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Global Well-Posedness for the One-Dimensional Euler–Fourier–Korteweg System

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Abstract

In this paper, we investigate the Cauchy problem of one-dimensional compressible Euler–Fourier–Korteweg system. The global unique strong solutions are established in the critical Besov spaces with small initial data close to a constant equilibrium state. This extends the recent work of Kawashima et al. (Commun Partial Differ Equ 47:378–400, 2022) on the dissipative structure of linear Euler–Fourier–Korteweg system to the non-linear system in critical space.

Keywords Euler–Fourier–Korteweg system · Besov spaces · Global well-posedness

Mathematics Subject Classification 35G25 · 35Q35 · 35B35

1 Introduction

It is known that the motion of the one-dimensional compressible non-isothermal viscous fluid with internal capillarity can be described by the Navier–Stokes–Korteweg system (see [16, 23] for the derivation of that model):

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$$\text{(NSK)} \quad \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) = \partial_x(\mu\partial_x u) + \partial_x\mathcal{K}, \\ \partial_t\left(\rho\left(e + \frac{|u|^2}{2}\right)\right) + \partial_x\left(\rho u\left(e + \frac{|u|^2}{2}\right)\right) \\ \quad = \partial_x\left(\tilde{\alpha}\partial_x\theta + u\mathcal{K} + \mu u\partial_x u + W\right). \end{cases}$$

As can be seen, the fluid is characterized by its density ρ, velocity field u, and temperature θ. Moreover, $E = e + \frac{|u|^2}{2}$ denotes the internal energy density of the system, and the potential energy e is given by the Helmholtz free energy density Ψ according to

$$e = \Psi(\rho, \phi, \theta) + \theta s; \quad s = -\partial_\theta\Psi,$$

where s denoting the entropy density and $\phi = |\partial_x\rho|^2$. The interstitial working W (Introduced by Dunn and Serrin [16]) and the Korteweg stress tensor $\mathcal{K}$ are defined as

$$W = -\kappa(\rho)\rho\partial_x\rho\partial_x u = \kappa(\rho)\partial_t\rho\left(\partial_t\rho + u\partial_x\rho\right), \; \mathcal{K} = \left(-\rho^2\partial_\rho\Psi + \rho\partial_x(\kappa(\rho)\partial_x\rho)\right) \\ -\kappa(\rho)(\partial_x\rho)^2,$$

with the capillary coefficient $\kappa(\rho) = 2\rho\partial_\phi\Psi$. As in [22], we set

$$\Psi(\rho, \phi, \theta) = \frac{\tilde{\kappa}(\rho)}{2\rho}\phi + \bar{\Psi}(\rho, \theta),$$

where $\theta > 0, \; \rho > 0, \; \partial_{\theta\theta}\bar{\Psi} < 0$. As a particular case of above, we consider that

$$\Psi = \frac{\tilde{\kappa}(\rho)}{2\rho}(\partial_x\rho)^2 + \theta ln\rho - \theta ln\theta.$$

Thus, the system (NSK) can be rewritten as the following form (see Appendix B):

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x P(\rho, \theta) = \partial_x(\mu\partial_x u) + \partial_x K, \\ \rho\partial_t\theta + \rho u\partial_x\theta + P\partial_x u - \tilde{\alpha}\partial_{xx}\theta = \mu(\partial_x u)^2 \end{cases} \tag{1.1}$$

with $P(\rho, \theta) = \rho\theta$ and $K = \rho\kappa(\rho)\partial_{xx}\rho + \frac{1}{2}\left(\rho\kappa'(\rho) - \kappa(\rho)\right)(\partial_x\rho)^2$. In this paper, we mainly focus on the case $\kappa(\rho) = k\rho^{-1}$, $(k > 0)$ that so-called quantum-type fluid, for the technical reasons. And the correspondingly Korteweg stress tensor is defined as

