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Traveling Wave Solutions in Temporally Discrete Lotka-Volterra Competitive Systems with Delays

Huaqin Peng[1,2] · Qing Zhu[1,2]

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Abstract

In this paper, we investigate the existence of traveling wave solution for temporally discrete Lotka Volterra competitive system with delays. By using the cross iteration method and Schauder's fixed point theorem, we reduce the existence of traveling wave solutions to the existence of a pair of upper and lower solutions. The obtained results makes up and improves the results of the existence of traveling wave solutions for this systems.

Keywords Traveling wave solution · Upper and lower solution · Schauder's fixed point theorem · Delay · Reaction diffusion system

Mathematics Subject Classification 35K57 · 35R10 · 92D25

1 Introduction

Traveling wave solutions have been widely investigated for reaction diffusion systems, such as Britton [1], Hosono [2], Guo and Liang [3], Huang and Han [4]and the references cited therein. For time delayed reaction diffusion equations such as [5–12]and the references cited therein. For discrete reaction diffusion equations [13–15] and the references cited therein. For delayed lattice differential equations, Wu and Zou [16]use iterative scheme and the upper-lower solution method to prove the existence of traveling wave fronts of lattice differential equation. The problems on traveling wave solutions for other types of spatio-temporal delays see [15, 17–20], and references therein.

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✉ Qing Zhu
zhuqingscnu@163.com

1 School of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China

2 The Center for Applied Mathematics of Guangxi (Guangxi Normal University), Guangxi Normal University, Guilin 541004, China

For some organisms of non-overlapping generations,temporally discrete and spatially continuous diffusion model will be more suitable than its corresponding time continuous diffusion model to study the dynamic behavior of a single species that living in a spatially continuous habitat in population ecology. Thus the study of the time discontinuous model is necessary, Lin and Li [21] used the same approach as [16] presented a temporally discrete reaction diffusion equation with delay

$$u_n(x) - u_{n-1}(x) = d\Delta u_n(x) + f(u_n(x), u_{n-\tau}(x)), \quad n \in \mathbb{N}, x \in \mathbb{R}, \tag{1}$$

which can be considered as a temporal discretization of the following differential equation

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x), u(t-\tau,x)) \;\; t \geq 0, \quad x \in \mathbb{R}. \tag{2}$$

Wang and Zhang [22] by using Schauder's fixed point theorem for (1) and establish the existence of traveling wave fronts. For more researches on this topic see [23–27].

In nature, the populations of both species are affected by their respective the effect of inherent growth rate, at the same time, in time of t and $t - \tau_i (i = 1, 2, 3, 4)$, as the population density of both its own and competing species increases, the growth rate of population density declined for both species.This suggests that the population density of both competing species is affected separately by their own time t,$t - \tau_i (i = 1, 3)$, and competing species density at time$t$,$t - \tau_i (i = 2, 4)$ is affected. the model is as follows:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D_1 \frac{\partial^2 u(x,t)}{\partial x^2} + r_1 u(x,t)[1 - a_1 u(x,t) - b_1 u(x, t-\tau_1) \\ \qquad\qquad - c_1 v(x,t) - d_1 v(x, t-\tau_2)], \\ \frac{\partial v(x,t)}{\partial t} = D_2 \frac{\partial^2 v(x,t)}{\partial x^2} + r_2 v(x,t)[1 - a_2 v(x,t) - b_2 v(x, t-\tau_3) \\ \qquad\qquad - c_2 u(x,t) - d_2 u(x, t-\tau_4)]. \end{cases} \tag{3}$$

In system (3), if $a_1 = 0$,$c_1 = 0$,$a_2 = 0$,$c_2 = 0$, the equations was considered by Li [11], they use a cross iteration scheme and Schauder's fixed point theorem established the existence of traveling wave solutions. Xia and Yu [23] apply nonstandard finite difference schemes and Euler's method to the models of [11] and obtain the existence of traveling wave solutions for a class of temporally discrete reaction-diffusion systems with delays.

Motivated by the above works, we apply nonstandard finite difference schemes and Euler's method to the models (3) and can obtain the discrete-time models

$$\begin{cases} u_n(x) - u_{n-1}(x) = D_1 \Delta u_n(x) + r_1 u_n(x)[1 - a_1 u_n(x) - b_1 u_{n-\tau_1}(x) \\ \qquad\qquad - c_1 v_n(x) - d_1 v_{n-\tau_2}(x)], \\ v_n(x) - v_{n-1}(x) = D_2 \Delta v_n(x) + r_2 v_n(x)[1 - a_2 v_n(x) - b_2 v_{n-\tau_3}(x) \\ \qquad\qquad - c_2 u_n(x) - d_2 u_{n-\tau_4}(x)], \end{cases} \tag{4}$$

where $u_n(x)$, $v_n(x)$ are the densities of populations of two species at time n and location x respectively, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. An interesting problem is that whether (3) and

(4) can have similar dynamical behavior. In this paper,we will consider the existence of traveling wave solution of (4) by using the cross iteration method and Schauder's fixed point theorem,which was used by [9, 10].

The organization of this paper is as follows. In section 2, we introduce abstract results and obtain the existence of traveling wave solutions for more general equations with discrete delays under some conditions. Section 3 is invoked to derive the existence of travelling waves by constructing a pair of upper and lower solutions for temporally discrete diffusion-competition systems (4).

2 Preliminaries

In this section, we will consider the existence of traveling wave solution for more general equations, its generalization with delays can be written as

$$\begin{cases} u_n(x) - u_{n-1}(x) = D_1 \Delta u_n(x) + f_1(u_{n-\tau_1}(x), v_{n-\tau_2}(x)), \\ v_n(x) - v_{n-1}(x) = D_2 \Delta v_n(x) + f_2(u_{n-\tau_3}(x), v_{n-\tau_4}(x)), \end{cases} \tag{5}$$

where $D_1, D_2 > 0$, $f_i : C([-c\tau, 0], \mathbb{R}^2) \to \mathbb{R}$ is a continuous function, $\tau = \max\limits_{1 \le i \le 4} \{\tau_i\}$.

