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ARTICLE

Why does production function take the Cobb–Douglas form?

Direct observation of production function using empirical data

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Abstract
We directly observed a Cobb–Douglas symmetric plane using the index of surface openness, which is used in geography, and successfully identified it. Based on this observation, we measured the capital shares (capital elasticity) and labor shares (labor elasticity) and compared them with the results of multiple regression analysis used in economics. We confirmed consistent agreement in seven countries: Japan, Germany, France, Spain, Italy, the UK, and the Netherlands. Thus, we show that the Cobb–Douglas production function can be clearly captured in empirical data as a geometric entity with a quasi-inverse symmetry of variables. Based on the above discussion, we theoretically clarified why the Cobb–Douglas production function is better fitted to empirical data in economics, because it uniquely derives the fact that its variable follows a power-law distribution.

Keywords Cobb–Douglas production function · Quasi-inverse symmetry · Surface openness · Power law

JEL Classification E0

1 Introduction

In economics, firms are regarded as economic entities that produce goods (Y) using capital, labor, and other resources, which are called production factors $\left(x_1, x_2, \dots, x_n\right)$. The production activity of firms is modeled as a function that inputs production factors and outputs the total production: $Y = F\left(x_1, x_2, \dots, x_n\right)$. This

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is called a production function. As a simplified model, economists have proposed various two-variable production functions in which the production factors are capital (K) and labor (L).

A typical example is the Cobb–Douglas production function: $Y(K, L) = AK^{\alpha}L^{\beta}$ (Cobb and Douglass 1928). Here, A is the total factor productivity, which is interpreted as a firm's production efficiency or technological capability that cannot be measured by labor or capital. α and β are called the capital share (capital elasticity) and labor share (labor elasticity), and represent production's capital or labor dependency. The constant elasticity of substitution (CES) type production function, $Y(K, L) = A\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-\frac{1}{\rho}}$, is an extension of the Cobb–Douglas production function (Solow 1956; Arrow et al. 1961). Here, δ and ρ are the distribution and substitution parameters. The CES production function matches the Cobb–Douglas production function at $\alpha + \beta = 1$ in the $\rho \to 0$ limit as well as the Leontief production function: $Y = A\min\{K, L\}$ in the limit of $\rho \to +\infty$. Furthermore, economists proposed a translog-type production function, $\log Y(K, L) = \log A + \alpha \log K + \beta \log L + \gamma_1 \log^2 K + \gamma_2 \log^2 L + \gamma_3 \log K \log L$, as an extended version that explicitly includes the Cobb–Douglas production function (Christensen et al. 1973).

In economics, various forms of production functions have been proposed and widely used by researchers in microeconomics and macroeconomics. Total factor productivity has been applied to industries around the world since the late 1950s as an indicator of productivity, and many empirical studies have addressed production functions. However, the conventional research is limited to comparisons of the superiority of the production function based on the regression's good fit to the shape of several production functions described above. Note that there is only a limited argument concerning why a production function takes a functional form, such as the Cobb–Douglas type. Among these studies, Houthakker focused on the similarity between the Cobb–Douglas production function and the power-law distribution, followed by its variables from the microeconomic foundation (Houthakker 1955). Unfortunately, however, his argument fails to reach the constant returns to scale discussed below.

In this context, as in Houthakker's work, where the Cobb–Douglas production function is a power-law function of variables (K, L, Y), we argue that the Cobb–Douglas production function can be interpreted as an inverse symmetric plane and the residual from it in the 3D space of variables (K, L, Y) (Mizuno et al. 2012; Ishikawa et al. 2013, 2014). Here, we observed the quasi-inverse symmetry in the joint probability density function (PDF) $P_{KLY}(K, L, Y)$ under variable exchange: $Y \leftrightarrow aK^{\alpha}L^{\beta}$, which is deeply related to the power-law distribution in the large-scale range of three variables (K, L, Y) and the log-normal distribution in the mid-scale range (Gibrat 1932; Sutton 1997). Importantly, the observed fact that each variable (K, L, Y) of the production function follows a power-law distribution is uniquely concluded from the form of the Cobb–Douglas production function. In previous studies, however, the validation of the above analytical arguments with empirical data was indirect. We have not confirmed whether our quasi-inverse symmetric plane is consistent with the Cobb–Douglas production function, which is treated in economics. The possibility that capital share (α) and labor share (β) are different between the quasi-inverse symmetric plane and the multiple regression plane cannot be denied.

In this study, we directly observed a Cobb–Douglas symmetric plane using the index of surface openness, which is used in geography, and successfully identified it. Based on this observation, we measured the capital share (capital elasticity) and labor share (labor elasticity), compared our results with the results of multiple regression analysis used in economics, and confirmed consistent agreement in seven countries: Japan, Germany, France, Spain, Italy, the UK, and The Netherlands. Thus, we showed that the Cobb–Douglas production function can be clearly captured in empirical data as a geometric entity with a quasi-inverse symmetry of variables. Our previous research was limited to indirect observation of the power-law region of the variables, but this current research is characterized by direct observation of the Cobb–Douglas symmetric plane in the whole region other than the power-law one. Based on the above discussion, we theoretically clarified why the Cobb–Douglas production function is better fitted to empirical data in economics.

The rest of this paper is structured as follows. Section 2 describes the data that we used in our analysis. Section 3 briefly reviews our previous work. In Sect. 4, we explain the method to measure the Cobb–Douglas symmetric plane using an index called surface openness. In Sect. 5, we identify the Cobb–Douglas symmetric plane by identifying a ridge in Sect. 4's analytical preparation and measuring the capital share (capital elasticity) and labor share (labor elasticity). In addition, we measure the capital share (capital elasticity) and labor share (labor elasticity) by multiple regression analysis and confirm that both shares are consistent in seven countries: Japan, Germany, France, Spain, Italy, the UK, and The Netherlands. Finally, in Sect. 6, we summarize the main points of this paper and present future perspectives.

2 Analysis data

We used the 2016 version of the ORBIS database provided by Bureau van Dijk (2020). It is the world's largest corporate financial database with approximately 200 million listed and unlisted firms collected worldwide from more than 120 local credit bureaus and information vendors, including from Asia, the Americas, Europe, the Middle East, and Africa. This database is characterized not only by the huge amount of data but also by the fact that its data are arranged in a standardized format and can be compared internationally. It includes the corporate financial data for 1,831,481 firms in Japan (JP), 3,458,922 firms in Italy (IT), 1,953,140 firms in France (FR), 1,204,584 firms in Germany (DE), 5,070,698 firms in the UK (GB), 1, 604, 553 firms in Spain (ES), and 3,213,808 firms in The Netherlands (NL). This study focuses on firms in these countries that have sufficient data for statistical analysis. Although the number of firms collected in the US is sufficient, we did not include it in this analysis because of an unnatural bias that may be attributable to the method of data collection by local credit bureaus or information vendors. Since there are about 1–2 million active firms in Japan (Statistics Bureau 2020), the data analyzed here are highly comprehensive. This situation is probably the same in other countries.

To make the most of the features of this database, we use sales as Y, assets as K, and number of employees as L. When economists consider a production function, they generally use Y for added value, K for fixed assets in manufacturing, K for liquid assets in non-manufacturing, and L for total labor hours or wages. On average, however, only around 1/10 of the firms in our database have detailed data to calculate added value, etc. If we use the same amount as economists, we lose the completeness of the data, and have to filter out firms that have detailed data to calculate value added, etc. What is most important in this study is that K, L, and Y follow power-law distributions in the large-scale ranges. This property is common to both the (K, L, Y) used in economics and the simple (K, L, Y) we use (Aoyama et al. 2008). If the argument of directly observing a quasi-inversion symmetric plane is self-consistent, either quantity may be used. Therefore, to ensure a statistical advantage with a sufficient amount of data, this study will proceed with the analysis using the quantities mentioned above.

