Bull. Malays. Math. Sci. Soc. (2024) 47:166
https://doi.org/10.1007/s40840-024-01762-9

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Check for updates

Existence of Nodal Solutions with Arbitrary Number of Nodes for Kirchhoff Type Equations

Tao Wang[1] · Jing Lai[1,2] · Hui Guo[1,3]

Received: 11 March 2023 / Revised: 30 July 2024 / Accepted: 12 August 2024

Abstract

In this paper, we are interested in the following Kirchhoff type equation

$$\begin{cases} \left[a+\lambda\left(\int_{\mathbb{R}^3}(|\nabla u|^2+V(|x|)u^2)dx\right)^{\alpha}\right]\left(-\Delta u+V(|x|)u\right)=|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{0.1}$$

where $a, \lambda > 0, \alpha \in (0, 2)$ and $p \in (2\alpha + 2, 6)$. The potential $V(|x|)$ is radial and bounded below by a positive number. By introducing the Gersgorin Disc's theorem, we prove that for each positive integer k, Eq. (0.1) has a radial nodal solution U_k^λ with exactly k nodes. Moreover, the energy of U_k^λ is strictly increasing in k and for any

Communicated by Shangjiang Guo.

Tao Wang: Supported by National Natural Science Foundation of China (Grant No. 12001188), the Natural Science Foundation of Hunan Province (Grant No. 2022JJ30235) and Research on Teaching Reform in Ordinary Undergraduate Universities of Hunan Province (Grant Nos. 202401000915, 202401001472). Hui Guo: Supported by Scientific Research Fund of Hunan Provincial Education Department (Grant Nos. 22B0484, 22C0601) and Natural Science Foundation of Hunan Province (Grant No. 2024JJ5214).

✉ Hui Guo
huiguo_math@163.com

Tao Wang
wt_61003@163.com

Jing Lai
15163995093@163.com

1 College of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan 411201, Hunan, People's Republic of China

2 Department of Mathematics, No.1 Middle School of Linyi Shandong (West Campus), Linyi 276002, Shandong, People's Republic of China

3 Department of Mathematics and Finance, Hunan University of Humanities Science and Technology, Loudi 417000, Hunan, People's Republic of China

Published online: 03 September 2024

sequence $\{\lambda_n\}$ with $\lambda_n \to 0^+$, up to a subsequence, $U_k^{\lambda_n}$ converges to U_k^0 in $H^1(\mathbb{R}^3)$, which is also a radial nodal solution with exactly k nodes to the classical Schrödinger equation

$$\begin{cases} -a\Delta u + aV(|x|)u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$

Our results can be viewed as an extension of Kirchhoff equation concerning the existence of nodal solutions with any prescribed numbers of nodes.

Keywords Kirchhoff-type equation · Nodal solutions · Gersgorin Disc's theorem

Mathematics Subject Classification 35A15 · 35J20 · 35J50

1 Introduction

In this paper, we consider the following Kirchhoff type problem

$$\begin{cases} \left[a+\lambda\left(\int_{\mathbb{R}^3}(|\nabla u|^2+V(|x|)u^2)dx\right)^{\alpha}\right]\left(-\Delta u+V(|x|)u\right)=|u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a, \lambda > 0, \alpha \in (0, 2), p \in (2\alpha + 2, 6)$ and the potential function $V \in C([0, \infty), \mathbb{R})$ is radial and bounded below by a positive number. When $\alpha = 1$ and $V(x) \equiv b > 0$, (1.1) is reduced to the following Kirchhoff problem

$$\left[a+\lambda\left(\int_{\mathbb{R}^3}(|\nabla u|^2+bu^2)dx\right)\right]\left(-\Delta u+bu\right)=|u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

which has been studied by Li et al. [17] on the existence of positive solutions, see also [3, 6] for more details about the problem (1.2).

In the last two decades, the existence of positive solutions, multiple solutions and sign-changing solutions to the following Kirchhoff type problem on an open bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$

$$\begin{cases} -\left[a+b\int_{\Omega}|\nabla u|^2\right]\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has been extensively investigated by making use of the variational method. One can refer to [4, 5, 12–15, 20–22, 25–27, 33, 34] and references therein. For the Kirchhoff type problem in the whole space $\mathbb{R}^N$, Li and Ye [19] considered

$$\begin{cases} -\left[a + b\int_{\mathbb{R}^3} |\nabla u|^2\right]\Delta u + V(|x|)u = f(x,u), \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, \end{cases} \tag{1.4}$$

where $f(x,u) = u^{p-2}u$ with $p \in (3,6)$. Under certain assumptions on the potential $V(x)$, they proved that (1.4) has a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma. For related problems like (1.4), we refer to [1, 7, 9, 16, 27, 29, 32] and references cited therein.

Recently, the existence of sign-changing solutions to the Kirchhoff type problem in $\mathbb{R}^N$ has attracted much attention. Deng et al. [8] and Guo et al. [10] obtained the existence and asymptotic behaviors of nodal solutions with a prescribed number of nodes for problem (1.4) under some suitable assumptions on the nonlinearity $f(x,u)$. Corresponding to the classical pure power nonlinearity model $f(x,u) = |u|^{p-2}u$, their main results in [8, 10] solve the following equation

$$\begin{cases} -\left[a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right]\Delta u + V(|x|)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3 \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.5}$$

for the case $p \in (4,6)$, see [18, 24, 30, 31] for more related results. However, the presence of nonlocal term $\lambda\left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)u^2)dx\right)^\alpha$ in (1.1) with $\alpha \in (0,2)$ makes this problem more complicated. Then a natural question arises: can one find nodal solutions with any prescribed number of nodes for problem (1.1)? In this paper, we shall answer this question. To the best of our knowledge, this problem still remains unsolved.

In order to illustrate our results clearly, we need the following notations. Throughout this paper, we set the radial Sobolev space $H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}$ and let the action space

$$H_V := \left\{u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)u^2)dx < +\infty\right\}$$

be endowed with norm $\|u\| = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)u^2)dx\right)^{1/2}$. As usual, the energy functional $I_\lambda : H_V \to \mathbb{R}$ associated with (1.1) is defined by

$$I_\lambda(u) := \frac{a}{2}\|u\|^2 + \frac{\lambda}{2\alpha+2}\|u\|^{2\alpha+2} - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p. \tag{1.6}$$

Obviously, $I_\lambda \in C^2(H_V, \mathbb{R})$ and

$$\langle I_\lambda'(u), u\rangle = a\|u\|^2 + \lambda\|u\|^{2\alpha+2} - \int_{\mathbb{R}^3}|u|^p.$$

Then we define the usual Nehari manifold

$$\mathcal{N} = \left\{u \in H_V \backslash \{0\} : \langle I_\lambda'(u), u\rangle = 0\right\}, \tag{1.7}$$

and the ground state energy

$$m := \inf_{\mathcal{N}} I_\lambda(u). \tag{1.8}$$

By [6, Theorem 1.1], there exists a ground state solution $U_0 \in \mathcal{N}$ of (1.1) such that

$$m = I_\lambda(U_0) > 0. \tag{1.9}$$

For $k \in \mathbb{N}^*$ and $0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := +\infty$, we denote by $\mathbf{r}_k = (r_1, \ldots, r_k)$ and

$$B_1^{\mathbf{r}_k} := \left\{x \in \mathbb{R}^3 : 0 \le |x| < r_1\right\},$$
$$B_i^{\mathbf{r}_k} := \left\{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_i\right\},\ i = 2, \ldots, k+1.$$

Obviously, $B_1^{\mathbf{r}_k}$ is a ball, $B_2^{\mathbf{r}_k}, \ldots, B_k^{\mathbf{r}_k}$ are annulus and $B_{k+1}^{\mathbf{r}_k}$ is the complement of a ball. Then we define the Nehari type set

$$\mathcal{N}_k = \left\{u \in H_V : \text{there exists } \mathbf{r}_k \text{ s.t. } u_i \neq 0 \text{ in } B_i^{\mathbf{r}_k}, \langle I_\lambda'(u), u_i\rangle = 0,\ i = 1, \ldots, k+1\right\}, \tag{1.10}$$

and the infimum level

$$c_k = \inf_{u \in \mathcal{N}_k} I_\lambda(u), \tag{1.11}$$

where $u_i = u$ in $B_i^{\mathbf{r}_k}$ and $u_i = 0$ on $\partial B_i^{\mathbf{r}_k}$.

Our existence result is as follows.

Theorem 1.1 *For each $k \in \mathbb{N}^*$, problem (1.1) admits a radial nodal solution $U_k \in \mathcal{N}_k$ which changes sign exactly k-times and $I_\lambda(U_k) = c_k$.*

The next result shows that the energy of U_k obtained in Theorem 1.1 increases with the number of nodes.

Theorem 1.2 *Under the hypotheses of Theorem 1.1, the energy of U_k is strictly increasing in k. Namely,*

$$I_\lambda(U_{k+1}) > I_\lambda(U_k) \quad \textit{for all } k \in \mathbb{N}^*.$$

Moreover, $I_\lambda(U_{k+1}) > (k+1)I_\lambda(U_0)$.

Since U_k obtained in Theorem 1.1 depends on λ, we denote U_k by U_k^λ to emphasize this dependence. The last result shows the asymptotic behavior of U_k^λ as $\lambda \to 0^+$.

Theorem 1.3 *Under the assumptions of Theorem 1.1, for any sequence $\{\lambda_n\}$ with $\lambda_n \to 0^+$ as $n \to \infty$, up to a subsequence, $U_k^{\lambda_n}$ converges to U_k^0 strongly in H_V as $n \to \infty$, where U_k^0 is a least energy radial nodal solution among all the nodal solutions having exactly k nodes to the following equation*

$$-a\Delta u + aV(|x|)u = |u|^{p-2}u. \tag{1.12}$$

This paper is organized as follows. In Sect. 2, we give the variational framework of problem (1.1) and some preliminary lemmas. Section 3 is devoted to the proof of the existence of nodal solutions with a prescribed number of nodes. In Sect. 4, we study the energy comparison and asymptotic behaviors of those nodal solutions of (1.1).

2 Preliminaries

In this section, we give some notations and recall some useful lemmas. For each $k \in \mathbb{N}^*$, we define

$$\Gamma_k = \left\{ \mathbf{r}_k = (r_1, \ldots, r_k) \in (0,\infty)^k \;\; 0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := \infty \right\}.$$

For a fixed $\mathbf{r}_k \in \Gamma_k$ and thereby a family of annulus $\{B_i^{\mathbf{r}_k}\}_{i=1}^{k+1}$, we define a Hilbert space

$$H_i^{\mathbf{r}_k} := \left\{ u \in H_0^1(B_i^{\mathbf{r}_k}) : \; u(x) = u(|x|), u(x) = 0 \text{ for } x \in \partial B_i^{\mathbf{r}_k} \right\}$$

endowed with the norm $\|u\|_i = \left(\int_{B_i^{\mathbf{r}_k}} (|\nabla u|^2 + V(|x|)u^2) dx \right)^{1/2}$. Now, let the product space be

$$\mathcal{H}_k^{\mathbf{r}_k} = H_1^{\mathbf{r}_k} \times \cdots \times H_{k+1}^{\mathbf{r}_k},$$

and we introduce an energy functional $E_\lambda : \mathcal{H}_k^{\mathbf{r}_k} \to \mathbb{R}$ defined by

$$E_\lambda(u_1, \ldots, u_{k+1}) := \frac{a}{2} \sum_{i=1}^{k+1} \|u_i\|_i^2 + \frac{\lambda}{2\alpha+2} \left(\sum_{i=1}^{k+1} \|u_i\|_i^2 \right)^{\alpha+1} - \frac{1}{p} \sum_{i=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} |u_i|^p. \tag{2.1}$$

It is obvious that

$$E_\lambda(u_1, \ldots, u_{k+1}) = I_\lambda \left(\sum_{i=1}^{k+1} u_i \right). \tag{2.2}$$

If $(u_1, \ldots, u_{k+1})$ is a critical point E_λ, then each component u_i satisfies

$$\begin{cases} \left[a + \lambda\Big(\sum_{j=1}^{k+1} \|u_j\|\Big)^{2\alpha}\right]\Big(- \Delta u_i + V(|x|)u_i\Big) = |u_i|^{p-2}u_i & x \in B_i^{\mathbf{r}_k}, \\ u_i = 0 & x \notin B_i^{\mathbf{r}_k}. \end{cases} \tag{2.3}$$

Note that

$$\langle E'_\lambda(u_1, \ldots, u_{k+1}), u_i\rangle = a\|u_i\|_i^2 + \lambda\|u_i\|_i^2\Big(\sum_{j=1}^{k+1} \|u_j\|_j^2\Big)^\alpha - \int_{B_i^{\mathbf{r}_k}} |u_i|^p.$$

For each $\mathbf{r}_k \in \Gamma_k$, we define another Nehari type set

$$\mathcal{M}_k^{\mathbf{r}_k} := \Big\{(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}\ u_i \neq 0, \langle E'_\lambda(u_1, \ldots, u_{k+1}), u_i\rangle = 0,\ i = 1, \ldots, k+1\Big\}. \tag{2.4}$$

In the following, we shall prove the non-empty of $\mathcal{M}_k^{\mathbf{r}_k}$ by introducing two important lemmas. The first lemma is a corollary of the Gersgorin Disc's Theorem [28].