A traveling wave solution of (5) is a special solution with the form $u_n(x) = \phi(x + cn)$, $v_n(x) = \psi(x + cn)$, where $\phi, \psi \in C^2(\mathbb{R}, \mathbb{R})$ and c is a positive constant corresponding to the wave speed. Substituting $u_n(x) = \phi(x+cn)$, $v_n(x) = \psi(x+cn)$ into (5) and denoting $\phi_s(t) = \phi(t + s)$,$\psi_s(t) = \psi(t + s)$ and $x + cn$ by t, we obtain the following system

$$\begin{cases} D_1 \phi''(t) - \phi(t) + \phi(t - c) + f_1(\phi_t(-c\tau_1), \psi_t(-c\tau_2)) = 0, \\ D_2 \psi''(t) - \psi(t) + \psi(t - c) + f_2(\phi_t(-c\tau_3), \psi_t(-c\tau_4)) = 0, \end{cases} \tag{6}$$

here, (ϕ, ψ) is called a profile of the traveling wave solution. Motivated by the background of traveling wave solution, we also require that $(\phi(t), \psi(t))$ satisfy the following asymptotic boundary conditions

$$\lim_{t \to -\infty} \phi(t) = \phi_-, \quad \lim_{t \to +\infty} \phi(t) = \phi_+, \quad \lim_{t \to -\infty} \psi(t) = \psi_-, \quad \lim_{t \to +\infty} \psi(t) = \psi_+, \tag{7}$$

where (ϕ_-, ψ_-) and (ϕ_+, ψ_+) are two equilibria of (5). Without loss of generality, we may assume that $\phi_- = 0$, $\psi_- = 0$ and $\phi_+ = k_1 > 0$, $\psi_+ = k_2 > 0$. Thus (7) reads as

$$\lim_{t \to -\infty} \phi(t) = 0, \quad \lim_{t \to +\infty} \phi(t) = k_1, \quad \lim_{t \to -\infty} \psi(t) = 0, \quad \lim_{t \to +\infty} \psi(t) = k_2. \tag{8}$$

For convenience of statements, we make the following hypothesis:

(**P1**) There exists $k=(k_1,k_2)$ with $k_i>0$ such that

$$f_i(0,0)=f_i(k_1,k_2)=0,\ \text{for } i=1,2.$$

(**P2**) There exist two positive constants $L_1>0$ and $L_2>0$ such that

$$|f_1(\phi_1,\psi_1)-f_1(\phi_2,\psi_2)|\le L_1\|\Phi-\Psi\|,$$

$$|f_2(\phi_1,\psi_1)-f_2(\phi_2,\psi_2)|\le L_2\|\Phi-\Psi\|,$$

for $\Phi=(\phi_1,\psi_1),\Psi=(\phi_2,\psi_2)\in C([-c\tau,0],\mathbb{R}^2)$ with $0\le\phi_i(s),\psi_i(s)\le M_i,\ s\in[-c\tau,0]$ for $M=(M_1,M_2),\quad M_j\ge k_j,\tau=\max_{1\le i\le 4}\{\tau_i\}$.

Denote

$$C_{[0,M]}(\mathbb{R},\mathbb{R}^2)=\{(\phi,\psi)\in C(\mathbb{R},\mathbb{R}^2):0\le\phi(s)\le M_1,0\le\psi(s)\le M_2,s\in\mathbb{R}\}.$$

For any given positive constants $\beta_1>0,\beta_2>0$, define the operator $H=(H_1,H_2): C_{[0,M]}(\mathbb{R},\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ by

$$H_1(\phi,\psi)(t)=f_1(\phi_t(-c\tau_1),\psi_t(-c\tau_2))+(\beta_1-1)\phi(t)+\phi(t-c), \tag{9}$$

$$H_2(\phi,\psi)(t)=f_2(\phi_t(-c\tau_3),\psi_t(-c\tau_4))+(\beta_2-1)\psi(t)+\psi(t-c). \tag{10}$$

In terms of the expressions of H_1 and H_2, system (5) can be rewritten as

$$\begin{cases} d_1\phi''(t)-\beta_1\phi(t)+H_1(\phi,\psi)(t)=0,\\ d_2\psi''(t)-\beta_2\psi(t)+H_2(\phi,\psi)(t)=0.\end{cases} \tag{11}$$

For $(\phi,\psi)\in C_{[0,M]}(\mathbb{R},\mathbb{R}^2)$, define $F=(F_1,F_2):C_{[0,M]}(\mathbb{R},\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{aligned} F_1(\phi,\psi)(t)&=\frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\int_{-\infty}^{t}e^{\lambda_1(t-s)}H_1(\phi,\psi)(s)ds\\ &\quad+\int_t^{\infty}e^{\lambda_2(t-s)}H_1(\phi,\psi)(s)ds\Big],\\ F_2(\phi,\psi)(t)&=\frac{1}{d_2(\lambda_4-\lambda_3)}\Big[\int_{-\infty}^{t}e^{\lambda_3(t-s)}H_2(\phi,\psi)(s)ds\\ &\quad+\int_t^{\infty}e^{\lambda_4(t-s)}H_2(\phi,\psi)(s)ds\Big],\end{aligned}$$

where

$$\lambda_1=-\sqrt{\frac{\beta_1}{d_1}},\ \ \lambda_2=\sqrt{\frac{\beta_1}{d_1}},\ \ \lambda_3=-\sqrt{\frac{\beta_2}{d_2}},\ \ \lambda_4=\sqrt{\frac{\beta_2}{d_2}}.$$

It is easy to show that $F = (F_1, F_2)$ is well defined and for any $(\phi, \psi) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^2)$, $F_1(\phi, \psi)$ and $F_2(\phi, \psi)$ satisfy

$$\begin{cases} d_1 F_1(\phi, \psi)''(t) - \beta_1 F_1(\phi, \psi)(t) + H_1(\phi, \psi)(t) = 0, \\ d_2 F_2(\phi, \psi)''(t) - \beta_2 F_2(\phi, \psi)(t) + H_2(\phi, \psi)(t) = 0. \end{cases} \tag{12}$$

Thus, if $F(\phi, \psi) = (F_1(\phi, \psi), F_2(\phi, \psi)) = (\phi, \psi)$, *i.e.*, (ϕ, ψ) is a fixed point of F, then (11) has a solution (ϕ, ψ). If this solution further satisfies the boundary condition (8), then it is a traveling wave solution of (5).

In order to obtain a fixed point of F,we propose a condition on the reaction terms:

(**P3**) There exist two positive constants β_1 and β_2 such that

$$f_1(\phi_1(-c\tau_1), \psi_1(-c\tau_2)) - f_1(\phi_2(-c\tau_1), \psi_1(-c\tau_2)) + (\beta_1 - 1)[\phi_1(0) - \phi_2(0)] \geq 0,$$

$$f_1(\phi_1(-c\tau_1), \psi_1(-c\tau_2)) - f_1(\phi_1(-c\tau_1), \psi_2(-c\tau_2)) \leq 0,$$

$$f_2(\phi_1(-c\tau_3), \psi_1(-c\tau_4)) - f_2(\phi_1(-c\tau_3), \psi_2(-c\tau_4)) + (\beta_2 - 1)[\psi_1(0) - \psi_2(0)] \geq 0,$$

$$f_2(\phi_1(-c\tau_3), \psi_1(-c\tau_4)) - f_2(\phi_2(-c\tau_3), \psi_1(-c\tau_4)) \leq 0,$$

for any $\phi_1(s),\ \phi_2(s),\ \psi_1(s),\ \psi_2(s) \in C([-c\tau, 0], \mathbb{R})$ with

(i) $0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_2, s \in [-c\tau, 0]$,

(ii) $e^{\beta_1 s}[\phi_1(s) - \phi_2(s)]$ and $e^{\beta_2 s}[\psi_1(s) - \psi_2(s)]$ are nondecreasing in $s \in [-c\tau, 0]$.