3 Previous research

Such variables as sales (Y), assets (K), and number of employees (L), all of which represent a firm's size, follow a power-law distribution in the large-scale range (Pareto 1897; Newman 2005; Clauset et al. 2009):

$$P_K(K) \propto K^{-\mu_K-1}, \tag{1}$$

$$P_L(L) \propto L^{-\mu_L-1}, \tag{2}$$

$$P_Y(Y) \propto Y^{-\mu_Y-1}. \tag{3}$$

Here, P is a PDF, and power-law exponent μ is also called a Pareto's index.

Fujiwara et al. (2003, 2004) showed that these power distributions (1)–(3) can be derived from the inverse symmetry observed between the variables of 1 year $\left(x_1\right)$ and those of the next $\left(x_2\right)$:

$$P_{12}\left(x_1, x_2\right)dx_1dx_2 = P_{12}\left(x_2, x_1\right)dx_2dx_1, \tag{4}$$

and Gibrat's law, where growth rate distribution $Q\left(R|x_1\right)$ conditioned by first-year size $\left(x_1\right)$ does not depend on the first-year variables (Gibrat 1932; Sutton 1997):

$$Q\left(R|x_1\right) = Q(R). \tag{5}$$

Here, x_1 represents (K, L, Y) in 1 year, and x_2 represents (K, L, Y) in the next year. P_{12} is the joint PDF, and $R = x_2/x_1$ is the growth rate. In inversion symmetry, note that joint PDF P_{12} has the same shape on both sides of Eq. (4) under the permutation of variables $x_1 \leftrightarrow x_2$. The derivation by Fujiwara et al. is effective when the power-law index does not change. This derivation is described in Appendix 1.

We extended the argument by introducing an extended growth rate of $R = x_2/ax_1^\theta$ in quasi-inverse symmetry under the permutation of $ax_1^\theta \leftrightarrow x_2$ variables:

$$P_{12}\left(x_1, x_2\right)dx_1dx_2 = P_{12}\left(\left(\frac{x_2}{a}\right)^{\frac{1}{\theta}}, ax_1^\theta\right)d\left(\left(\frac{x_2}{a}\right)^{\frac{1}{\theta}}\right)d\left(ax_1^\theta\right). \tag{6}$$

We also showed that the power-law distribution can be derived from quasi-inverse symmetry (6) and Gibrat's law (5) even when the power-law index varies quasi-statically. Here, a and θ are parameters of quasi-inverse symmetry and are related with power exponents $\left(\mu_1, \mu_2\right)$ of variable $\left(x_1, x_2\right)$ as follows:

$$\theta = \frac{\mu_1}{\mu_2}. \tag{7}$$

This derivation is described in Appendix 2.

First, we confirmed this relation by the temporal change of power-law index μ, which can be observed in Japanese land prices (Ishikawa 2006, 2009). We then showed that this quasi-inverse symmetry can be similarly observed among such different firm size variables as assets, number of employees, and sales (K, L, Y) at the same time and confirmed the relation (7) among their power-law indices (Mizuno et al. 2012; Ishikawa et al. 2013, 2014). In this case, we consider $\left(x_1, x_2\right)$ to be (K, L), (K, Y), and (L, Y).

We extended the concept of quasi-inverse symmetry and the growth rate to three simultaneous variables (K, L, Y) and introduced growth rate $R = Y/aK^\alpha L^\beta$ into a system with quasi-inverse symmetry under the variable substitution of $aK^\alpha L^\beta \leftrightarrow Y$:

$$\begin{aligned} &P_{KLY}(K, L, Y)dKdLdY \\ &\quad = P_{KLY}\left(\left(\frac{Y}{aL^\beta}\right)^{\frac{1}{\alpha}}, \left(\frac{Y}{aK^\beta}\right)^{\frac{1}{\beta}}, aK^\alpha L^\beta\right) \\ &\qquad d\left(\left(\frac{Y}{aL^\beta}\right)^{\frac{1}{\alpha}}\right)d\left(\left(\frac{Y}{aK^\alpha}\right)^{\frac{1}{\beta}}\right)d\left(aK^\alpha L^\beta\right). \end{aligned} \tag{8}$$

The quasi-inverse symmetry (8) of the three variables and Gibrat's law of the three variables:

$$Q(R|K, L) = Q(R) \tag{9}$$

lead to power-law distributions of K, L, and Y, as in the case of two variables. The derivation of this conclusion is described below.

Note here that from the definition of the three-variable growth rate, if total factor productivity A is residual aR from the following quasi-inverse symmetric plane:

$$\log Y = \alpha \log K + \beta \log L + \log a, \tag{10}$$

then we can interpret the Cobb–Douglas production function as the quasi-inverse symmetry in the three-variable space (K, L, Y). In the course of its derivation, we discussed the following in our previous work (Ishikawa et al. 2014).

Three-variable quasi-inverse symmetry (8) is rewritten in three variables (K, L, R) as follows:

$$
\begin{aligned}
&P_{KLR}(K, L, R)dKdLdR \\
&= P_{KLR}\left(R^{\frac{1}{\alpha}}K, R^{\frac{1}{\beta}}L, R^{-1}\right)d\left(R^{\frac{1}{\alpha}}K\right)d\left(R^{\frac{1}{\beta}}L\right)d(R^{-1}).
\end{aligned}
\tag{11}
$$

This idea can be expressed as follows:

$$
P_{KLR}(K, L, R) = R^{\frac{1}{\alpha}+\frac{1}{\beta}-2}P_{KLR}\left(R^{\frac{1}{\alpha}}K, R^{\frac{1}{\beta}}L, R^{-1}\right). \tag{12}
$$

From the definition of a conditional PDF, $Q(R|K, L) = P_{KLR}(K, L, R)/P_{KL}(K, L)$, this equation can be modified:

$$
\begin{aligned}
\frac{P_{KL}(K, L)}{P_{KL}\left(R^{\frac{1}{\alpha}}K, R^{\frac{1}{\beta}}L\right)} &= R^{\frac{1}{\alpha}+\frac{1}{\beta}-2}\frac{Q\left(R^{-1}|R^{\frac{1}{\alpha}}K, R^{\frac{1}{\beta}}L\right)}{Q(R|K, L)} \\
&= R^{\frac{1}{\alpha}+\frac{1}{\beta}-2}\frac{Q\left(R^{-1}\right)}{Q(R)}.
\end{aligned}
\tag{13}
$$

In the modification from the 2nd to the 3rd equation, we used a three-variable Gibrat's law (9). Since the right-hand side of Eq. (13) is only a function of R, if we express it as $G(R)$, the expression (13) can be written:

$$
P_{KL}(K, L) = G(R)P_{KL}\left(R^{\frac{1}{\alpha}}K, R^{\frac{1}{\beta}}L\right). \tag{14}
$$

Expanding the right-hand side of Eq. (14) by $R = 1 + \epsilon$ $(0 < \epsilon \ll 1)$, the zeroth order of ϵ is an obvious expression, and the first order becomes the following differential equation:

$$
\left[G'(1) + \frac{K}{\alpha}\frac{\partial}{\partial K} + \frac{L}{\beta}\frac{\partial}{\partial L}\right]P_{KL}(K, L) = 0. \tag{15}
$$

Here, $G'(\cdot)$ represents the differential of $G(\cdot)$ by R. Also, no more useful information is available than the 2nd-order of ϵ.

