} \\ \vdots \\ \mathbf{a}_{0n} \end{bmatrix} + sW_\pi\begin{bmatrix} \mathbf{a}_{01} \\ \mathbf{a}_{02} \\ \vdots \\ \mathbf{a}_{0n} \end{bmatrix}$$

[11] The so-called total labor includes direct labor (living labor) and indirect labor (dead labor).

Table 2 Misunderstanding of the exchange between two departments (Bortkiewicz, Morishima, etc.)

Department	I	II
I	c_1	c_2
II	v_1	v_2

Source: created by the author

Here, $\mathbf{m}' = [m_i] = (m_1, m_2, \dots, m_n)'$ is the column vector of minimum-subsistence goods needed as real wage to cover the cost of production and reproduction of labor.[12]

Since $\boldsymbol{\pi}\mathbf{m} = W$, $\mathbf{A}_0(0)(1 + s)\mathbf{m} = 1$, and, thus,

$$s = \frac{1}{\mathbf{A}_0(0)\mathbf{m}} - 1$$

Then, because $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{m}\mathbf{A}_0(0)(1 + s)$, and $\frac{1}{1+s}$ is the eigenvalue of $\mathbf{m}\mathbf{A}_0(0)$, $\boldsymbol{\pi}$ is its eigenvector, and (9) is solvable. Hence, Samuelson believed that the physical quantities $\mathbf{a}_0$, $\mathbf{a}$, and $\mathbf{m}$ determine the rate of surplus value s and the commodity value vector $\boldsymbol{\pi}$ through $\boldsymbol{\pi} = W\mathbf{a}_0 + \boldsymbol{\pi}\mathbf{a} + sW\mathbf{a}_0$ (although he did not discuss the uniqueness of the solution).

After using the physical quantity system to solve the problem of value determination, he constructed his n-department production price equations as follows:

$$\mathbf{P} = (W\mathbf{a}_0 + \mathbf{P}\mathbf{a})(1 + r) \tag{10}$$

Here, r is the average profit rate, and $\mathbf{P} = [P_i] = (P_1, P_2, \dots, P_n)$ is the production price vector.

Regarding the solution of (10). Samuelson first considered how to solve for r. For this reason, he then he defined $\mathbf{A}_0(r)$ to be the vector of all labor consumption coefficients under the condition of average profit, namely,

$$\mathbf{A}_0(r) = \mathbf{a}_0(1 + r)[\mathbf{I} - \mathbf{a}(1 + r)]^{-1} = (1 + r)(\mathbf{a}_{01}, \mathbf{a}_{02}, \dots, \mathbf{a}_{0n}) \begin{bmatrix} 1-(1+r)\mathbf{a}_{11} & -(1+r)\mathbf{a}_{12} & \cdots & -(1+r)\mathbf{a}_{1n} \\ -(1+r)\mathbf{a}_{21} & 1-(1+r)\mathbf{a}_{22} & \cdots & -(1+r)\mathbf{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -(1+r)\mathbf{a}_{n1} & -(1+r)\mathbf{a}_{n2} & \cdots & 1-(1+r)\mathbf{a}_{nn} \end{bmatrix}^{-1}$$

Since $W\mathbf{a}_0(1 + r)[\mathbf{I} - \mathbf{a}(1 + r)]^{-1} = W\mathbf{A}_0(r)$, $\mathbf{Pm} = W$, so

$$\left.\begin{aligned} \mathbf{P} &= W\mathbf{A}_0(r) \\ \mathbf{Pm} &= W \end{aligned}\right\} \tag{11}$$

Then, he used $\mathbf{Pm} = W$ to get $\mathbf{A}_0(r)\mathbf{m} = 1$, so

[12] See Samuelson (1970). It is wrong to understand m as the minimum survival commodity vector required by laborers. Samuelson does not seem to know Engels' critique of Lassalle's "Iron law of wages". See Zhang (2008).

Table 3 Samuelson's correction of the expression of the exchange relationship between two departments

Department	I	II
I	c_1	c_2
II	0	0

Source: created by the author from Samuelson (1957)

$$\mathbf{a}_0(1 + r)[\mathbf{I} - \mathbf{a}(1 + r)]^{-1}\mathbf{m} = 1 \quad (12)$$

Equation (12) is a one-dimensional higher-order equation with unknown r. By substituting its solution into $\mathbf{P} = W\mathbf{A}_0(r)$, $\mathbf{P}$ can be obtained.[13] This shows that the production price is determined by a physical quantity system.

In short, Samuelson thought "the transformation problem" was merely "the problem of comparing and contrasting the mutually exclusive alternative of 'value' and 'price'".

His conclusion was that the so-called "transformation problem" is nothing but a problem of the transformation of "two alternative and discordant systems": "Write down one. Now, transform by taking an eraser and rubbing it out. Then, we fill in the other one. Voila! You have completed your transformation algorithm. By this technique, one can 'transform' from phlogiston to entropy; from Ptolemy to Copernicus; from Newton to Einstein; from Genesis to Darwin—and from entropy to phlogiston".[14] Samuelson proved that the production price system can be separated from the value system and determined independently by the physical quantity system. Therefore, he believed that the transformation of value into production price is nothing but a "returning from the unnecessary detour",[15] thus completely denying the necessity and significance of the transformation of value into production price.

Samuelson then began to use his method to analyze the Bortkiewicz model. He rewrote the Bortkiewicz model as[16]

[13] Steedman simplifies the solution of (12). He substitutes $\mathbf{Pm} = W$ into (10) and eliminates W, then (10) becomes.

$$\mathbf{P} = \mathbf{P}\left(\mathbf{m}\mathbf{a}_0 + \mathbf{a}\right)(1 + r) \quad (13)$$

In this way, $1/(1 + r)$ becomes the eigenvalue of matrix $\left(\mathbf{m}\mathbf{a}_0 + \mathbf{a}\right)$, and its calculation becomes quite easy. See Steedman (1977), pp. 50–52.