$$K = \partial_{xx}\rho - \rho^{-1}(\partial_x\rho)^2.$$

There have been many mathematical results on the compressible Navier–Stokes equations of Korteweg type. Hattori and Li [21] proved the local existence and global existence of smooth solutions. Hou et al. [22] investigated the global well-posedness of classical solutions. Chen et al. [11] proved the global smooth solutions to the Cauchy problem of one-dimensional non-isentropic system with large initial data. Recently, the authors and Song [37] studied the large-time behavior in L^p-type Besov space. For isothermal fluid, Chen et al. [10] studied the global stability for the large solutions around constant states. Charve and Haspot [20] proved the global existence of large strong solution for $\mu(\rho) = \varepsilon\rho$ and $\kappa(\rho) = \varepsilon^2\rho^{-1}$ in $\mathbb{R}$. Yang et al. [42] studied the asymptotic limits of Navier–Stokes equations with quantum effects. Another interesting and challenging problem is to study the stability of the compressible Navier–Stokes-Korteweg equation in the half space. Chen and Li [12] discussed the time-asymptotic behaviour of strong solutions to the initial-boundary value problem on the half-line $\mathbb{R}^+$, and showed the strong solution converges to the rarefaction wave as $t \to \infty$ for the impermeable wall problem under large initial perturbation. Li and Zhu [32] showed the existence and stability of stationary solution to an outflow problem (see also [31] for more information to outflow problem) with constant viscosity and capillarity coefficients, respectively. Li and Chen [28] studied the large-time behavior of solutions to an inflow problem. In two or three space dimensions, Tan and Wang [39] established global existence and optimal L^2 decay rates in Sobolev spaces. Li, Chen and Luo, and Li and Luo showed stability of the planar rarefaction wave to two- and three-dimensional compressible Navier–Stokes–Korteweg equations in [29, 30], respectively. In the Besov space, Danchin and Desjardins [15] investigated the global well-posedness in L^2-type critical spaces for initial data close enough to stable equilibria. Later, those results are improved by Charve et al. [13] and shown in more general critical L^p framework, and also the optimal time-decay estimates is established by Danchin and Xu [25]. Recently, Bresch et al. [9] studied the weak-strong uniqueness of the quantum fluids models.

It is well known that when the viscosity coefficient $\mu \equiv 0$, the Navier–Stokes model would reduced to the Euler model (see [17, 39]) that may develop singularities (shock waves) in finite time (see [34]). Looking for conditions that guarantee global existence of strong solutions of the Euler model is a nature challenging questions, which goes back to the researches on the partially dissipative hyperbolic systems of Shizuta and Kawashima [35], the thesis of Kawashima [24] and, more recently, to the paper of Yong [41]. A classic example of a partially dissipative system is the Euler damped system (see [17, 38]). Recently, Kawashima et al. [26] researched the dissipative structure for a class of symmetric hyperbolic-parabolic systems with Korteweg-type dispersion (containing the the non-isothermal Euler–Fourier–Korteweg linear) and established a new Craftsmanship conditions.

Motivate by the above work, in this paper, we devote ourself to the following Euler–Fourier–Korteweg system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x P(\rho, \theta) = \partial_x K, \\ \rho\partial_t\theta + \rho u\partial_x\theta + \rho\theta\partial_x u - \tilde{\alpha}\partial_{xx}\theta = 0, \end{cases} \tag{1.2}$$

which governs the evolution of the one-dimensional compressible non-isothermal non-viscous fluid with internal capillarity.

When $\theta \equiv 0$, the system (1.2) reduce to Euler–Korteweg system. Gavage et al. [18, 19] proved local well-posedness by reducing the system to a quasi-linear Schrödinger equation and studied the dispersive properties. Audiard [1] studied the local dispersive smoothing, precisely, $(\partial_x \rho, u) \in \mathcal{C}(0, T; H^s)$ and $(\partial_x \rho, u)/\langle x\rangle^{(1+\varepsilon)/2} \in L^2(0, T; H^{s+\frac{1}{2}})$. Berti et al. [8] proved the local existence for the classical solutions in the torus $\mathbb{T}^d$. Audiard and Haspot [2, 3] proved the global existence for small irrotational initial data in $\mathbb{R}^d$. However, their method and theory are effective for only the case $d \geq 3$. To our knowledge, the global well-posedness remains open in even $d = 2$.

For the non-isotherm case, the dissipation provide by the coupled temperature equation help us study global well-posedness in $\mathbb{R}$ *from the perspective of dissipation*. To our knowledge, this is the first attempt to consider non-isotherm Euler-Korteweg system in Besov spaces, which might fills a gap in the global result of Euler-Korteweg system for 1-D.

The main difficulty lies on the processing of nonlinear terms. The analysis of the dissipative structure in [26] implies that the density ρ and the velocity u are mainly affected by *damping* in the high-frequencies. That is in stark contrast to the Navier–Stokes–Korteweg equations [25], in which the density and the velocity are mainly affected by *heat kernel*. In other words, there is a regularity loss brought by Korteweg term because the lost of parabolic smoothness. To overcome this difficult, we make an innovative symmetric transformation of the system and obtain the first global result of one-dimensional Euler-Korteweg system in critical Besov space using commutative estimate and classical product estimates.

