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ARTICLE

Keynesian and classical theories: static and dynamic perspectives

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Abstract
This paper reexamines the fundamental difference between the Keynesian and classical theories from both static and dynamic perspectives. It is shown that the rigidity of wages plays a pivotal role in the distinction between these theories in statics and that they can be differentiated in terms of long-run stability in dynamics.

Keywords Classical theory · Keynesian theory · Long-run stability · Wage flexibility

JEL Classification C62 · E12 · E13 · E24

1 Introduction

It is no exaggeration to say that the development of macroeconomics has been stimulated by controversies between the Keynesian and classical theories. In general, the Keynesian theory allows for the existence of involuntary unemployment, while the classical theory does not. There have been a lot of arguments on the similarities and dissimilarities between them. The main topic of these arguments is the fundamental cause of the difference in conclusion between them.

The purpose of this paper is to reconsider the fundamental difference between the Keynesian and classical theories from both static and dynamic perspectives. This paper is organized as follows. Section 2 summarizes the static properties of the Keynesian and classical theories from a static viewpoint. It confirms Modigliani's (1944) claim that the fundamental difference between them in statics is due to the rigidity of nominal wages. Section 3 presents a different view on the Keynesian and classical theories from a dynamic (and long-term) perspective. In this section, two dynamic systems with Keynesian and classical features are examined to consider the

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difference in view on long-run stability between these theories. It is demonstrated that the same long-run equilibrium has totally different characteristics in stability between the Keynesian and classical systems and that the fundamental difference between the theories lies in the difference in stability properties of long-run equilibrium. Section 4 concludes this paper.

2 Keynesian and classical theories in statics

This section examines the Keynesian and classical theories from static perspectives. We compare the Keynesian and classical theories in statics and look at the differences in assumption and conclusion between them.

2.1 Keynesian theory in statics

The static Keynesian theory can be illustrated by the well-established IS-LM (or AD-AS) system.[1] The IS-LM system examined in this section is the following one proposed by Modigliani (1944, 1963):[2]

$$\frac{M}{P} = L(r, Y), \tag{1}$$

$$I = I(r, Y), \tag{2}$$

$$S = S(r, Y), \tag{3}$$

$$S = I, \tag{4}$$

$$Y = F(N), \tag{5}$$

$$\frac{W}{P} = F'(N), \tag{6}$$

[1] The formal theory developed in Keynes' General Theory is substantially static. Since the view is widespread that the IS-LM system provides a "distorted" interpretation of Keynes' General Theory, it may be useful to argue against this view. Keynes did not oppose but endorse Hicks (1937) as a summary of his General Theory. This fact can be confirmed by the following correspondence between Keynes and Hicks:

> At last long I have caught up with my reading and have been through the enclosed [Hicks (1937)]. I found it very interesting and really have next to nothing to say by way of criticism. (the letter from Keynes to Hicks in Keynes (1973, p. 80)).

[2] The IS-LM system is taken as an expression of the static Keynesian system in this paper, but as recognized in Modigliani (1944), it can be adapted to a dynamic context if time lags are appropriately introduced. For a recent study on the dynamic effects of time lags concerning consumption and investment in the IS-LM system, see Murakami (2017).

where the notations are given as follows: Y, aggregate real income or output; r, the rate of interest; I, aggregate investment or the investment function; S, aggregate saving or the saving function; L, aggregate real demand for money or the liquidity preference function; M, aggregate supply of money; N, aggregate employment of labor; F, the short-run production or aggregate supply function; P, the price level; W, the nominal or money wage. In this system, aggregate supply of money M is treated as given, while the other variables, $Y,\ r,\ I,\ S,\ N,\ P$ and $W,$ are endogenous variables.

Equation (1) represents the equilibrium condition for the money or assets market, whereby the rate of interest is determined, and illustrates the liquidity preference theory of interest (Keynes 1936, Chap, 15).

Equations (2) and (3) reflect the Keynesian theory of consumption and investment (Keynes 1936, chaps. 8–12).[3]

Equation (4) has a great importance in the Keynesian theory because it represents not only the equilibrium condition for the goods-services market but also the principle of effective demand (Keynes, Chap, 3).[4] It should, in the Keynesian theory, be interpreted to mean that aggregate saving, defined as the difference between aggregate income and aggregate consumption, is determined by aggregate investment, the sum of which and aggregate consumption is aggregate effective demand, not the other way round.

Equation (5) is nothing but the aggregate supply function (Keynes 1936, Chap, 3) and specifies the short-run relationship between aggregate employment and aggregate output with aggregate stock of capital kept constant. In the Keynesian theory, aggregate employment is adjusted by this equation to that level which is consistent with aggregate output determined by aggregate effective demand.

Finally, we would like to have a closer look at (6), because it may invoke some confusion; its meaning is different between the Keynesian and classical theories. It represents the "first postulate" of the classical theory, adopted as a realistic hypothesis in Keynes' General Theory (Keynes 1936, Chap, 2). In the classical theory, this postulate has the implication that aggregate demand for labor or aggregate employment is adjusted to that level at which the marginal productivity of labor is equalized to the given real wage. Thus, in the classical theory, Eq. (6) describes the (aggregate)

[3] Although it is argued in Keynes' General Theory that aggregate investment is determined by the marginal efficiency of capital (or the expected profit on capital) and the rate of interest, Eq. (2) may be regarded as consistent with his theory of investment because expected effective demand, which is a major determinant of the marginal efficiency of capital, is influenced largely by current effective demand or aggregate output. Equation (2) is also consistent with the profit principle of investment postulated in the Keynesian theories of business cycles of Kalecki (1935, 1939) and Kaldor (1940, 1951).

[4] It seems that the reason why the investment and saving functions and the equilibrium condition for the goods-services market are separately posited in Modigliani's (1944, 1963) system is that he appreciated the difference between ex ante investment and saving. The following statement is made in Modigliani (1944):

> In our case, the equilibrium of the "money market" is a condition of short-run equilibrium (that determines the rate of interest for each period) because it is the result of decisions that can be carried out into effect immediately. The condition saving = investment, on the other hand, is a condition of long-run equilibrium because the equality of ex ante saving and investment cannot be brought about instantaneously. This is a different way of stating the familiar proposition that the multiplier takes time to work out in its full effect. (p. 62).

labor demand function, which relates aggregate employment to the given real wage. In the Keynesian theory, on the other hand, aggregate employment or aggregate demand for labor is determined by aggregate effective demand for goods and services, as indicated by (4) and (5), and hence Eq. (6) should not be interpreted to represent the (aggregate) labor demand function. Indeed, it should be interpreted as the price setting function, which represents the price level as a function of aggregate employment for the given nominal wage;[5] otherwise it is obviously inconsistent with the principle of effective demand.[6] Thus, in the Keynesian theory, it describes the determination mechanism not of aggregate employment but of the price level.[7]

It is easily seen that the number of Eqs. (1)–(6) is one less than that of endogenous variables Y, r, I, S, N, P and W. Then, (at least) one of the endogenous variables cannot be solved. In other words, there is indeterminacy in this system. As argued in Modigliani (1944, 1963), however, the indeterminacy problem in the IS-LM system can be solved by fixing the nominal wage W to some predetermined

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[5] Equation (6) is equivalent to

$$P = \frac{W}{F'(N)}.$$

It can, thus, be taken as a behavioral equation for firms' price setting. Also, if the curvature of the production function is gradual, then the price level is almost proportionate to the nominal wage and, hence, almost fixed for the given nominal wage.