The two variables (K, L) are strongly correlated and are not independent. Therefore, the differential equation (15) cannot be solved as it is. In our previous study (Ishikawa et al. 2014), we converted variables (K, L) into normalized variables (k, l):

$$
\log k = \frac{\log K - m_K}{\sigma_K}, \quad \log l = \frac{\log L - m_L}{\sigma_L}, \tag{16}
$$

and rotated them as $-\pi/4$ and finally obtained orthogonal independent variables $\left(z_1, z_2\right)$:

$$\log z_1 = \frac{1}{\sqrt{2}}(\log k + \log l), \quad \log z_2 = \frac{1}{\sqrt{2}}(-\log k + \log l). \tag{17}$$

Here, $\left(m_K, m_L\right)$ and $\left(\sigma_K, \sigma_L\right)$ are the mean and standard deviation of $(\log K, \log L)$ in the power-law region of variables (K, L). In Eq. (16), the distribution's width is normalized by dividing it by $\left(\sigma_K, \sigma_L\right)$ after subtracting the average $\left(m_K, m_L\right)$ from $(\log K, \log L)$. In our previous study (Ishikawa et al. 2014), we numerically showed that this operation yields orthogonal independent variables $\left(z_1, z_2\right)$. We rewrote the differential equation (15) with the independent variables $\left(z_1, z_2\right)$ and solved it to analytically show that variable $\left(z_1, z_2\right)$ obeys the power-law distribution and numerically confirmed the result in empirical data.

Finally, we analytically derived the distribution function followed by variable (K, L) by the above inverse transformation and confirmed it numerically. This discussion is based on the fact that the Cobb–Douglas production function is a quasi-inverse symmetric plane in (K, L, Y) space. Therefore, the above numerical consistency indirectly confirms our interpretation of the Cobb–Douglas production function.

4 Analytical preparation for direct observation of the Cobb–Douglas quasi-inverse symmetric plane

In the previous section, variables (K, L) were normalized by dividing them by logarithmic standard deviations, which are indicators of the width of the distribution in the large-scale range where the distribution follows the power law. On the other hand, the width of the distribution of the variables following the power-law distribution of exponent μ is also given as $1/\mu$ on a logarithmic scale. Thus, the normalization of $(\log K, \log L)$ divided by distribution width $\left(\sigma_K, \sigma_L\right)$ in Eq. (16) is equivalent to the operation of dividing by $\left(1/\mu_K, 1/\mu_L\right)$ or multiplying by $\left(\mu_K, \mu_L\right)$:

$$\log k = \mu_K\left(\log K - \log K_{\mathrm{up}}\right), \quad \log l = \mu_L\left(\log L - \log L_{\mathrm{up}}\right). \tag{18}$$

Here, the origin is shifted by subtracting the logarithm of the power-law ranges' upper limits $(\log K_{\mathrm{up}}, \log L_{\mathrm{up}})$.

We first consider the quasi-inverse symmetry of two variables (K, L). Because the normalized widths of the k and l distributions are equal, a two-variable quasi-inverse symmetric line, $\log L = \theta \log K + \log a$ (where $x_1 = K, x_2 = L$), is converted to a slope 1 inverse symmetric line: $\log l = \log k + \log a'$. This is consistent with Eq. (7) where the relationship between slope θ of the quasi-inverse symmetric line and exponents μ_K and μ_L is given:

$$\frac{1}{\mu_L} = \theta \frac{1}{\mu_K}. \tag{19}$$

Here $\mu_1 = \mu_K$, and $\mu_2 = \mu_L$.

Extending this concept to three variables (K, L, Y), the following relationship is established between the capital and labor share (α, β) of the three-variable quasi-inverse symmetric plane (10) and the power-law exponents $\left(\mu_K, \mu_L, \mu_Y\right)$:

$$\frac{1}{\mu_Y} = \alpha\frac{1}{\mu_K} + \beta\frac{1}{\mu_L}. \tag{20}$$

This is an extension of the constant returns to scale:

$$1 = \alpha + \beta. \tag{21}$$

Using the following normalized notation:

$$\alpha' = \alpha\frac{\mu_Y}{\mu_K}, \quad \beta' = \beta\frac{\mu_Y}{\mu_L}. \tag{22}$$

Equation (20) is simply expressed as:

$$1 = \alpha' + \beta'. \tag{23}$$

In Fig. 1, we compared the accuracy of Eqs. (21) and (23) using α_M, β_M, α'_M, and β'_M evaluated by multiple regression analysis used in economics in Japanese firms from 2010 to 2014. The database that we used was created using a collection method that reduced the number of firms in the past. Therefore, Fig. 1 shows data from 2014 to the previous five years. Data for 2015 were not included, because they were still being collected when we obtained the database. From Fig. 1, we can confirm that the accuracy of the constant returns to scale, normalized by the power-law index (23),

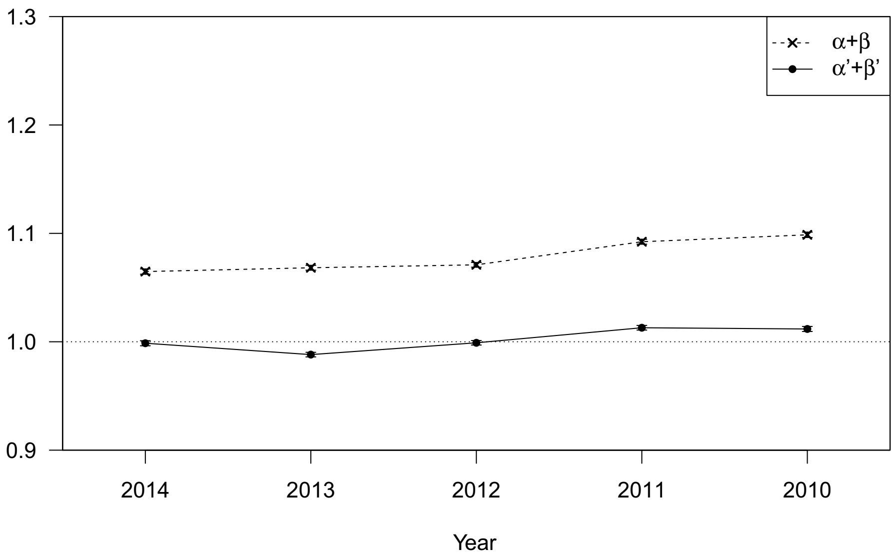


Fig. 1 Comparison of constant returns to scale ($\alpha + \beta = 1$) of Japanese firms from 2014 to 2000 and standardized one ($\alpha' + \beta' = 1$)

and is higher than the conventional one (21). Results similar to those of Japanese firms can be confirmed for firms in the countries that we analyzed.

More interestingly, the normalized capital share (capital elasticity) α' and the labor share (labor elasticity) β' can be directly estimated by observing the Cobb–Douglas symmetric plane, as described below.

In the previous section's discussion, we rewrote the quasi-inverse symmetry of the (K, L, Y) space to the quasi-inverse one of the (K, L, R) space and limited the discussion to the power-law region of (K, L) where Gibrat's law holds, and derived a differential equation for (K, L) by expanding growth rate R around "1". In this discussion, we only investigate the neighborhood of the quasi-inverse symmetric plane. To directly observe the Cobb–Douglas symmetric plane, the entire space of the three variables must be observed without adopting this method.