The rest of this paper unfolds as follows. In Sect. 2, we present an reformulated system and linearize it, then give out the main results of this paper. In Sect. 3, we devote ourself to the a-priori estimate. In Sect. 4, we prove the global existence and uniqueness of solutions. For the convenience of reader, in Appendix, we present the basic tools and estimates that will be needed, and the derivation of model (1.1).

2 Reformulated system and main result

In this section, we are going to reformulate (1.2). In order to overcome the difficulty of regularity loss, we introduce so-called kinetic energy a new unknown $m \triangleq \sqrt{\rho}u$, which is common especially in the vacuum problems (see [27, 33]). For convenience, we set $\varrho \triangleq \sqrt{4\rho}$, and the system (1.2) can be reformulated as

$$\begin{cases} \partial_t \varrho + \dfrac{1}{\varrho} m \partial_x \varrho + \partial_x m = 0, \\ \partial_t m + \dfrac{3}{\varrho} m \partial_x m - \dfrac{1}{\varrho^2} m^2 \partial_x \varrho + \theta \partial_x \varrho + \dfrac{1}{2} \varrho \partial_x \theta = \partial_x^3 \varrho - \dfrac{1}{\varrho} \partial_x \varrho \partial_{xx} \varrho, \\ \partial_t \theta + \dfrac{2}{\varrho} m \partial_x \theta + \dfrac{2}{\varrho} \theta \partial_x m - \dfrac{2}{\varrho^2} \theta m \partial_x \varrho - \dfrac{4\widetilde{\alpha}}{\varrho^2} \partial_{xx} \theta = 0, \end{cases} \tag{2.1}$$

with the evolution equation of $\partial_x \varrho$,

$$\partial_{xt}\varrho = -\partial_{xx}m - \partial_x\left(\frac{1}{\varrho}m\partial_x\varrho\right),$$

and the initial data

$$(\varrho, m, \theta)|_{t=0} = (\varrho_0, m_0, \theta_0). \tag{2.2}$$

For simplicity, we assume the heat transfer coefficient $\widetilde{\alpha}$ to be 1 and we focus on the case where the density and temperature goes to 2 and 1 at ∞. Setting $a \triangleq \varrho - 2$ and $\mathcal{T} \triangleq \theta - 1$, and looking for reasonably smooth solutions with positive density, and (2.1) is equivalent to

$$\begin{cases} \partial_t a + \partial_x m = F, \\ \partial_t m + \partial_x a + \partial_x \mathcal{T} - \partial_x^3 a = G, \\ \partial_t \mathcal{T} + \partial_x m - \partial_{xx}\mathcal{T} = H. \end{cases} \tag{2.3}$$

Accordingly, the development equation of $\partial_x\varrho$ is rewritten as

$$\partial_{xt}a + \partial_{xx}m = \partial_x F. \tag{2.4}$$

Defining $\widetilde{K}_i(a) = \int_0^a K_i(\tilde{a})d\tilde{a},\ K_1(a) = \frac{a}{a+2},\ K_2(a) = \frac{a^2+2a}{(a+1)^2}$ and

$$F = F_1 + F_2,\ G = \sum_{i=1}^{8} G_i,\ H = \sum_{i=1}^{10} H_i,$$

the non-linear term have the following concrete forms

$$\begin{aligned}
&F_1 = -\frac{1}{2}\partial_x am,\ F_2 = \frac{1}{2}\partial_x(\widetilde{K}_1(a))m;\ G_1 = -\frac{1}{2}\partial_x a\partial_{xx}a,\ G_2 = \frac{1}{2}\partial_x(\widetilde{K}_1(a))\partial_{xx}a,\\
&G_3 = -\frac{3}{2}m\partial_x m,\ G_4 = -\frac{1}{2}a\partial_x\mathcal{T},\ G_5 = \frac{1}{4}\partial_x am^2,\ G_6 = -\partial_x a\mathcal{T},\ G_7 = -\frac{1}{4}\partial_x(\widetilde{K}_2(a))m^2,\\
&G_8 = \frac{3}{4}K_1(a)\partial_x(m^2),\ H_1 = -K_2(a)\partial_{xx}\mathcal{T},\ H_2 = -m\partial_x\mathcal{T},\ H_3 = K_1(a)\partial_x m,\\
&H_4 = -\mathcal{T}\partial_x m,\ H_5 = \frac{1}{2}m\partial_x a,\ H_6 = K_1(a)m\partial_x\mathcal{T},\ H_7 = K_1(a)\partial_x m\mathcal{T},\\
&H_8 = -\frac{1}{2}\partial_x(\widetilde{K}_2(a))m,\\
&H_9 = \frac{1}{2}m\mathcal{T}\partial_x a,\ H_{10} = -\frac{1}{2}\partial_x(\widetilde{K}_2(a))m\mathcal{T}.
\end{aligned}$$