[14] Samuelson (1971), p. 400.

[15] Samuelson (1971), p. 421.

[16] Samuelson was completely unaware that he had actually corrected an error in the body equations in the Bortkiewicz model. If we faithfully rewrite Bortkiewicz's main equations according to Samuelson's notation, we should write:

$$y_j\pi_j = \left(y_2 W\mathbf{a}_{0j} + \sum_{i=1}^{3} y_1\pi_1\mathbf{a}_{1j}\right)(1 + r), \quad (j = 1, 2, 3)$$

$$\left.\begin{aligned} y_j\pi_j &= \left(W\mathbf{a}_{0j} + \sum_{i=1}^{n} y_i\pi_i\mathbf{a}_{ij}\right)(1+r) \quad (j = 1,2,\ldots,n) \\ \sum_{j=1}^{n} y_j\pi_j m_j &= W \end{aligned}\right\}. \tag{14}$$

However, (14) is precisely that of (11) with $P_i = y_i\pi_i$, $y_i = P_i/\pi_i (i = 1, 2, \ldots, n)$.

Although Samuelson partially corrected the main equations in the Bortkiewicz model, a fatal error remained—that is, Samuelson unreasonably assumes that the wage rate W remained unchanged after the transformation of value into production price. This means that $\boldsymbol{\pi}\mathbf{m} = W = \mathbf{P}\mathbf{m}$. However, unless the vector $\boldsymbol{\pi} = \mathbf{P}$, $\boldsymbol{\pi}\mathbf{m} = \mathbf{P}\mathbf{m}$ is usually impossible.

Moreover, Samuelson forcibly replaced Marx's two constraints (double invariance) with one constraint $\left(\sum_{j=1}^{n} y_j\pi_j m_j = W\right)$. Although this replacement has an approximate effect, it is not equivalent. Therefore, the production price derived from value using the Samuelson model cannot be accurate unless the exogenous data of $\mathbf{m}$ can meet certain conditions.

Therefore, except under special circumstances, the production price calculated according to the Samuelson model could not meet the two constraints of Marx. However, ironically, this calculation error made by Samuelson let him deny the Marxian transformation problem.

Despite the abovementioned errors, Samuelson's model was the closest to the Zhang (2000) model. Let us discuss the Zhang model in detail below.

3 Construction and examples of static transformation model

In the first section of this paper, we listed 6 problems that need to be solved to construct a static transformation model. Zhang (2000) provides answers to them as follows:

(1) We adhere to Marx's original framework for studying the transformation problem.

(2) The precondition of the static transformation model is to assume that the turnover rate of constant and variable capital is 1 year and that the technology remains unchanged.

(3) The mathematical relation between value and production price is reflected by the deviation rate of the production price from the value.

(4) The number of endogenous variables (unknowns) is n + 2, including the n deviation rates of production price to value, the deviation rate of wage rate under production price to the wage rate under value, and the average profit rate.

(5) The main equations reflecting the relationship between value and production price after the transformation is constructed with reference to the input–output analysis.

(6) The two constraints (the sum of production prices = the sum of values; the sum of average profit = the sum of surplus value) of Marx and the main equations mentioned above need to be combined to form a complete transformation model.

The symbols are specified below. First, let's set signs concerning the value system.

Exogenous variables of the static transforming model:c_i, v_i, m_i, and w_i represent the constant capital, variable capital, surplus value, and total value of the ith department under the value system, respectively. Here, $c_i + v_i + m_i = w_i (i = 1, 2, \ldots, n)$, which is Marx's value Eq. (1). However, what c_i represents here is only the sum of the constant capital used by the ith department. In reality, constant capital comes from various departments—that is, $c_i = \sum_{j=1}^{n} c_{ij} \quad (i = 1, 2, \ldots, n)$. Measuring unit of all exogenous variables in labor time.

Marx's value system is presented in Table 4.

Second, there are signs concerning the production price system. C_i, V_i, R_i, and P_i represents the constant capital, variable capital, average profit, and total production price of the department under the production price system, respectively. Here, $C_i + V_i + R_i = P_i (i = 1, 2, \ldots, n)$. Further, $C_i = \sum_{j=1}^{n} C_{ij} \quad (i = 1, 2, \ldots, n)$. Let r represent the average profit rate; then, $R_i = r(C_i + V_i)(i = 1, 2, \cdots, n)$. Thus,$(1 + r)\left(C_i + V_i\right) = P_i (i = 1, 2, \ldots, n)$. This is the production price equation commonly used by Marx. Marx's production price system is presented in the Table 5 below.

Then, we establish a mathematical connection between value and production price.

Endogenous variables of the static transforming model: the deviation rate of production price of the ith department to value is x_i, the deviation rate of variable capital under production price to variable capital under the value is y.[17] Add the average profit rate r.and we obtain a total of $n + 2$ endogenous variables. Obviously, $C_{ij} = x_j c_{ij} \, (i, j = 1, 2, \ldots, n)$, $V_i = y v_i$, $P_i = x_i w_i (i = 1, 2, \ldots, n)$; thus,

$$x_i w_i = P_i = (1 + r)\left(C_i + V_i\right) = (1 + r)\left(\sum_{j=1}^{n} C_{ij} + V_i\right) = (1 + r)\left(\sum_{j=1}^{n} x_j c_{ij} + y v_i\right)$$

Here, $P_i = x_i w_i$ clearly shows the conversion relationship of value w_i to production price P_i.

If we consider $x_i \, (i = 1, 2, \ldots, n)$, y, and r as unknowns, we obtain the following n equations with $n + 2$ unknowns:

$$(1 + r)\left(\sum_{j=1}^{n} c_{ij} x_j + v_i y\right) = w_i x_i \quad (i = 1, 2, \ldots, n) \tag{15}$$

[17] The deviation rate y of variable capital is very important. If $y = 1$, it means that variable capital has not been converted into production price, and problems in Samuelson's model will arise.