[6] According to Keynes' interpretation, the first postulate just describes the correlation between aggregate employment and the real wage, as the following statement indicates:

> It [the first postulate] means that, with a given organisation, equipment and technique, real wages and the volume of output (and hence of employment) are uniquely correlated, so that, in general, an increase in employment can only occur to the accompaniment of a decline in the rate of real wages. Thus I am not disputing this vital fact which the classical economists have (rightly) asserted as indefeasible. In a given state of organisation, equipment and technique, the real wage earned by a unit of labour has a unique (inverse) correlation with the volume of employment. Thus if employment increases, in the short period, the reward per unit of labour in terms of wage-goods must, in general, decline and profits increase. (Keynes 1936, p. 17)

His interpretation is more explicit in his debate with Dunlop and Tarshis on real wages:

> In the passage quoted above [Keynes (Keynes 1936, pp. 9–10)] I was dealing with the relation of real wages to changes in *output*, and had in mind situations where changes in real and money wages are reflection of changes in the level of employment caused by changes in effective demand. (Keynes 1939, p. 35)

[7] It seems that Keynes should have abandoned (6) and replaced it with a naive but realistic mark-up pricing hypothesis to establish the principle of effective demand as a foundation of his theory. In the Keynesian system based on the principle of effective demand ((4) and (5)), Eq. (6) may be better taken as a mark-up pricing rule than as a first order condition for optimality. Indeed, it can be written as

$$P = \frac{NF(N)}{F'(N)}\frac{WN}{Y} = \frac{1}{\beta}\frac{W}{Y/N},$$

where β stands for the employment elasticity of production. If β is approximately constant (and less than unity), this equation is consistent with the mark-up principle. In this sense, Kalecki's (1939, 1971) theory, based on the mark-up principle, is more faithful to the principle of effective demand. Indeed, in examining the Keynesian theory from dynamic perspectives (in the next section), we will adopt a hypothesis on price settings based on Kalecki's theory rather than Keynes', because the former is more "Keynesian" than the latter.

level. With the nominal wage fixed, the other variables can be determined by the system, (1)–(6). Aggregate employment so determined does not necessarily correspond to full employment because aggregate supply of labor is not taken into account in this system. Put in a different way, *involuntary unemployment* can arise in this system. Thus, the existence of involuntary unemployment or underemployment equilibrium can be verified in the presence of rigid nominal wages. Due to this, it is widely thought that the rigidity of nominal wages is the main cause of involuntary unemployment.[8] On these grounds, the Keynesian theory, in statics, is viewed as a macroeconomic theory based on the assumption of rigid nominal wages.[9]

2.2 Classical theory in statics

Now, we proceed to explore the properties of the (neo-)classical theory from static perspectives. The static classical theory can substantially be delineated by the same IS-LM system, (1)–(6), with one missing equation. The one missing equation is given by

$$N = N_S\left(\frac{W}{P}\right), \tag{7}$$

where N_S is the labor supply function.

To examine the differences between the (static) Keynesian and classical theories, we consider the meanings of (1)–(6), as well as of (7), in the classical theory. For this purpose, we first look at (6) and (7) and then turn to (5), (2)–(4) and (1) in order.

Equation (6) truly expresses the first postulate of the classical theory, which implies that aggregate demand for labor is determined by the real wage. In this sense, as we have argued above, it plays the role of the labor demand function in the classical theory. Equation (7), on the other hand, indicates that aggregate employment is equal to aggregate supply of labor corresponding to the given real wage. It reflects the "second postulate" of the classical theory, which states that the marginal disutility of labor, divided by the marginal utility of consumption, is equal to the real wage (Keynes 1936, Chap, 2). Thus, Eqs. (6) and (7) imply that aggregate employment, which appears as a symbol in both of them, is equal to aggregate demand for and supply of labor and hence that the labor market is in equilibrium in the sense that there is no involuntary unemployment. In other words, aggregate employment

[8] This view is asserted in Modigliani (1944) as follows:

> It is usually considered as one of the most important achievements of the Keynesian theory that it explains the consistency with the presence of involuntary unemployment. It is, however, not sufficiently recognized that, except in a limiting case to be considered later, this result it due entirely to the assumption of "rigid wages" and not to the Keynesian liquidity preference. (p. 65) The liquidity-preference theory is not necessary to explain underemployment equilibrium; it is sufficiently only in a limiting case; the "Keynesian case." In the general case it is neither necessary nor sufficient; it can explain this phenomenon only with the additional assumption or rigid wages. (pp. 75–76).

[9] As noted in the outset of Modigliani's (1944) analysis, this view on the Keynesian theory is only valid in a static context. See Sect. 2.3.

derived from (6) and (7) corresponds to full employment. The presence of (7) characterizes the classical theory because it means full employment.[10]

Equation (5) describes the short-run technique of production, but its role in the classical theory is different from that in the Keynesian theory. In the Keynesian theory, it derives aggregate employment from aggregate real output corresponding to aggregate effective demand; in the classical theory, it derives aggregate real output from aggregate employment corresponding to full employment. It implies different causal relationships between aggregate employment and output in these theories.

Naturally, Eqs. (2)–(4), in the classical theory, play a role different from that in the Keynesian theory. Since aggregate real output is determined by (5), aggregate investment and saving can be reduced to functions of the rate of interest alone by (2) and (3). Then, what Eq. (4) determines is not aggregate output but the rate of interest. It reflects the classical (or loanable funds) theory of interest.

Equation (1) is, of course, the equilibrium condition of the money market but does not describe the determination mechanism of the rate of interest. Since aggregate output and the rate of interest are already determined by (5) and (4), respectively, the right-hand side of (1), or aggregate real demand for money, is constant. It then follows from (1) that the price level is proportionate to aggregate supply of money, which implies that the quantity theory of money holds. Equation (1) gives the price level in response to the given quantity of money.

In the static classical theory, aggregate supply of labor is taken into consideration by (7). This is the main difference from the static Keynesian theory. Because of it, the real wage is allowed to be adjusted to the level at which the labor market is in equilibrium in the sense that aggregate demand for and supply of labor equal each other; the flexibility of (real) wages is implicitly assumed in the classical theory. As a result of this flexibility, aggregate output is fixed at the level corresponding to full employment in the labor market and hence determined independently from the quantity of money. In this respect, the classical dichotomy holds. This is reflected in the conclusion that the price level is determined entirely by the quantity of money. Thus, the flexibility of wages, as well as the labor supply function, plays a vital role for establishing full-employment equilibrium.

[10] The equivalence of the second postulate and full employment is recognized in Keynes' General Theory:

At different points in this chapter, we have made the classical theory to depend in succession on the assumptions:

(1) that the real wage is equal to the marginal disutility of the existing employment;

(2) that there is no such thing as involuntary unemployment in the strict sense;

(3) that supply creates its own demand in the sense that the aggregate demand price is equal to the aggregate supply price for all levels of output and employment.