One key step in proving global results is a refined analysis of the linearized system (2.3), and this work is mainly inspired by the work of Kawashima et al. [26]. For readers' convenience, we recall the main results in [26] and firstly rewrite the linearized part of (2.3) as following hyperbolic-parabolic systems

$$A^0 U_t + AU_x = BU_{xx} + DU_{xxx} \tag{2.5}$$

with $U = (a, m, \mathcal{T})^T$ and

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking the Fourier transform with respect to x, then the linearized system translates into

$$A^0 \widehat{U}_t + i\xi A\widehat{U} - (i\xi)^2 B\widehat{U} - i\xi^3 D\widehat{U} = 0 \tag{2.6}$$

with $\xi \in \mathbb{R}$. And the corresponding eigenvalue problem is

$$\{\lambda A^0 + i\xi A - (i\xi)^2 B - (i\xi)^3 D\}\mathcal{V} = 0.$$

Then they proved that the system is of "standard type", that is

$$Re(\lambda(i\xi)) \leq C\frac{-|\xi|^2}{1+|\xi|^2}. \tag{2.7}$$

Applying the perturbation theory of one-parameter family of matrices, the following proposition can be obtained.

Proposition 2.1 (see[26]) *Assume* $\lambda = \lambda_j(i\xi),\ j = 1, 2, 3$ *are the eigenvalues of (2.3) then there are the following asymptotic expansions as* $\xi \to 0$ *and* $|\xi| \to \infty$

$$\lambda_j(i\xi) = \sum_{n=1}^{\infty} (i\xi)^n \lambda_j^{(n)}, \quad \xi \to 0, \quad \textit{and} \ \ \lambda_j(i\xi) = \sum_{n=1}^{3} (i\xi)^{3-n} \tilde{\lambda}_j^{(3-n)} + \sum_{n=0}^{\infty} (i\xi)^{-n} \lambda_j^{(-n)}, \quad |\xi| \to \infty,$$

where

$$\lambda_{1,3}^{(1)} = \pm\sqrt{2},\ \lambda_2^{(1)} = 0;\ \lambda_{1,2,3}^{(2)} = \frac{1}{4},\ \tilde{\lambda}_{1,2,3}^{(3)} = 0;\ \tilde{\lambda}_{1,2}^{(2)} = \pm i,\ \tilde{\lambda}_3^{(2)} = \frac{1}{2};\ \tilde{\lambda}_{1,2}^{(1)} = 0;$$

and

$$\tilde{\lambda}_{1,2}^{(0)} = -\frac{1}{2}\left\{\frac{1/2}{1/4+1} \pm i\left(1 + \frac{1}{1/4+1}\right)\right\}.$$

Notice that $\lambda_j^{(2)} > 0$ *for* $j = 1, 2, 3$. *Also,* $\tilde{\lambda}_3^{(2)} > 0$, $Re\tilde{\lambda}_k^{(2)} = 0$ *and* $Re\tilde{\lambda}_k^{(0)} < 0$ *for* $k = 1, 2$. *Therefore these asymptotic expansions suggest the optimality of the characterization (2.7) for the dissipative structure.*

From which, we can observe that there are a real and two complex conjugated eigenvalues coexist both in high and low frequencies and the solution might verifies in some way a Schrödinger equation. That is to say the classical method of "effective velocity" (see [20] and [5]) is not effect. Moreover in high frequencies, the treatment of the regularity loss term depends on the symmetry of the system which is also bring us a difficult when we research in L^p framework. Therefore, we only discuss the problem in L^2 framework.

Our main result is stated as follows:

Theorem 2.1 *Suppose the initial data* $(a_0, m_0, \mathcal{T}_0)^h \in \dot{B}^{\frac{5}{2}}_{2,1} \times \left(\dot{B}^{\frac{3}{2}}_{2,1}\right)^2$, $(a_0, m_0, \mathcal{T}_0)^\ell \in \left(\dot{B}^{-\frac{1}{2}}_{2,\infty}\right)^3$, *and the data satisfy for* $\delta_0 << 1$

$$\mathcal{X}_0 \overset{def}{=} \|(a_0, m_0, \mathcal{T}_0)\|^\ell_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} + \|a_0\|^h_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|m_0\|^h_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\mathcal{T}_0\|^h_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq \delta_0, \tag{2.8}$$

then system (2.3) *associated to the initial data* $(a_0, m_0, \mathcal{T}_0)$ *admits a unique global-in-time solution* $(a, m, \mathcal{T})$ *in the space* $\mathcal{X}$ *defined by*