Thus, in addition to Eq. (18), we consider the standardization of Y:

$$\log y = \mu_Y(\log Y - \log Y_{\mathrm{up}}). \tag{24}$$

For the two variables (K, L), the direction vector of the quasi-inverse symmetric axis is $\left(1/\mu_K, 1/\mu_L\right)$, and the normalized direction vector is $(1, 1)$ in two-variable space (k, l). Using Eq. (17), we rotated it to be $\left(\sqrt{2}, 0\right)$, resulting in independent variables $\left(z_1, z_2\right)$. For the three variables (K, L, Y), the densest direction vector in the quasi-inverse symmetric plane is $\left(1/\mu_K, 1/\mu_L, 1/\mu_Y\right)$. The rotation of normalized direction vector $(1, 1, 1)$ in three-variable space (k, l, y) to $\left(\sqrt{3}, 0, 0\right)$ is given by:

$$\log z_1 = \frac{1}{\sqrt{3}}(\log k + \log l + \log y), \tag{25}$$

$$\log z_2 = \frac{1}{\sqrt{2}}(-\log k + \log l), \tag{26}$$

$$\log z_3 = \frac{1}{\sqrt{6}}(-\log k - \log l + 2\log y). \tag{27}$$

This rotation caused the densest directional vector in the Cobb–Douglas quasi-inverse symmetric plane to overlap the z_1 axis, as shown in Fig. 2.

To align a Cobb–Douglas quasi-inverse symmetric plane with the $w_3 = Const.$ plane, we need to rotate it further about the z_1 axis. The rotation, expressed using parameter ψ, is as follows (see Fig. 3):

$$\log w_2 = \frac{1}{\sqrt{\psi^2 + 1}}\left(\log z_2 + \psi \log z_3\right), \tag{28}$$

$$\log w_3 = \frac{1}{\sqrt{\psi^2 + 1}}\left(-\psi \log z_2 + \log z_3\right). \tag{29}$$

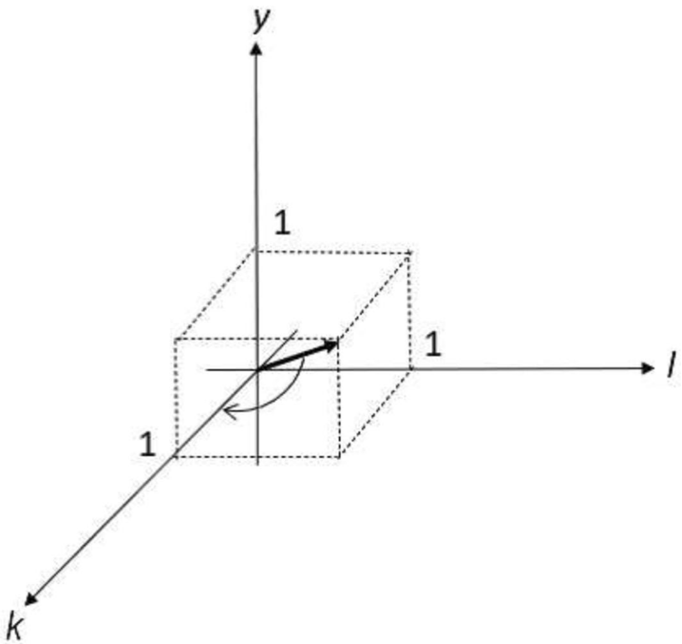


Fig. 2 The rotation (25)–(27) causes the densest directional vector in the Cobb–Douglas quasi-inverse symmetric plane to overlap the z_1-axis

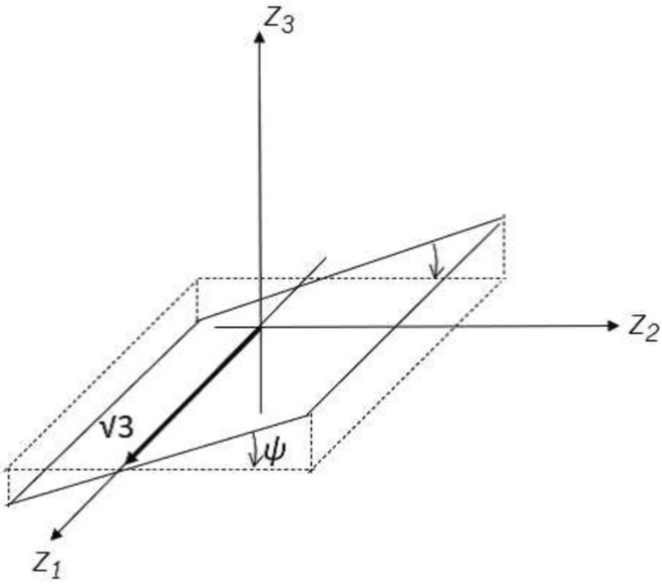


Fig. 3 The rotation (28), (29) aligns a Cobb–Douglas quasi-inverse symmetric plane with the w_3 = Const. plane. For clarity, the figure is written as Const. = 0

Due to the above rotations in Eqs. (25)–(29), the Cobb–Douglas quasi-inverse symmetric plane overlaps the w_3 = Const. plane. Here, if the w_3 = Const. plane is expressed by the quasi-inverse symmetric plane in the (k, l, y) space by inverse transformation, it can be expressed:

$$\log y = \frac{1 - \sqrt{3}\psi}{2} \log k + \frac{1 + \sqrt{3}\psi}{2} \log l + \text{Const.} \tag{30}$$

From the above equation, using parameter ψ that represents the slope of the Cobb–Douglas quasi-inverse symmetric plane, normalized capital share (capital elasticity) α' and labor share (labor elasticity) β' can be expressed:

$$\alpha' = \frac{1 - \sqrt{3}\psi}{2}, \quad \beta' = \frac{1 + \sqrt{3}\psi}{2}. \tag{31}$$

5 Direct observation of the Cobb–Douglas quasi-inverse symmetric plane

In the previous section, we analytically discussed the following procedures to simplify observing the Cobb–Douglas quasi-inverse symmetric plane in three-variable space (K, L, Y). First, we converted three variables (K, L, Y) into variables (k, l, y), which are normalized, such as the values of the power-law exponents in the power-law region to be "1". Next, we consider a rotation (25)–(27) that converts the densest direction vector $(1, 1, 1)$ in a Cobb–Douglas quasi-inverse symmetric plane into a vector of only the first component and obtain variables $\left(z_1, z_2, z_3\right)$.

If 3D data are projected onto the $z_1 z_2$ plane in the z_3 direction, then the densest direction vector should overlap the z_1 axis. If 3D data are projected onto the $z_1 z_3$ plane in the z_2 direction, the densest direction vector should still overlap the z_1 axis. When 3D data are then projected onto the $z_2 z_3$ plane in the z_1 direction, the densest direction vector is observed, and its slope is given as ψ. In this section, we first confirm the above analytical discussions by directly observing empirical data. In direct observation, identifying the densest directional vector in a 2D plane is critical. We observed a dense directional vector as a ridge using the geographical index of surface openness, as in previous studies (Ishikawa et al. 2014). The method is briefly described below.

z_1 and z_2 data points are scattered in the $z_1 z_2$-plane. For example, Fig. 4 is a scatter plot of Japanese firms in 2014. To clearly comprehend the density, we divided it into logarithmically equal-sized cells and expressed the amount of data points in the cells by different shades. For example, Fig. 5 is a diagram of Japanese firms in 2014 created in this way. The logarithm of a cell's density is its height. Then, as stated above, the ridge must be observed horizontally in the $z_1 z_2$ plane. As the steepest-ascent line in the profit space, a ridge was previously discussed (Souma 2007; Aoyama et al. 2008). In our previous study (Ishikawa et al. 2014), we determined the cells that constitute the ridge using the surface openness defined as follows (Yokoyama et al. 1999, 2002; Prima et al. 2006).