In the following, we give the definition of weak upper and lower solutions of system (5).

Definition 1 A pair of continuous functions $\overline{\Phi}(t) = (\overline{\phi}(t), \overline{\psi}(t))$ and $\underline{\Phi}(t) = (\underline{\phi}(t), \underline{\psi}(t))$ are called a pair of weak upper and lower solutions of (5), respectively, if there exist constants T_i $(i = 1, 2, \cdots, m)$ such that $\overline{\Phi}$ and $\underline{\Phi}$ are two continuously differentiable functions in $\mathbb{R} \setminus \{T_i, i = 1, 2, \cdots, m\}$ and satisfy

$$\begin{cases} d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t - c) + f_1(\overline{\phi}_t(-c\tau_1), \underline{\psi}_t(-c\tau_2)) \\ \quad \leq 0, t \in \mathbb{R} \setminus \{T_i, i = 1, 2, \cdots, m\}, \\ d_2\overline{\psi}''(t) - \overline{\psi}(t) + \overline{\psi}(t - c) + f_2(\overline{\phi}_t(-c\tau_3), \overline{\psi}_t(-c\tau_4)) \\ \quad \leq 0, t \in \mathbb{R} \setminus \{T_i, i = 1, 2, \cdots, m\}, \end{cases}$$

and

$$\begin{cases} d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t - c) + f_1(\underline{\phi}_t(-c\tau_1), \overline{\psi}_t(-c\tau_2)) \\ \quad \geq 0, t \in \mathbb{R} \setminus \{T_i, i = 1, 2, \cdots, m\}, \\ d_2\underline{\psi}''(t) - \underline{\psi}(t) + \underline{\psi}(t - c) + f_2(\underline{\phi}_t(-c\tau_3), \underline{\psi}_t(-c\tau_4)) \\ \quad \geq 0, t \in \mathbb{R} \setminus \{T_i, i = 1, 2, \cdots, m\}. \end{cases}$$

For $\mu \in (0, \min\{\lambda_2, \lambda_4\})$, $C(\mathbb{R}, \mathbb{R}^2)$ can be equipped with the exponential decay norm defined by $|\Phi|_\mu = \sup\limits_{t\in\mathbb{R}} |\Phi(t)|e^{-\mu|t|}$. Let

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = \{\Phi \in C(\mathbb{R}, \mathbb{R}^2) : \ |\Phi|_\mu < \infty\}.$$

Then it is easy to check that $(B_\mu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\mu)$ is a Banach space.

In what follows, we assume that there exists a pair of weak upper and lower solutions $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ to (5) satisfying

(A1) $(0, 0) \le (\underline{\phi}(t), \underline{\psi}(t)) \le (\overline{\phi}(t), \overline{\psi}(t)) \le (M_1, M_2)$,

(A2) $\lim\limits_{t\to-\infty}(\overline{\phi}(t), \overline{\psi}(t)) = (0, 0)$, $\lim\limits_{t\to\infty}(\underline{\phi}(t), \underline{\psi}(t)) = \lim\limits_{t\to\infty}(\overline{\phi}(t), \overline{\psi}(t)) = (k_1, k_2)$,

(A3) $\overline{\phi}'(t^+) \le \overline{\phi}'(t^-)$, $\overline{\psi}'(t^+) \le \overline{\psi}'(t^-)$, $\quad \underline{\phi}'(t^+) \ge \underline{\phi}'(t^-)$, $\underline{\psi}'(t^+) \ge \underline{\psi}'(t^-)$,

Define the set of profiles by

$$\Gamma := \{(\phi, \psi) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^2) : \ \underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t) \text{ and } \underline{\psi}(t) \le \psi(t) \le \overline{\psi}(t)\}.$$

It is easy to see that Γ is nonempty. In fact, by **(A1)**, we know that $(\underline{\phi}(t), \underline{\psi}(t))$ and $(\overline{\phi}(t), \overline{\psi}(t))$ satisfy. Moreover, it is obvious that Γ is convex, closed and bounded in $B_\mu(\mathbb{R}, \mathbb{R}^2)$.

Now we explore some basic properties of the operator H and F.

Lemma 1 *Assume that (**P1**),(**P2**) and (**P3**) hold. Then we have*

$$\begin{cases} H_1(\phi_2, \psi_1)(t) \le H_1(\phi_1, \psi_1)(t), \\ H_1(\phi_1, \psi_1)(t) \le H_1(\phi_1, \psi_2)(t), \\ H_2(\phi_2, \psi_2)(t) \le H_2(\phi_1, \psi_1)(t), \end{cases}$$

and

$$\begin{cases} F_1(\phi_2, \psi_1)(t) \le F_1(\phi_1, \psi_1)(t), \\ F_1(\phi_1, \psi_1)(t) \le F_1(\phi_1, \psi_2)(t), \\ F_2(\phi_2, \psi_2)(t) \le F_2(\phi_1, \psi_1)(t), \end{cases}$$

for any $\phi_i, \psi_i \in C([-c\tau, 0], \mathbb{R})$, $i = 1, 2$, *with* $0 \le \phi_2(s) \le \phi_1(s) \le M_1$, $\ 0 \le \psi_2(s) \le \psi_1(s) \le M_2$.

Next, we further explore the profile of the operator F.

Lemma 2 *Assume (**P1**),(**P2**) holds, then* $F = (F_1, F_2)$ *is continuous with respect to the norm* $|\cdot|_\mu$ *in* $B_\mu(\mathbb{R}, \mathbb{R}^2)$.

In order to apply Schauder's fixed point theorem, we must prove

Lemma 3 *If (**P3**) hold, then* $F(\Gamma) \subset \Gamma$.

Lemma 4 *If (**P3**) hold, then the operator* $F : \Gamma \to \Gamma$ *is compact with respect to the delay norm* $|\cdot|_\mu$.

Since the proofs of lemma1–4 are similar to [11, 20, 23], so we omit it here.

Now, we are in the position to state and prove the following existence theorem.

Theorem 5 *Suppose that* (**P1**)*,*(**P2**) *and* (**P3**) *hold. If* (5) *has a pair of weak upper* $(\overline{\phi}, \overline{\psi})$ *and weak lower solutions* $(\underline{\phi}, \underline{\psi})$ *satisfying* (**A1**)*-*(**A3**)*, then system* (5) *has a traveling wave solution.*

Proof It is easy to verify that Γ is a nonempty, closed and convex subset of $B_\mu(\mathbb{R}, \mathbb{R}^2)$, combining lemma 1–4 with schauder's fixed theorem, we know that there exists a fixed point $(\phi^*(t), \psi^*(t))$ of F in Γ. In order to show this fixed point is traveling wave solution, we need to verify the boundary condition (8).