These three assumptions, however, all amount to the same thing in the sense that they all stand and fall together, any one of them logically involving the other two. (Keynes 1936, pp. 21–22)

2.3 Keynesian and classical theories in statics: summary

We have so far discussed the Keynesian and classical theories from static viewpoints. We can now conclude that whether the nominal wage is flexible or not makes a great difference between the Keynesian and classical theories. It then follows from this conclusion that the existence of underemployment equilibrium with *involuntary unemployment* is only the outcome of the rigidity of nominal wages. According to this view, the Keynesian theory reduces to the classical theory without the nominal-wage rigidity. This is the view shared among a lot of economists.[11]

The rigidity of nominal wages plays an essential role for verifying the Keynesian theory or the presence of involuntary unemployment in statics; it is a sufficient (not necessary) condition for the existence of equilibrium with involuntary unemployment. In statics, phenomena do not possess practical relevance unless they are shown to occur in equilibrium. In this sense, the widespread view on the Keynesian theory is right in a static context.

3 Keynesian and classical theories in dynamics

In statics, as we have confirmed in the last section, the rigidity of nominal wages or prices is of great importance in explaining *involuntary unemployment*; it guarantees the existence of *underemployment equilibrium*. Involuntary unemployment, however, need not be characterized as a phenomenon occurring at *equilibrium*. It can be described as a disequilibrium phenomenon in a dynamic context. Indeed, if the economy stays away from the full-employment equilibrium for a long while, it undergoes prolonged involuntary unemployment in disequilibrium or underemployment disequilibrium. From this point of view, we do not necessarily have to assume the rigidity of nominal wages or prices to explain the phenomenon of underemployment equilibrium. We may thus argue that what matters for the practical relevance of the Keynesian theory is not the existence of underemployment equilibrium but the (in-)stability of full-employment equilibrium.[12]

[11] This view is made explicit in Ball et al. (1988) as follows:

> According to the Keynesian view, fluctuations in output arise largely from fluctuations in nominal aggregate demand. These changes in demand have real effects because nominal wages and prices are rigid. (p. 1)

[12] This position is taken in Patinkin (1965) as the following statements indicate:

> Thus Keynesian economics is the economics of unemployment *dis*equilibrium. It argues that as a result of interest-inelasticity, on the one hand, and distribution and expectation effects, on the other, the dynamic process of Chapter XIII:3—even when aided by monetary policy—is unlikely to converge either smoothly or rapidly to the full-employment position. Indeed, if these influences are sufficiently strong, they may even render this process unstable. In such a case the return to full employment would have to await the fortunate advent of some exogenous force that would expand aggregate demand sufficiently. (pp. 337–338)
>
> While our interpretation takes off the analytical edge of Keynesian economics in one direction, it sharpens it in another, more vital one. It makes unmistakably clear—what should always have been clear—that the involuntary unemployment of the General Theory need not have its origin in wage rigidities. Indeed, in this respect we are more Keynesian than Keynes. For by unequivocally placing the center of emphasis on the inadequacy of aggregate demand in the commodity market, and by

This section discusses the Keynesian and classical theories in dynamics from the viewpoint of stability of full-employment equilibrium. For this purpose, we examine two dynamic systems, one of which is with Keynesian features and the other with classical features; they possess the same equilibrium corresponding to the natural rate of unemployment, identified with full employment, but differ in dynamic adjustments in the price level and aggregate output. The issue in this section is the differences in stability properties of the same equilibrium between the Keynesian and classical systems.

The subsections that follow explore Keynesian and classical systems in a long-run context. Section 3.1 sets up the basic framework common between our Keynesian and classical systems. Sections 3.2 and 3.3 formalize and examine dynamic systems with Keynesian and classical properties, respectively.

3.1 Aggregate demand and aggregate supply

This subsection presents behavioral equations concerning aggregate demand and aggregate supply. In what follows, the notations used in the last section are kept except when special mention is made.

We first consider aggregate demand. Aggregate saving is assumed to be represented by

$$S = sY, \tag{8}$$

where s is a positive constant which represents the (marginal and average) propensity to save.

We postulate that aggregate (gross) investment is described by the following equation:

$$\frac{I}{K} = f\left(\frac{Y}{K}, r - \pi^e\right) = f(x, \rho^e), \tag{9}$$

where K and π^e stand for aggregate stock of capital and the expected rate of inflation, respectively; x and ρ^e represent the output–capital ratio and the real rate of interest,

Footnote 12 (continued)

recognizing the resulting involuntary unemployment to be phenomenon of economic dynamics, we have freed ourselves from the necessity of static analysis to connect decreases in employment with increases in the real wage rate. We have been able to explain the existence of involuntary unemployment without placing any restrictions on the movement of the real wage rate. (pp. 340–341)

A similar argument is made in Tobin (1975):

Very likely Keynes chose the wrong battlefield. Equilibrium analysis and comparative statics were the tools to which he naturally turned to express his ideas, but they were probably not the best tools for his purpose. (p. 195)

The real issue is not the existence of a long-run static *equilibrium* with unemployment, but the possibility of protracted unemployment which the natural adjustments of a market economy remedy very slowly if at all. So what if, within the recherché rules of the contest, Keynes failed to establish an “underemployment equilibrium”? The phenomena he described are better regarded as disequilibrium dynamics. Keynes’s comparative statics were an awkward analytical language unequal to the shrewd observations and intuitions he was trying to embody. (pp. 195–196) respectively; f is the capital formation function which is assumed to be twice continuously differentiable with

$$f_x \equiv \frac{\partial f}{\partial x} > 0, \quad f_{\rho^e} \equiv \frac{\partial f}{\partial \rho^e} < 0. \tag{10}$$

This equation reflects the Keynesian theory of investment.[13]

We postulate that the nominal rate of interest is represented by the following equation:

$$r = r\left(\frac{PY}{M}\right) = r(v) \tag{11}$$

where v represents the ratio between aggregate nominal income to aggregate nominal supply of money; r is interest rate function which is assumed to be twice continuously differentiable with

$$r_v \equiv r'(v) > 0, \tag{12}$$

This relation can be derived from the money-market equilibrium condition given by

$$\frac{M}{P} = l(r)Y, \tag{13}$$

where l is the liquidity preference function with

$$l'(r) < 0. \tag{14}$$

We turn to aggregate supply. Aggregate real output is determined by the following Cobb–Douglas technique:

$$Y = K^{1-\beta}(AN)^{\beta}, \tag{15}$$

where A stands for the labor-augmenting technology; β is a positive constant less than unity.

With the capital formation and interest rate functions (10) and (11), aggregate capital formation can be described by[14]

$$\frac{\dot{K}}{K} = f\left(\frac{Y}{K}, r\left(\frac{PY}{M}\right) - \pi^e\right) - \delta, \tag{16}$$

where δ is a positive constant which represents the rate of capital depreciation.[15]

[13] Equation (9) is consistent with the marginal efficiency theory of investment (Keynes 1936, Chap, 11) and also with the profit principle of investment (cf. Kalecki 1935, 1939; Kaldor 1940, 1951). For an intertemporal microeconomic foundation on the Keynesian theory of investment, see Murakami (2016a).

[14] Throughout this paper, $\dot{q}$ denotes the time derivative of q, i.e., $\dot{q} = \mathrm{d}q/\mathrm{d}t$.

[15] Since aggregate ex ante investment and saving are distinguished, aggregate ex post capital formation may not be equal to aggregate ex ante investment net of capital depreciation. For simplicity, however, the difference between ex ante and ex post investment is ignored in (16). For a more general formalization of capital formation, see Stein (1969); for a further generalization, see Murakami (2014).