Lemma 2.1 [11, Lemma 2.3] *For any $a_{ij} = a_{ji} > 0$ with $i \neq j$ and $s_i > 0$ with $i = 1, \ldots, m$, if the matrix $B := (b_{ij})_{m\times m}$ is defined by*

$$b_{ij} = \begin{cases} -\sum_{l\neq i} \frac{s_l a_{il}}{s_i} & i = j, \\ a_{ij} > 0 & i \neq j, \end{cases}$$

then $(b_{ij})_{m\times m}$ is a negative semi-definite symmetric matrix.

Lemma 2.2 [30, Lemma 2.3] *If $f \in C^2(\mathbb{R}^m, \mathbb{R})$ is a strictly concave function and has a critical point $\bar{\mathbf{s}} := (\bar{s}_1, \ldots, \bar{s}_m)$ in $\mathbb{R}^m$, then $\bar{\mathbf{s}}$ is the unique critical point of f in $\mathbb{R}^m$.*

Now we are ready to prove the non-empty of the set $\mathcal{M}_k^{\mathbf{r}_k}$.

Lemma 2.3 *For each $(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ with $u_i \neq 0$ for $i = 1, \ldots, k+1$, there exists a unique $(k+1)$ tuple $(t_1, \ldots, t_{k+1})$ of positive numbers such that $(t_1u_1, \ldots, t_{k+1}u_{k+1}) \in \mathcal{M}_k^{\mathbf{r}_k}$.*

Proof For fixed $(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ with $u_i \neq 0$, $(t_1u_1, \ldots, t_{k+1}u_{k+1})$ belongs to $\mathcal{M}_k^{\mathbf{r}_k}$ if and only if

$$at_i^2\|u_i\|_i^2 + \lambda t_i^2\|u_i\|_i^2\left(\sum_{j=1}^{k+1} t_j^2\|u_j\|_j^2\right)^\alpha - t_i^p\int_{B_i^{\mathbf{r}_k}} |u_i|^p = 0 \tag{2.5}$$

for each $i = 1, \dots, k+1$. Hence, it suffices to verify that there is a unique $(k+1)$ tuple $(t_1, \dots, t_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$ satisfying (2.5).

Define a new function $g : (\mathbb{R}_{>0})^{k+1} \to \mathbb{R}$ by

$$
\begin{aligned}
g(s_1, \dots, s_{k+1}) :&= E\left(s_1^{\frac{1}{p}} u_1, \dots, s_{k+1}^{\frac{1}{p}} u_{k+1}\right) \\
&= \frac{a}{2}\sum_{i=1}^{k+1} s_i^{\frac{2}{p}} \|u_i\|_i^2 + \frac{\lambda}{2(\alpha+1)}\left(\sum_{i=1}^{k+1} s_i^{\frac{2}{p}} \|u_i\|_i^2\right)^{\alpha+1} \\
&\quad - \frac{1}{p}\sum_{i=1}^{k+1} s_i \int_{B_i^{\mathbf{r}_k}} |u_i|^p.
\end{aligned} \tag{2.6}
$$

According to (2.6), we see that $g(s_1, \dots, s_{k+1}) \to -\infty$ uniformly as $|(s_1, \dots, s_{k+1})| \to \infty$, and $g(s_1, \dots, s_{k+1}) \to 0$ uniformly as $|(s_1, \dots, s_{k+1})| \to 0$.

Some direct computations show that the partial derivatives of g satisfy

$$
\begin{aligned}
g'_{s_i}(s_1, \dots, s_{k+1}) &= \frac{a}{p} s_i^{\frac{2-p}{p}} \|u_i\|_i^2 + \frac{\lambda}{p} s_i^{\frac{2-p}{p}} \left(\sum_{j=1}^{k+1} s_j^{\frac{2}{p}} \|u_j\|_j^2\right)^{\alpha} \|u_i\|_i^2 - \frac{1}{p}\int_{B_i^{\mathbf{r}_k}} |u_i|^p, \\
g''_{s_i s_i}(s_1, \dots, s_{k+1}) &= \frac{a(2-p)}{p} s_i^{\frac{2-2p}{p}} \|u_i\|_i^2 + s_i^{\frac{2-2p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right)^{\alpha-1} \\
&\quad \left[\left(\frac{2\lambda(1+\alpha)}{p^2} - \frac{\lambda}{p}\right)\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right] \\
&\quad - \frac{2\lambda\alpha}{p^2} s_i^{\frac{2-2p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right)^{\alpha-1} \left(\sum_{j\neq i}^{k+1} s_j^{\frac{2}{p}} \|u_j\|_j^2\right), \\
g''_{s_i s_j}(s_1, \dots, s_{k+1}) &= \frac{2\lambda\alpha}{p^2} s_i^{\frac{2-p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right)^{\alpha-1} s_j^{\frac{2-p}{p}} \|u_j\|_j^2.
\end{aligned} \tag{2.7}
$$

Let

$$
\begin{aligned}
A_{ii} &= \frac{a(2-p)}{p} s_i^{\frac{2-2p}{p}} \|u_i\|_i^2 + s_i^{\frac{2-2p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right)^{\alpha-1} \\
&\quad \left[\left(\frac{2\lambda(1+\alpha)}{p^2} - \frac{\lambda}{p}\right)\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right], \\
B_{ii} &= -\frac{2\lambda\alpha}{p^2} s_i^{\frac{2-2p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2\right)^{\alpha-1} \left(\sum_{j\neq i}^{k+1} s_j^{\frac{2}{p}} \|u_j\|_j^2\right)
\end{aligned}
$$

$$= -\sum_{j\neq i}^{k+1} \frac{s_j}{s_i} \left(\frac{2\lambda\alpha}{p^2} s_i^{\frac{2-p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2 \right)^{\alpha-1} s_j^{\frac{2-p}{p}} \|u_j\|_j^2 \right),$$

$$A_{ij} = 0 \ and \ B_{ij} = \frac{2\lambda\alpha}{p^2} s_i^{\frac{2-p}{p}} \|u_i\|_i^2 \left(\sum_{l=1}^{k+1} s_l^{\frac{2}{p}} \|u_l\|_l^2 \right)^{\alpha-1} s_j^{\frac{2-p}{p}} \|u_j\|_j^2 \ while \ i \neq j.$$

Then the matrix

$$(g''_{s_i s_j}(s_1, \ldots, s_{k+1}))_{(k+1)\times(k+1)} = (A_{ij})_{(k+1)\times(k+1)} + (B_{ij})_{(k+1)\times(k+1)}.$$

Moreover, it follows from Lemma 2.1 that the matrix $(g''_{s_i s_j}(s_1, \ldots, s_{k+1}))_{(k+1)\times(k+1)}$ is negative definite at each point $(s_1, \ldots, s_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$. So g is a strictly concave function in $(\mathbb{R}_{>0})^{k+1}$. By Lemma 2.2, we deduce that g has a unique critical point $(\bar{s}_1, \ldots, \bar{s}_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$. Letting $\bar{s}_i = t_i^p$, we conclude from (2.5) and (2.7) that

$$\langle E'_\lambda(t_1u_1, \ldots, t_{k+1}u_{k+1}), t_iu_i \rangle = pt_i^p g_{s_i}(t_1^p, \ldots, t_{k+1}^p) = 0.$$

The proof is finished. □

We define $\phi : (\mathbb{R}_{\geq 0})^{k+1} \to \mathbb{R}$ by $\phi(c_1, \ldots, c_{k+1}) = E_\lambda(c_1u_1, \ldots, c_{k+1}u_{k+1})$, where $(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$. Then we get the following corollary.

Corollary 2.4 *For fixed* $(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ *with* $u_i \neq 0$ *for* $i = 1, \ldots, k+1$, ϕ *has a unique maximum point* $(t_1, \ldots, t_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$. *Moreover,* $\frac{\partial \phi}{\partial c_i}(t_1, \ldots, t_{i-1}, c_i, t_{i+1}, t_{k+1}) > 0$ *if* $c_i < t_i$ *and* $\frac{\partial \phi}{\partial c_i}(t_1, \ldots, t_{i-1}, c_i, t_{i+1}, \ldots, t_{k+1}) < 0$ *if* $c_i > t_i$.

Proof We see that

$$\begin{aligned} \phi(c_1, \ldots, c_{k+1}) :&= E_\lambda(c_1u_1, \ldots, c_{k+1}u_{k+1}) \\ &= \frac{a}{2}\sum_{i=1}^{k+1} \|c_iu_i\|_i^2 + \frac{\lambda}{2(\alpha+1)} \left(\sum_{i=1}^{k+1} \|c_iu_i\|_i^2 \right)^{\alpha+1} \\ &\quad - \frac{1}{p}\sum_{i=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} |c_iu_i|^p. \end{aligned} \tag{2.8}$$

Obviously, ϕ is continuous. From the proof of Lemma 2.3, $(t_1, \ldots, t_{k+1})$ is the unique critical point of ϕ in $(\mathbb{R}_{>0})^{k+1}$. Due to the fact that $p \in (2+2\alpha, 6)$, we have $\phi(c_1, \ldots, c_{k+1}) \to -\infty$ as $|(c_1, \ldots, c_{k+1})| \to \infty$ and $\phi(c_1, \ldots, c_{k+1}) \to 0$ as $|(c_1, \ldots, c_{k+1})| \to 0$. This implies that ϕ admits a unique maximum point $(t_1, \ldots, t_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$. Then we obtain that for each i,

$$
\begin{aligned}
&\frac{\partial \phi}{\partial c_i}(t_1, \ldots, c_i, \ldots, t_{k+1}) \\
&\quad = a c_i \|u_i\|_i^2 + \lambda \left(\sum_{j \neq i}^{k+1} t_j^2 \|u_j\|_j^2 + c_i^2 \|u_i\|_i^2 \right)^{\alpha} c_i \|u_i\|_i^2 - c_i^{p-1} \int_{B_i^{\mathbf{r}_k}} |u_i|^p \\
&\quad = c_i^{p-1} \left[a c_i^{2-p} \|u_i\|_i^2 + \lambda \left(\sum_{j \neq i}^{k+1} c_i^{\frac{2-p}{\alpha}} t_j^2 \|u_j\|_j^2 + c_i^{2+\frac{2-p}{\alpha}} \|u_i\|_i^2 \right)^{\alpha} \|u_i\|_i^2 \right] \\
&\qquad - c_i^{p-1} \int_{B_i^{\mathbf{r}_k}} |u_i|^p,
\end{aligned}
$$

which implies that $\frac{\partial \phi}{\partial c_i}(t_1, \ldots, c_i, \ldots, t_{k+1}) > 0$ if $c_i < t_i$ and $\frac{\partial \phi}{\partial c_i}(t_1, \ldots, c_i, \ldots, t_{k+1}) < 0$ if $c_i > t_i$. □

We define $\mathbf{F} = (F_1, \ldots, F_{k+1}) : \mathcal{H}_k^{\mathbf{r}_k} \to \mathbb{R}^{k+1}$ by

$$
F_i(u_1, \ldots, u_{k+1}) := \langle \partial_{u_i} E'_\lambda(u_1, \ldots, u_{k+1}), u_i \rangle \tag{2.9}
$$

for $i = 1, \ldots, k+1$. Then we have the following lemma.