Table 4 Value system of n departments

Department	1	2	…	n
1	c_{11}	c_{12}	…	c_{1n}
2	c_{21}	c_{22}	…	c_{2n}
…	…	…	…	…
n	c_{n1}	c_{n2}	…	c_{nn}
V	v_1	v_2	…	v_n
M	m_1	m_2	…	m_n
Total	w_1	w_2	…	w_n

Source: Created by the author

This is the main equation set of the general static transformation model proposed by Zhang at the 48th annual conference of the Japan Society of Political Economy in October 2000.[18]

According to the analysis above, as long as Marx's two constraints (double invariance) are added to this main equation system, a general static transformation model is obtained. The first constraint (the sum of production prices = the sum of values) is relatively simple and can be expressed as:

$$\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i \tag{16}$$

However, the second constraint can be expressed in three ways. The first shows that the sum of the average profit is the sum of the surplus value. This can be expressed as follows:

$$r \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} x_j + v_i y \right) = \sum_{i=1}^{n} m_i \tag{17}$$

The second shows that the total cost price under the value is equal to the total cost price under the production price. This can be expressed as follows:

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} x_j + v_i y \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} + v_i \right) \tag{18}$$

Third, the average profit rate is determined by Marx's formula for the average profit rate. This can be derived from formulas (17) and (18). The formula is as follows:

$$r = \sum_{i=1}^{n} m_i / \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} + v_i \right) \tag{19}$$

18 See Zhang (2000, 2001).

Table 5 The production price system of n departments

Department	1	2	…	n
1	C_{11}	C_{12}	…	C_{1n}
2	C_{21}	C_{22}	…	C_{2n}
…	…	…	…	…
n	C_{n1}	C_{n2}	…	C_{nn}
V	V_1	V_1	…	V_1
R	R_1	R_2	…	R_n
Total	P_1	P_2	…	P_n

Source: Created by the author

It should be noted that (19) is the same formula as (3) in this paper, but the denominator is expressed in a different way. Zhang's (2000) model adopted the third mode of expression for the second constraint, while Zhang (2008) argued that the second constraint of the model is more standardized in the second mode of expression, because it enables us to omit the exogenous variable m_i as well as make r endogenous. Hence, the Zhang's present transformation model is as follows:

$$\left.\begin{aligned} &(1+r)\left(\sum_{j=1}^{n} c_{ij}x_j + v_i y\right) = w_i x_i \quad (i = 1, 2, \cdots, n) \\ &\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i \\ &\sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}x_j + v_i y\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij} + v_i\right) \end{aligned}\right\} \tag{20}$$

It can be proved that, under the premise that the exogenous variables can satisfy $(1+r)c_i = (1+r)\sum_{j=1}^{n} c_{ij} < w_i \quad (i = 1, 2, \ldots, n)$, the solutions of model (20) are unique and greater than zero.[19]

Need to emphasize, in Japan, scholars such as Makoto Itoh, one of the representatives of the Uno School, have always insisted that value and price belong to different dimensions. A fundamental divergence arises from this: is the transformation problem the problem of converting value (measured as labor time) to production price (measured in labor time) to production price (measured also in labor time) to price (measured in money)[20]? In this regard, we think that, although "production price is the result of deviation from value", its deviation is that of quantitative nature, and its essence is not changed. The essence of production price and value are the same. Both are measured in labor time.

It needs to be emphasized that the problem of the conversion of value to pecuniary price is another problem that is different from "the transformation problem".

[19] See Huan et al. (2005).

[20] See Itoh (1981), p. 324.

Originally, price is the pecuniary expression of value, so the conversion of value to price is very simple. When value is transformed into production price, then the price becomes the pecuniary expression of production price. At this time, the problem of "the transformation of value to price" becomes the problem of value transformation to price (of the pecuniary expression of production price). If we import the concept of unit commodity value, we can also include these two different aspects of the transformation process through model (20) demonstrations or simulations. In particular, we can simulate that the matter of value transformation to price (of the pecuniary expression of production price), through the model (20).[21] This result may be able to fit the disagreement between this paper and Itoh et al.

To avoid unnecessary misunderstandings, we emphasize that in *Capital* profit and its rate are terms applicable to both value and production price. In model (20), the average profit rate r exists as the solution of the model. It is an endogenous variable determined by the model and is not an exogenous independent variable.[22]

[21] The conversion of value (in labor time) to price (in money) contains two aspects. Let π indicate the value of unit commodities, q the physical quantity, then $w_i = w_i^{(q)}\pi_i$, $c_{ij} = c_{ij}^{(q)}\pi_j$, $v_i = \omega l_i$; the wage rate here is $\omega = \frac{v_i}{l_i}(\forall i)$. Thus, (20) can be rewritten as:

$$\left.\begin{aligned} &(1+r)\left(\sum_{j=1}^{n} c_{ij}^{(q)}\pi_j x_j + v_i y\right) = w_i^{(q)}\pi_i x_i \quad (i = 1,2,\cdots,n) \\ &\sum_{i=1}^{n} w_i^{(q)}\pi_i x_i = \sum_{i=1}^{n} w_i^{(q)}\pi_i \\ &\sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}^{(q)}\pi_j x_j + \omega l_i y\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}^{(q)}\pi_j + \omega l_i\right) \end{aligned}\right\} \tag{21}$$