By (**A2**) and the fact that $0 \le (\underline{\phi}(t), \underline{\psi}(t)) \le (\phi^*(t), \psi^*(t)) \le (\overline{\phi}(t), \overline{\psi}(t)) \le (M_1, M_2)$, we know that $\lim_{t\to-\infty}(\phi^*(t), \psi^*(t)) = (0, 0)$, and $\lim_{t\to+\infty}(\phi^*(t), \psi^*(t)) = (k_1, k_2)$. Therefore, the fixed points $(\phi^*(t), \psi^*(t))$ satisfies the boundary conditions. This completes the proof. □

3 Traveling Wave Solutions of (4)

In the section, we shall apply the result of section 2 to temporally discrete for competitive system (4). Let $a_1 + b_1 > c_2 + d_2$, $a_2 + b_2 > c_1 + d_1$, system (4) have four steady states namely $E_1(0,0)$,$E_2(0, \frac{1}{a_2+b_2})$, $E_3(\frac{1}{a_1+b_1}, 0)$, $E_4(k_1, k_2)$, where $k_1 = \frac{(a_2+b_2)-(c_1+d_1)}{(a_1+b_1)(a_2+b_2)-(c_1+d_1)(c_2+d_2)} > 0$, $k_2 = \frac{(a_1+b_1)-(c_2+d_2)}{(a_1+b_1)(a_2+b_2)-(c_1+d_1)(c_2+d_2)} > 0$. Assume that $c > 0$, and substituting

$$u_n(x) = \phi(x + cn) = \phi(t), \quad v_n(x) = \psi(x + cn) = \psi(t), \quad t = x + cn,$$

into equation (4) then the corresponding wave system is

$$\begin{cases} D_1\phi''(t) - \phi(t) + \phi(t-c) + r_1\phi(t)[1 - a_1\phi(t) - b_1\phi(t - c\tau_1) \\ \quad - c_1\psi(t) - d_1\psi(t - c\tau_2)] = 0, \\ D_2\psi''(t) - \psi(t) + \psi(t-c) + r_2\psi(t)[1 - a_2\psi(t) - b_2\psi(t - c\tau_3) \\ \quad - c_2\phi(t) - d_2\phi(t - c\tau_4)] = 0, \end{cases} \tag{13}$$

where, $\phi, \psi \in c([-c\tau, 0], \mathbb{R})$,$\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$. We are interested in solution of (13) satisfy the following asymptotic boundary condition

$$\lim_{t\to-\infty}\phi(t) = 0, \quad \lim_{t\to+\infty}\phi(t) = k_1, \quad \lim_{t\to-\infty}\psi(t) = 0, \quad \lim_{t\to+\infty}\psi(t) = k_2. \tag{14}$$

Define $f(\phi, \psi) = (f_1(\phi, \psi), f_2(\phi, \psi))$ by

$$f_1(\phi, \psi) = r_1\phi(0)[1 - a_1\phi(0) - b_1\phi(-c\tau_1) - c_1\psi(0) - d_1\psi(-c\tau_2)],$$

$$f_2(\phi, \psi) = r_2\psi(0)[1 - a_2\psi(0) - b_2\psi(-c\tau_3) - c_2\phi(0) - d_2\phi(-c\tau_4)].$$

Obviously, f_1 and f_2 satisfies (**P1**) and (**P2**). Now, let us prove that $f(\phi, \psi)$ satisfies (**P3**).

Lemma 6 *When τ_1, τ_3 are small enough, the function $f(\phi, \psi)$ satisfies (**P3**).*

Proof For any $0 \le \phi_2(s) \le \phi_1(s) \le M_1, 0 \le \psi_2(s) \le \psi_1(s) \le M_2$,$s \in [-c\tau, 0]$,where $M_1 > k_1$,$M_2 > k_2$.

$$\begin{aligned}
& f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) \\
& = [r_1 - r_1 a_1(\phi_1(0) + \phi_2(0)) - r_1 c_1 \psi_1(0)](\phi_1(0) - \phi_2(0)) \\
& \quad - d_1 r_1 \psi_1(-c\tau_2)[\phi_1(0) - \phi_2(0)] - b_1 r_1[\phi_1(0)\phi_1(-c\tau_1) - \phi_2(0)\phi_2(-c\tau_1)] \\
& \ge [r_1 - 2r_1 a_1 M_1 - r_1 c_1 M_2 - d_1 r_1 M_2](\phi_1(0) - \phi_2(0)) \\
& \quad - b_1 r_1 \phi_1(-c\tau_1)[\phi_1(0) - \phi_2(0)] - b_1 r_1[\phi_1(-c\tau_1) - \phi_2(-c\tau_1)] \\
& \ge r_1[1 - 2a_1 M_1 - b_1 M_1 - c_1 M_2 - d_1 M_2](\phi_1(0) - \phi_2(0)) \\
& \quad - b_1 r_1 \phi_2(0) e^{\beta_1 c\tau_1} e^{-\beta_1 c\tau_1}[\phi_1(-c\tau_1) - \phi_2(-c\tau_1)] \\
& \ge r_1[1 - 2a_1 M_1 - b_1 M_1 - c_1 M_2 - d_1 M_2 - b_1 M_1 e^{\beta_1 c\tau_1}](\phi_1(0) - \phi_2(0)).
\end{aligned}$$

If we choose $\beta_1 > 0$, such that

$$\beta_1 - 1 > r_1[2a_1 M_1 + 2b_1 M_1 + c_1 M_2 + d_1 M_2 - 1],$$

then, for τ_1 small enough, we obtain

$$\beta_1 - 1 > r_1[2a_1 M_1 + b_1 M_1 + c_1 M_2 + d_1 M_2 b_1 M_1 e^{\beta_1 c\tau_1} - 1].$$

Thus

$$f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) + (\beta_1 - 1)(\phi_1(0) - \phi_2(0)) \ge 0.$$

On the other hand,

$$f_1(\phi_1, \psi_1) - f_1(\phi_1, \psi_2) \le 0.$$

In a similar way, there exists $\beta_2 > 0$ such that $f_2(\phi_1, \psi_1) - f_2(\phi_1, \psi_2) + (\beta_2 - 1)(\phi_1(0) - \psi_2(0)) \ge 0$ and $f_2(\phi_1, \psi_1) - f_2(\phi_2, \psi_1) \le 0$. The proof is completed. □

we need to construct a weak upper and a weak lower solution of (13)satisfying the conditions in Theorem 2.1.