Lemma 2.5 *For any $(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ with nonzero components such that $F_i(u_1, \ldots, u_{k+1}) < 0$ for each $i = 1, \ldots, k+1$, the $(k+1)$ tuple $(t_1, \ldots, t_{k+1})$ of positive numbers obtained in Lemma 2.3 satisfies $t_i \leq 1$ for each i.*

Proof By Lemma 2.3, $(t_1 u_1, \ldots, t_{k+1} u_{k+1}) \in \mathcal{M}_k^{\mathbf{r}_k}$, then for each $i = 1, \ldots, k+1$,

$$
a t_i^2 \|u_i\|_i^2 + \lambda t_i^2 \|u_i\|_i^2 \left(\sum_{j=1}^{k+1} t_j^2 \|u_j\|_j^2 \right)^{\alpha} = t_i^p \int_{B_i^{\mathbf{r}_k}} |u_i|^p. \tag{2.10}
$$

Without loss of generality, we assume that $t_{i_0} = \max\{t_1, \ldots, t_{k+1}\}$. Then

$$
a t_{i_0}^2 \|u_{i_0}\|_{i_0}^2 + \lambda t_{i_0}^{2+2\alpha} \|u_{i_0}\|_{i_0}^2 \left(\sum_{j=1}^{k+1} \|u_j\|_j^2 \right)^{\alpha} > t_{i_0}^p \int_{B_{i_0}^{\mathbf{r}_k}} |u_{i_0}|^p. \tag{2.11}
$$

Since $F_i(u_1, \ldots, u_{k+1}) < 0$, we have

$$
a \|u_{i_0}\|_{i_0}^2 + \lambda \|u_{i_0}\|_{i_0}^2 \left(\sum_{j=1}^{k+1} \|u_j\|_j^2 \right)^{\alpha} < \int_{B_{i_0}^{\mathbf{r}_k}} |u_{i_0}|^p. \tag{2.12}
$$

By combining (2.11) and (2.12), we obtain

$$
\left(\frac{a}{t_{i_0}^{2\alpha}} - a \right) \|u_{i_0}\|_{i_0}^2 \geq (t_{i_0}^{p-2\alpha-2} - 1) \int_{B_i^{\mathbf{r}_k}} |u_{i_0}|^p.
$$

If $t_{i_0} > 1$, the left side of this inequality is negative, but the right side is positive, which leads to a contradiction. Hence, we have $t_i \leq 1$ for each i. The proof is completed. □

Notice that

$$\mathcal{M}_k^{\mathbf{r}_k} := \left\{ (u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k} \;\; u_i \neq 0 \mid \mathbf{F}(u_1, \ldots, u_{k+1}) = 0 \right\},$$

where $\mathbf{F}(u_1, \ldots, u_{k+1})$ is defined in (2.9). Hereafter, we say that $\mathcal{M}_k^{\mathbf{r}_k}$ is a differentiable manifold in $\mathcal{H}_k^{\mathbf{r}_k}$, means that the matrix

$$N := (N_{ij})_{(k+1)\times(k+1)} = \langle \partial_{u_i} F_j'(u_1, \ldots, u_{k+1}), u_i \rangle, \quad i, j = 1, \ldots, k+1$$

is nonsingular at each point $(u_1, \ldots, u_{k+1}) \in \mathcal{M}_k^{\mathbf{r}_k}$.

Lemma 2.6 *$\mathcal{M}_k^{\mathbf{r}_k}$ is a differentiable manifold in $\mathcal{H}_k^{\mathbf{r}_k}$. Moreover, a minimizer $(u_1, \ldots, u_{k+1})$ of E_λ on $\mathcal{M}_k^{\mathbf{r}_k}$ is a critical point of E_λ in $\mathcal{H}_k^{\mathbf{r}_k}$ with nonzero components.*

Proof By some calculations, we have

$$\begin{aligned} N_{ii} &= 2a\|u_i\|_i^2 + 2\lambda\|u_i\|_i^2 \left(\sum_{l=1}^{k+1} \|u_l\|_l^2 \right)^{\alpha} \\ &\quad + 2\lambda\alpha\|u_i\|_i^2 \left(\sum_{l=1}^{k+1} \|u_l\|_l^2 \right)^{\alpha-1} \|u_i\|_i^2 - p\int_{B_i^{\mathbf{r}_k}} |u_i|^p, \\ N_{ij} &= 2\lambda\alpha\|u_i\|_i^2 \left(\sum_{l=1}^{k+1} \|u_l\|_l^2 \right)^{\alpha-1} \|u_j\|_j^2, \quad \text{for} \quad j \neq i, \quad i, j = 1, \ldots, k+1. \end{aligned}$$

Due to the fact that $p \in (2+2\alpha, 6)$, we obtain

$$\begin{aligned} N_{ii} + \sum_{j\neq i}^{k+1} N_{ij} &= 2a\|u_i\|_i^2 + 2\lambda\|u_i\|_i^2 \left(\sum_{l=j}^{k+1} \|u_j\|_j^2 \right)^{\alpha} \\ &\quad + 2\lambda\alpha\|u_i\|_i^2 \left(\sum_{j=1}^{k+1} \|u_j\|_j^2 \right)^{\alpha} - p\int_{B_i^{\mathbf{r}_k}} |u_i|^p \\ &= 2a\|u_i\|_i^2 + (2\alpha+2)\left(\int_{B_i^{\mathbf{r}_k}} |u_i|^p - a\|u_i\|_i^2 \right) - p\int_{B_i^{\mathbf{r}_k}} |u_i|^p \\ &= -2\alpha a\|u_i\|_i^2 + (2+2\alpha-p)\int_{B_i^{\mathbf{r}_k}} |u_i|^p < 0. \end{aligned}$$

So

$$N_{ii} < -\sum_{j\neq i}^{k+1} N_{ij} < 0 \Rightarrow |N_{ii}| > \sum_{j\neq i}^{k+1} |N_{ij}|.$$

Then the matrix $N = (N_{ij})$ is diagonally dominant, and thereby it is nonsingular and $\det N \neq 0$.

If $(u_1, \ldots, u_{k+1})$ is a minimizer of $E_\lambda|_{\mathcal{M}_k^{\mathbf{r}_k}}$, then there is a Lagrangian multiplier $(\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{R}^{k+1}$ such that

$$\lambda_1 F_1'(u_1, \ldots, u_{k+1}) + \cdots + \lambda_{k+1} F_{k+1}'(u_1, \ldots, u_{k+1}) = E_\lambda'(u_1, \ldots, u_{k+1}) \tag{2.13}$$

Applying $(u_1, 0, \ldots, 0), (0, u_2, \ldots, 0), \ldots, (0, \ldots, 0, u_{k+1})$ to the identity (2.13), we get

$$N_{ij} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, $\lambda_1, \ldots, \lambda_{k+1}$ are all zeros and $(u_1, \ldots, u_{k+1})$ is a critical point of E_λ in $\mathcal{H}_k^{\mathbf{r}_k}$.

Finally, for any $(u_1, \ldots, u_{k+1}) \in \mathcal{M}_k^{\mathbf{r}_k}$, we have

$$a\|u_i\|_i^2 \leq \int_{B_i^{\mathbf{r}_k}} |u_i|^p \leq C\|u_i\|_i^p \ \textit{and} \ 0 < \delta := (\frac{a}{C})^{\frac{1}{p-2}} \leq \|u_i\|_i. \tag{2.14}$$

Then each u_i is bounded away from zero. Thus minimizers of E_λ in $\mathcal{M}_k^{\mathbf{r}_k}$ cannot have any zero components. The proof is completed. □

Lemma 2.7 *For fixed* $\mathbf{r}_k = (r_1, \ldots, r_{k+1}) \in \Gamma_k$*, there exists a minimizer* $(w_1, \ldots, w_{k+1})$ *of* $E_\lambda|_{\mathcal{M}_k^{\mathbf{r}_k}}$ *such that each* $(-1)^{i+1} w_i$ *is positive on* $B_i^{\mathbf{r}_k}$ *for* $i = 1, \ldots, k+1$*. Moreover,* $(w_1, \ldots, w_{k+1})$ *satisfies* (2.3).

Proof For $(u_1, \ldots, u_{k+1}) \in \mathcal{M}_k^{\mathbf{r}_k}$, it holds that

$$\begin{aligned} E_\lambda(u_1, \ldots, u_{k+1}) &= \left(\frac{a}{2} - \frac{a}{2\alpha+2}\right) \sum_{i=1}^{k+1} \|u_i\|_i^2 + \left(\frac{1}{2\alpha+2} - \frac{1}{p}\right) \sum_{i=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} |u_i|^p \\ &\geq \left(\frac{a}{2} - \frac{a}{2\alpha+2}\right) \sum_{i=1}^{k+1} \|u_i\|_i^2 > \delta, \end{aligned} \tag{2.15}$$

where δ is defined in (2.14). Then there exists a minimizing sequence $\{(u_1^n, \ldots, u_{k+1}^n)\}_{n=1}^\infty \subset \mathcal{M}_k^{\mathbf{r}_k}$ such that $E_\lambda(u_1^n, \ldots, u_{k+1}^n) \to \min\limits_{\mathcal{M}_k^{\mathbf{r}_k}} E_\lambda$ as $n \to \infty$.

By combining with (1.9), we know that

$$
\begin{aligned}
m+1 > & E_\lambda(u_1^n,\ldots,u_{k+1}^n) \\
& = \left(\frac{a}{2}-\frac{a}{2\alpha+2}\right)\sum_{i=1}^{k+1}\|u_i^n\|_i^2 + \left(\frac{1}{2\alpha+2}-\frac{1}{p}\right)\sum_{i=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}|u_i^n|^p \\
& \ge \left(\frac{a}{2}-\frac{a}{2\alpha+2}\right)\sum_{i=1}^{k+1}\|u_i^n\|_i^2.
\end{aligned}
\tag{2.16}
$$

Hence, $\{u_i^n\}_{n\ge1}$ is bounded in $H_i^{\mathbf{r}_k}$ for each $i=1,\ldots,k+1$. Up to a subsequence, there exists $(u_1^0,\ldots,u_{k+1}^0)\in\mathcal{H}_k^{\mathbf{r}_k}$ such that $u_i^n\rightharpoonup u_i^0$ in $H_i^{\mathbf{r}_k}$ and $u_i^n\to u_i^0$ in $L^p(B_i^{\mathbf{r}_k})$ with $p\in(2,6)$. Since $(u_1^n,\ldots,u_{k+1}^n)\subset\mathcal{M}_k^{\mathbf{r}_k}$, we have

$$
\begin{aligned}
0<\delta \le a\liminf_{n\to\infty}\|u_i^n\|_i^2 &< \liminf_{n\to\infty}\int_{B_i^{\mathbf{r}_k}}|u_i^n|^p. \\
&= \|u_i^0\|_i^p.
\end{aligned}
\tag{2.17}
$$

This implies that $u_i^0\ne0$ for each $i=1,\ldots,k+1$.