If the fth department is gold, the unit product value of fth department is π_f, and the conversion method of value to price can be expressed as $\frac{\pi_i}{\pi_f}$ $(\forall i)$ and $1/\pi_f$ is the value price transform coefficient. After value transformation to production price, i.e., the monetization of production price, it can be expressed as:

$$\left.\begin{aligned} &(1+r)\left(\sum_{j=1}^{n} c_{ij}^{(q)}\frac{\pi_j}{\pi_f} x_j + v_i y\right) = w_i^{(q)}\frac{\pi_i}{\pi_f} x_i \quad (i = 1,2,\cdots,n) \\ &\sum_{i=1}^{n} w_i^{(q)}\frac{\pi_i}{\pi_f} x_i = \sum_{i=1}^{n} w_i^{(q)}\frac{\pi_i}{\pi_f} \\ &\sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}^{(q)}\frac{\pi_j}{\pi_f} x_j + \omega l_i y\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}^{(q)}\frac{\pi_j}{\pi_f} + \omega l_i\right) \end{aligned}\right\} \tag{22}$$

[22] The average rate of profit r solved from the model (20) is exactly the same as the formula (19), which just shows that the concept of rate of profit is applicable to both the value system and the production price system. As the solution of the model, the average profit rate r is determined by the exogenous variables of the model. It is not independent and cannot increase indefinitely. From a mathematical point of view, the condition of the model (20) is $(1+r)c_i = (1+r)\sum_{j=1}^{n} c_{ij} < w_i \quad (i = 1,2,\ldots,n)$, i.e., the average profit rate r has an upper limit: $r < \frac{w_i}{c_i} - 1$. Thus, we cannot increase r sufficiently, that is, it is impossible to increase $\frac{\mathbf{X}}{y}$ infinitely. Therefore, it is impossible to make all elements of **X** greater than one and y smaller than one. Furthermore, as a transformation problem of the redistribution of surplus value, it is impossible for all elements of **X** to be greater than one to appear.

Next, let us examine a famous example on the transformation problem that Bortkiewicz (1907) and Sweezy (1942) have both analyzed. Samuelson (1970, 1971) adopted Eqs. (9) and (11) under the assumption $r = \frac{1}{4}$, $W = 1$ to derive the value system and the production price system, respectively. The data of the value system for this example are listed in Table 6.

Samuelson (1971) uses formula (11) to derive the production price system that is consistent with Sweezy's (1942) calculation result of this example. For the sake of convenience, Samuelson reduced it by 15 /16. In the calculation result of Samuelson's model, the total production price is 937.5, which is not equal to the total value of 875, and the total average profit of 187.5 is not equal to the total surplus value of 200 (as shown in Table 7).[23]

There is no doubt that Samuelson's production price calculation was wrong, as we have already determined in the previous section.

Now, we use Zhang's model and formula (11) to calculate the same example to see the result. For the convenience of calculation, Zhang first calculates the average profit rate $r = \frac{200}{375+300} = \frac{8}{27}$ according to formula (6) (note: r is not $\frac{1}{4}$ subjectively stipulated by Samuelson). In this way, by substituting the data into formula (11), the following equations are obtained:

$$\left(1 + \frac{8}{27}\right)\left(225x_1 + 100y\right) = 375x_1$$

$$\left(1 + \frac{8}{27}\right)\left(90x_1 + 120y\right) = 300x_2$$

$$\left(1 + \frac{8}{27}\right)\left(60x_1 + 80y\right) = 200x_3$$

$$375x_1 + 300x_2 + 200x_3 = 875$$

The solution of this equation system is $x_1 = 1.145$, $x_2 = 0.919$, $x_3 = 0.848$, and $y = 0.818$. Thus, we obtain Table 8. Here, we see that the total production price 875.000 is equal to the total value 875, and the average total profit 200.000 is equal to the total surplus value of 200.

Let us examine another generalized example. The value system data are shown in Table 9.

Now, we use formula (11) to calculate this example. After calculating the average profit rate $r = \frac{1}{4}$ according to formula (6) and substituting the data into formula (11), we obtain the following equations:

$$\left(1 + \frac{1}{4}\right)\left(105x_1 + 205x_2 + 70x_3 + 100x_4 + 300y\right) = 1050x_1$$

[23] See Samuelson (1971), p. 425.

Table 6 Bortkiewicz's value system (Example)

Department	c	v	m	w
I	225	90	60	375
II	100	120	80	300
III	50	90	60	200
Total	375	300	200	875

Source: author in the past

Table 7 Samuelson's production price system derived from Bortkiewicz's value system

Department	C	V	R	P
I	270	90	90	450
II	120	120	60	300
III	60	90	37.5	187.5
Total	450	300	187.5	937.5

Source: Author in the past

$$\left(1+\frac{1}{4}\right)\left(250x_1+130x_2+235x_3+75x_4+150y\right)=960x_2$$

$$\left(1+\frac{1}{4}\right)\left(60x_1+110x_2+55x_3+210x_4+200y\right)=785x_3$$

$$\left(1+\frac{1}{4}\right)\left(135x_1+75x_2+95x_3+80x_4+160y\right)=705x_4$$

$$1050x_1+960x_2+785x_3+705x_4=3500$$

The solution of this equation system is $x_1=0.939$, $x_2=1.088$, $x_3=1.009$, $x_4=0.960$, $y=1.002$. Thus, we obtain Table 10. Here, we see that the total production price 3500.000 is equal to the total value 3500, and the average total profit 700.000 is also equal to the total surplus value 700.

4 Several related issues

From a mathematical perspective, the static transformation model that satisfies Marx's two constraints can be constructed in several ways. For example, Zhang (2004) indirectly derived the following transformation model[24] from the three diagrams of Goodwin (1983)[25]:

[24] See Zhang (2002, 2004), p. 137.

[25] See Goodwin (1983), pp. 133–139.