Define

$$\Delta_{1c}(\lambda) = D_1 \lambda^2 + e^{-\lambda c} + r_1 - 1,$$
$$\Delta_{2c}(\lambda) = D_2 \lambda^2 + e^{-\lambda c} + r_2 - 1,$$

then it easy obtain the following lemma

Lemma 7 *Let $0 < r_1 < 1$, $0 < r_2 < 1$. Then there exists $c^* > 0$ such that for $c > c^*$, $\Delta_{1c}(\lambda)$, $\Delta_{2c}(\lambda)$, respectively, has two positive real roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ with $\lambda_1 < \lambda_2$, $\lambda_3 < \lambda_4$. Moreover,*

$$\Delta_{1c}(\lambda) = \begin{cases} > 0, & \text{for } \lambda < \lambda_1; \\ < 0, & \text{for } \lambda \in (\lambda_1, \lambda_2); \\ > 0, & \text{for } \lambda > \lambda_2, \end{cases} \quad \text{and} \quad \Delta_{2c}(\lambda) = \begin{cases} > 0, & \text{for } \lambda < \lambda_3; \\ < 0, & \text{for } \lambda \in (\lambda_3, \lambda_4); \\ > 0, & \text{for } \lambda > \lambda_4. \end{cases}$$

The proof of the Lemma is easy and we omit it.

From now on, we assume that $c > c^*$ with c^* given by lemma 7.

For fixed $\eta \in (1, \min\{2, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_3}, \frac{\lambda_1+\lambda_3}{\lambda_1} \frac{\lambda_1+\lambda_3}{\lambda_3}\})$, and a large constant $q > 0$, we define two functions $g_1(t) = e^{\lambda_1 t} - qe^{\eta\lambda_1 t}$ and $g_2(t) = e^{\lambda_3 t} - qe^{\eta\lambda_3 t}$, then it easily learn that $g_1(t)$ and $g_2(t)$ have global maximum m_1, m_2, respectively, and there exist $t_1 = \frac{1}{\lambda_1(\eta-1)} \ln \frac{1}{\eta q} < 0$ and $t_3 = \frac{1}{\lambda_3(\eta-1)} \ln \frac{1}{\eta q} < 0$ with $e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1} = m_1$, $e^{\lambda_3 t_3} - qe^{\eta\lambda_3 t_3} = m_2$. Therefore, for any given $\lambda > 0$ there exist $\varepsilon_2 > 0$ and $\varepsilon_4 > 0$ such that $k_1 - \varepsilon_2 e^{-\lambda t_1} = m_1$, $k_3 - \varepsilon_4 e^{-\lambda t_3} = m_2$.

Since $a_1 + b_1 > c_2 + d_2$, $a_2 + b_2 > c_1 + d_1$, there exist $\varepsilon_0 > 0$,$\varepsilon_1 > 0$,$\varepsilon_3 > 0$ such that

$$(a_1 + b_1)\varepsilon_1 - (c_1 + d_1)\varepsilon_4 > \varepsilon_0, (a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3 > \varepsilon_0, \tag{15}$$

$$(a_2 + b_2)\varepsilon_3 - (c_2 + d_2)\varepsilon_2 > \varepsilon_0, (a_2 + b_2)\varepsilon_4 - (c_2 + d_2)\varepsilon_1 > \varepsilon_0, \tag{16}$$

Let $q > 0$ large enough and $\lambda > 0$ small enough be given, for the above constants and suitable constants t_2, t_4, define the following continuous functions

$$\overline{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \le t_2, \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t \ge t_2, \end{cases} \qquad \overline{\psi}(t) = \begin{cases} e^{\lambda_3 t}, & t \le t_4, \\ k_2 + \varepsilon_3 e^{-\lambda t}, & t \ge t_4, \end{cases}$$

$$\underline{\phi}(t) = \begin{cases} e^{\lambda_1 t} - qe^{\eta\lambda_1 t}, & t \le t_1, \\ k_1 - \varepsilon_2 e^{-\lambda t}, & t \ge t_1, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} e^{\lambda_3 t} - qe^{\eta\lambda_3 t}, & t \le t_3, \\ k_2 - \varepsilon_4 e^{-\lambda t}, & t \ge t_3. \end{cases}$$

obviously, $M_1 = \sup_{t\in\mathbb{R}} \overline{\phi}(t) > k_1$, $M_2 = \sup_{t\in\mathbb{R}} \overline{\psi}(t) > k_2$, and $\overline{\phi}(t)$, $\underline{\phi}(t)$, $\overline{\psi}(t)$, $\underline{\psi}(t)$ satisfy (**A1**)-(**A3**), and

$$\min\{t_2, t_4\} - c\max\{1, \tau_1, \tau_2, \tau_3, \tau_4\} \ge \max\{t_1, t_3\}.$$

In the following, we prove that $(\overline{\phi}(t), \overline{\psi}(t))$, $(\underline{\phi}(t), \underline{\psi}(t))$ are a pair of weak upper and weak lower solutions of (6), respectively.

Lemma 8 *Let* $0 < r_1 < 1$, $0 < r_2 < 1$ *and suppose that* (15), (16) *are satisfied. Then* $(\overline{\phi}(t), \overline{\psi}(t))$ *is a weak upper solution of* (13).

Proof we first prove $\overline{\phi}(t)$ is a weak upper solution.

(i) For $t \le t_2$, $\overline{\phi}(t) = e^{\lambda_1 t}$, $\overline{\phi}(t - c) = e^{\lambda_1(t-c)}$, $\overline{\phi}(t - c\tau_1) \ge 0$,$\underline{\psi}(t) \ge 0$ $\underline{\psi}(t - c\tau_2) \ge 0$, we have

$$\begin{aligned}
&D_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t - c) \\
&\quad + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\overline{\phi}(t - c\tau_1) - c_1\underline{\psi}(t) - d_1\underline{\psi}(t - c\tau_2)] \\
&\le D_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t - c) + r_1\overline{\phi}(t) \\
&= e^{\lambda_1 t}(D_1\lambda_1^2 + e^{-\lambda_1 c} + r_1 - 1) = 0.
\end{aligned}$$

(ii) For $t_2 \le t \le t_2 + c\tau_1$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t - c\tau_1) = e^{\lambda_1(t-c\tau_1)}$, $\overline{\phi}(t - c) = e^{\lambda_1(t-c)}$, $\underline{\psi}(t) = k_2 - \varepsilon_4 e^{-\lambda t}$, $\underline{\psi}(t - c\tau_2) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}$, note that $k_1 + \varepsilon_1 e^{-\lambda t_2} = e^{\lambda_1 t_2}$, we obtain