The natural-rate level of aggregate employment, proportional to aggregate supply of labor,[16] and the labor-augmenting technology are assumed to change at constant rates as follows:

$$\frac{\dot{N}^*}{N^*} = n, \tag{17}$$

$$\frac{\dot{A}}{A} = a, \tag{18}$$

where N^* stands for the natural-rate level of aggregate employment; n is a real constant which represents the rate of change in aggregate supply of labor; a is a non-negative constant which represents the rate of technical change with $\delta + n + a > 0$.[17]

Given the production technique described by (15), the natural-rate level of aggregate real output is defined by the natural-rate level of aggregate employment as follows:

$$Y^* = K^{1-\beta}(AN^*)^{\beta}, \tag{19}$$

where Y^* stands for the natural-rate level of aggregate real output. For this variable, the following three ratios can be defined:

$$y = \frac{Y}{Y^*}, \tag{20}$$

$$k = \frac{K}{Y^*}, \tag{21}$$

$$m = \frac{M}{PY^*}, \tag{22}$$

where y, k and m are called the rate of utilization, the capital coefficient and the money coefficient, which corresponds to the Marshallian k at the natural rate of employment, respectively.

It follows from (19) that the rate of change in the natural-rate level of aggregate output can be written as

$$\frac{\dot{Y}^*}{Y^*} = (1-\beta)\frac{\dot{K}}{K} + \beta\left(\frac{\dot{A}}{A} + \frac{\dot{N}^*}{N^*}\right),$$

which can, by (16)–(18) and (20)–(22), be reduced to

[16] If the natural rate of unemployment is constant over time, the natural-rate level of aggregate employment is proportional to aggregate supply of labor.

[17] Provided that this condition holds, the rate of change in aggregate supply of labor, which can be identified with the rate of change in population, can be negative.

$$\frac{\dot{Y}^*}{Y^*} = (1-\beta)\left[f\left(\frac{y}{k}, r-\pi^e\right) - \delta\right] + \beta(n+a), \tag{23}$$

With (16)–(18) and (20)–(22), the rate of change in the capital coefficient can also be given by

$$\frac{\dot{k}}{k} = \beta\left[f\left(\frac{y}{k}, r\left(\frac{y}{m} - \pi^e\right)\right) - (\delta + n + a)\right]. \tag{24}$$

This is a dynamic equation common to our Keynesian and classical systems explored in the next subsections.

As regards the labor market, it is seen from (15), (19) and (20) that the ratio between actual and natural rates of employment can be related to the rate of utilization as follows:

$$\frac{N}{N^*} = y^{1/\beta}. \tag{25}$$

The rate of change in the nominal wage is supposed to be determined by the actual and natural rates of employment and the expected rate of inflation in the following fashion:

$$\frac{\dot{W}}{W} = \gamma\left(\frac{N-N^*}{N^*}\right) + \pi^e + a,$$

where γ is a positive parameter that measures the speed of adjustment in the labor market and also the degree of wage flexibility. This equation is a natural-rate version of the Phillips curve augmented with the rate of change in labor-augmenting technology. It then follows from (25) that this Phillips curve can be written as

$$\frac{\dot{W}}{W} = \gamma(y^{1/\beta} - 1) + \pi^e + a. \tag{26}$$

We suppose that the monetary authority changes aggregate supply of money at a constant rate in the following way:

$$\frac{\dot{M}}{M} = \mu, \tag{27}$$

where μ is a real constant.

Finally, it is assumed that the expected rate of inflation is revised adaptively in the following fashion:

$$\dot{\pi}^e = \epsilon\left(\frac{\dot{P}}{P} - \pi^e\right), \tag{28}$$

where ϵ is a positive constant which represents the speed of revisions of inflation expectations.[18]

3.2 Keynesian theory in dynamics

In dynamics, the Keynesian and classical theories can be distinguished in terms of adjustments in the price level and aggregate output. In the Keynesian theory, as the principle of effective demand implies, aggregate output is changed in response to fluctuations in aggregate effective demand; in the classical theory, the gap between aggregate demand and output is filled by adjustments in the price level.[19] Following this view, this and the next subsections consider the Keynesian and classical theories in a long-run and dynamic context to discuss the difference in view on long-run stability between them.

The principle of effective demand is no doubt the core of the Keynesian theory. In statics, it maintains that aggregate real output is equalized to aggregate effective demand. In dynamics, it is interpreted to mean that aggregate output is adjusted to meet aggregate effective demand in the short run. In other words, it can be regarded as the hypothesis that it is quantity, rather than prices, that is adjusted in response to the gap between (aggregate) demand and supply. Given that a "long term" is a consequence of a "short term," the "quantity adjustment" should also characterize the long-term Keynesian theory.

Reflecting this view, we may postulate for dynamic analysis that aggregate output is adjusted to aggregate demand. In the long term, however, the level of potential output, identified with the natural-rate level of output, increases through changes in the supply side (e.g., capital formation, population changes and technical progress). In this respect, it is reasonable to suppose that aggregate output is determined relative to its potential level in response to changes in the demand side. In other words, the rate of utilization, the ratio between the two, should be taken as the adjusting variable in a long-term version of Keynesian quantity adjustment.

Thus, we assume that the rate of utilization is changed in response to the discrepancy between aggregate effective demand and supply in the following fashion:

[18] As verified by Muth (1961), the "adaptive expectations" rule is "rational" (or efficient) if the actual rate of inflation is composed of both permanent and transitory disturbances.

[19] This is a common view as observed in Leijonhufvud (1968):

> In general equilibrium flow models, prices are the only endogenous variables which enter as arguments into the demand and supply functions of individual households. Tastes and initial resource endowments are parametric. In "Keynesian" flow models the corresponding arguments are real income and the interest rate. Of these, real income is a measure of quantity, not of prices. On a highly abstract level, the fundamental distinction between general equilibrium and Keynesian models lies in the appearance of this quantity variable in the excess demand relation to the latter. The difference is due to the assumptions made about the adjustment behavior of the two systems. In the short run, the "Classical" system adjusts to changes in money expenditures by means of price-level movements; the Keynesian adjusts primarily by way of real income movements. (p. 51)

$$\dot{y} = \alpha\left(\frac{E - Y}{Y^*}\right) = \alpha\left(\frac{I - S}{Y^*}\right), \tag{29}$$

where E stands for aggregate effective demand equal to the sum of aggregate consumption and investment;[20] α is a positive constant which represents the speed of adjustment. This equation says that the rate of utilization is varied in response to the ratio between aggregate excess demand and the natural-rate level of aggregate output. It may be regarded as a long-term version of quantity adjustment.

Substituting (8), (9), (11) and (20)–(22) in (29), we obtain the following:

$$\dot{y} = \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]. \tag{30}$$

This equation, as well as (24), constitutes our Keynesian system.