Now we claim that up to a subsequence, u_i^n converges to u_i^0 strongly in $H_i^{\mathbf{r}_k}$. Notice that $u_i^n\rightharpoonup u_i^0$ weakly in $H_i^{\mathbf{r}_k}$. We may suppose on the contrary that $\|u_i^0\|_i < \liminf_{n\to\infty}\|u_i^n\|_i$ for at least one $i\in\{1,\ldots,k+1\}$. Since each component of $(u_1^0,\ldots,u_{k+1}^0)$ is nonzero, by Lemma 2.3, one can find $(t_1^0,\ldots,t_{k+1}^0)\in(\mathbb{R}_{>0})^{k+1}$ such that $(t_1^0u_1^0,\ldots,t_{k+1}^0u_{k+1}^0)\in\mathcal{M}_k^{\mathbf{r}_k}$. However, in this situation, Corollary 2.4 implies that

$$
\begin{aligned}
&\inf_{(u_1,\ldots,u_{k+1})\in\mathcal{M}_k^{\mathbf{r}_k}} E_\lambda(u_1,\ldots,u_{k+1}) \\
&\le E_\lambda(t_1^0u_1^0,\ldots,t_{k+1}^0u_{k+1}^0) \\
&= \left(\frac{a}{2}-\frac{a}{2\alpha+2}\right)\sum_{i=1}^{k+1}\left((t_i^0)^2\|u_i^0\|_i^2\right) \\
&\quad -\left(\frac{1}{2\alpha+2}-\frac{1}{p}\right)\sum_{i=1}^{k+1}(t_i^0)^p\liminf_{n\to\infty}\int_{B_i^{\mathbf{r}_k}}|u_i^n|^p \\
&< \left(\frac{a}{2}-\frac{a}{2\alpha+2}\right)\sum_{i=1}^{k+1}\left((t_i^0)^2\liminf_{n\to\infty}\|u_i^n\|_i^2\right) \\
&\quad -\left(\frac{1}{2\alpha+2}-\frac{1}{p}\right)\sum_{i=1}^{k+1}(t_i^0)^p\liminf_{n\to\infty}\int_{B_i^{\mathbf{r}_k}}|u_i^n|^p \\
&\le \liminf_{n\to\infty}E_\lambda(u_1^n,\ldots,u_{k+1}^n)
\end{aligned}
$$

$$= \inf_{(u_1,\ldots,u_{k+1})\in\mathcal{M}_k^{\mathbf{r}_k}} E_\lambda(u_1,\ldots,u_{k+1}).$$

This is a contradiction. Thus the claim holds, and going if necessary to a subsequence, $(u_1^n,\ldots,u_{k+1}^n) \to (u_1^0,\ldots,u_{k+1}^0)$ in $\mathcal{H}_k^{\mathbf{r}_k}$.

Therefore, $(u_1^0,\ldots,u_{k+1}^0)$ is contained in $\mathcal{M}_k^{\mathbf{r}_k}$ and is a minimizer of $E_\lambda|_{\mathcal{M}_k^{\mathbf{r}_k}}$. Obviously,

$$(w_1,\ldots,w_{k+1}) := (|u_1^0|, -|u_2^0|,\ldots,(-1)^{k+2}|u_{k+1}^0|)$$

is also a minimizer of $E_\lambda|_{\mathcal{M}_k^{\mathbf{r}_k}}$. Hence it is a critical point of E_λ by Lemma 2.6 and satisfies (2.3). Then by the standard elliptic regularity theory, all $w_i \in C^2(B_i^{\mathbf{r}_k})$. Furthermore, since $(-1)^{i+1}w_i \geq 0$, by applying the strong maximum principle to (2.3), it follows immediately $(-1)^{i+1}w_i > 0$. The proof is completed. □

3 Existence of Nodal Solutions

In this section, we are devoted to the proof of Theorem 1.1. In view of Lemma 2.7, we can define a function $\Psi : \Gamma_k \to \mathbb{R}$ by

$$\begin{aligned}\Psi(\mathbf{r}_k) = \Psi(r_1,\ldots,r_{k+1}) &= E_\lambda(w_1^{\mathbf{r}_k},\ldots,w_{k+1}^{\mathbf{r}_k})\\ &= \inf_{(u_1^{\mathbf{r}_k},\ldots,u_{k+1}^{\mathbf{r}_k})\in\mathcal{M}_k^{\mathbf{r}_k}} E_\lambda(u_1^{\mathbf{r}_k},\ldots,u_{k+1}^{\mathbf{r}_k}).\end{aligned} \tag{3.1}$$

Then we shall give the following lemma which shows some properties of $\Psi(\mathbf{r}_k)$.

Lemma 3.1 *For any positive integer k, let $\boldsymbol{r}_k = (r_1,\ldots,r_k) \in \Gamma_k$. Then the following statements are true.*

(i) *If $r_i - r_{i-1} \to 0$ for some $i \in \{1,\ldots,k\}$, then $\Psi(\boldsymbol{r}_k) \to +\infty$.*
(ii) *If $r_k \to \infty$, then $\Psi(\boldsymbol{r}_k) \to +\infty$.*
(iii) *Ψ is continuous in Γ_k. Moreover, there exists a minimum point $\tilde{\boldsymbol{r}}_k \in \Gamma_k$ such that $\Psi(\tilde{\boldsymbol{r}}_k) = \min_{\boldsymbol{r}_k\in\Gamma_k} \Psi(\boldsymbol{r}_k)$.*

Proof (i) Assume that $r_{i_0} - r_{i_0-1} \to 0$ for some $i_0 \in \{1,\ldots,k+1\}$. Since $(w_1^{\mathbf{r}_k},\ldots,w_{k+1}^{\mathbf{r}_k}) \in M_k^{\mathbf{r}_k}$, by using the Hölder inequality and Sobolev inequality, we obtain that

$$\|w_{i_0}^{\mathbf{r}_k}\|_{i_0}^2 \leq \int_{B_i^{\mathbf{r}_k}} |w_{i_0}^{\mathbf{r}_k}|^p \leq \left(\int_{B_{i_0}^{\mathbf{r}_k}} |w_{i_0}^{\mathbf{r}_k}|^6\right)^{\frac{p}{6}} |B_{i_0}^{\mathbf{r}_k}|^{1-\frac{p}{6}} \leq C\|w_{i_0}^{\mathbf{r}_k}\|_{i_0}^p |B_{i_0}^{\mathbf{r}_k}|^{1-\frac{p}{6}}, \tag{3.2}$$

where $C > 0$ is a positive constant. Note that $2\alpha + 2 < p < 6$. Then $\|w_{i_0}^{\mathbf{r}_k}\|_{i_0} \to \infty$. We see that

$$
\begin{aligned}
&E_\lambda(w_1^{\mathbf{r}_k}, \ldots, w_{k+1}^{\mathbf{r}_k})\\
&= \sum_{i=1}^{k+1} \left[\left(\frac{a}{2} - \frac{a}{p}\right) \|w_i^{\mathbf{r}_k}\|_i^2 + \left(\frac{\lambda}{2\alpha+2} - \frac{\lambda}{p}\right) \|w_i^{\mathbf{r}_k}\|_i^2 \left(\sum_{j=1}^{k+1} \|w_j^{\mathbf{r}_k}\|_j^2\right)^{\alpha} \right]\\
&\geq \sum_{i=1}^{k+1} \left(\frac{a}{2} - \frac{a}{p}\right) \|w_i^{\mathbf{r}_k}\|_i^2\\
&\geq \left(\frac{a}{2} - \frac{a}{p}\right) \|w_{i_0}^{\mathbf{r}_k}\|_{i_0}^2.
\end{aligned}
\tag{3.3}
$$

This combined with (3.2), implies that

$$
\Psi(\mathbf{r}_k) \to +\infty, \quad \text{if } r_i - r_{i-1} \to 0.
$$

Thus (i) follows.

(ii) Recall the Strauss inequality [23], for any $u \in H_r^1(\mathbb{R}^3)$, there exists a constant $C > 0$ such that $|u(x)| \leq C\frac{\|u\|}{|x|}$, a.e in $\mathbb{R}^3$. Then we obtain

$$
\begin{aligned}
\|w_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 \leq \int_{B_{k+1}^{\mathbf{r}_k}} |w_{k+1}^{\mathbf{r}_k}|^p &\leq C \int_{B_{k+1}^{\mathbf{r}_k}} \frac{\|w_{k+1}^{\mathbf{r}_k}\|_{k+1}^{p-2} |w_{k+1}^{\mathbf{r}_k}|^2}{|x|^{p-2}} dx\\
&\leq C \frac{\|w_{k+1}^{\mathbf{r}_k}\|_{k+1}^{p-2}}{r_k^{p-2}} \|w_{k+1}^{\mathbf{r}_k}\|_{k+1}^2\\
&= C r_k^{2-p} \|w_{k+1}^{\mathbf{r}_k}\|_{k+1}^p.
\end{aligned}
\tag{3.4}
$$

This yields that $r_k^{p-2} \leq C\|w_{k+1}^{\mathbf{r}_k}\|_{k+1}^{p-2}$. Therefore, the conclusion follows from (3.3).

(iii) Take a sequence $\{\mathbf{r}_k^n\}_{n=1}^\infty = \{(r_1^n, \ldots, r_k^n)\}_{n=1}^\infty \subset \Gamma_k$ converging to $\bar{\mathbf{r}}_k = (\bar{r}_1, \ldots, \bar{r}_k) \in \Gamma_k$. It suffices to prove that $\Psi(\mathbf{r}_k^n) \to \Psi(\bar{\mathbf{r}}_k)$. By Lemma 2.7, we assume that $(w_1^{\mathbf{r}_k^n}, \ldots, w_{k+1}^{\mathbf{r}_k^n})$ and $(w_1^{\bar{\mathbf{r}}_k}, \ldots, w_{k+1}^{\bar{\mathbf{r}}_k})$ are minimizers of $E_\lambda|_{\mathcal{M}_k^{\mathbf{r}_k^n}}$ and $E_\lambda|_{\mathcal{M}_k^{\bar{\mathbf{r}}_k}}$, respectively. In the sequel, we shall prove that

$$
\Psi(\bar{\mathbf{r}}_k) \geq \limsup_{n\to\infty} \Psi(\mathbf{r}_k^n) \quad \text{and} \quad \Psi(\bar{\mathbf{r}}_k) \leq \liminf_{n\to\infty} \Psi(\mathbf{r}_k^n). \tag{3.5}
$$

First, we prove that $\Psi(\bar{\mathbf{r}}_k) \geq \limsup_{n\to\infty} \Psi(\mathbf{r}_k^n)$. Define $v_i^{\mathbf{r}_k^n} : [r_{i-1}^n, r_i^n] \to \mathbb{R}$ such that

$$v_i^{\mathbf{r}_k^n}(r) = \alpha_i^n w_i^{\bar{\mathbf{r}}_k}(\frac{\bar{r}_i - \bar{r}_{i-1}}{r_i^n - r_{i-1}^n}(r - r_{i-1}^n) + \bar{r}_{i-1}), \quad \text{for} \quad i = 1, \dots, k,$$
$$v_{k+1}^{\mathbf{r}_k^n}(r) = \alpha_{k+1}^n w_{k+1}^{\bar{\mathbf{r}}_k}(\frac{\bar{r}_k}{r_k^n} r),$$

where $r_0^n = 0, r_{k+1}^n = \infty$ and each $(\alpha_1^n, \dots, \alpha_{k+1}^n)$ is a unique (k+1)-tuple of positive real number such that $(v_1^{\mathbf{r}_k^n}, \dots, v_{k+1}^{\mathbf{r}_k^n}) \in M_k^{\mathbf{r}_k^n}$. Then by the definition of $(w_1^{\mathbf{r}_k^n}, \dots, w_{k+1}^{\mathbf{r}_k^n})$, we have

$$E_\lambda(v_1^{\mathbf{r}_k^n}, \dots, v_{k+1}^{\mathbf{r}_k^n}) \geq E_\lambda(w_1^{\mathbf{r}_k^n}, \dots, w_{k+1}^{\mathbf{r}_k^n}) = \Psi(\mathbf{r}_k^n).$$