$$
\begin{aligned}
& D_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) \\
&\quad + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\overline{\phi}(t - c\tau_1) - c_1\underline{\psi}(t) - d_1\underline{\psi}(t - c\tau_2)] \\
&= D_1\varepsilon_1\lambda^2 e^{-\lambda t} - (k_1 + \varepsilon_1 e^{-\lambda t}) + e^{\lambda_1(t-c)} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) \\
&\quad - b_1 e^{\lambda_1(t-c\tau_1)} - c_1(k_2 - \varepsilon_4 e^{-\lambda t}) - d_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&\le D_1\varepsilon_1\lambda^2 e^{-\lambda t} - (k_1 + \varepsilon_1 e^{-\lambda t}) + e^{\lambda_1 t_2} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) \\
&\quad - b_1 e^{\lambda_1(t_2-c\tau_1)} - c_1(k_2 - \varepsilon_4 e^{-\lambda t}) - d_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&= D_1\varepsilon_1\lambda^2 e^{-\lambda t} - \varepsilon_1 e^{-\lambda t} + \varepsilon_1 e^{-\lambda t_2} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[b_1 k_1 - a_1\varepsilon_1 e^{-\lambda t} \\
&\quad - b_1 e^{-\lambda_1 c\tau_1}(k_1 + \varepsilon_1 e^{-\lambda t_2}) + c_1\varepsilon_4 e^{-\lambda t} + d_1\varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&:= I_1(\lambda),
\end{aligned}
$$

since τ_1 small enough, there exist $\varepsilon^*(0 < \varepsilon^* < \frac{\varepsilon_0}{b_1(k_1+\varepsilon_1)})$, such that $1 - \varepsilon^* < e^{-\lambda_1 c\tau_1}$, it follows that $(a_1 + b_1)\varepsilon_1 - (c_1 + d_1)\varepsilon_4 > \varepsilon_0$, we have

$$
\begin{aligned}
I_1(0) &\le r_1(k_1 + \varepsilon_1)[b_1 k_1 - a_1\varepsilon_1 - b_1(1 - \varepsilon^*)(k_1 + \varepsilon_1) + c_1\varepsilon_4 + d_1\varepsilon_4] \\
&= r_1(k_1 + \varepsilon_1)[c_1\varepsilon_4 + d_1\varepsilon_4 - a_1\varepsilon_1 - b_1\varepsilon_1 + b_1\varepsilon^*(k_1 + \varepsilon_1)] \\
&< 0,
\end{aligned}
$$

therefore, there exist a $\lambda_1^* > 0$ such that $I_1(\lambda) < 0$ for $\lambda \in (0, \lambda_1^*)$.

(iii) For $t_2 + c\tau_1 \le t \le t_2 + c$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t - c\tau_1) = k_1 + \varepsilon_1 e^{\lambda_1(t-c\tau_1)}$, $\overline{\phi}(t - c) = e^{\lambda_1(t-c)}$, $\underline{\psi}(t) = k_2 - \varepsilon_4 e^{-\lambda t}$, $\underline{\psi}(t - c\tau_2) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}$, we have

$$
\begin{aligned}
& D_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) \\
&\quad + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\overline{\phi}(t - c\tau_1) - c_1\underline{\psi}(t) - d_1\underline{\psi}(t - c\tau_2)] \\
&= D_1\varepsilon_1\lambda^2 e^{-\lambda t} - (k_1 + \varepsilon_1 e^{-\lambda t}) + e^{\lambda_1(t-c)} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) \\
&\quad - b_1(k_1 + \varepsilon_1 e^{\lambda_1(t-c\tau_1)}) - c_1(k_2 - \varepsilon_4 e^{-\lambda t}) - d_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&\le D_1\varepsilon_1\lambda^2 e^{-\lambda t} - (k_1 + \varepsilon_1 e^{-\lambda t}) + e^{\lambda_1 t_2} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[c_1\varepsilon_4 e^{-\lambda t} + d_1\varepsilon_4 e^{-\lambda(t-c\tau_2)} \\
&\quad - a_1\varepsilon_1 e^{-\lambda t} - b_1\varepsilon_1 e^{-\lambda(t-c\tau_1)}] \\
&= D_1\varepsilon_1\lambda^2 e^{-\lambda t} - \varepsilon_1 e^{-\lambda t} + \varepsilon_1 e^{-\lambda t_2} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[c_1\varepsilon_4 e^{-\lambda t} + d_1\varepsilon_4 e^{-\lambda(t-c\tau_2)} \\
&\quad - a_1\varepsilon_1 e^{-\lambda t} - b_1\varepsilon_1 e^{-\lambda(t-c\tau_1)}] \\
&:= I_2(\lambda),
\end{aligned}
$$

since $(a_1 + b_1)\varepsilon_1 - (c_1 + d_1)\varepsilon_4 > \varepsilon_0$, we have $I_2(0) = r_1(k_1 + \varepsilon_1)[(c_1 + d_1)\varepsilon_4 - (a_1 + b_1)\varepsilon_1] < 0$, therefore, there exist a $\lambda_2^* > 0$ such that $I_2(\lambda) < 0$ for $\lambda \in (0, \lambda_2^*)$.

(iv)For $t \geq t_2 + c$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t-c) = k_1 + \varepsilon_1 e^{-\lambda(t-c)}$, $\overline{\phi}(t - c\tau_1) = k_1 + \varepsilon_1 e^{-\lambda(t-c\tau_1)}$, $\underline{\psi}(t) = k_2 - \varepsilon_4 e^{-\lambda t}$, $\underline{\psi}(t - c\tau_2) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}$, we obtain

$$\begin{aligned}
&D_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c)\\
&\quad + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\overline{\phi}(t - c\tau_1) - c_1\underline{\psi}(t) - d_1\underline{\psi}(t - c\tau_2)]\\
&= D_1\varepsilon_1\lambda^2 e^{-\lambda t} - \varepsilon_1 e^{-\lambda t} + \varepsilon_1 e^{-\lambda(t-c)} + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[c_1\varepsilon_4 e^{-\lambda t} + d_1\varepsilon_4 e^{-\lambda(t-c\tau_2)}\\
&\quad - a_1\varepsilon_1 e^{-\lambda t} - b_1\varepsilon_1 e^{-\lambda(t-c\tau_1)}]\\
&:= I_3(\lambda),
\end{aligned}$$

it follows that $(a_1 + b_1)\varepsilon_1 - (c_1 + d_1)\varepsilon_4 > \varepsilon_0$, we have $I_3(0) = r_1(k_1 + \varepsilon_1)[(c_1 + d_1)\varepsilon_4 - (a_1 + b_1)\varepsilon_1] < 0$, which implies that there exist a $\lambda_3^* > 0$ such that $I_3(\lambda) < 0$ for $\lambda \in (0, \lambda_3^*)$.