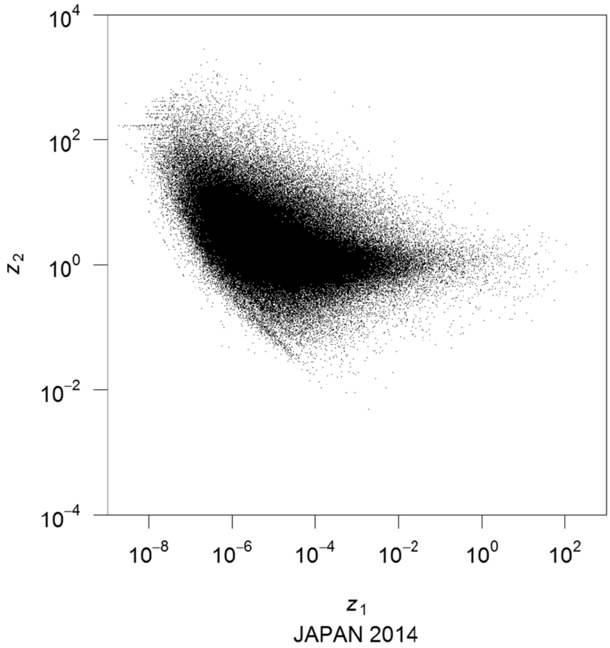


Fig. 4 Scatter plot of (z_1, z_2) with Japanese firm data for 2014 projected on z_1z_2-plane in z_3 direction

Figure 6 depicts the grid linked by the center points of the cells. From grid point A in Fig. 6, we counterclockwisely represent each azimuth as $D = 1, 2, \ldots, 8$. As shown in Fig. 7, the minimum zenith and nadir angles at grid point A within distance L in azimuth D are represented by ${}_D\phi_L$ and ${}_D\psi_L$. Positive openness Φ_L is defined by the mean value of ${}_D\phi_L$ along the eight azimuths, and negative openness Ψ_L is the corresponding mean of ${}_D\psi_L$. The surface openness is defined by the following difference:

$$\Phi_L - \Psi_L = \frac{1}{8}\sum_{D=1}^{8} {}_D\phi_L - \frac{1}{8}\sum_{D=1}^{8} {}_D\psi_L. \tag{32}$$

The surface openness takes a negative value at the depressions and the valleys, zero at the level surface, the saddle point, and the uniform slope, and positive values at the ridge (see Fig. 8) and the summit (see Fig. 9). The shading in Fig. 5 shows only absolutely high and low points. However, using the surface openness, points that are relatively higher than the circumference (a ridge or a summit) or points that are relatively lower than the circumference (a valley or a depression) can be easily extracted. This is the advantage of the surface openness.

In this analysis, by setting $L = 5$, we estimate the surface openness for each cell and extract the cells of the openness that exceed 0.9, the value of which appropriately determines the ridge in this data set. In Fig. 5, the cells are expressed by black dots. As expected, the series of densest cells observed as a ridge overlap the z_1 axis.

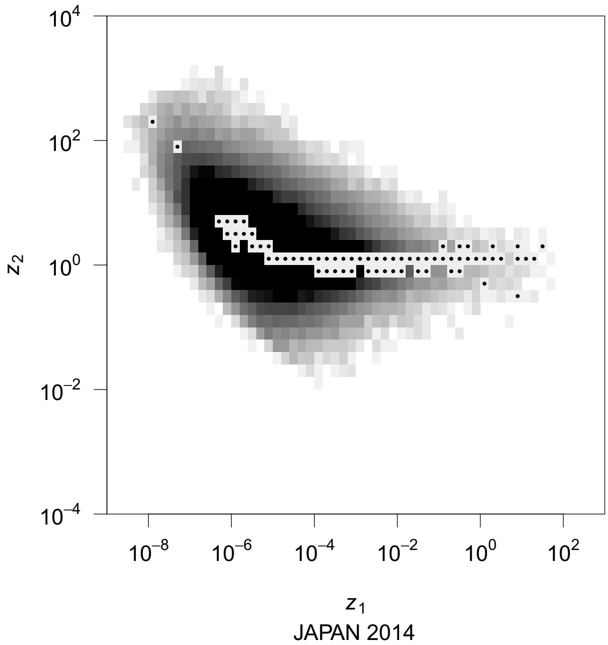


Fig. 5 Data points in scatter plot of Fig. 2 are placed in cells separated at equal logarithmic intervals, and amount of data in each cell is represented by shading. In addition, cell's center, judged to form a ridge by surface openness, is indicated by black circles

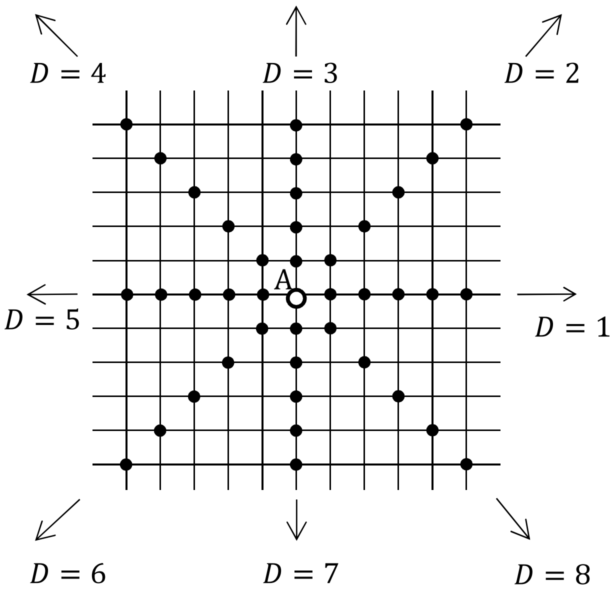


Fig. 6 Grid linked by center points of cells: From a grid point, we counterclockwisely represent each azimuth as $D = 1, 2, \ldots, 8$

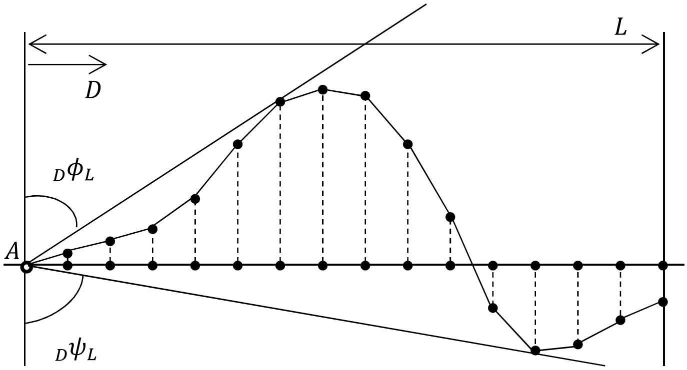
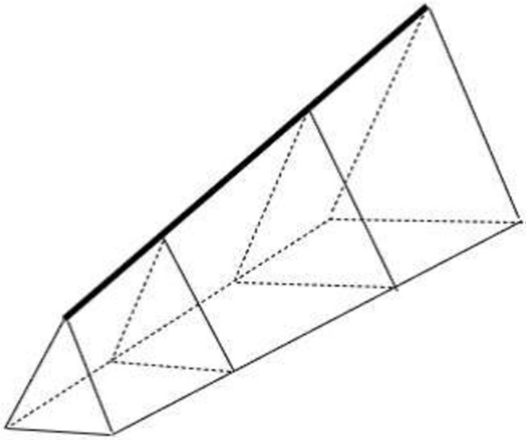


Fig. 7 Dots represent height of cells within distance L in azimuth D. From grid point A within distance L, we estimate zenith angles and denote minimum as $_D\phi_L$. Similarly, minimum nadir angle is expressed by $_D\psi_L$

Fig. 8 A schematic illustration of a ridge. In the ridge indicated by the thick lines, the surface openness is positive, because the surrounding area is often lower than the height of the ridge



A scatter plot in which 3D data are projected onto the z_1z_3 plane in the z_2 direction is similarly processed to obtain Fig. 10, which also shows that, as expected, the series of densest cells observed as a ridge overlap the z_1-axis.