Table 8 Use formula (11) to derive the production price system from Bortkiewicz's value system

Dept	C	V	R	P
I	257.727	73.636	98.182	429.545
II	114.545	98.182	63.030	275.758
III	57.273	73.636	38.788	169.697
Total	429.545	245.455	200.000	875.000

Source: Author in the past

Table 9 A general example of value system

Dept	I	II	III	IV	Total
I	105	250	60	135	550
II	205	130	110	75	520
III	70	235	55	95	455
IV	100	75	210	80	465
v	300	150	200	160	810
m	270	120	150	160	700
w	1050	960	785	705	3500

Source: Created by the author

Table 10 The production price system transformed from the general example of the value system

Dept	I	II	III	IV	Total
I	98.635	234.844	56.363	126.816	516.657
II	223.096	141.476	119.71	81.621	565.903
III	70.63	237.116	55.495	95.855	459.096
IV	96.006	72.005	201.613	76.805	446.429
V	300.709	150.355	200.473	160.378	811.915
R	197.269	208.949	158.413	135.369	700.000
P	986.346	1044.744	792.067	676.844	3500.000

Source: Created by the author

$$\left.\begin{array}{ll} c_i + v_i x_i + m_i = w_i y_i & (i = 1, 2, \ldots, n) \\ \dfrac{m_i}{c_i + v_i x_i} = \dfrac{m_j}{c_j + v_j x_j} & (i \neq j) \\ \sum v_i x_i = \sum v_i & \end{array}\right\} \quad (23)$$

Although (23) meets Marx's two constraints, it fails to reflect Marx's requirement for redistribution of surplus value, because this model is completed through the redistribution of variable capital. In addition, Zhang (2004) also introduced the construction of a transformation model similar to Goodwin (1983) in 1997[26]:

[26] See Zhang (2004), p. 137.

$$\left.\begin{array}{l} (c_i + v_i)x_i + m_i = w_i y_i \quad (i = 1, 2, \ldots, n) \\ \dfrac{m_i}{(c_i + v_i)x_i} = \dfrac{m_j}{(c_j + v_j)x_j} \qquad (i \neq j) \\ \sum (c_i + v_i)x_i = \sum (c_i + v_i) \end{array}\right\} \tag{24}$$

Although (24) also meets Marx's two constraints, it fails to reflect Marx's requirement for redistribution of surplus value, because the model is obtained through the redistribution of cost prices.

In general, the problem of converting value into production price can be called the direct transformation problem, and the problem of converting production price into value is called the inverse transformation problem. Mathematically, the inverse transformation is equivalent to the inverse function of the direct transformation.

Zhang (2004), under the premise that the rate of surplus value is equal, first provided a static inverse transformation model that meets Marx's two constraints as follows[27]:

$$\left.\begin{array}{l} \sum_{j=1}^{n} C_{ij}X_j + (1+e)V_iY = P_iX_i \quad (i = 1, 2, \ldots, n) \\ \sum_{i=1}^{n} P_iX_i = \sum_{i=1}^{n} P_i \\ e = \sum_{i=1}^{n} S_i / \sum_{i=1}^{n} V_iY \end{array}\right\} \tag{25}$$

Here, X_i represents the deviation rate of the value w_i from the production price P_i, that is $w_i = P_iX_i$ $(i=1,2,\ldots,n)$, and Y represents the deviation rate of the variable capital V_i under the production price, that is, $v_i = YV_i$ $(i=1,2,\ldots,n)$. The average profit rate r is still given by formulas (19).

In a mathematical sense, Zhang (2004) directly derived the reversal model from the aforementioned three diagrams of Goodwin (1983), which satisfies Marx's two constraints[28]:

[27] See Zhang (2004), p. 170.

[28] See Zhang (2004), p. 136.

$$\left.\begin{array}{l} C_i + V_iX_i + R_i = P_iY_i \quad (i = 1, 2, \dots, n) \\ \dfrac{R_i}{V_iX_i} = \dfrac{R_j}{V_jX_j} \quad (i \neq j) \\ \sum V_iX_i = \sum V_i \end{array}\right\} \tag{26}$$

Regarding the dynamic transformation model, Zhang (2003) provided two department transformation models as follows[29]:

$$\left.\begin{array}{l} \dfrac{w_1}{c_1 + v_1\frac{\beta_1^t}{\lambda^t}} = 1 + r \\ \dfrac{w_2}{\left(c_2\frac{1}{\beta_1^t} + v_2\frac{1}{\lambda^t}\right)\beta_2^t} = 1 + r \\ c_1\beta_1^t + c_2\beta_2^t - w_1\beta_1^{t-1} = 0 \\ (w_1 + w_2)\beta^t - w_1\beta_1^t - w_2\beta_2^t = 0 \end{array}\right\} \tag{27}$$

It is assumed here that the two departments grow with deviation according to β_1 and β_2,[30] respectively, starting from the first year. The expanded reproduction system in the tth year is $c_1\beta_1^t + v_1\beta_1^t + m_1\beta_1^t = w_1\beta_1^t$, $c_2\beta_2^t + v_2\beta_2^t + m_2\beta_2^t = w_2\beta_2^t$.

Let β^t denote the growth index of the whole society in the tth year; the growth deviation degree of the ith department value in the tth year is $\frac{\beta^t}{\beta_1^t}$, and the growth deviation degree of variable capital is represented by $\frac{\beta_2^t}{\lambda^t}$. The model (27) is solvable; therefore,[31] it is unnecessary to reinvestigate it here.

5 The unity of static transformation and dynamic transformation

In this section, we first extend Zhang's (2003) 2-department dynamic transformation model to the n-department to build a more general dynamic transformation model. However, prior to this, we must emphasize a theoretical premise. When we construct a static transformation model, we follow the premise that the transformation of the value to the production price is the result of the residual value redistribution. Because Marx described it in Chapter 9 of vol. III of *Capital*, Okishio (1977) clearly recognized this.[32] Now, we build a dynamic transformation model of the n-department based on these principles.

[29] See Zhang (2003).

[30] $\beta_j = 1 + \frac{e\alpha_j}{1+k_j} (j = 1, 2)$. α_j and k_j are the accumulation rate of the jth department and the organic composition of capita, respectively.

[31] See Zhang (2004) pp. 253–262.

[32] See Marx (1966) p. 200, and Okishio (1977) p. 36.