If n is large enough, we can calculate that for each $i, j = 1, \dots, k+1$,

$$\begin{aligned}
&\|v_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}} \\
&= \int_{r_{i-1}^n}^{r_i^n} |\nabla v_i^{\mathbf{r}_k^n}|^2 \beta(N) r^2 dr + \int_{r_{i-1}^n}^{r_i^n} V(v_i^{\mathbf{r}_k^n})^2 \beta(N) r^2 dr \\
&= (\alpha_i^n)^2 \int_{r_{i-1}^n}^{r_i^n} |\nabla w_i^{\bar{\mathbf{r}}_k}(\frac{\bar{r}_i - \bar{r}_{i-1}}{r_i^n - r_{i-1}^n}(r - r_{i-1}^n) + \bar{r}_{i-1})|^2 \beta(N) r^2 dr \\
&\quad + (\alpha_i^n)^2 \int_{r_{i-1}^n}^{r_i^n} V|w_i^{\bar{\mathbf{r}}_k}\left(\frac{\bar{r}_i - \bar{r}_{i-1}}{r_i^n - r_{i-1}^n}(r - r_{i-1}^n) + \bar{r}_{i-1}\right)|^2 \beta(N) r^2 dr \\
&= \beta(N)(\alpha_i^n)^2 \frac{\bar{r}_i - \bar{r}_{i-1}}{r_i^n - r_{i-1}^n} \int_{r_{i-1}^n}^{r_i^n} |\nabla w_i^{\bar{\mathbf{r}}_k}(t)|^2 \left(\frac{r_i^n - r_{i-1}^n}{\bar{r}_i - \bar{r}_{i-1}}(t - \bar{r}_{i-1}) + r_{i-1}^n\right)^2 \left(\frac{r_i^n - r_{i-1}^n}{\bar{r}_i - \bar{r}_{i-1}}\right) dt \\
&\quad + \beta(N)(\alpha_i^n)^2 \int_{r_{i-1}^n}^{r_i^n} V|w_i^{\bar{\mathbf{r}}_k}(t)|^2 \left(\frac{r_i^n - r_{i-1}^n}{\bar{r}_i - \bar{r}_{i-1}}(t - \bar{r}_{i-1}) + r_{i-1}^n\right)^2 (\frac{r_i^n - r_{i-1}^n}{\bar{r}_i - \bar{r}_{i-1}}) dt \\
&= (\alpha_i^n)^2 \|w_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\bar{\mathbf{r}}_k}} + o(1),
\end{aligned}$$

where $\beta(N)$ indicates the surface area of the unit sphere in $\mathbb{R}^N$. Similarly,

$$\|v_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}} \left(\sum_{j=1}^{k+1} \|v_j^{\mathbf{r}_k^n}\|^2_{B_j^{\mathbf{r}_k^n}}\right)^\alpha = (\alpha_i^n)^2 \|w_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\bar{\mathbf{r}}_k}} \left(\sum_{j=1}^{k+1} (\alpha_j^n)^2 \|w_j^{\bar{\mathbf{r}}_k}\|^2_{B_j^{\bar{\mathbf{r}}_k}}\right)^\alpha + o(1)$$

and

$$\int_{B_i^{\mathbf{r}_k^n}} |v_i^{\mathbf{r}_k^n}|^p = (\alpha_i^n)^p \int_{B_i^{\bar{\mathbf{r}}_k}} |w_i^{\bar{\mathbf{r}}_k}|^p + o(1).$$

This combined with the fact that $(v_1^{\mathbf{r}_k^n}, \dots, v_{k+1}^{\mathbf{r}_k^n}) \in M_k^{\mathbf{r}_k^n}$, yields

$$a(\alpha_i^n)^2\|w_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\bar{\mathbf{r}}_k}}+\lambda(\alpha_i^n)^2\|w_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\bar{\mathbf{r}}_k}}\left(\sum_{j=1}^{k+1}(\alpha_j^n)^2\|w_j^{\bar{\mathbf{r}}_k}\|^2_{B_j^{\bar{\mathbf{r}}_k}}\right)^{\alpha}-(\alpha_i^n)^p\int_{B_i^{\bar{\mathbf{r}}_k}}|w_i^{\bar{\mathbf{r}}_k}|^p=o(1)$$

for each $i=1,\dots,k+1$. In addition,

$$a\|w_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\bar{\mathbf{r}}_k}}+\lambda\|w_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\bar{\mathbf{r}}_k}}\left(\sum_{j=1}^{k+1}\|w_j^{\bar{\mathbf{r}}_k}\|^2_{B_j^{\bar{\mathbf{r}}_k}}\right)^{\alpha}-\int_{B_i^{\bar{\mathbf{r}}_k}}|w_i^{\bar{\mathbf{r}}_k}|^p=0$$

for each i, and this gives that $\lim_{n\to\infty}\alpha_i^n=1$ for all i. Therefore, we get

$$\Psi(\bar{\mathbf{r}}_k)=\limsup_{n\to\infty}E_\lambda(v_1^{\mathbf{r}_k^n},\dots,v_{k+1}^{\mathbf{r}_k^n})\geq\limsup_{n\to\infty}E_\lambda(w_1^{\mathbf{r}_k^n},\dots,w_{k+1}^{\mathbf{r}_k^n})=\limsup_{n\to\infty}\Psi(\mathbf{r}_k^n).$$

On the other hand, we prove $\Psi(\bar{\mathbf{r}}_k)\leq\liminf_{n\to\infty}\Psi(\mathbf{r}_k^n)$. Similarly as the former case, define $u_i^{\mathbf{r}_k^n}:[\bar{r}_{i-1},\bar{r}_i]\to\mathbb{R}$ such that

$$u_i^{\mathbf{r}_k^n}(t)=t_i^n w_i^{\mathbf{r}_k^n}\left(\frac{r_i^n-r_{i-1}^n}{\bar{r}_i-\bar{r}_{i-1}}(t-\bar{r}_{i-1})+r_{i-1}^n\right),\quad\text{for}\quad i=1,\dots,k,$$
$$u_{k+1}^{\mathbf{r}_k^n}(t)=t_{k+1}^n w_{k+1}^{\mathbf{r}_k^n}\left(\frac{r_k^n}{\bar{r}_k}t\right).$$

where $r_0^n=0$, $r_{k+1}^n=\infty$ and each $(t_1^n,\dots,t_{k+1}^n)$ is a unique (k+1)-tuple of positive real number such that $(u_1^{\mathbf{r}_k^n},\dots,u_{k+1}^{\mathbf{r}_k^n})\in M_k^{\bar{\mathbf{r}}_k}$. Then it also follows that

$$a(t_i^n)^2\|w_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}}+\lambda(t_i^n)^2\|w_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}}\left(\sum_{j=1}^{k+1}(t_j^n)^2\|w_j^{\mathbf{r}_k^n}\|^2_{B_j^{\mathbf{r}_k^n}}\right)^{\alpha}-(t_i^n)^p\int_{B_i^{\mathbf{r}_k^n}}|w_i^{\mathbf{r}_k^n}|^p=o(1)$$

and

$$a\|w_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}}+\lambda\|w_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}}\left(\sum_{j=1}^{k+1}\|w_j^{\mathbf{r}_k^n}\|^2_{B_j^{\mathbf{r}_k^n}}\right)^{\alpha}-\int_{B_i^{\mathbf{r}_k^n}}|w_i^{\mathbf{r}_k^n}|^p=0$$

for each i. Since $\liminf_{n\to\infty}\|w_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}}$ is strictly positive, we conclude that $t_i^n\to1$ as $n\to\infty$ for all i. Thus

$$\Psi(\bar{\mathbf{r}}_k)\leq\liminf_{n\to\infty}E_\lambda(v_1^{\mathbf{r}_k^n},\dots,v_{k+1}^{\mathbf{r}_k^n})=\liminf_{n\to\infty}E_\lambda(w_1^{\mathbf{r}_k^n},\dots,w_{k+1}^{\mathbf{r}_k^n})=\liminf_{n\to\infty}\Psi(\mathbf{r}_k^n).$$

This completes the proof of (iii). Finally, by (i)–(iii), we can conclude that there exists a minimum point $\tilde{\mathbf{r}}_k=(\tilde{r}_1,\dots,\tilde{r}_k)\in\Gamma_k$ of Ψ. □

According to Lemmas 2.7 and 3.1, there exists $(w_1^{\tilde{\mathbf{r}}_k}, \dots, w_{k+1}^{\tilde{\mathbf{r}}_k})$ satisfying (2.3) and

$$E_\lambda(w_1^{\tilde{\mathbf{r}}_k}, \dots, w_{k+1}^{\tilde{\mathbf{r}}_k}) = \inf_{\mathbf{r}_k \in \Gamma_k} \Psi(\mathbf{r}_k). \tag{3.6}$$

Now we are in position to show that $\sum_{i=1}^{k+1} w_i^{\tilde{\mathbf{r}}_k}$ is a desired nodal solution of (1.1) which changes sign exactly k times.

Proof of Theorem 1.1 We shall argue it by contradiction. Suppose on the contrary that $\sum_{i=1}^{k+1} w_i^{\tilde{\mathbf{r}}_k}$ is not the solution of (1.1). In other words, suppose that there is $l \in \{1, \dots, k\}$ such that

$$w_- := \lim_{t \to \tilde{r}_{l-}} \frac{dw_l^{\tilde{\mathbf{r}}_k}(t)}{dt} \neq \lim_{t \to \tilde{r}_{l+}} \frac{dw_{l+1}^{\tilde{\mathbf{r}}_k}(t)}{dt} =: w_+$$

We define a $(k+1)$-tuple of function $(\tilde{z}_1, \dots, \tilde{z}_{k+1})$ as follows. Given a small positive number δ, set

$$\tilde{y}(t) = \begin{cases} w_l^{\tilde{\mathbf{r}}_k}(t), & \text{for } t \in (\tilde{r}_{l-1}, \tilde{r}_l - \delta), \\ w_l^{\tilde{\mathbf{r}}_k}(\tilde{r}_l - \delta) + \dfrac{w_{l+1}^{\tilde{\mathbf{r}}_k}(\tilde{r}_l + \delta) - w_l^{\tilde{\mathbf{r}}_k}(\tilde{r}_l - \delta)}{2\delta}(t - \tilde{r}_l + \delta), & \text{for } t \in (\tilde{r}_l - \delta, \tilde{r}_l + \delta), \\ w_{l+1}^{\tilde{\mathbf{r}}_k}(t), & \text{for } t \in (\tilde{r}_l + \delta, \tilde{r}_{l+1}). \end{cases}$$

Then $\tilde{y}$ has a unique zero point $\tilde{s}_l$ in $(\tilde{r}_{l-1}, \tilde{r}_{l+1})$. Let

$$\tilde{z}_l(t) = \tilde{y}(t) \text{ in } (\tilde{r}_{l-1}, \tilde{s}_l), \quad \tilde{z}_{l+1}(t) = \tilde{y}(t) \text{ in } (\tilde{s}_l, \tilde{r}_{l+1}) \text{ and}$$
$$\tilde{z}_i(t) = w_i^{\tilde{\mathbf{r}}_k}(t) \text{ for } (\tilde{r}_{i-1}, \tilde{r}_i), \quad i \neq l, l+1.$$

By Lemma 2.3, there exists $(a_1, \dots, a_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$ such that $(z_1, \dots, z_{k+1}) := (a_1\tilde{z}_1, \dots, a_{k+1}\tilde{z}_{k+1}) \in M_k^{\bar{\mathbf{r}}_k}$ with $\bar{r}_k = (\tilde{r}_1, \dots, \tilde{r}_{l-1}, \tilde{s}_l, \tilde{r}_{l+1}, \dots, \tilde{r}_{k+1})$. In addition, we have

$$(a_1, \dots, a_{k+1}) \to (1, \dots, 1) \quad \text{as} \quad \delta \to 0 \tag{3.7}$$

and

$$I_\lambda(W) = E_\lambda(w_1^{\tilde{\mathbf{r}}_k}, \dots, w_{k+1}^{\tilde{\mathbf{r}}_k}) \leq E_\lambda(z_1^{\tilde{\mathbf{r}}_k}, \dots, z_{k+1}^{\tilde{\mathbf{r}}_k}) = I_\lambda(Z), \tag{3.8}$$

where $W(t) = \sum_{i=1}^{k+1} w_i^{\tilde{\mathbf{r}}_k}(t)$ and $Z(t) = \sum_{i=1}^{k+1} z_i^{\tilde{\mathbf{r}}_k}(t)$.