A scatter plot in which 3D data are projected onto the z_2z_3 plane in the z_1 direction is similarly processed to obtain Fig. 11, which shows that the ridge, which is a series of the densest cells, is located from the upper right to the lower left. The angle between this ridge and the z_2 axis is ψ, which we measured, assuming that a ridge is a straight line that regresses the center of cells with a surface openness greater than 0.9, which was appropriately determined to be a ridge cell. For example, for Japanese firms in 2014, $\psi = 0.39 \pm 0.04$. In this case, using Eq. (31), we can estimate $\alpha'_R = 0.16 \pm 0.03$ and $\beta'_R = 0.84 \pm 0.16$. On the other hand, with the same data and multiple regression analysis, we also estimate $\alpha'_M = 0.16 \pm 0.0$ and $\beta'_M = 0.84 \pm 0.0$; both values are in good agreement. Figure 12 shows

Fig. 9 A schematic illustration of a summit. At the summit, the surface openness is positive, because all the surroundings are lower than the summit

measurements taken by Japanese firms from 2014 to 2010 and by firms in Germany, France, Spain, Italy, and The Netherlands during the 5-year period for which data were available (from 2013 to 2009). Among the measurements, those in the year with the largest number of firms in each country are plotted in Fig. 13.

From Figs. 12 and 13, we confirmed that capital elasticity $\alpha_R{}'$, measured by the ridge specified by the surface openness and capital elasticity $\alpha_M{}'$ measured by the multiple regression analysis used in economics, are in good agreement with the 5-year firm data of the seven countries.

6 Conclusion and discussion

Our previous study argued that the Cobb–Douglas production function, which is the core concept of economics, uniquely derived power-law distributions of its variables. In this study, we show that the Cobb–Douglas production function can be directly observed using empirical data. If firms' assets, labor, and production are expressed as a set of points in a 3D space, the Cobb–Douglas production function can be interpreted as a quasi-inverse symmetric plane in 3D space and a residual from it. From this viewpoint, we rotated the Cobb–Douglas quasi-inverse symmetric plane in 3D space to simplify observation and projected the data in 3D space onto the 2D plane. With the geographic index of surface openness, we identified the densest axis in the plane as the ridge.

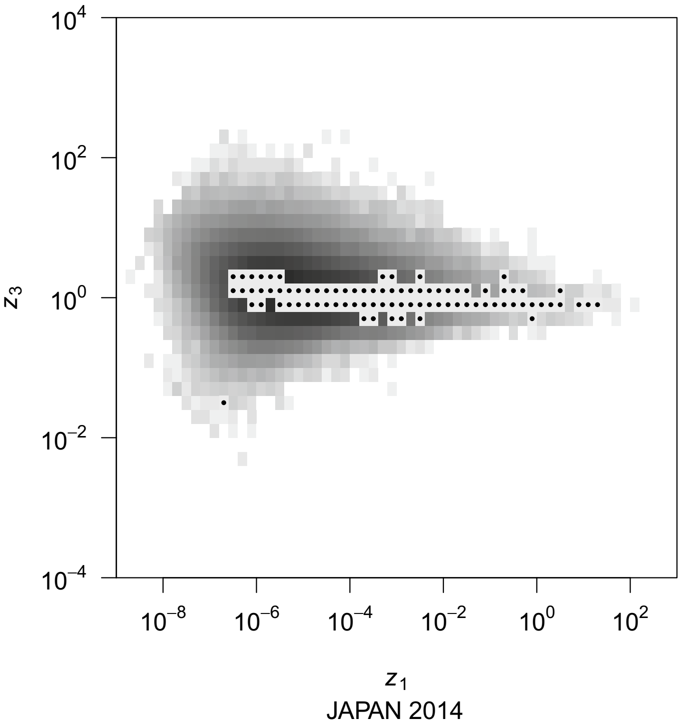


Fig. 10 Japanese firms' data for 2014 in database are projected on z_1z_3 plane in z_2-direction. $\left(z_1, z_3\right)$ data points are placed in logarithmically evenly spaced cells, and amount of data is represented by shading. In addition, cell's center, judged to form a ridge by surface openness, is indicated by black circles

Identifying this axis is difficult when the dispersion is large in the 2D plane. For example, by dividing the data on a 2D plane into equally spaced bins in a certain direction and calculating the average value or the logarithmic average value in the vertical direction for each bin, we can combine them and find the axis. However, since the average value or the logarithmic average value varies depending on the selection method in a certain direction, the axis obtained by combining the average value or the logarithmic average value is not fixed to one axis. What is not affected by such arbitrariness is the advantage of the index of surface openness adopted in this paper. Using it, we successfully and directly observed the Cobb–Douglas production function as a quasi-inverse symmetric plane.

The approach proposed by Hildenbrnad (1981) and developed by Dosi et al. (2016) shares our arguments and perspectives on firms' productivity, but uses a very different approach to measuring it. They theoretically showed that measurement of productivity is possible without depending on the shape of the production function by the volume of a polyhedron composed of points in 3D space and the angle formed with the axis by the main diagonal, and confirmed

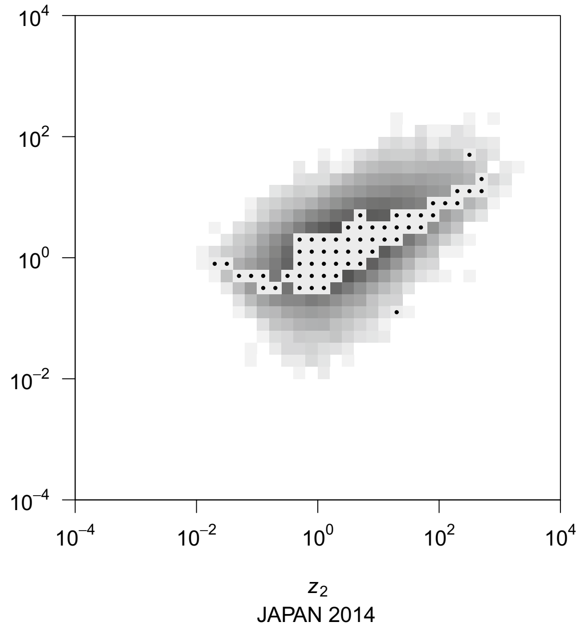


Fig. 11 Japanese firms' data for 2014 in database are projected on z_2z_3 plane in z_1-direction. $\left(z_2, z_3\right)$ data points are placed in logarithmically evenly spaced cells, and data amount is represented by shading. In addition, cell's center, which is judged to form a ridge by surface openness, is indicated by black circles

it by actual data. Their approach to measuring productivity is very innovative. The advantage of our study, on the other hand, is that it geometrically identifies the form of the production function that they avoided. This makes it possible to discuss specifically how production relates to assets and labor as follows.

Interestingly, the capital and labor elasticities of the Cobb–Douglas production function can be expressed using the slope of a quasi-inverse symmetric plane in 3D space. By comparing the measured values of the capital elasticity observed in this way with those by the multiple regression analysis conventionally used in economics, we showed that the accuracy of these measured values agrees with the accuracy using the data for 5 years of seven countries with sufficient data quantity, including Japan.