According to the principle of effective demand, the gap between aggregate demand and supply is filled by quantity adjustment rather than price adjustment. In the Keynesian theory, the price level should then be determined by a mechanism different from price adjustment. In Keynes' General Theory, as we have observed, the "marginal" principle plays a role in price determination; "[t]he general price-level depends partly on the rate of renumeration of the factors of production which enter into marginal cost and partly on the scale of output as a whole" (Keynes 1936, p. 294). As we have argued, however, the mark-up principle, proposed by Kalecki (1939, 1971), is more suited to the Keynesian theory. In Kalecki's (1939; 1971) theory, the mark-up ratio is influenced by the degree of monopoly, and aggregate share of labor is determined by the (average) mark-up ratio. As argued by him, aggregate share of labor so determined is subject to fluctuations in the short run but is fairly constant in the long run.[21]

Reflecting the Kaleckian theory, we may postulate that the price level is determined so that aggregate share of labor would be constant at the natural rate of unemployment:

$$P = (1 + \theta)\frac{WN^*}{Y^*}, \tag{31}$$

where θ is a positive constant which represents the mark-up ratio. This is consistent with the short-run fluctuations and long-run stability of aggregate share of labor.

It follows from (31) that the rate of inflation is given by

$$\frac{\dot{P}}{P} = \frac{\dot{W}}{W} + \frac{\dot{N}^*}{N^*} - \frac{\dot{Y}^*}{Y^*},$$

which can, with (17), (23) and (26), be reduced to

[20] Government expenditure or net exports can be included in aggregate effective demand as exogenous factors, provided that they are proportional to the natural-rate level of aggregate output.

[21] The long-run stability of aggregate share of labor has been confirmed by Jones (2016). According to him, the U.S. share of capital was almost constant (about 34.2 percent) until around 2000, though it has recently been rising (to 38.7 percent by 2012).

$$\frac{\dot{P}}{P} = \gamma(y^{1/\beta} - 1) - (1-\beta)\Big[f\Big(\frac{y}{k}, r\Big(\frac{y}{m}\Big) - \pi^e\Big) - (\delta + n + a)\Big] + \pi^e. \tag{32}$$

The rate of change in the money coefficient (defined by (22)) can then be calculated as

$$\frac{\dot{m}}{m} = \frac{\dot{M}}{M} - \frac{\dot{P}}{P} - \frac{\dot{Y}^*}{Y^*},$$

which can, by (23) and (27), be reduced to

$$\frac{\dot{m}}{m} = \mu - n - a - \gamma(y^{1/\beta} - 1) - \pi^e. \tag{33}$$

It is seen from (28) and (32) that the expected rate of inflation is revised as follows:

$$\dot{\pi}^e = \epsilon\Big\{\gamma(y^{1/\beta} - 1) - (1-\beta)\Big[f\Big(\frac{y}{k}, r\Big(\frac{y}{m}\Big) - \pi^e\Big) - (\delta + n + a)\Big]\Big\}. \tag{34}$$

Thus, we can obtain the following system of equations:

$$\dot{y} = \alpha\Big[f\Big(\frac{y}{k}, r\Big(\frac{y}{m}\Big) - \pi^e\Big)k - sy\Big], \tag{30}$$

$$\frac{\dot{k}}{k} = \beta\Big[f\Big(\frac{y}{k}, r\Big(\frac{y}{m}\Big) - \pi^e\Big) - (\delta + n + a)\Big], \tag{24}$$

$$\frac{\dot{m}}{m} = \mu - n - a - \gamma(y^{1/\beta} - 1) - \pi^e. \tag{33}$$

$$\dot{\pi}^e = \epsilon\Big\{\gamma(y^{1/\beta} - 1) - (1-\beta)\Big[f\Big(\frac{y}{k}, r\Big(\frac{y}{m}\Big) - \pi^e\Big) - (\delta + n + a)\Big]\Big\}. \tag{34}$$

We call the system of these equations "System (K)" (to mean Keynesian) and examine it to look at the dynamic characteristics of the Keynesian theory.[22]

The long-run equilibrium values of y, k, m and π^e, denoted respectively by y^*, k^*, m^* and π^*, are solutions of the following simultaneous equations:[23]

[22] System (K) has a lot in common with the Kaldor–Tobin models (synthesizing Kaldor (1940) and Tobin (1975)) of Asada (1991), Chiarella and Flaschel (2000, Chap, 6), Chiarella et al. (2013, Chap, 13) and Murakami and Asada (2018) but differs from them in the following points: (i) the rate of utilization is not identified with the output–capital ratio; (ii) the price level is determined based on the natural-rate (long-run) levels (not actual levels) of employment and output (cf. (31)). It may also be viewed as a long-term extension of the short-term Keynesian model of Flaschel et al. (1997, Chap, 7) and of the medium-term Keynesian models of Murakami (2014, 2016b). The purpose of our analysis is not to present a generalized Keynesian model integrating the related ones but to consider the difference in view on stability between the Keynesian and classical theories making use of fairly standard models.

[23] $k = 0$ or $m = 0$ is ruled out as an equilibrium value of k or m.

$$0 = f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy,$$
$$0 = f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a),$$
$$0 = \mu - n - a - \gamma(y^{1/\beta} - 1) - \pi^e,$$
$$0 = \gamma(y^{1/\beta} - 1) - (1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right].$$

It follows that y^*, k^* and π^* can be calculated as

$$y^* = 1, \tag{35}$$

$$k^* = \frac{s}{\delta + n + a}, \tag{36}$$

$$\pi^* = \mu - n - a. \tag{37}$$

It is also seen that m^* is given by

$$f\left(\frac{\delta + n + a}{s}, r\left(\frac{1}{m^*}\right) - \mu + n + a\right) = \delta + n + a. \tag{38}$$

It can be found from (35) that the natural-rate level of aggregate output is attained at the long-run equilibrium and the long-run expected (and actual) rate of inflation is determined by the rate of change in the quantity of money.

To guarantee the existence (and hence uniqueness) of long-run equilibrium, it is assumed that[24]

$$\begin{aligned} &\lim_{m \to +0} f\left(\frac{\delta + n + a}{s}, r\left(\frac{1}{m}\right) - \mu + n + a\right) < \delta + n + a \\ &< \lim_{m \to \infty} f\left(\frac{\delta + n + a}{s}, r\left(\frac{1}{m}\right) - \mu + n + a\right). \end{aligned} \tag{39}$$

Due to (10) and (51), this ensures the existence and uniqueness of $m^* > 0$ that satisfies (38).

The long-run stability of system (*K*) can be discussed by examining the Jacobian matrix evaluated at the long-run equilibrium, denoted by J_K^*:

$$J_K^* = \begin{pmatrix} \alpha(f_x^* - s + k^* f_{\rho^e}^* r_v^*/m^*) & \alpha(s - f_x^*)/k^* & -\alpha k^* f_{\rho^e}^* r_v^*/(m^*)^2 & -\alpha k^* f_{\rho^e}^* \\ \beta(f_x^* + k^* f_{\rho^e}^* r_v^*/m^*) & -\beta f_x^*/k^* & -\beta k^* f_{\rho^e}^* r_v^*/(m^*)^2 & -\beta k^* f_{\rho^e}^* \\ -\gamma m^*/\beta & 0 & 0 & -m^* \\ \epsilon[\gamma/\beta - (1-\beta)(f_x^*/k^* + f_{\rho^e}^* r_v^*/m^*)] & (1-\beta)\epsilon f_x^*/(k^*)^2 & (1-\beta)\epsilon f_{\rho^e}^* r_v^*/(m^*)^2 & (1-\beta)\epsilon f_{\rho^e}^* \end{pmatrix},$$

where $*$ signifies the value evaluated at the long-run equilibrium. The characteristic equation associated with the Jacobian matrix J_K^* is given by

[24] A similar assumption is made for the same purpose in Asada (1991).