On the other hand, since $Z, W > 0$, we can check that

$$\frac{1}{p}|Z|^p \geq \frac{1}{p}|W|^p + \frac{Z^2 - W^2}{2}|W|^{p-2}. \tag{3.9}$$

Then there holds that

$$
\begin{aligned}
& I_\lambda(Z) - I_\lambda(W) \\
& \le \left(\int_0^{\tilde{r}_l-\delta} + \int_{\tilde{r}_l+\delta}^{\infty} \right) \left(\frac{a}{2} Z'^2 + \frac{a}{2} V(t) Z^2 - \frac{1}{p} |W|^p - \frac{Z^2 - W^2}{2} |W|^{p-2} \right) t^2 dt \\
& \quad - \left(\int_0^{\tilde{r}_l-\delta} + \int_{\tilde{r}_l+\delta}^{\infty} \right) \left(\frac{a}{2} W'^2 + \frac{a}{2} V(t) W^2 - \frac{1}{p} |W|^p \right) t^2 dt \\
& \quad + \int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta} \left(\frac{a}{2} Z'^2 + \frac{a}{2} V(t) Z^2 - \frac{1}{p} |Z|^p \right) t^2 dt \\
& \quad - \int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta} \left(\frac{a}{2} W'^2 + \frac{a}{2} V(t) W^2 - \frac{1}{p} |W|^p \right) t^2 dt \\
& \quad + \frac{\lambda}{2\alpha+2} \left(\int_0^{\infty} (Z'^2 + V(t) Z^2) t^2 dt \right)^{\alpha+1} \\
& \quad - \frac{\lambda}{2\alpha+2} \left(\int_0^{\infty} (W'^2 + V(t) W^2) t^2 dt \right)^{\alpha+1}.
\end{aligned}
$$

Furthermore, by the definition of W, we have

$$
\begin{aligned}
& \int_0^{\infty} \left(a W'^2 + a V(t) W^2 \right) t^2 dt + \lambda \left(\int_0^{\infty} (W'^2 + V(t) W^2) t^2 dt \right)^{\alpha+1} \\
& = \int_0^{\infty} |W|^p t^2 dt.
\end{aligned} \tag{3.10}
$$

Set $A := \left(\int_0^{\infty} (W'^2 + V(t) W^2) t^2 dt \right)^{\alpha}$. Then we conclude from (3.9)–(3.10) that

$$
\begin{aligned}
& I_\lambda(Z) - I_\lambda(W) \\
& \le \underbrace{\left(\int_0^{\tilde{r}_l-\delta} + \int_{\tilde{r}_l+\delta}^{\infty} \right) \left(\frac{a}{2} Z'^2 + \frac{a}{2} V(t) Z^2 - \frac{Z^2}{2} |W|^{p-2} + \frac{\lambda A}{2} W'^2 + \frac{\lambda A}{2} V(t) W^2 \right) t^2 dt}_{\mathbf{A}} \\
& \quad \underbrace{+ \int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta} \left(\frac{a}{2} Z'^2 + \frac{a}{2} V(t) Z^2 - \frac{1}{p} |Z|^p + \frac{1}{p} |W|^p + \frac{\lambda A}{2} W'^2 + \frac{\lambda A}{2} V(t) W^2 \right) t^2 dt}_{\mathbf{B}} \\
& \quad \underbrace{+ \frac{\lambda}{2\alpha+2} \left(\int_0^{\infty} (Z'^2 + V(t) Z^2) t^2 dt \right)^{\alpha+1} - \frac{\lambda}{2\alpha+2} \left(\int_0^{\infty} (W'^2 + V(t) W^2) t^2 dt \right)^{\alpha+1}}_{\mathbf{C}}.
\end{aligned} \tag{3.11}
$$

We consider the first part **(A)**. Notice that W satisfies

$$\left[a+\lambda\left(\int_0^\infty (W'^2+V(t)W^2)t^2dt\right)^\alpha\right]\left(-(t^2W')'+V(t)Wt^2\right)$$
$$=|W|^{p-2}Wt^2, \quad \tilde{r}_{l-1}\le t\le \tilde{r}_l. \tag{3.12}$$

Since $W(\tilde{r}_l)=0$, by Taylor formula, we have $W(\tilde{r}_l)=W(\tilde{r}_l-\delta)+W'(\tilde{r}_l-\delta)\delta+o(\delta)$ and $W(\tilde{r}_l-\delta)=-\delta\cdot w_-+o(\delta)$. Moreover, by (3.12) and Taylor formula again, we have $(t^2W')'(\tilde{r}_l)=0$ and

$$(\tilde{r}_l-\delta)^2W'(\tilde{r}_l-\delta)-\tilde{r}_l^2W'(\tilde{r}_l)=-\delta(\tilde{r}_l^2W'(\tilde{r}_l))'+o(\delta).$$

So

$$(\tilde{r}_l-\delta)^2W'(\tilde{r}_l-\delta)=\tilde{r}_l^2w_-+o(\delta). \tag{3.13}$$

By multiplying both sides of (3.12) by W, we obtain

$$\begin{aligned}\int_0^{\tilde{r}_l-\delta}|W|^pt^2dt&=\int_0^{\tilde{r}_l-\delta}(a+\lambda A)(-(t^2W')'W+V(t)W^2t^2)dt\\&=(a+\lambda A)\left(\int_0^{\tilde{r}_l-\delta}(-(t^2W')'W+\int_0^{\tilde{r}_l-\delta}V(t)W^2t^2\right)\\&=(a+\lambda A)\left(-t^2W'W\Big|_0^{\tilde{r}_l-\delta}+\int_0^{\tilde{r}_l-\delta}\left(t^2W'^2+V(t)W^2t^2\right)\right).\end{aligned} \tag{3.14}$$

Thus,

$$\begin{aligned}\int_0^{\tilde{r}_l-\delta}\frac{|W|^p}{2}t^2dt=&-\frac{(a+\lambda A)}{2}(\tilde{r}_l-\delta)^2W'(\tilde{r}_l-\delta)W(\tilde{r}_l-\delta)\\&+\int_0^{\tilde{r}_l-\delta}\left(\frac{(a+\lambda A)}{2}W'^2+\frac{(a+\lambda A)}{2}V(t)W^2\right)t^2dt.\end{aligned} \tag{3.15}$$

This combined with (3.13), implies that

$$\begin{aligned}&\int_0^{\tilde{r}_l-\delta}\left(\frac{a}{2}Z'^2+\frac{a}{2}V(t)Z^2-\frac{Z^2}{2}|W|^{p-2}+\frac{\lambda A}{2}W'^2+\frac{\lambda A}{2}V(t)W^2\right)t^2dt\\&=(1+o(1))\int_0^{\tilde{r}_l-\delta}\left(\frac{a}{2}W'^2+\frac{a}{2}V(t)W^2-\frac{|W|^p}{2}+\frac{\lambda A}{2}W'^2+\frac{\lambda A}{2}V(t)W^2\right)t^2dt\\&=(1+o(1))\frac{(a+\lambda A)}{2}(\tilde{r}_l-\delta)^2W'(\tilde{r}_l-\delta)W(\tilde{r}_l-\delta)\\&=-\frac{(a+\lambda A)}{2}\tilde{r}_l^2(w_-)^2\delta+o(\delta).\end{aligned} \tag{3.16}$$

By the same method, we obtain

$$\begin{aligned}&\int_{\tilde{r}_l+\delta}^{\infty}\left(\frac{a}{2}Z'^2+\frac{a}{2}V(t)Z^2-\frac{Z^2}{2}|W|^{p-2}+\frac{\lambda A}{2}W'^2+\frac{\lambda A}{2}V(t)W^2\right)t^2dt\\&\quad=-\frac{(a+\lambda A)}{2}\tilde{r}_l^2(w_+)^2\delta+o(\delta).\end{aligned}\tag{3.17}$$

Next, we consider the second part (**B**). Indeed,

$$\int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta}\left(\frac{a}{2}V(t)Z^2-\frac{1}{p}|Z|^p+\frac{1}{p}|W|^p+\frac{\lambda A}{2}V(t)W^2\right)t^2dt=o(\delta),\tag{3.18}$$

and

$$\begin{aligned}\int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta}\left(\frac{a}{2}Z'^2+\frac{\lambda A}{2}W'^2\right)t^2dt&=(1+o(1))\int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta}\left(\frac{a+\lambda A}{2}W'^2\right)t^2dt\\&=\frac{a+\lambda A}{4}(w_++w_-)^2\tilde{r}_l^2\delta+o(\delta).\end{aligned}\tag{3.19}$$

Finally, we consider the third part (**C**). Notice that

$$\begin{aligned}&\frac{\lambda}{2\alpha+2}\left(\int_0^\infty(Z'^2+V(t)Z^2)t^2dt\right)^{\alpha+1}-\frac{\lambda}{2\alpha+2}\left(\int_0^\infty(W'^2+V(t)W^2)t^2dt\right)^{\alpha+1}\\&=o(1)\left(\int_0^\infty(W'^2+V(t)W^2)t^2dt\right)^{\alpha+1}=o(\delta).\end{aligned}\tag{3.20}$$

Consequently, we conclude from (3.16)–(3.20) that

$$I_\lambda(Z)-I_\lambda(W)\leq-\frac{(a+\lambda A)}{4}(w_+-w_-)^2\tilde{r}_l^2\delta+o(\delta).\tag{3.21}$$

By taking $\delta>0$ small enough, we have $I_\lambda(Z)-I_\lambda(W)<0$, which is a contradiction with (3.8). This completes the proof. □

4 Energy Comparison and Asymptotic Behaviors

In this section, we are going to prove Theorems 1.2 and 1.3 by establishing subtle energy estimates.