$$\begin{aligned}
&(a, m, \mathcal{T})^\ell \in \tilde{\mathcal{C}}(\mathbb{R}_+; \dot{B}^{-\frac{1}{2}}_{2,\infty}(\mathbb{R})) \cap \widetilde{L}^1(\mathbb{R}_+; \dot{B}^{\frac{3}{2}}_{2,\infty}(\mathbb{R})) \cap \widetilde{L}^2(\mathbb{R}_+; \dot{B}^{\frac{1}{2}}_{2,\infty}(\mathbb{R})),\\
&a^h \in \tilde{\mathcal{C}}(\mathbb{R}_+; \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R})) \cap \widetilde{L}^1(\mathbb{R}_+; \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R})), \quad m^h \in \tilde{\mathcal{C}}(\mathbb{R}_+; \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R})) \cap \widetilde{L}^1(\mathbb{R}_+; \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R})),\\
&\mathcal{T}^h \in \tilde{\mathcal{C}}(\mathbb{R}_+; \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R})) \cap \widetilde{L}^1(\mathbb{R}_+; \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R})).
\end{aligned}$$

Moreover, the following inequality holds

$$\|(a, m, \mathcal{T})\|_{\mathcal{X}} \leq C\mathcal{X}_0. \tag{2.9}$$

Remark 2.1 This result in fact reveal the dissipative structure of Euler–Fourier–Korteweg system more precisely. Comparing the results in [26], it seems that it is more suitable to work with the same regularity for $\partial_{xx}a$, $\partial_x m$, $\mathcal{T}$ rather than $\partial_{xx}a$, $\partial_x m$, $\partial_x \mathcal{T}$ in high-frequencies region (see Proposition 4.3. in [26]). Otherwise, the damping effect of a, m and the parabolic effect of $\mathcal{T}$ may not be represented in the L^1 framework of time.

3 Global a priori estimates

For convenience, we define by $E(T)$ the energy functional and by $D(T)$ the corresponding dissipation functional:

$$E(T) := E^\ell(T) + E^h(T) = \|(a, m, \mathcal{T})\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\partial_{xx}a, \partial_x m, \mathcal{T})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}$$

and

$$\begin{aligned} D(T) := D^\ell(T) + D^h(T) = \|(a, m, \mathcal{T})\|^\ell_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})\cap\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,\infty})} \\ + \|(\partial_{xx}a, \partial_x m, \partial_{xx}\mathcal{T})\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})}, \end{aligned}$$

where the low frequencies and high frequencies part defined as (5.3). And then we give the key a-priori estimates leading to the global existence of solutions for (2.1).

Proposition 3.1 *Suppose* $(a, m, \mathcal{T})$ *is a solution of* (2.3) *for* $T > 0$*, with*

$$\|(a, m, \mathcal{T})\|_{L^\infty_t(L^\infty)} << 1. \tag{3.1}$$

Then, for all $0 \leq t < T$*, it holds that*

$$E(T) + D(T) \leq C(\mathcal{X}_0 + E(T)D(T)), \tag{3.2}$$

where $C > 0$ *is a universal constant and* $\mathcal{X}_0$ *is defined by* (2.8).

For clarify, we divide the proof of Proposition 3.1 into two cases: the high-frequencies and low-frequencies estimates.

3.1 High-frequencies estimates

In this subsection, we establish a priori estimates in high-frequencies region ($j \geq j_0+1$) and we always assume j_0 large in this paper. And finally establish the following Proposition.

Proposition 3.2 *Assume* $(a, m, \mathcal{T})$ *is a solution of* (2.1) *satisfying* (3.1) *then*

$$\begin{aligned} &\|a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\mathcal{T}\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|m\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{2}}_{2,1})} + \|(a, \mathcal{T})\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} + \|m\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,1})} \\ &\lesssim E(T)D(T) + \mathcal{X}_0. \end{aligned} \tag{3.3}$$

To prove the above Proposition, we first consider the temperature equation and devote to obtain the dissipation for $\mathcal{T}$ and finally establish the following Lemma.