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$a_1 = \alpha\left(s - f_x^* - \frac{k^*}{m^*} f_{\rho^e}^* r_v^*\right) + \beta\frac{f_x^*}{k^*} - (1-\beta) f_{\rho^e}^* \epsilon, \tag{40}$$

$$a_2 = \left\{\alpha\left[\frac{k^*}{\beta}\gamma\epsilon - \frac{1}{\beta}\frac{k^*}{m^*} r_v^*\gamma - (1-\beta)s\epsilon - \beta s\frac{r_v^*}{m^*}\right] + (1-\beta)\frac{r_v^*}{m^*}\epsilon\right\} f_{\rho^e}^*, \tag{41}$$

$$a_3 = \alpha\left[\left(s - \frac{1}{\beta}\frac{k^*}{m^*} r_v^*\right)\gamma\epsilon - s\frac{r_v^*}{m^*}\gamma + (1-\beta)s\frac{r_v^*}{m^*}\epsilon\right] f_{\rho^e}^*, \tag{42}$$

$$a_4 = -\alpha\gamma\epsilon s f_{\rho^e}^* \frac{r_v^*}{m^*} > 0, \tag{43}$$

The inequality holds under (10) and (51).

According to the Routh–Hurwitz criterion, the following is the necessary and sufficient condition for the local asymptotic stability of the long-run equilibrium of system (K):

$$a_1 > 0,\ a_3 > 0,\ a_4 > 0,$$
$$(a_1a_2 - a_3)a_3 - a_1^2 a_4 > 0.$$

It then follows from (36), (41) and (42) that necessary conditions for the stability is given by

$$a_2 > 0,\ a_3 > 0,$$

or

$$\alpha s\left[\frac{1}{\beta(\delta+n+a)}\gamma\epsilon - \frac{1}{\beta(\delta+n+a)}\frac{r^*}{\eta_r^*}\gamma - (1-\beta)\epsilon - \beta\frac{r^*}{\eta_r^*}\right] + (1-\beta)\frac{r^*}{\eta_r^*}\epsilon < 0, \tag{44}$$

$$\left[1 - \frac{1}{\beta(\delta+n+a)}\frac{r^*}{\eta_r^*}\right]\gamma\epsilon - \frac{r^*}{\eta_r^*}\gamma - (1-\beta)\frac{r^*}{\eta_r^*}\epsilon < 0, \tag{45}$$

where

$$r^* = r\left(\frac{1}{m^*}\right), \tag{46}$$

$$\eta_r^* = \frac{m^* r^*}{r_v^*}. \tag{47}$$

It can be seen from (13), (14), (35) and (46) that η_r^*, defined by (47), is equal to the interest elasticity of liquidity preference evaluated at the long-run equilibrium.[25]

Conditions (44) and (45) are examined to discuss the long-run stability in our Keynesian system. As we have observed, the principle of effective demand implies that quantity adjustment is carried out fast enough. In this respect, the speed of quantity adjustment α should be large enough in the Keynesian theory.[26] If both the degree of wage flexibility γ and the speed of revisions of expectations ϵ are large enough, condition (44) is violated and hence, the long-run equilibrium is unstable, provided that α is large enough. Also, if the interest elasticity of liquidity preference η_r^* (evaluated at the long-run equilibrium) is sufficiently large or if the long-run equilibrium is stuck at the "liquidity trap," both conditions (44) and (45) are violated and hence, the long-run stability is lost, as long as both γ and ϵ are large enough.[27] Thus, in the Keynesian theory, characterized by fast quantity adjustment or by the "liquidity trap," the long-run equilibrium is more likely to be unstable, as the nominal wage is more flexible and expectations on inflation are more frequently revised. In particular, a high degree of wage flexibility is not conducive to stability in the Keynesian theory.[28] This conclusion is contrasted with the classical conclusion on the effect of wage flexibility on stability.

It is possible to explain the mechanism of instability of system (K) in a plain way. Assume that the state of the economy was originally at the long-run equilibrium but the rate of utilization y suddenly falls. If the degree of wage flexibility γ and the speed of revisions of expectation ϵ are large enough, the response to this fall can be described by the following diagram:

$$y \downarrow \begin{cases} \xrightarrow{(33)} & m \uparrow \xrightarrow{(30)} \quad y \uparrow \\ \xrightarrow{(34)} & \pi^e \downarrow\downarrow \xrightarrow{(30)} \quad y \downarrow\downarrow \end{cases},$$

where $\uparrow$, $\downarrow$ and $\downarrow\downarrow$ indicate a rise, decline and sharp decline in the variable on the left side, respectively. This diagram illustrates the destabilizing effects of wage flexibility and of expectations on deflation in our Keynesian system.[29]

It may be helpful to discuss the similarities and dissimilarities between Harrod's (1939) Keynesian system and our Keynesian system. In the Harrodian system,

[25] The interest elasticity of liquidity preference is given by

$$\eta_r = -\frac{rl'(r)}{l(r)} = \frac{m}{y}\frac{r}{r_v} > 0.$$

[26] The principle of effective demand in statics, represented by (4), can be taken as the limiting case of $\alpha \to \infty$ in the quantity adjustment described by (30).

[27] This is consistent with the conclusion in Yoshikawa's (1981) dynamic Keynesian model abstracting from capital formation.

[28] This conclusion supports the following view on the flexibility of wages, presented in Keynes' General Theory:

> To suppose that a flexible wage policy is a right and proper adjunct of a system which on the whole is one of laissez-faire, is the opposite of truth. (Keynes 1936, p. 269)

[29] For discussions on the existence (and uniqueness) of persistent business cycles in related Keynesian models, see Murakami (2014, 2018, 2019, 2020) and Murakami and Asada (2018).

unlike in our system, aggregate investment is assumed to be determined by a change in aggregate income:

$$I = v\dot{Y}, \tag{48}$$

where v is a positive constant which represents the "capital coefficient." This reflects the acceleration principle of investment, different from the profit principle adopted in our analysis. Based on the Harrodian postulate that the difference between ex ante investment and ex post investment (or saving) induces a change in the rate of economic growth (cf. Alexander 1950), we may suppose in a long-run context that[30]

$$\dot{g} = \kappa\left(\frac{I - S}{Y*}\right),$$

where g stands for the rate of economic growth, i.e., $g = \dot{Y}/Y$ and κ is a positive constant that measures the speed of adjustment. It then follows from (8) and (48) that

$$\dot{g} = \kappa\left(v\frac{\dot{Y}}{Y*} - sy\right) = \kappa v\left(g - \frac{s}{v}\right)y. \tag{49}$$

It is easily seen from (49) that the rate of economic growth g does not converges to the "warranted rate of growth" s/v, unless its initial value $g(0)$ happens to be equal to this value (provided that the rate of utilization y is positive).[31] This implies that the Harrodian system is unstable irrespective of the degree of wage or price flexibility. In this respect, our Keynesian system can be said to be "more stable," compared with the Harrodian system, though its relative stability is lost when the degree of wage flexibility is high enough.

3.3 Classical theory in dynamics

In the classical theory, it is not aggregate output but the price level that is changed in response to the difference between aggregate demand and output. In the presence of expectations on inflation or deflation, it is also reasonable to assume that they are passed on to price changes.