It is not obvious that the planes of the multiple regression analysis and the quasi-inverse symmetry become identical. If all the data in the 3D space exist on a plane, they coincide. In practice, however, the data are widely dispersed. If they are spread vertically about the plane based on a log-normal distribution and are not concentrated on the symmetry plane, clearly identifying the ridge is difficult where the data are concentrated, even if with quasi-inverse symmetry. This paper clarified that such a ridge exists in empirical data using the index of surface openness in 2D distribution in which 3D distribution is projected.

The fact that these observations are consistent in the seven countries with sufficient data means that the 3D space composed of capital, labor, and production

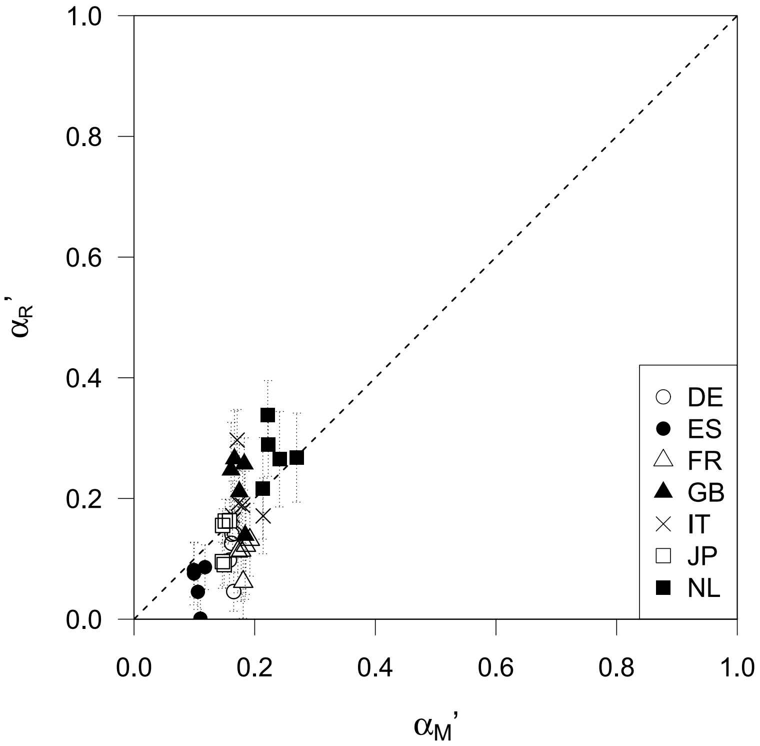


Fig. 12 Normalized capital elasticity ($\alpha_M{}'$), calculated by multiple regression analysis for firms of each country in database of Japan, Italy, the UK, and The Netherlands from 2010 to 2014 and France, Germany, and Spain from 2009 to 2013, is compared with normalized capital elasticity ($\alpha_R{}'$) calculated from ridge judged by surface openness

has a ridge where the data are concentrated as a distinct entity. This idea can probably be applied to countries other than the seven countries surveyed in this paper. Therefore, if sufficient data exist, the results in this paper can be reproduced. As a result, we clarified from empirical data that the Cobb–Douglas production function, which has been discussed in economics, can be interpreted as the quasi-inverse symmetry of three variables and the residual from it. This study theoretically clarified why the Cobb–Douglas production function fits the empirical data better than a simple comparison of candidate production functions due to the goodness of fit of the data.

In the process, we also carried out the following new consideration on the constant returns to scale, which is a feature of the Cobb–Douglas production function. In economics, when the variables of the production function (K and L in this case) are multiplied by λ and production (Y) is also multiplied by λ such as $\alpha + \beta = 1$, the production function is called the constant returns to scale. The increasing returns if it exceeds λ times such as $\alpha + \beta > 1$, and the diminishing returns if it is less than λ times, such as $\alpha + \beta < 1$. These three cases of returns

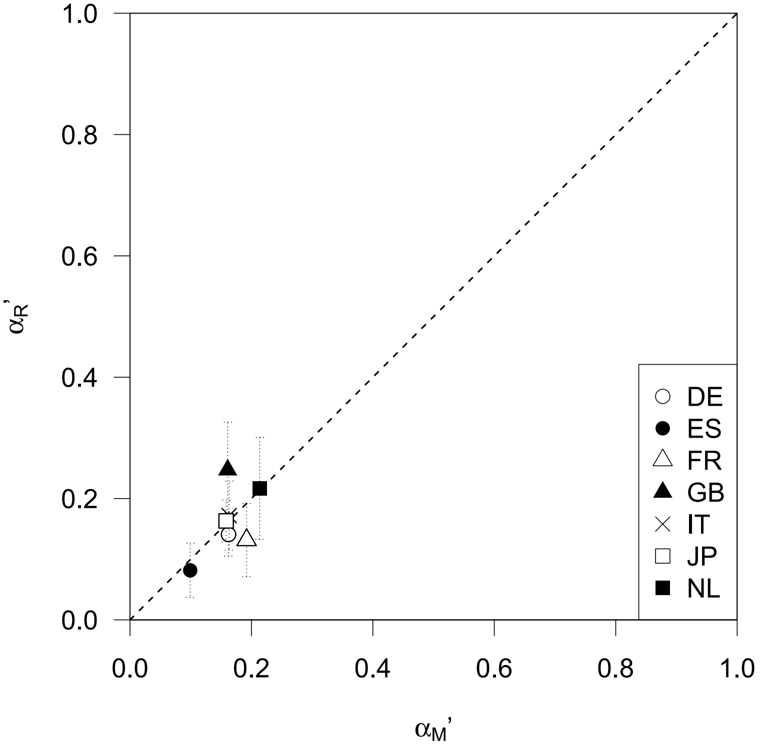


Fig. 13 Comparison between $\alpha_M{}'$ and $\alpha_R{}'$in Fig. 8 for Japan, Italy, the UK, and The Netherlands in 2014 and France, Germany, and Spain in 2013

to scale are easy to understand by considering the following simple examples. When the number of factories is multiplied by λ, the assets and labor of a firm are each multiplied by λ if the increased factories are the same size. The case where there is no change in the production efficiency and the production quantity becomes λ times is called the constant returns to scale; the case where the production efficiency improves and the production quantity becomes larger than λ times is called the increasing returns; the case where the production efficiency deteriorates and the production quantity becomes smaller than λ times is called the diminishing returns. Economic arguments often assume constant returns to scale in the production function, and in fact, this is often observed approximately in various empirical data analyses.

These observations indicate that, on average, a constant return to scale is achieved in firm production. In this study, we showed by direct observation that the Cobb–Douglas production function can be understood as a quasi-inverse symmetric plane in the space created by three variables (K, L, Y) and the residual from it. We concluded that the constant returns to scale is strictly satisfied with α' and β' normalized by the power-law indices. Since the values of power-law exponents μ_K, μ_L, and μ_Y are all close to “1”, the constant returns to scale was approximately observed even in the non-standardized capital share (capital elasticity) α and labor share (labor elasticity) β.

In this study, as in our previous study, we did not limit the data area in which the Cobb–Douglas production function was observed to the large-scale region where the power law holds. As our previous study showed, there is also quasi-inverse symmetry in the mid-scale region. The collapse of the power-law distribution observed in the large-scale region reflects that Gibrat's law does not hold in the mid-scale region. This concept is independent of quasi-inverse symmetry. On the other hand, serious questions remain about the completeness of the data in the small scale. The current study did not exclude small data. If it can be implemented in a reliable way, the numerical consistency between our proposed method and multiple regression analysis may be higher. This is a future issue.