Let us consider a general dynamic transformation model, i. e. a transformation process that takes k years to complete. To simplify the explanation, we assume that, in the tth$(0 < t \leq k)$ year, the growth rate of the jth department's total value is $\delta_j^{(t)}$ (exogenous parameter)[33]; when $t = 0$, there is no economic growth, so$\delta_j^{(0)} = 0$; under the same premise, the growth rate of constant capital and variable capital of the jth department is $\theta_j^{(t)}$ (exogenous parameter), and, in the process of transforming value to production price, the deviation rate of the value of this department is $x_j^{(t)}$, and the deviation rate of the variable capital is$y^{(t)}$. $k_j^{(t)}$ (exogenous parameter) represents the redistribution ratio of the surplus value of the jth department in the tth year, and $k_j^{(0)} = 0$, $\sum_{s=0}^{k} k_j^{(s)} = 1$. In the tth year, the redistribution of the surplus value of the i^{th} department is:

$$\sum_{s=0}^{t} k_i^{(s)} \left\{ r\left(\sum_{j=1}^{n} \left[\prod_{s=0}^{t} (1 + \delta_j^{(s)}) c_{ij} x_j^{(t)} \right] + \prod_{s=0}^{t} (1 + \theta^{(s)}) v_i y^{(t)} \right) - m_i \right\}$$

In this way, we get the general dynamic transformation model $f(t)(\forall 0 \leq t \leq k)$ as follows:

$$f(t) = \left\{ \begin{array}{l} \left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{t} \left(1 + \delta_j^{(s)}\right) c_{ij} x_j^{(t)} \right] + \prod_{s=0}^{t} (1 + \theta^{(s)}) v_i y^{(t)} \right\} + \lambda^{(t)} S_i^{(t)} = \prod_{s=0}^{t} \left(1 + \delta_j^{(s)}\right) w_i x_i^{(t)} \quad (i = 1, 2, \cdots, n) \\ \sum_{i=1}^{n} \left[\prod_{s=0}^{t} \left(1 + \delta_j^{(s)}\right) w_i x_i^{(t)} \right] = \sum_{i=1}^{n} \left[\left(1 + \delta_j^{(s)}\right) w_i \right] \\ \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{t} \left(1 + \delta_j^{(s)}\right) c_{ij} x_j^{(t)} \right] + \prod_{s=0}^{t} (1 + \theta^{(s)}) v_i y^{(t)} \right\} = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{t} \left(1 + \delta_j^{(s)}\right) c_{ij} \right] + \prod_{s=0}^{t} (1 + \theta^{(s)}) v_i \right\} \end{array} \right\} \tag{28}$$

Here, the variable $\lambda^{(t)}$ is used to adjust the redistribution of surplus value in the $t^{th}(0 < t < k)$ year.[34] Additionally,

$$S_i^{(t)} = m_i + \sum_{s=0}^{t} k_i^{(s)} \left\{ r\left(\sum_{j=1}^{n} \left[\prod_{s=0}^{t} (1 + \delta_j^{(s)}) c_{ij} x_j^{(t)} \right] + \prod_{s=0}^{t} (1 + \theta^{(s)}) v_i y^{(t)} \right) - m_i \right\}$$

which reflects the change in the surplus value in the process of profit averaging.

Note that, since $S_i^{(t)}$ contains $\lambda^{(t)}$, we now have $n+3$ unknowns: x_i $(i = 1, 2, \cdots, n)$, y, r, and $\lambda^{(t)}$, but as we have only $n+2$ equations there is no way to solve them. To solve this problem, let us consider how model (28) will change in the kth year.

The transformation process is completed in the kth year. At this point, the redistribution of surplus value ends, and $S_i^{(k)}$ transforms itself to average profit, that is, $S_i^{(k)} = R_i^{(k)}(i = 1, 2, \cdots, n)$. Here, $R_i^{(k)}$ represents the average profit of the ith department.

$$R_i^{(k)} = r\left(\sum_{j=1}^{n} \left[\prod_{s=0}^{k} (1 + \delta_j^{(s)}) c_{ij} x_j^{(t)} \right] + \prod_{s=0}^{k} (1 + \theta^{(s)}) v_i y^{(t)} \right)$$

33 If we assume a same growth rate for all departments, $\delta_j^{(t)} = \delta^{(t)}(\forall j)$, If we assume a same growth rate for each year too, $\delta_j^{(t)} = \delta(\forall j, t)$, We can acquire the conditions of the balanced growth.

34 When $t = k$, the transformation process is completed and $\lambda^{(\mathrm{k})}$ is 1. Thus, it is no longer a variable.

Thereby,

$$S_i^{(k)} = r\left(\sum_{j=1}^{n} \left[\prod_{s=0}^{t} \left(1+\delta_j^{(s)}\right) c_{ij} x_j^{(t)} \right] + \prod_{s=0}^{t} \left(1+\theta^{(s)}\right) v_i y^{(t)} \right)$$

This conclusion can also be derived from $\sum_{s=0}^{k} k_j^{(s)} = 1$ and $\lambda^{(k)} = 1$. From this, we get

$$f(k) = \left\{ \begin{array}{l} (1+r^*)\left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) c_{ij} x_j^{(k)} \right] + \prod_{s=0}^{k} \left(1+\theta^{(s)}\right) v_i y^{(k)} \right\} = \prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) w_i x_i^{(k)} \quad (i = 1, 2, \cdots, n) \\ \sum_{i=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) w_i x_i^{(k)} \right] = \sum_{i=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) w_i \right] \\ \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) c_{ij} x_j^{(k)} \right] + \prod_{s=0}^{k} \left(1+\theta^{(s)}\right) v_i y^{(k)} \right\} = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) c_{ij} \right] + \prod_{s=0}^{k} \left(1+\theta^{(s)}\right) v_i \right\} \end{array} \right\} \tag{29}$$