Proof of Theorem 1.2 By applying Theorem 1.1, we can assume that for any fixed positive integer k, equation (1.1) has a radial nodal solution U_{k+1} with exactly $k+1$ nodes $0<\bar{r}_1<\cdots<\bar{r}_{k+1}<+\infty$, and $I_\lambda(U_{k+1})=E_\lambda(w_1^{\bar{\mathbf{r}}_{k+1}},\ldots,w_{k+2}^{\bar{\mathbf{r}}_{k+1}})=\inf_{\mathbf{r}_{k+1}\in\Gamma_{k+1}}\Psi(\mathbf{r}_{k+1})$. Set

$$\tilde{\mathbf{r}}_{k+1}:=(\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_{k+1})$$

and

$$w_i^{\tilde{\mathbf{r}}_{k+1}} := \chi_{B_i^{\tilde{\mathbf{r}}_{k+1}}} U_{k+1},$$

where $\chi_{B_i^{\tilde{\mathbf{r}}_{k+1}}}$ is the characteristic function on $B_i^{\tilde{\mathbf{r}}_{k+1}}$. Obviously, $(w_1^{\tilde{\mathbf{r}}_{k+1}}, \ldots, w_{k+2}^{\tilde{\mathbf{r}}_{k+1}})$ satisfies

$$\begin{cases} \left[a + \lambda\left(\sum_{j=1}^{k+1} \|w_j^{\tilde{\mathbf{r}}_{k+1}}\|_j^2\right)^\alpha\right]\left(-\Delta w_i^{\tilde{\mathbf{r}}_{k+1}} + V(|x|) w_i^{\tilde{\mathbf{r}}_{k+1}}\right) = |w_i^{\tilde{\mathbf{r}}_{k+1}}|^{p-2} w_i^{\tilde{\mathbf{r}}_{k+1}}, \ x \in B_i^{\tilde{\mathbf{r}}_{k+1}}, \\ w_i^{\tilde{\mathbf{r}}_{k+1}} = 0, \quad x \notin B_i^{\tilde{\mathbf{r}}_{k+1}}. \end{cases} \tag{4.1}$$

Next, let $\hat{\mathbf{r}}_k := (\bar{r}_2, \ldots, \bar{r}_{k+1})$. Clearly, $\hat{\mathbf{r}}_k \in \Gamma_k$. By using Lemma 2.7, there is a minimizer $(w_1^{\hat{\mathbf{r}}_k}, \ldots, w_{k+1}^{\hat{\mathbf{r}}_k})$ of its corresponding energy $E_\lambda|_{\mathcal{M}_k^{\hat{\mathbf{r}}_k}}$, i.e.

$$E_\lambda(w_1^{\hat{\mathbf{r}}_k}, \ldots, w_{k+1}^{\hat{\mathbf{r}}_k}) = \inf_{(u_1,\ldots,u_{k+1}) \in \mathcal{M}_k^{\hat{\mathbf{r}}_k}} E_\lambda(u_1, \ldots, u_{k+1}). \tag{4.2}$$

Then, by Lemma 2.3, there exists a unique $(k+1)-$tuple $(t_1, t_3, \ldots, t_{k+2})$ of positive numbers such that

$$(t_1 w_1^{\tilde{\mathbf{r}}_{k+1}}, t_3 w_3^{\tilde{\mathbf{r}}_{k+1}}, \ldots, t_{k+2} w_{k+2}^{\tilde{\mathbf{r}}_{k+1}}) \in \mathcal{M}_k^{\hat{\mathbf{r}}_k}.$$

This combined with (4.2), implies that

$$E_\lambda(w_1^{\hat{\mathbf{r}}_k}, \ldots, w_{k+1}^{\hat{\mathbf{r}}_k}) \le E_\lambda(t_1 w_1^{\tilde{\mathbf{r}}_{k+1}}, t_3 w_3^{\tilde{\mathbf{r}}_{k+1}}, \ldots, t_{k+2} w_{k+2}^{\tilde{\mathbf{r}}_{k+1}}). \tag{4.3}$$

By letting $s > 0$ be small enough, we obtain

$$\begin{aligned} &E_\lambda(t_1 w_1^{\tilde{\mathbf{r}}_{k+1}}, t_3 w_3^{\tilde{\mathbf{r}}_{k+1}}, \ldots, t_{k+2} w_{k+2}^{\tilde{\mathbf{r}}_{k+1}}) \\ &\quad < E_\lambda(t_1 w_1^{\tilde{\mathbf{r}}_{k+1}}, s w_2^{\tilde{\mathbf{r}}_{k+1}}, t_3 w_3^{\tilde{\mathbf{r}}_{k+1}}, \ldots, t_{k+2} w_{k+2}^{\tilde{\mathbf{r}}_{k+1}}). \end{aligned} \tag{4.4}$$

On the other hand, Corollary 2.4 gives that

$$\begin{aligned} &E_\lambda(t_1 w_1^{\tilde{\mathbf{r}}_{k+1}}, s w_2^{\tilde{\mathbf{r}}_{k+1}}, t_3 w_3^{\tilde{\mathbf{r}}_{k+1}}, \ldots, t_{k+2} w_{k+2}^{\tilde{\mathbf{r}}_{k+1}}) \\ &\quad < E_\lambda(w_1^{\tilde{\mathbf{r}}_{k+1}}, w_2^{\tilde{\mathbf{r}}_{k+1}}, \ldots, w_{k+2}^{\tilde{\mathbf{r}}_{k+1}}) = I_\lambda(U_{k+1}). \end{aligned} \tag{4.5}$$

Since $E_\lambda(w_1^{\tilde{\mathbf{r}}_k}, \ldots, w_{k+1}^{\tilde{\mathbf{r}}_k}) = \inf_{\mathbf{r}_k \in \Gamma_k} \Psi(\mathbf{r}_k)$, we deduce from Lemma 3.1 that

$$I_\lambda(U_k) = E_\lambda(w_1^{\tilde{\mathbf{r}}_k}, \ldots, w_{k+1}^{\tilde{\mathbf{r}}_k}) < E_\lambda(w_1^{\hat{\mathbf{r}}_k}, \ldots, w_{k+1}^{\hat{\mathbf{r}}_k}) \tag{4.6}$$

Then, it follows from (4.3)–(4.5) that

$$I_\lambda(U_k) < I_\lambda(U_{k+1}).$$

Thus $I_\lambda(U_k)$ is strictly increasing with respect to k.

Finally, we claim that $I_\lambda(U_k) > (k+1)I_\lambda(U_0)$. In fact, since $\langle I'_\lambda(U_k), w_i^{\tilde{\mathbf{r}}_k}\rangle = 0$, we have

$$\langle I'_\lambda(w_i^{\tilde{\mathbf{r}}_k}), w_i^{\tilde{\mathbf{r}}_k}\rangle = a\|w_i^{\tilde{\mathbf{r}}_k}\|_i^2 + \lambda\|w_i^{\tilde{\mathbf{r}}_k}\|_i^{2\alpha+2} - \int_{B_i^{\tilde{\mathbf{r}}_k}} |w_i^{\tilde{\mathbf{r}}_k}|^p dx < 0.$$

By Lemma 2.3, there is a unique $\bar{\delta}_i \in (0,1)$ such that $\bar{\delta}_i w_i^{\tilde{\mathbf{r}}_k} \in \mathcal{N}$, where $\mathcal{N}$ is defined in (1.7). Hence, $I_\lambda(\bar{\delta}_i w_i^{\tilde{\mathbf{r}}_k}) \geq I_\lambda(U_0)$ and

$$\begin{aligned}
(k+1)I_\lambda(U_0) &\leq \sum_{i=1}^{k+1}\left(I_\lambda(\bar{\delta}_i w_i^{\tilde{\mathbf{r}}_k}) - \frac{1}{2\alpha+2}\langle I'_\lambda(\bar{\delta}_i w_i^{\tilde{\mathbf{r}}_k}), \bar{\delta}_i w_i^{\tilde{\mathbf{r}}_k}\rangle\right)\\
&= \sum_{i=1}^{k+1}\left(\left(\frac{a}{2} - \frac{a}{2\alpha+2}\right)\bar{\delta}_i^2\|w_i^{\tilde{\mathbf{r}}_k}\|_i^2 + \left(\frac{1}{2\alpha+2} - \frac{1}{p}\right)\bar{\delta}_i^p \int_{B_i^{\tilde{\mathbf{r}}_k}} |w_i^{\tilde{\mathbf{r}}_k}|^p\right)\\
&< \sum_{i=1}^{k+1}\left(\left(\frac{a}{2} - \frac{a}{2\alpha+2}\right)\|w_i^{\tilde{\mathbf{r}}_k}\|_i^2 + \left(\frac{1}{2\alpha+2} - \frac{1}{p}\right)\int_{B_i^{\tilde{\mathbf{r}}_k}} |w_i^{\tilde{\mathbf{r}}_k}|^p\right)\\
&= I_\lambda\left(\sum_{i=1}^{k+1} w_i^{\tilde{\mathbf{r}}_k}\right) - \frac{1}{2\alpha+2}\left\langle I'_\lambda\left(\sum_{i=1}^{k+1} w_i^{\tilde{\mathbf{r}}_k}\right), \sum_{i=1}^{k+1} w_i^{\tilde{\mathbf{r}}_k}\right\rangle\\
&= I_\lambda(U_k).
\end{aligned}$$

The claim hods and we complete the proof. □

Hereafter, we denote U_k by U_k^λ in order to emphasize the dependence on λ. Analogically, set $\mathbf{r}_{k,\lambda} = (\bar{r}_{1,\lambda}, \ldots, \bar{r}_{k,\lambda})$ and $U_k^\lambda = \sum_{i=1}^{k+1} w_i^{\mathbf{r}_{k,\lambda}} \in H_V$ obtained in Theorem 1.1. In the following, we shall show the asymptotic behaviors of U_k^λ as $\lambda \to 0^+$.

Proof of Theorem 1.3 We divide the whole proof into three steps.

Step 1. We claim that for any sequence $\{\lambda_n\}$ with $\lambda_n \to 0^+$ as $n \to \infty$, $\{U_k^{\lambda_n}\}_n$ is bounded in H_V. In fact, for fixed $\mathbf{r}_k \in \Gamma_k$, we take nonzero radial functions $\varphi_i \in C_c^\infty(B_i^{\mathbf{r}_k})$, $i = 1, \ldots, k+1$. Then for any $\lambda \in [0,1]$, there exists a $k+1$ tuple $(b_1, \ldots, b_{k+1})$ of positive numbers such that

$$F_i(b_1\varphi_1, \ldots, b_{k+1}\varphi_{k+1}) < 0, \quad \text{for} \quad i = 1, \ldots, k+1.$$

By Lemmas 2.3 and 2.5, there is a $k+1$ tuple $(a_1(\lambda), \ldots, a_{k+1}(\lambda)) \in (0,1]^{k+1}$ depending on λ such that

$$(\bar{\varphi}_1, \ldots, \bar{\varphi}_{k+1}) := (a_1(\lambda)b_1\varphi_i, \ldots, a_{k+1}(\lambda)b_{k+1}\varphi_i) \in \mathcal{M}_k^{\mathbf{r}_k}.$$

Then there is $C_0 > 0$ such that for n large enough,

$$
\begin{aligned}
& I_{\lambda_n}(U_k^{\lambda_n}) - \frac{1}{2\alpha+2}\langle I'_{\lambda_n}(U_k^{\lambda_n}), U_k^{\lambda_n}\rangle \\
&= \left(\frac{a}{2} - \frac{a}{2\alpha+2}\right)\|U_k^{\lambda_n}\|^2 + \left(\frac{a}{2} - \frac{1}{2\alpha+2}\right)\int_{\mathbb{R}^3}|U_k^{\lambda_n}|^p \\
&\le I_{\lambda_n}\left(\sum_{i=1}^{k+1}\bar{\varphi}_i(x)\right) = I_{\lambda_n}\left(\sum_{i=1}^{k+1}\bar{\varphi}_i(x)\right) - \frac{1}{2\alpha+2}\langle I'_{\lambda_n}\left(\sum_{i=1}^{k+1}\bar{\varphi}_i(x)\right), \bar{\varphi}_i(x)\rangle \\
&= \sum_{i=1}^{k+1}\left(\left(\frac{a}{2} - \frac{a}{2\alpha+2}\right)\|\bar{\varphi}_i(x)\|^2 + \left(\frac{1}{2\alpha+2} - \frac{1}{p}\right)\int_{B_i^{\mathbf{r}_k}}|\bar{\varphi}_i(x)|^p\right) \\
&\le \sum_{i=1}^{k+1}\left(\left(\frac{a}{2} - \frac{a}{2\alpha+2}\right)\|b_i\varphi_i(x)\|^2 + \left(\frac{1}{2\alpha+2} - \frac{1}{p}\right)\int_{B_i^{\mathbf{r}_k}}|b_i\varphi_i(x)|^p\right) \\
&=: C_0.
\end{aligned} \tag{4.7}
$$

This implies that $\{U_k^{\lambda_n}\}_n$ is bounded in H_V. So the claim follows.