Thus, we postulate that the rate of inflation is determined as follows:

$$\frac{\dot{P}}{P} = \alpha\left(\frac{E - Y}{Y*}\right) + \pi^e,$$

which can be written as follows:

[30] For different formalizations of Harrod's (1939) theory, see Yoshida (1999) and Sportelli (2000).

[31] The solution of Eq. (49) can be given as

$$g(t) = \frac{s}{v} + \left[g(0) - \frac{s}{v}\right]\exp\left(\kappa v\int_0^t y(s)ds\right).$$

$$\frac{\dot{P}}{P} = \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right] + \pi^e, \tag{50}$$

where α is a positive constant which is here interpreted as the speed of price adjustment. This is a formulation of the "classical" version of price dynamics.[32]

In the static classical system (cf. Sect. 2.2), it is not the price level but the rate of interest that varies to equalize aggregate investment and saving, as the loanable funds theory of interest indicates. Reflecting this view, it may be reasonable to assume, in a long run context, that the rate of interest is changed in the following fashion:

$$\dot{r} = \tilde{\alpha}\left(\frac{E - Y}{Y^*}\right) = \tilde{\alpha}\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right], \tag{51}$$

where $\tilde{\alpha}$ is a positive constant. In this respect, Eq. (50) may not seem consistent with the classical theory, but it is. According to the equilibrium condition of the money market (13), the rate of inflation can be expressed as follows:

$$\frac{\dot{P}}{P} = -\frac{l'(r)}{l(r)}\dot{r} + \frac{\dot{M}}{M} - \frac{\dot{Y}}{Y},$$

where Eq. (27) is taken into account. This equation can be approximated around the long-run equilibrium values (35)–(38) as follows:

$$\frac{\dot{P}}{P} \approx -\frac{l'(r^*)}{l(r^*)}\dot{r} + \mu - \frac{\dot{Y}^*}{Y^*} \approx -\frac{l'(r^*)}{l(r^*)}\dot{r} + \mu - (n + a),$$

because

$$\frac{\dot{Y}}{Y} \approx \frac{\dot{Y}^*}{Y^*} \approx (1 - \beta)\left[f\left(\frac{y^*}{k^*}, r^* - \pi^*\right) - \delta\right] + \beta(n + a) = n + a.$$

Since $\pi^e \approx \pi^* = \mu - (n + a)$ around the long-run equilibrium, it then follows from (51) that

$$\frac{\dot{P}}{P} \approx -\frac{l'(r^*)}{l(r^*)}\dot{r} + \pi^e = \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right] + \pi^e,$$

where $\alpha = -\tilde{\alpha}l'(r^*)/l(r^*) > 0$. This is identical with (50). It is thus confirmed that Eq. (50) is consistent with the classical theory.

In the classical theory, aggregate employment is determined by the first postulate. Given the Cobb–Douglas production function (15), the first postulate implies that aggregate employment and output are related in the following fashion:

32 Equation (50) is a modified version of the law of price dynamics in the "Keynes–Wicksell" model (cf. Stein 1969; Fischer 1972).

$$\frac{WN}{PY} = \beta.$$

It then follows that

$$\frac{\dot{W}}{W} + \frac{\dot{N}}{N} - \frac{\dot{P}}{P} - \frac{\dot{Y}}{Y} = 0,$$

which can, by (20) and (25), be reduced to

$$\frac{\dot{W}}{W} + \left(\frac{1}{\beta}\frac{\dot{y}}{y} + \frac{\dot{N}^*}{N^*}\right) - \frac{\dot{P}}{P} - \left(\frac{\dot{y}}{y} + \frac{\dot{Y}^*}{Y^*}\right) = 0.$$

It is seen from (17), (23), (26) and (50) that

$$\begin{aligned}\frac{\dot{y}}{y} =& \frac{\beta}{1-\beta}\left\{\alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]\right.\\ &\left.+(1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right] - \gamma(y^{1/\beta} - 1)\right\}.\end{aligned} \tag{52}$$

This describes quantity dynamics in the classical theory.

Given the definition of the money coefficient (38), its rate of change can be derived as follows:

$$\frac{\dot{m}}{m} = \frac{\dot{M}}{M} - \frac{\dot{P}}{P} - \frac{\dot{Y}^*}{Y^*}.$$

It then follows from (23), (27) and (50) that

$$\begin{aligned}\frac{\dot{m}}{m} =& \mu - n - a - \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]\\ &- (1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right] - \pi^e.\end{aligned} \tag{53}$$

It is seen from (28) and (50) that revisions of the expected rate of inflation can be described by

$$\dot{\pi}^e = \alpha\epsilon\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]. \tag{54}$$

Thus, we obtain the following system for the classical theory in dynamics:

$$\begin{aligned}\frac{\dot{y}}{y} =& \frac{\beta}{1-\beta}\left\{\alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]\right.\\ &\left.+(1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right] - \gamma(y^{1/\beta} - 1)\right\},\end{aligned} \tag{52}$$

$$\frac{\dot{k}}{k} = \beta\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right], \tag{24}$$

$$\begin{aligned}\frac{\dot{m}}{m} =& \mu - n - a - \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]\\ &- (1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right] - \pi^e,\end{aligned} \tag{53}$$

$$\dot{\pi}^e = \alpha\epsilon\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]. \tag{54}$$

The system of these equations is denoted by "System (C)" (to mean "Classical").[33]

The long-run equilibrium values of $y,\ k,\ m$ and π^e are given by

$$\begin{aligned}0 =& \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]\\ &+ (1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right] - \gamma(y^{1/\beta} - 1),\\ 0 =& f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a),\\ 0 =& \mu - n - a - \alpha\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy\right]\\ &- (1-\beta)\left[f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right) - (\delta + n + a)\right] - \pi^e,\\ 0 =& f\left(\frac{y}{k}, r\left(\frac{y}{m}\right) - \pi^e\right)k - sy.\end{aligned}$$

It is seen that these equilibrium values in system (C) are identical with those in system (K), characterized by (35)–(38). The existence and uniqueness of long-run equilibrium can then be guaranteed by (39) in system (C).

The stability of system (C) can be examined in the same way as that of system (K). The Jacobian matrix of system (C) evaluated at the long-run equilibrium, denoted by J_C^*, can be given by

$$J_C^* = \begin{pmatrix} \beta j_{21} + (\beta j_{41} - \gamma)/(1-\beta) & \beta[j_{22} + j_{42}/(1-\beta)] & \beta[j_{23} + j_{43}/(1-\beta)] & \beta[j_{24} + j_{44}/(1-\beta)] \\ \beta k^* j_{21} & \beta k^* j_{22} & \beta k^* j_{23} & \beta k^* j_{24} \\ -[(1-\beta)j_{21} + j_{41}]m^* & -[(1-\beta)j_{22} + j_{42}]m^* & -[(1-\beta)j_{23} + j_{43}]m^* & -[(1-\beta)j_{24} + j_{44}]m^* \\ \epsilon j_{41} & \epsilon j_{42} & \epsilon j_{43} & \epsilon j_{44} \end{pmatrix},$$

where

[33] System (C) may be regarded as a long-term extension of Tobin's (1975) M (Marshall) model. In most studies on the dynamic Keynesian theory (cf. Flaschel et al. 1997, Chap, 7; Chiarella and Flaschel 2000, Chap, 6; Chiarella et al. 2013, Chap, 13; Murakami 2014, 2016b; Murakami and Asada 2018), the effects of price or wage flexibility on the Keynesian system are examined in detail, but the differences in stability properties between the Keynesian and classical systems are not thoroughly studied (especially in a long-run context). Our analysis studies the stabilizing effect of wage flexibility in the classical system to elucidate the fundamental difference between the Keynesian and classical systems.