Thus, in (29), the average profit rate r^* is determined as follows[35]:

$$r^* = \frac{\sum_{i=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) w_i \right]}{\sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left[\prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) c_{ij} \right] + \prod_{s=0}^{k} \left(1+\theta^{(s)}\right) v_i \right\}} - 1 \tag{30}$$

If we make $c_{ij}^* = \prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) c_{ij}$, $v_i^* = \prod_{s=0}^{k} \left(1+\theta^{(s)}\right) v_i$, $w_i^* = \prod_{s=0}^{k} \left(1+\delta_j^{(s)}\right) w_i$, then (30) can also be rewritten as

$$r^* = \frac{\sum_{i=1}^{n} w_i^*}{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(c_{ij}^* + v_i^*\right)} - 1 \tag{31}$$

Therefore, we can further rewrite (29) as

$$f(k) = \left\{ \begin{array}{l} (1+r^*)\left\{ \sum_{j=1}^{n} c_{ij}^* x_j^{(k)} + v_i^* y^{(k)} \right\} = w_i^* x_i^{(k)} \quad (i = 1, 2, \cdots, n) \\ \sum_{i=1}^{n} w_i^* x_i^{(k)} = \sum_{i=1}^{n} w_i^* \\ \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} c_{ij}^* x_j^{(k)} + v_i^* y^{(k)} \right\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(c_{ij}^* + v_i^*\right) \end{array} \right\} \tag{32}$$

[35] There is no essential difference between this formula and the previous (19); (30) only introduces the growth rate δ and θ.

Table 11 The dynamic transformation results of Bortkiewicz's value system ($k = 2$, simple reproduction)

0^{th} year Value

Dept.	*c*	*v*	*m*	*w*
I	225	90	60	375
II	100	120	80	300
III	50	90	60	200
Total	375	300	200	875

1^{st} year Value

Dept.	*c*	*v*	*m*	*w*
I	225	90	60	375
II	100	120	80	300
III	50	90	60	200
Total	375	300	200	875

1^{st} year

Dept.	c^*	v^*	m^*	w^*
I	237.91	83.55	79.09	400.54
II	105.74	111.39	71.52	288.65
III	52.87	83.55	49.39	185.81
Total	396.51	278.49	200.00	875.00

2^{nd} year Value

Dept.	*c*	*v*	*m*	*w*
I	225	90	60	375
II	100	120	80	300
III	50	90	60	200
Total	375	300	200	875

2^{nd} year Production price

Dept.	*C*	*V*	*R*	*P*
I	257.73	73.64	98.18	429.55
II	114.55	98.18	63.03	275.76
III	57.27	73.64	38.79	169.70
Total	429.55	245.45	200.00	875.00

Source: Created by the author

Table 12 The dynamic transformation results of Bortkiewicz's value system ($k = 2$, expanded reproduction)

0^{th} year Value

Dept.	c	v	m	w
I	225	90	60	375
II	100	120	80	300
III	50	90	60	200
Total	375	300	200	875

1^{st} year Value

Dept.	c	v	m	w
I	231.75	93.15	62.10	387.00
II	103.00	124.20	82.80	310.00
III	51.50	93.15	62.10	206.75
Total	386.25	310.50	207.00	903.75

1^{st} year

Dept.	c^*	v^*	m^*	w^*
I	248.61	84.72	81.82	415.15
II	110.49	112.96	74.04	297.49
III	55.25	84.72	51.14	191.11
Total	414.35	282.40	207.00	903.75

2^{nd} year Value

Dept.	c	v	m	w
I	238.70	96.41	64.27	399.39
II	106.09	128.55	85.70	320.34
III	53.05	96.41	64.27	213.73
Total	397.84	321.37	214.25	933.45

2^{nd} year Production price

Dept.	C	V	R	P
I	273.42	79.05	105.00	457.47
II	121.52	105.40	67.60	294.52
III	60.76	79.05	41.65	181.46
Total	455.69	263.51	214.25	933.45

Source: Created by the author

Thus, (32) seems to degenerate into a general static transformation model (20). In fact, the general static transformation model (20) should be regarded as the completion of the general dynamic transformation model (28). Compared with (20), (29) or (32) is more general. In this way, we can harmonize static transformation with dynamic transformation. Moreover, this end point satisfies "the double invariance".

Now, we use model (28) to re-examine the famous example of Bortkiewicz (1907), which we discussed in Sect. 3 of this paper (the data of the value system are in Table 5). We analyzed two situations.

1) Assuming that the initial year is the 0th year and the transformation is completed in the 2nd year. It is also assumed that the transformation is performed under conditions of simple reproduction such that $\delta_j^{(t)} = 0$ and $\theta^{(t)} = 0$. Assume that $k_j^{(1)} = 0.5(j = 1, 2, 3)$. Therefore, according to model (28), we obtain Table 11 as follows. Here, we see that the scale of production remains unchanged, and the value of the 2nd year is transformed into the production price. The total production price of 875.00 is equal to the total value of 875, and the average total profit of 200.00 is also equal to the total surplus value of 200.
2) We still assume that the initial year is the 0th year and that the transformation is completed in the 2nd year. However, it is assumed that the transformation is carried out under conditions of expanded reproduction. Here, it is assumed that $\delta_j^{(1)} = 3\%(j = 1, 2, 3), \theta^{(1)} = 3.5\%$, and $k_j^{(1)} = 0.5(j = 1, 2, 3)$. Therefore, we obtain Table 12 as follows, according to Model (28). Here, the scale of production has expanded, and the value has been converted into production price in the 2nd year. The total production price 933.45 in the 2nd year is equal to the total value 933.45 and the total average profit 214.25 is equal to the total surplus value 214.25.