Taking $\lambda^* = \min\{\lambda_1^*, \lambda_2^*, \lambda_3^*\}$, then $\lambda \in (0, \lambda^*)$, we have

$$\begin{aligned}
&D_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t)\\
&\quad - b_1\overline{\phi}(t - c\tau_1) - c_1\underline{\psi}(t) - d_1\underline{\psi}(t - c\tau_2)] \leq 0.
\end{aligned}$$

In a similar way, we can find a $\lambda^{**} > 0$,then $\lambda \in (0, \lambda^{**})$, we have

$$\begin{aligned}
&D_2\overline{\psi}''(t) - \overline{\psi}(t) + \overline{\psi}(t-c)\\
&\quad + r_2\overline{\psi}(t)[1 - a_2\overline{\psi}(t) - b_2\overline{\psi}(t - c\tau_3) - c_2\underline{\phi}(t) - d_2\underline{\phi}(t - c\tau_4)] \leq 0.
\end{aligned}$$

The proof is completed. □

Lemma 9 *Let $0 < r_1 < 1$, $0 < r_2 < 1$. Suppose that (15),(16) are satisfied. Then $(\underline{\phi}(t), \underline{\psi}(t))$ is a weak lower solution of (13).*

Proof (i) For $t \leq t_1$, $\underline{\phi}(t) = e^{\lambda_1 t} - q e^{\eta\lambda_1 t}$, $\underline{\phi}(t-c) = e^{\lambda_1(t-c)} - q e^{\eta\lambda_1(t-c)}$, $\underline{\phi}(t - c\tau_1) = e^{\lambda_1(t-c\tau_1)} - q e^{\eta\lambda_1(t-c\tau_1)}$, $\overline{\psi}(t) = e^{\lambda_3 t}$, $\overline{\psi}(t - c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, we obtain

$$\begin{aligned}
&D_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)[1 - a_1\underline{\phi}(t) - b_1\underline{\phi}(t - c\tau_1)\\
&\quad - c_1\overline{\psi}(t) - d_1\overline{\psi}(t - c\tau_2)]\\
&= e^{\lambda_1 t}(D_1\lambda_1^2 + e^{-\lambda_1 c} + r_1 - 1) - q e^{\eta\lambda_1 t}[D_1(\eta\lambda_1)^2 + e^{-\eta\lambda_1 c} + r_1 - 1] - r_1(e^{\lambda_1 t}\\
&\quad - q e^{\eta\lambda_1 t})\{a_1(e^{\lambda_1 t} - q e^{\eta\lambda_1 t}) + b_1[e^{\lambda_1(t-c\tau_1)} - q e^{\eta\lambda_1(t-c\tau_1)}] + c_1 e^{\lambda_3 t} + d_1 e^{\lambda_3(t-c\tau_2)}\}\\
&\quad \geq -q e^{\eta\lambda_1 t}\Delta_{1c}(\eta\lambda_1) - r_1 e^{\lambda_1 t}[(a_1 + b_1)e^{\lambda_1 t} + (c_1 + d_1)e^{\lambda_3 t}]\\
&\quad = -q e^{\eta\lambda_1 t}[\Delta_{1c}(\eta\lambda_1) + \frac{r_1(a_1 + b_1)}{q}e^{(2\lambda_1 - \eta\lambda_1)t} + \frac{r_1(c_1 + d_1)}{q}e^{(\lambda_1 + \lambda_3 - \eta\lambda_1)t}]\\
&\quad \geq -q e^{\eta\lambda_1 t}[\Delta_{1c}(\eta\lambda_1) + \frac{r_1(a_1 + b_1)}{q} + \frac{r_1(c_1 + d_1)}{q}],
\end{aligned}$$

choose a sufficiently large number $q > 0$ such that $q > -\frac{r_1}{\Delta_{1c}(\eta\lambda_1)}(a_1 + b_1 + c_1 + d_1)$, then $-q e^{\eta\lambda_1 t}[\Delta_{1c}(\eta\lambda_1) + \frac{r_1(a_1+b_1)}{q} + \frac{r_1(c_1+d_1)}{q}] > 0$.

(ii) For $t_1 \le t \le t_1+c\tau_1$, $\underline{\phi}(t) = k_1-\varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t-c\tau_1) = e^{\lambda_1(t-c\tau_1)}-qe^{\eta\lambda_1(t-c\tau_1)}$, $\underline{\phi}(t-c) = e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}$, $\overline{\psi}(t) = e^{\lambda_3 t}$, $\overline{\psi}(t-c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, note that $e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1} = k_1 - \varepsilon_2 e^{-\lambda t_1}$, $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$ and $t_1 + c \le t_4$, we get

$$\begin{aligned}
&D_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c)\\
&\quad +r_1\underline{\phi}(t)[1 - a_1\underline{\phi}(t) - b_1\underline{\phi}(t-c\tau_1) - c_1\overline{\psi}(t) - d_1\overline{\psi}(t-c\tau_2)]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + (e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}) + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1\\
&\quad -a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(e^{\lambda_1(t-c\tau_1)} - qe^{\eta\lambda_1(t-c\tau_1)}) - c_1 e^{\lambda_3 t} - d_1 e^{\lambda_3(t-c\tau_2)}]\\
&\ge -D_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda(t_1+c)}) + (e^{\lambda_1(t_1-c)} - e^{\lambda_1 t_1}) + (e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1})\\
&\quad +r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_1 - \varepsilon_2 e^{-\lambda t_1}) - c_1 e^{\lambda_3 t_4} - d_1 e^{\lambda_3 t_4}]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda(t_1+c)} + e^{\lambda_1 t_1}(e^{-\lambda_1 c} - 1) - \varepsilon_2 e^{-\lambda t_1}\\
&\quad +r_1(k_1 - \varepsilon_2 e^{-\lambda t})[a_1\varepsilon_2 e^{-\lambda t} + b_1\varepsilon_2 e^{-\lambda t_1} - (c_1 + d_1)\varepsilon_3 e^{-\lambda t_4}]\\
&:= I_4(\lambda),
\end{aligned}$$

we know that $-t_1$ is large enough if q is large enough by the definition of t_1, therefore, from $(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3 > \varepsilon_0$, it is obvious that $I_4(0) = e^{\lambda_1 t_1}(e^{-\lambda_1 c} - 1) + r_1(k_1 - \varepsilon_2)[(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3] > 0$ for large enough q, thus, there exists a $\lambda_4^* > 0$ such that $I_7(\lambda) > 0$ for $\lambda \in (0, \lambda_4^*)$.

(iii) For $t_1 + c\tau_1 \le t \le t_1 + c$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t-c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t-c) = e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}$, $\overline{\psi}(t) = e^{\lambda_3 t}$, $\overline{\psi}(t-c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, note that $e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1} = k_1 - \varepsilon_2 e^{-\lambda t_1}$, $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$ and $t_1 + c \le t_4$, we get