$$j_{21} = \frac{1}{k^*}f_x^* + \frac{1}{m^*}f_{\rho^e}^* r_v^*,\ j_{22} = -\frac{1}{(k^*)^2}f_x^*,\ j_{23} = -\frac{1}{(m^*)^2}f_{\rho^e}^* r_v^*,\ j_{24} = -f_{\rho^e}^*,$$
$$j_{41} = \alpha\left(f_x^* - s + \frac{k^*}{m^*}f_{\rho^e}^* r_v^*\right),\ j_{42} = \alpha\frac{1}{k^*}(s - f_x^*),\ j_{43} = -\alpha\frac{k^*}{(m^*)^2}f_{\rho^e}^* r_v^*,\ j_{44} = -\alpha k^* f_{\rho^e}^*.$$

The characteristic equation associated with J_C^* can be calculated as follows:

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0,$$

where

$$b_1 = \frac{\gamma}{1-\beta} - \alpha\left[\frac{\beta}{1-\beta}(s - f_x^*) + k^* f_{\rho^e}^*\left(\epsilon - \frac{\beta}{1-\beta}\frac{r_v^*}{m^*}\right)\right] - f_{\rho^e}^*\frac{r_v^*}{m^*}, \tag{55}$$

$$b_2 = \left[\frac{\alpha}{1-\beta}k^* f_{\rho^e}^*\left(\epsilon - \frac{r_v^*}{m^*}\right) + \frac{\beta}{1-\beta}\frac{f_x^*}{k^*} - f_{\rho^e}^*\frac{r_v^*}{m^*}\right]\gamma - \alpha f_{\rho^e}^*\frac{r_v^*}{m^*}\left(\frac{\beta}{1-\beta}s + \epsilon\right), \tag{56}$$

$$b_3 = \frac{\alpha\gamma}{1-\beta}f_{\rho^e}^*\left[\epsilon\left(\beta s - k^*\frac{r_v^*}{m^*}\right) - \beta s\frac{r_v^*}{m^*}\right], \tag{57}$$

$$b_4 = -\frac{\alpha\beta}{1-\beta}\gamma\epsilon s f_{\rho^e}^*\frac{r_v^*}{m^*} > 0. \tag{58}$$

The last inequality holds under (10) and (51).

The necessary and sufficient condition for the local asymptotic stability in system (C) can be obtained in the same way as in system (K):

$$b_1 > 0,\ b_3 > 0,\ b_4 > 0,$$
$$(b_1 b_2 - b_3)b_3 - b_1^2 b_4 > 0.$$

With (46), (47) and (55)–(58), they can be reduced to[34]

$$\alpha\left[\frac{\beta}{1-\beta}(s - f_x^*) + k^* f_{\rho^e}^*\left(\epsilon - \frac{\beta}{1-\beta}\frac{r^*}{\eta_r^*}\right)\right] + \frac{\gamma}{1-\beta} - f_{\rho^e}^*\frac{r^*}{\eta_r^*} > 0, \tag{59}$$

$$\epsilon\left(\beta s - k^*\frac{r^*}{\eta_r^*}\right) - \beta s\frac{r^*}{\eta_r^*} < 0, \tag{60}$$

[34] The exact values of g_1 and g_2 in (61) are not necessary for our analysis.

$$g(\gamma)(= (1-\beta)^3[(b_1b_2-b_3)b_3-b_1^2b_4]/(\alpha\gamma f_{\rho^e}^*))$$
$$= \left\{ \left[\frac{\alpha}{1-\beta}k^* f_{\rho^e}^*\left(\epsilon - \frac{r^*}{\eta_r^*}\right) + \frac{\beta}{1-\beta}\frac{f_x^*}{k^*} - f_{\rho^e}^*\frac{r^*}{\eta_r^*}\right] \right.$$
$$\left. \left[\epsilon\left(\beta s - k^*\frac{r^*}{\eta_r^*}\right) - \beta s\frac{r^*}{\eta_r^*}\right] + \epsilon s\frac{r^*}{\eta_r^*}\right\}\gamma^2 + g_1\gamma + g_2 < 0 \tag{61}$$

To discuss the stability in our classical system, we make the "classical" assumption that the interest elasticity of liquidity preference (evaluated at the long-run equilibrium), η_r^*, is nearly equal to zero (as in the purest form of the quantity theory of money). Under this classical assumption, conditions (59) and (60) are satisfied. Moreover, condition (61) is also fulfilled if the degree of wage flexibility γ is large enough, as long as the speed of revisions of expectations ϵ is not so large. This confirms the stabilizing effect of wage flexibility in the classical theory. As we have observed, it is in the effect of wage flexibility on stability that the classical theory can be distinguished most from the Keynesian theory.

The mechanism of stability of system (C) can be explained with a schematic diagram. Assume that the state of the economy was originally at the long-run equilibrium but the rate of utilization y suddenly falls. If the degree of wage flexibility γ is large enough, the response to this fall can be described by the following diagram:

$$y\downarrow \begin{cases} \xrightarrow{(52)} & y\uparrow\uparrow \\ \xrightarrow{(54)} & \pi^e\downarrow \xrightarrow{(52)} \quad y\downarrow \end{cases}.$$

This illustrates the stabilizing effect of wage flexibility in our classical system.[35]

3.4 Keynesian and classical theories in dynamics: summary

We have explored the dynamic properties of two dynamic systems with Keynesian features and classical features. Our Keynesian and classical systems possess the same long-run equilibrium with full employment (or the natural rate of unemployment) but are characterized by different dynamic adjustments. Because of the difference in dynamics, the same long-run equilibrium can have totally different stability properties between the Keynesian and classical systems. Indeed, the flexibility of wages is conducive to stability in our classical system while it is not in our Keynesian system.

As we have argued, it is not the existence of underemployment equilibrium but the (in-)stability of full-employment equilibrium that makes the fundamental difference between the Keynesian and classical systems. Our analysis has confirmed this view in a long-run context.

[35] For the possibility of persistent cyclical fluctuations in a classical model (with public capital), see Murakami and Sasaki (2020).

4 Conclusion

This paper has reexamined the fundamental difference between the Keynesian and classical theories from static and dynamic perspectives. As we have seen, the rigidity of nominal wages plays an essential role in establishing involuntary unemployment as an equilibrium phenomenon. For this reason, the Keynesian theory is widely considered a theory built on the rigidity of nominal wages. If involuntary unemployment is taken as a *disequilibrium* phenomenon in contrast to full-employment equilibrium, however, wage rigidity is no longer a prerequisite for it; unless the long-run equilibrium has stability, full employment does not possess practical relevance and involuntary unemployment prevails for a long while. Indeed, the (same) long-run equilibrium is more likely to be stable in our classical system, while it is more likely to be unstable in our Keynesian system, as the nominal wage is more flexible. Our analysis shows that the distinction between the Keynesian and classical theories in terms of wage rigidity may not be appropriate even in a long-term context.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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