Table 13 Assumed values of exogenous parameters

Department	I (%)	II (%)	III (%)	IV (%)
$\delta_j^{(t)}$				
1st year	3.0	2.0	3.1	4.1
2nd year	2.5	2.7	4.0	2.9
3rd year	1.3	2.1	4.2	2.0
$\theta_j^{(t)}$				
1st year	3.0	2.0	3.6	4.2
2nd year	2.5	3.0	3.5	3.1
3rd year	1.3	2.5	4.0	2.5
$k_j^{(t)}$				
1st year	35	25	29	41
2nd year	25	41	32	26
3rd year	40	34	39	33

Source: Created by the author

Let us look at a more general example and use the data in Table 9. Assuming that the initial year is the 0th year, the transformation is completed in the 3rd year. It is assumed that the transformation is carried out under expanded reproduction. The assumed exogenous parameters are listed in Table 13.

Table 14 The dynamic transformation results of the value system of 4 departments ($k = 3$, expanded reproduction)

0th year	Dept.	I	II	III	IV	Total
Value	I	105	250	60	135	550
	II	205	130	110	75	520
	III	70	235	55	95	455
	IV	100	75	210	80	465
	v	300	150	200	160	810
	m	270	120	150	160	700
	w	1,050	960	785	705	3,500

1st year	Dept.	I	II	III	IV	Total
Value	I	108.150	257.500	61.800	139.050	566.500
	II	209.100	132.600	112.200	76.500	530.400
	III	72.170	242.285	56.705	97.945	469.105
	IV	104.100	78.075	218.610	83.280	484.065
	v	309.000	153.000	207.200	166.720	835.920
	m	278.100	122.400	155.400	166.720	722.620
	w	1,080.620	985.860	811.915	730.215	3,608.610

⇨

1st year	Dept.	I	II	III	IV	Total
Value*	I	106.088	252.590	60.622	136.398	555.697
	II	214.184	135.824	114.928	78.360	543.296
	III	72.577	243.652	57.025	98.498	471.752
	IV	102.967	77.226	216.232	82.374	478.798
	v*	309.195	153.096	207.330	166.825	836.446
	m*	255.003	147.442	160.360	159.815	722.620
	w*	1,060.014	1,009.830	816.497	722.270	3,608.610

2nd year	Dept.	I	II	III	IV	Total
Value	I	110.854	263.938	63.345	142.526	580.663
	II	214.746	136.180	115.229	78.566	544.721
	III	75.057	251.976	58.973	101.863	487.869
	IV	107.119	80.339	224.950	85.695	498.103
	v	316.725	157.590	214.452	171.888	860.655
	m	285.053	126.072	160.839	171.888	743.852
	w	1,109.553	1,016.095	837.788	752.426	3,715.863

⇨

2nd year	Dept.	I	II	III	IV	Total
Value*	I	106.675	253.987	60.957	137.153	558.772
	II	226.843	143.851	121.720	82.991	575.406
	III	75.412	253.170	59.252	102.345	490.179
	IV	104.360	78.270	219.157	83.488	485.276
	v*	317.359	157.905	214.881	172.232	862.378
	m*	237.075	186.150	165.788	154.840	743.852
	w*	1,067.724	1,073.334	841.756	733.050	3,715.863

3rd year	Dept.	I	II	III	IV	Total
Value	I	112.295	267.369	64.168	144.379	588.211
	II	219.255	139.040	117.649	80.215	556.160
	III	78.209	262.559	61.450	106.141	508.360
	IV	109.261	81.946	229.449	87.409	508.065
	v	320.842	161.530	223.030	176.186	881.588
	m	288.758	129.224	167.273	176.186	761.440
	w	1,128.621	1,041.668	863.019	770.516	3,803.824

⇨

3rd year	Dept.	I	II	III	IV	Total
Production price	I	105.726	251.728	60.415	135.933	553.802
	II	238.909	151.504	128.195	87.406	606.014
	III	78.639	264.003	61.788	106.725	511.155
	IV	104.709	78.532	219.890	83.768	486.899
	V	321.907	162.066	223.770	176.770	884.513
	R	212.709	227.210	173.707	147.814	761.440
	P	1,062.600	1,135.043	867.765	738.416	3,803.824

Source: Created by the author

Table 15 Solutions of various variables in each year

	1st year	2nd year	3rd year
x_1	0.980931	0.962301	0.941502
x_2	1.024313	1.056332	1.089640
x_3	1.005643	1.004735	1.005499
x_4	0.989120	0.974248	0.958340
y	1.000630	1.002001	1.003318
λ	0.988199	0.945736	
r			0.250277

Source: Created by the author

Therefore, according to Model (28), we obtain Table 14 as follows. Here, we see that the scale of production has expanded, and the value is converted into production price in the 3rd year.

The solutions of various variables used in calculating Table 14 for each year are shown in Table 15:

6 Conclusion

From a mathematical perspective, the transformation problem in the narrow sense is mainly about how to construct a mathematical model that can reflect the transformation process of value to production price under certain constraints.

From a historical perspective, according to Marx's logic, there is value first and then it is followed by production price. When the commodity economy reaches the point where surplus value is redistributed according to the law of "equal sums of capital demand equal profits", the value is transformed into production price. This conversion process may take a long time. Thus, the conversion of value to production price is represented as a process, and the mathematical description of this process should assume the form of a dynamic model.

After being transformed into production price, value of commodities is stabilized in the form of production price. The production price becomes prominent whereas the value (labor value) retreats behind the scenes. However, the link between value and production price remains. The static transformation model reflects the relation between the value and production price.

Using simulations, this study attempts to construct a computable mathematical model that connects dynamic transformation and static transformation. This simulation may have many flaws, but it can demonstrate the unity of two modes of the transformation. Hopefully, our study will stimulate the construction of a more complete transformation model.

In short, from a mathematical perspective, not only can the transformation problem be solved, but there may also be multiple solutions. The general static transformation model given by Zhang (2000) and the general dynamic transformation model given in this paper are not necessarily the best, but they are both effective. Regardless of whether a better solution should appear in the future, we are able to declare that the transformation problem can be solved mathematically.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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