In our method, a firm's assets (K), number of employees (L), and production (Y) are regarded as one point in three-variable space (K, L, Y), and we focus on how those several million groups are distributed. This is an approach to microfoundations that attempts to derive the macroscopic properties of their distribution from the microscopic objects of individual points. This is an important issue that has been studied in economics for many years. Finally, we describe the relationship between our discussion and Aoki et al., a pioneering study as a physical approach to this problem.

Aoki and Yoshikawa hypothesized a multisystem in which the total number of employees and the total amount of production of a company are fixed, and found that the randomness (entropy) is greatest in a society in equilibrium. In that case, they used physics methodologies to claim that the average number of employees in firms follows the Boltzmann distribution (Aoki and Yoshikawa 2007). Aoyama et al. noticed that there is actually a large fluctuation in the total amount of production and incorporated the effect by superimposing the Boltzmann distribution. As a result, it was theoretically shown that labor productivity of (Y/L) follows the power-law distribution in the large-scale range and the Boltzmann distribution with negative temperature in the middle- and low-scale ranges, and this was confirmed by the empirical data (Aoyama et al. 2009, 2009, 2010, 2015, 2017; Souma et al. 2009; Iyetomi 2012).

This can be described geometrically as follows. First, consider the 2D plane of the logarithmic axes of labor productivity (Y/L) on the horizontal axis and number of employees (L) on the vertical axis. The horizontal axis is divided into logarithmically equal-sized bins, and the vertical average of the data in the bins is plotted. The distribution followed by these points can be described by superposition of the Boltzmann distribution.

In this paper, we show the following. Total factor productivity ($Y/K^{\alpha}L^{\beta}$) is divided into logarithmically equal-sized bins, and the distribution of the number of data in the bins is observed. We can determine (α, β, a), so that the distribution is symmetric with respect to a constant a. Thus, our observations differ from those of Aoki et al. To unite the two, we need to extend the discussion of Aoyama et al. to total factor productivity, including capital (K) . If this can be carried out, it will be possible to discuss the distribution of productivity that firms' capital as well as labor, and to analyze firms in various industries in terms of both labor and capital. This is a major challenge for the future.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this article.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

Appendix 1

Here, we show that under the inverse symmetry (4), according to Fujiwara et al. (2003, 2004), the power laws of x_1 and x_2 with the same Pareto index are derived from Gibrat's law (5). Using growth rate $R = x_2/x_1$, (4) is rewritten by variables x_1, R as follows:

$$P_{12}(x_1, R) = R^{-1}\, P_{12}(Rx_1, R^{-1}). \tag{33}$$

Using conditional PDF $Q(R|x_1) = P_{12}(x_1, R)/P(x_1)$ and Gibrat's law (5), this can be reduced:

$$\frac{P(x_1)}{P(Rx_1)} = \frac{1}{R}\frac{Q(R^{-1}|Rx_1)}{Q(R|x_1)} = \frac{1}{R}\frac{Q(R^{-1})}{Q(R)}. \tag{34}$$

Since the system has inverse symmetry, Gibrat's law (5) is also established under the inverse transformation of $x_1 \leftrightarrow Rx_1\ (= x_2)$. The right side of Eq. (34) is only a function of R, and so, we signify it by $G(R)$ and expand Eq. (34) by $R = 1 + \epsilon$ $(\epsilon \ll 1)$ in ϵ. The zeroth order term of ϵ is trivial, and the first-order term yields the following differential equation:

$$G'(1)P(x_1) + x_1 \frac{d}{dx_1} P(x_1) = 0. \tag{35}$$

Here, $G'(\cdot)$ means the R differentiation of $G(\cdot)$. No more useful information can be obtained from the second-order and higher order terms of ϵ. The solution to this equation is uniquely given:

$$P(x_1) \propto {x_1}^{-G'(1)}. \tag{36}$$

This solution satisfies Eq. (34) even if R is not near $R=1$, when $Q(R)=R^{-G'(1)-1}\,Q(R)$ holds (this is called a reflection law). Reflection law has been confirmed with various actual data (Fujiwara et al. 2004). Finally, if we set $G'(1)=\mu+1$, Eq. (36) matches the power laws (1), (2), and (3) described by variables at 1 year x_1. Since this system has symmetry $x_1 \leftrightarrow x_2$, the power laws holds for the same Pareto index μ even at time 2.

Appendix 2

Here, we show that according to Ishikawa (2006, 2009), Mizuno et al. (2012), and Ishikawa et al. (2013, 2014), under the quasi-inverse symmetry (6), the power laws of x_1 and x_2 with different Pareto indices are derived from Gibrat's law (5). Using extended growth rate $R = x_{T+1}/a{x_T}^{\theta}$, (6) is rewritten by variables x_1, R as follows:

$$P_{12}(x_1,R) = R^{1/\theta-2}\,P_{12}(R^{1/\theta}x_1, R^{-1}). \tag{37}$$

Equation (37) is reduced to Eq. (33) at $\theta=1$. Using the conditional PDF $Q(R|x_T)$ and Gibrat's law (5), this is reduced to:

$$\frac{P(x_1)}{P(R^{1/\theta}x_1)} = R^{1/\theta-2}\frac{Q(R^{-1}|R^{1/\theta}x_1)}{Q(R|x_1)} = R^{1/\theta-2}\frac{Q(R^{-1})}{Q(R)}. \tag{38}$$

Here, we assume that Gibrat's law (5) holds under a transformation: $x_1 \leftrightarrow R^{1/\theta}x_1$ $(=(x_2/a)^{1/\theta})$. This is valid in a system that has quasi-inverse symmetry. Since the last term in Eq. (38) is only a function of R, we signify it by $G_\theta(R)$ and expand Eq. (38) to R near 1 as $R=1+\epsilon$ $(\epsilon \ll 1)$. The zeroth order of ϵ is trivial, and the first-order term yields the following differential equation:

$${G_\theta}'(1)P(x_1) + \frac{x_1}{\theta}\frac{\mathrm{d}}{\mathrm{d}x_1}P(x_1) = 0. \tag{39}$$

Here, ${G_\theta}'(\cdot)$ denotes the R differentiation of $G_\theta(\cdot)$. No more useful information can be obtained from the second-order and higher order terms of ϵ. The solution to this equation is uniquely given:

$$P(x_1) \propto {x_1}^{-\theta {G_\theta}'(1)}. \tag{40}$$

Similar to Appendix 1, this solution satisfies Eq. (38) even if R is not near $R=1$, when $Q(R)=R^{-{G_\theta}'(1)-1}\,Q(R)$ holds.

Next, in quasi-static system (x_1, x_2), we identify distribution $P(x_2)$. Actually, we should write $P_{x_1}(x_1)$, $P_{x_2}(x_2)$; however, because function forms are complicated, they are collectively written as P. From Eq. (40) and $P(x_1)dx_1 = P(x_2)dx_2$, $P(x_2)$ can be expressed:

$$P(x_2) = P(x_1)\frac{\mathrm{d}x_1}{\mathrm{d}x_2} \propto {x_2}^{-{G_\theta}'(1)+1/\theta-1}. \tag{41}$$

Here, we signify Pareto indices at year 1, 2 by μ_1, μ_2 and represent $P(x_1)$, $P(x_2)$ as follows:

$$P(x_1) \propto {x_1}^{-\mu_1 - 1}, \quad P(x_2) \propto {x_2}^{-\mu_2 - 1}. \tag{42}$$

Comparing Eqs. (40) and (41) to Eq. (42), we obtain $\theta {G_\theta}'(1) = \mu_1 + 1$, ${G_\theta}'(1) - 1/\theta + 1 = \mu_2 + 1$ and conclude the relation among μ_1, μ_2, and θ as follows:

$$\theta = \frac{\mu_1}{\mu_2}. \tag{43}$$

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