Lemma 3.1 (The dissipation for $\mathcal{T}$) *Assume* $(a, m, \mathcal{T})$ *is a solution of* (2.1) *and* (2.2), *the initial data satisfying* (2.8)*, then*

$$\|\mathcal{T}\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\mathcal{T}\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} \lesssim E(T)D(T) + \|m\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\mathcal{T}_0\|^h_{\dot{B}^{\frac{1}{2}}_{2,1}}. \tag{3.4}$$

Proof In fact, the third equation in (2.3)

$$\partial_t \mathcal{T} - \partial_{xx}\mathcal{T} = \partial_x m + H$$

is a parabolic equation, and the standard estimates (see [7, 15]) implies

$$\|\mathcal{T}\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\mathcal{T}\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} \lesssim \|\mathcal{T}_0\|^h_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|m\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,1})} + \|H\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})}. \tag{3.5}$$

Next, we seriatim bound the non-linear part. Making use of the product law (5.6) and the para-linearization theorem (5.3), we have

$$\begin{aligned}\|H_1\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} &\lesssim \|K_1(a)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|\partial_{xx}\mathcal{T}\|_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \|a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|\mathcal{T}\|_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})}\\ &\lesssim E(T)D(T).\end{aligned}$$

In the last inequality, we used (5.5) and deduced the fact that

$$\begin{aligned}\|a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} &\lesssim \|a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})},\\ \|\mathcal{T}\|_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} &\lesssim \|\mathcal{T}\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\mathcal{T}\|^\ell_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})}.\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}&\|H_3\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|H_4\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|H_7\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})}\\ &\lesssim \|(a,\mathcal{T})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|m\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,1})} + \|a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|\mathcal{T}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|m\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,1})}\\ &\lesssim E(T)D(T).\end{aligned}$$

For H_2, it follows from the product law (5.6) and Hölder inequality

$$\|H_2\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \|m\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|\mathcal{T}\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim E(T)D(T).$$

In the last inequality, we used the interpolation inequality (5.2) and deduced

$$\|\mathcal{T}\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim \|\mathcal{T}\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\|\mathcal{T}\|^{\frac{1}{2}}_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} \lesssim \sqrt{E(T)D(T)},$$

$$\|m\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \|m\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,1})} \|m\|^{\frac{1}{2}}_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim \sqrt{E(T)D(T)}.$$

And very similarly, we have

$$\|H_6\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \|a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} \|m\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \|\mathcal{T}\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim E(T)D(T).$$

In fact, the interpolation inequality (5.2) also implies

$$\|a\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim \|a\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,\infty})} \|a\|^{\frac{1}{2}}_{\widetilde{L}^1_T(\dot{B}^{\frac{5}{2}}_{2,\infty})} \lesssim \sqrt{E(T)D(T)}.$$

And therefor we can further deduce by the product law (5.6)

$$\begin{aligned}
&\|H_5\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|H_8\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \|m\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \|a\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim E(T)D(T),\\
&\|H_9\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|H_{10}\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \|\mathcal{T}\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \|m\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \|a\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim E(T)D(T).
\end{aligned}$$

Hence

$$\|H\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim E(T)D(T). \tag{3.6}$$

The proof of Lemma 3.1 is finished. □

Next, we devote to obtain the dissipation for m and a. And the corresponding Lemma we established is stated as following.

Lemma 3.2 (The dissipation for m and a) *Assume $(a, m, \mathcal{T})$ is a solution of (2.1) and (2.2), the initial data satisfying (2.8), then*

$$\|(\partial_{xx}a, \partial_x m, \mathcal{T})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|(\partial_{xx}a, \partial_x m, \mathcal{T})\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim E(T)D(T) + \mathcal{X}_0. \tag{3.7}$$

Proof Applying the operator $\dot{\Delta}_j$ to the equations (2.3) gives

$$\begin{cases}
\dot{\Delta}_j \partial_t a + \dot{\Delta}_j \partial_x m = \dot{\Delta}_j F,\\
\dot{\Delta}_j \partial_t m + \dot{\Delta}_j \partial_x a + \dot{\Delta}_j \partial_x \mathcal{T} - \dot{\Delta}_j \partial_x^3 a = \dot{\Delta}_j G,\\
\dot{\Delta}_j \partial_t \mathcal{T} + \dot{\Delta}_j \partial_x m - \dot{\Delta}_j \partial_{xx} \mathcal{T} = \dot{\Delta}_j H.
\end{cases} \tag{3.8}$$

Multiplying the second and the third equation in (3.8) by $\dot{\Delta}_j\partial_x\mathcal{T}$ and $\dot{\Delta}_j\partial_x m$ respectively and then adding them together, we can obtain

$$\begin{aligned}&\frac{d}{dt}\int\dot{\Delta}_j\mathcal{T}\dot{\Delta}_j\partial_x m\,dx+\|\dot{\Delta}_j\partial_x m\|_{L^2}^2-\|\dot{\Delta}_j\partial_x\mathcal{T}\|_{L^2}^2-\int\dot{\Delta}_j\partial_{xx}\mathcal{T}\dot{\Delta}_j\partial_x m\,dx\\&\quad+\int\dot{\Delta}_j\partial_x^3a\dot{\Delta}_j\partial_x\mathcal{T}\,dx=\int\dot{\Delta}_j\partial_x a\dot{\Delta}_j\partial_x\mathcal{T}-\dot{\Delta}_j\partial_x G\dot{\Delta}_j\mathcal{T}+\dot{\Delta}_j H\dot{\Delta}_j\partial_x m\,dx.\end{aligned}\tag{3.9}$$