$$\begin{aligned}
&D_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c)\\
&\quad +r_1\underline{\phi}(t)[1 - a_1\underline{\phi}(t) - b_1\underline{\phi}(t-c\tau_1) - c_1\overline{\psi}(t) - d_1\overline{\psi}(t-c\tau_2)]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + (e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}) + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1\\
&\quad -a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_1 - \varepsilon_2 e^{-\lambda_1(t-c\tau_1)}) - c_1 e^{\lambda_3 t} - d_1 e^{\lambda_3(t-c\tau_2)}]\\
&\ge -D_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda(t_1+c)}) + (e^{\lambda_1(t_1+c\tau_1-c)} - e^{\lambda_1 t_1}) + (e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1})\\
&\quad +r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda(t_1+c-c\tau_1)}) - b_1(k_1 - \varepsilon_2 e^{-\lambda(t_1+c-c\tau_1)})\\
&\quad -c_1 e^{\lambda_3 t_4} - d_1 e^{\lambda_3 t_4}]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda(t_1+c)} + e^{\lambda_1 t_1}(e^{-\lambda_1(c-c\tau_1)} - 1) - \varepsilon_2 e^{-\lambda t_1}\\
&\quad +r_1(k_1 - \varepsilon_2 e^{-\lambda t})[(a_1 + b_1)\varepsilon_2 e^{-\lambda(t_1+c-c\tau_1)} - (c_1 + d_1)\varepsilon_3 e^{-\lambda t_4}]\\
&:= I_5(\lambda),
\end{aligned}$$

we know that $-t_1$ is large enough if q is large enough by the definition of t_1, therefore, from $(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3 > \varepsilon_0$, it is obvious that $I_5(0) = e^{\lambda_1 t_1}(e^{\lambda_1(c\tau_1-c)} - 1) + r_1(k_1 - \varepsilon_2)[(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3] > 0$ for large enough q, thus, there exists a $\lambda_5^* > 0$ such that $I_5(\lambda) > 0$ for $\lambda \in (0, \lambda_5^*)$.

(iv) For $t_1 + c \le t \le t_4$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t-c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda_1(t-c\tau_1)}$, $\underline{\phi}(t-c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t) = e^{\lambda_3 t}$, $\overline{\psi}(t-c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, in view of $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$, we get

$$\begin{aligned}
&D_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)[1 - a_1\underline{\phi}(t) - b_1\underline{\phi}(t - c\tau_1\\
&\quad -c_1\overline{\psi}(t) - d_1\overline{\psi}(t - c\tau_2)]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + (k_1 - \varepsilon_2 e^{-\lambda(t-c)}) + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1\\
&\quad -a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_1 - \varepsilon_2 e^{-\lambda_1(t-c\tau_1)}) - c_1 e^{\lambda_3 t} - d_1 e^{\lambda_3(t-c\tau_2)}]\\
&\geq -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\\
&\quad [1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}) - c_1 e^{\lambda_3 t_4} - d_1 e^{\lambda_3 t_4}]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\\
&\quad [a_1\varepsilon_2 e^{-\lambda t} + b_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - (c_1 + d_1)\varepsilon_3 e^{-\lambda t_4}]\\
&:= I_6(\lambda),
\end{aligned}$$

therefore, from $(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3 > \varepsilon_0$, it is obvious that $I_6(0) = r_1(k_1 - \varepsilon_2)[(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3] > 0$, thus, there exists a $\lambda_6^* > 0$ such that $I_6(\lambda)$ for $\lambda \in (0, \lambda_6^*)$.

(v) For $t_4 \leq t \leq t_4 + c\tau_2$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda_1(t-c\tau_1)}$, $\underline{\phi}(t-c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t) = k_2 + \varepsilon_3 e^{-\lambda t}$, $\overline{\psi}(t - c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, in view of $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$, we get

$$\begin{aligned}
&D_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)[1 - a_1\underline{\phi}(t) - b_1\underline{\phi}(t - c\tau_1)\\
&\quad -c_1\overline{\psi}(t) - d_1\overline{\psi}(t - c\tau_2)]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + (k_1 - \varepsilon_2 e^{-\lambda(t-c)}) + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1\\
&\quad -a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_1 - \varepsilon_2 e^{-\lambda_1(t-c\tau_1)}) - c_1(k_2 + \varepsilon_3 e^{-\lambda t}) - d_1 e^{\lambda_3(t-c\tau_2)}]\\
&\geq -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\\
&\quad [1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}) - c_1(k_2 + \varepsilon_3 e^{-\lambda t}) - d_1 e^{\lambda_3 t_4}]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\\
&\quad [a_1\varepsilon_2 e^{-\lambda t} + b_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - c_1\varepsilon_3 e^{-\lambda t} - d_1\varepsilon_3 e^{-\lambda t_4}]\\
&:= I_7(\lambda),
\end{aligned}$$

therefore, from $(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3 > \varepsilon_0$, it is obvious that $I_7(0) = r_1(k_1 - \varepsilon_2)[(a_1 + b_1)\varepsilon_2 - (c_1 + d_1)\varepsilon_3] > 0$, thus, there exists a $\lambda_7^* > 0$ such that $I_7(\lambda)$ for $\lambda \in (0, \lambda_7^*)$.

(vi) For $t > t_4 + c\tau_2$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t-c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t) = k_2 + \varepsilon_3 e^{-\lambda t}$, $\overline{\psi}(t - c\tau_2) = k_2 + \varepsilon_3 e^{-\lambda(t-c\tau_2)}$, we get

$$\begin{aligned}
&D_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)[1 - a_1\underline{\phi}(t) - b_1\underline{\phi}(t - c\tau_1)\\
&\quad -c_1\overline{\psi}(t) - d_1\overline{\psi}(t - c\tau_2)]\\
&= -D_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\\
&\quad [a_1\varepsilon_2 e^{-\lambda t} + b_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - c_1\varepsilon_3 e^{-\lambda t} - d_1\varepsilon_3 e^{-\lambda(t-c\tau_2)})]\\
&:= I_8(\lambda),
\end{aligned}$$

therefore, from $(a_1+b_1)\varepsilon_2-(c_1+d_1)\varepsilon_3>\varepsilon_0$, it is obvious that $I_8(0)=r_1(k_1-\varepsilon_2)[(a_1+b_1)\varepsilon_2-(c_1+d_1)\varepsilon_3]>0$, thus, there exists a $\lambda_8^*>0$ such that $I_8(\lambda)$ for $\lambda\in(0,\lambda_8^*)$.

Taking $\lambda^{***}=\min\{\lambda_4^*,\lambda_5^*,\lambda_6^*,\lambda_7^*,\lambda_8^*\}$, then $\lambda\in(0,\lambda^{***})$, we have

$$D_1\underline{\phi}''(t)-\underline{\phi}(t)+\underline{\phi}(t-c)+r_1\underline{\phi}(t)[1-a_1\underline{\phi}(t)-b_1\underline{\phi}(t-c\tau_1)-c_1\overline{\psi}(t)-d_1\overline{\psi}(t-c\tau_2)]\geq 0.$$

Similarly, the rest inequalities are satisfied.The proof is completed. □

Therefore, by Theorem 2.1, we can get the following result.

Theorem 10 *Let* $0<r_1<1$, $0<r_2<1$, τ_1,τ_3 *are small enough and suppose that* (15), (16) *are satisfied. Then* (13) *has a traveling wave solution* $(\phi(x+cn),\psi(x+cn))$ *with wave speed c which connects* $(0,0)$ *and* (k_1,k_2).

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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