Step 2. According to **Step 1**, there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ and $U_k^0 \in H_V$ such that $U_k^{\lambda_{n_j}} \rightharpoonup U_k^0$ and $(U_k^{\lambda_{n_j}})_i \rightharpoonup (U_k^0)_i$ weakly in H_V as $n_j \to +\infty$. Then U_k^0 is a weak solution of (1.12). It suffices prove that U_k^0 is a radial nodal solution of (1.12) with exactly $k+1$ nodal domains.

In fact, by the compact embedding $H_V \hookrightarrow L^s(\mathbb{R}^3)$ with $2 < s < 6$, it follows that

$$
\begin{aligned}
& \|U_k^{\lambda_{n_j}} - U_k^0\|^2 \\
&= \langle I'_{\lambda_{n_j}}(U_k^{\lambda_{n_j}}) - I'_0(U_k^0), U_k^{\lambda_{n_j}} - U_k^0\rangle \\
&\quad + \int_{\mathbb{R}^3}\left(|U_k^{\lambda_{n_j}}|^{p-2}U_k^{\lambda_{n_j}} - |U_k^0|^{p-2}U_k^0\right)(U_k^{\lambda_{n_j}} - U_k^0)dx \\
&\quad - \lambda_{n_j}\|U_k^{\lambda_{n_j}}\|^{2\alpha}\int_{\mathbb{R}^3}\left(\nabla U_k^{\lambda_{n_j}}\nabla(U_k^{\lambda_{n_j}} - U_k^0) + V(|x|)U_k^{\lambda_{n_j}}(U_k^{\lambda_{n_j}} - U_k^0)\right) \\
&\quad \to 0, \qquad as\ j \to \infty.
\end{aligned}
$$

Then $U_k^{\lambda_{n_j}} \to U_k^0$ strongly in H_V.

Next, we prove $(U_k^0)_i \ne 0$. Since $\langle I'_{\lambda_{n_j}}(U_k^{\lambda_{n_j}}), (U_k^{\lambda_{n_j}})_i\rangle = 0$, there is a number $\delta > 0$ such that

$$\liminf_{j\to\infty}\|(U_k^{\lambda_{n_j}})_i\|_i \ge \delta > 0.$$

This together with the compact embedding $H_V \hookrightarrow L^s(\mathbb{R}^3)$, gives that

$$\delta^2 \leq \|(U_k^{\lambda_{n_j}})_i\|_i^2 \leq \int_{\mathbb{R}^3} |(U_k^{\lambda_{n_j}})_i|^p \to \int_{\mathbb{R}^3} |(U_k^0)_i|^p,$$

which shows that $(U_k^0)_i \neq 0$. Thus, U_k^0 is a radial nodal solution of (1.12) with exactly $k+1$ nodal domains.

Step 3. We prove that U_k^0 is a least energy radial solution of (1.12) among all the radial solutions changing sign exactly k times.

In fact, according to [2, Theorem 2.1], we assume that there is $\bar{\mathbf{r}}_k \in \Gamma_k$ and $V_k := \sum_{i=1}^{k+1} v_i$ is a least energy radial solution of (1.12) among all the nodal solutions changing sign exactly k times, where v_i is supported on annuli $B_i^{\bar{\mathbf{r}}_k}$. We assume that $U_k^{\lambda_n} := w_1^{\lambda_n} + \cdots + w_{k+1}^{\lambda_n}$.

By Lemma 2.3, for each $\lambda_n > 0$, there is a unique $(k+1)-$tuple $(t_1(\lambda_n), \ldots, t_{k+1}(\lambda_n))$ of positive numbers such that

$$(t_1(\lambda_n)v_1, \ldots, t_{k+1}(\lambda_n)v_{k+1}) \in \mathcal{M}_k^{\mathbf{r}_k}.$$

Then, for $i = 1, \ldots, k+1$, we have

$$a(t_i(\lambda_n))^2\|v_i\|_i^2 + \lambda_n(t_i(\lambda_n))^2\|v_i\|_i^2\left(\sum_{j=1}^{k+1}(t_j(\lambda_n))^2\|v_j\|_i^2\right)^\alpha = \int_{B_i^{\mathbf{r}_k}} (t_i(\lambda_n))^p|v_i|_i^p. \tag{4.8}$$

Recall that v_i satisfies $a\|v_i\|_i^2 = \int_{B_i^{\mathbf{r}_k}} |v_i|_i^p$. One can easily check that

$$(t_1(\lambda_n), \ldots, t_{k+1}(\lambda_n)) \to (1, \ldots, 1), \quad \text{as} \quad n \to \infty. \tag{4.9}$$

From (4.8)–(4.9), we have

$$\begin{aligned} I_0(V_k) \leq I_0(U_k^0) = \lim_{n\to\infty} I_{\lambda_n}(U_k^{\lambda_n}) &= \lim_{n\to\infty} E_{\lambda_n}(w_1^{\lambda_n}, \ldots, w_{k+1}^{\lambda_n}) \\ &\leq \lim_{n\to\infty} E_{\lambda_n}(t_1(\lambda_n)v_1 + \cdots + t_{k+1}(\lambda_n)v_{k+1}) \\ &= E_0(v_1 + \cdots + v_{k+1}) \\ &= I_0(V_k). \end{aligned}$$

Therefore, U_k^0 is a least energy radial solution of (1.12) which changes sign exactly k times. □

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

References

1. Azzollini, A.: The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity. Differ. Integral Equ. **25**, 543–554 (2012)
2. Bartsch, T., Willem, M.: Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$. Arch. Ration. Mech. Anal. **124**(3), 261–276 (1993)
3. Cheng, X., Dai, G.: Positive solutions for p-Kirchhoff type problems on $\mathbb{R}^N$. Math. Methods Appl. Sci. **38**, 2650–2662 (2015)
4. Chen, C., Kuo, Y., Wu, T.: The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions. J. Differ. Equ. **250**, 1876–1908 (2011)
5. Chen, S., Tang, X.H., Wei, J.: Improved results on planar Kirchhoff-type elliptic problems with critical exponential growth. Z. Angew. Math. Phys. **72**(38), 1–18 (2021)
6. Chen, C., Zhu, Q.: Existence of positive solutions to p-Kirchhoff-type problem without compactness conditions. Appl. Math. Lett. **28**, 82–87 (2014)
7. Chen, S., Zhang, B., Tang, X.: Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity. Adv. Nonlinear Anal. **9**(1), 148–167 (2020)
8. Deng, Y., Peng, S., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. **269**, 3500–3527 (2015)
9. Guo, Z.: Ground states for Kirchhoff equations without compact condition. J. Differ. Equ. **259**(7), 2884–2902 (2015)
10. Guo, H., Tang, R., Wang, T.: Infinitely many nodal solutions with a prescribed number of nodes for the Kirchhoff-type equations. J. Math. Anal. Appl. **505**, 125519 (2022)
11. Guo, H., Wu, D.: Nodal solutions for the Schrödinger–Poisson equations with convolution terms. Nonlinear Anal. **196**, 111781 (2020)
12. He, X., Zou, W.: Infinitely many positive solutions for Kirchhoff-type problems. Nonlinear Anal. **70**, 1407–1414 (2009)
13. He, X., Zou, W.: Multiplicity of solutions for a class of Kirchhoff type problems. Acta Math. Appl. Sin. Engl. Ser. **26**, 387–394 (2010)
14. Jeanjean, L.: Local condition insuring bifurcation from the continuous spectrum. Math. Z. **232**, 651–664 (1999)
15. Jin, F., Yan, B.: The sign-changing solutions for nonlinear elliptic problem with Carrier type. J. Math. Anal. Appl. **487**, 124002 (2020)
16. Liu, Z., Guo, S.: Existence of positive ground state solutions for Kirchhoff type problems. Nonlinear Anal. **120**, 1–13 (2015)
17. Li, Y., Li, F., Shi, J.: Existence of a positive solution to Kirchhoff type problems without compactness conditions. J. Differ. Equ. **253**(7), 2285–2294 (2012)
18. Liu, Z., Lou, Y., Zhang, J.: A perturbation approach to studying sign-changing solutions of Kirchhoff equations with a general nonlinearity. Ann. Mat. Pur. Appl. **201**, 1229–1255 (2022)
19. Li, G., Ye, H.: Existence of positive ground state solutions for the nonlinear Kirchhoff-type equations in $\mathbb{R}^3$. J. Differ. Equ. **257**(2), 566–600 (2014)
20. Mao, A., Zhang, Z.: Sign-changing and multiple solutions of Kirchhoff-type problems without the P.S. condition. Nonlinear Anal. **70**(3), 1275–1287 (2009)
21. Perera, K., Zhang, Z.: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. **221**, 246–255 (2006)
22. Shuai, W.: Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. J. Differ. Equ. **259**(4), 1256–1274 (2015)
23. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**(2), 149–162 (1977)
24. Sun, J., Li, L., Cencelj, M., Gabrovsek, B.: Infinitely many sign-changing solutions for Kirchhoff type problems in $\mathbb{R}^3$. Nonlinear Anal. **186**, 33–54 (2018)
25. Sun, J., Tang, C.: Existence and multiplicity of solutions for Kirchhoff type equations. Nonlinear Anal. **74**, 1212–1222 (2011)
26. Tang, X.H., Cheng, B.: Ground state sign-changing solutions for Kirchhoff-type problems in bounded domains. J. Differ. Equ. **261**(4), 2384–2402 (2016)
27. Tang, X.H., Chen, S.T.: Ground state solutions of Nehari–Pohozaev type for Kirchhoff type problems with general potentials. Calc. Var. **56**, 110 (2017)
28. Varga, R.S.: Gersgorin and His Circles. Springer, Berlin (2010)

29. Wu, X.: Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in $\mathbb{R}^N$. Nonlinear Anal. Real World Appl. **12**, 1278–1287 (2011)
30. Wang, T., Yang, Y., Guo, H.: Multiple nodal solutions of the Kirchhoff-type problem with a cubic term. Adv. Nonlinear Anal. **11**(1), 1030–1047 (2022)
31. Wang, L., Zhang, B., Cheng, K.: Ground state sign-changing solutions for the Schrödinger–Kirchhoff equation in $\mathbb{R}^3$. J. Math. Anal. Appl. **466**(2), 1545–1569 (2018)
32. Xie, Q., Ma, S., Zhang, X.: Bound state solutions of Kirchhoff type problems with critical exponent. J. Differ. Equ. **261**(2), 890–924 (2016)
33. Yang, Y., Zhang, J.: Nontrivial solutions of a class of nonlocal problems via local linking theory. Appl. Math. Lett. **23**, 377–380 (2010)
34. Zhang, Z., Perera, K.: Sign-changing solutions of Kirchhoff-type problems via invariant sets of descent flow. J. Math. Anal. Appl. **317**(2), 456–463 (2006)