Deriving the second equation in (3.8) with respect to x and then multiplying by $\dot{\Delta}_j\partial_x m$, we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\partial_x m\|_{L^2}^2+\int\dot{\Delta}_j\partial_{xx}\mathcal{T}\dot{\Delta}_j\partial_x m+\dot{\Delta}_j\partial_{xx}a\dot{\Delta}_j\partial_x m-\dot{\Delta}_j\partial_x^4a\dot{\Delta}_j\partial_x m\,dx\\&\quad=\int\dot{\Delta}_j\partial_x G\dot{\Delta}_j\partial_x m\,dx.\end{aligned}$$

And similarly, we can deduce by the first equation in (3.8) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\dot{\Delta}_j\partial_x a,\dot{\Delta}_j\partial_{xx}a)\|_{L^2}^2-\int\dot{\Delta}_j\partial_x m\dot{\Delta}_j\partial_{xx}a\,dx+\int\dot{\Delta}_j\partial_x m\dot{\Delta}_j\partial_x^4a\,dx\\&\quad=\int\dot{\Delta}_j\partial_x F\dot{\Delta}_j\partial_x a+\dot{\Delta}_j\partial_{xx}F\dot{\Delta}_j\partial_{xx}a\,dx.\end{aligned}$$

Adding the above two equations together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\dot{\Delta}_j\partial_x a,\dot{\Delta}_j\partial_{xx}a,\dot{\Delta}_j\partial_x m)\|_{L^2}^2+\int\dot{\Delta}_j\partial_{xx}\mathcal{T}\dot{\Delta}_j\partial_x m\,dx\\&\quad=\int\dot{\Delta}_j\partial_x G\dot{\Delta}_j\partial_x m+\dot{\Delta}_j\partial_x F\dot{\Delta}_j\partial_x a+\dot{\Delta}_j\partial_{xx}F\dot{\Delta}_j\partial_{xx}a\,dx.\end{aligned}$$

On the other hand, multiplying the first and the third equation in (3.8) by $\dot{\Delta}_j\partial_{xx}\mathcal{T}$ and $\dot{\Delta}_j\partial_{xx}a$ respectively and then adding them together, we can obtain

$$\begin{aligned}&\frac{d}{dt}\int\dot{\Delta}_j\partial_{xx}a\dot{\Delta}_j\mathcal{T}\,dx+\int\dot{\Delta}_j\partial_{xx}\mathcal{T}\dot{\Delta}_j\partial_x m+\dot{\Delta}_j\partial_x^3a\dot{\Delta}_j\partial_x\mathcal{T}\,dx\\&\quad+\int\dot{\Delta}_j\partial_x m\dot{\Delta}_j\partial_{xx}a\,dx\\&\quad=\int\dot{\Delta}_j F\dot{\Delta}_j\partial_{xx}T+\dot{\Delta}_j H\dot{\Delta}_j\partial_{xx}a\,dx.\end{aligned}$$

The first equation in (3.8) implies that

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\partial_x a\|_{L^2}^2-\int\dot{\Delta}_j\partial_x m\dot{\Delta}_j\partial_{xx}a\,dx=\int\dot{\Delta}_j\partial_x F\dot{\Delta}_j\partial_x a\,dx.\tag{3.10}$$

Adding the above two equations together, we have

$$\frac{d}{dt}\left(\int \dot\Delta_j\partial_{xx}a\dot\Delta_j\mathcal{T}\,dx+\frac{1}{2}\|\dot\Delta_j\partial_x a\|_{L^2}^2\right)+\int \dot\Delta_j\partial_{xx}\mathcal{T}\dot\Delta_j\partial_x m+\dot\Delta_j\partial_x^3 a\dot\Delta_j\partial_x\mathcal{T}\,dx$$
$$=\int \dot\Delta_j F\dot\Delta_j\partial_{xx}T+\dot\Delta_j H\dot\Delta_j\partial_{xx}a+\dot\Delta_j\partial_x F\dot\Delta_j\partial_x a\,dx. \tag{3.11}$$

Define

$$L_j^2 := \|(\dot\Delta_j\partial_x a,\dot\Delta_j\partial_{xx}a,\dot\Delta_j\partial_x m)\|_{L^2}^2$$
$$+\int \dot\Delta_j\mathcal{T}\dot\Delta_j\partial_x m-\dot\Delta_j\partial_{xx}a\dot\Delta_j\mathcal{T}\,dx-\frac{1}{2}\|\dot\Delta_j\partial_x a\|_{L^2}^2,$$

then we can deduce by (3.9), (3.1) and (3.11)

$$\begin{aligned}
&\frac{d}{dt}L_j^2+\|\dot\Delta_j\partial_x m\|_{L^2}^2-\|\dot\Delta_j\partial_x\mathcal{T}\|_{L^2}^2\\
&\quad=\int \dot\Delta_j\partial_x a\left(\dot\Delta_j\partial_x\mathcal{T}+\dot\Delta_j\partial_x F\right)-\dot\Delta_j\partial_x G\dot\Delta_j\mathcal{T}+\dot\Delta_j H\dot\Delta_j\partial_x m\,dx\\
&\quad+2\int \dot\Delta_j\partial_x G\dot\Delta_j\partial_x m+\dot\Delta_j\partial_{xx}F\dot\Delta_j\partial_{xx}a\,dx\\
&\quad-\int \dot\Delta_j F\dot\Delta_j\partial_{xx}T+\dot\Delta_j H\dot\Delta_j\partial_{xx}a\,dx.
\end{aligned}\tag{3.12}$$

Multiplying the second and third equations in (3.8) by $\dot\Delta_j m$ and $\dot\Delta_j\mathcal{T}$ respectively, and then we can obtain

$$\frac{1}{2}\frac{d}{dt}\|(\dot\Delta_j m,\dot\Delta_j\mathcal{T})\|_{L^2}^2+\|\dot\Delta_j\partial_x\mathcal{T}\|_{L^2}^2+\int \dot\Delta_j\partial_x a\dot\Delta_j m dx-\int \dot\Delta_j\partial_x^3 a\dot\Delta_j m\,dx$$
$$=\int \dot\Delta_j G\dot\Delta_j m+\dot\Delta_j H\dot\Delta_j\mathcal{T}\,dx.$$

Adding together with (3.10), we can deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\dot\Delta_j m,\dot\Delta_j\mathcal{T},\dot\Delta_j\partial_x a)\|_{L^2}^2+\|\dot\Delta_j\partial_x\mathcal{T}\|_{L^2}^2+\int \dot\Delta_j\partial_x a\dot\Delta_j m\,dx\\
&\quad=\int \dot\Delta_j G\dot\Delta_j m+\dot\Delta_j H\dot\Delta_j\mathcal{T}+\dot\Delta_j\partial_x F\dot\Delta_j\partial_x a\,dx.
\end{aligned}\tag{3.13}$$

Summing (3.12) and (3.13) up, we can obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_j^2+\|\dot\Delta_j\partial_x m\|_{L^2}^2+\|\dot\Delta_j\partial_x\mathcal{T}\|_{L^2}^2\\&\quad\lesssim\|(\dot\Delta_j\partial_{xx}a,\dot\Delta_j\partial_x m,\dot\Delta_j\mathcal{T})\|_{L^2}\|(\dot\Delta_j F,\dot\Delta_j G,\dot\Delta_j H)\|_{L^2}+\|\dot\Delta_j\partial_x a\|_{L^2}^2\\&\qquad+\int\dot\Delta_j\partial_x G\dot\Delta_j\partial_x m+\dot\Delta_j\partial_{xx}F\dot\Delta_j\partial_{xx}a\,dx-\int\dot\Delta_j\partial_x G\dot\Delta_j\mathcal{T}+\dot\Delta_j F\dot\Delta_j\partial_{xx}\mathcal{T}\,dx.\end{aligned}\tag{3.14}$$

And here $\mathcal{L}_j^2:=L_j^2+\|(\dot\Delta_j m,\dot\Delta_j\mathcal{T},\dot\Delta_j\partial_x a)\|_{L^2}^2$. Multiplying the first and the second equation in (3.8) by $-\dot\Delta_j m_x$ and $\dot\Delta_j a_x$ respectively, and then adding them together we can get the dissipation of a

$$\begin{aligned}&\frac{d}{dt}\int\dot\Delta_j m\dot\Delta_j\partial_x a\,dx+\|\dot\Delta_j\partial_x a\|_{L^2}^2+\|\dot\Delta_j\partial_{xx}a\|_{L^2}^2-\|\dot\Delta_j\partial_x m\|_{L^2}^2\\&\quad=\int\dot\Delta_j G\dot\Delta_j\partial_x a-\dot\Delta_j F\dot\Delta_j\partial_x m-\dot\Delta_j\partial_x\mathcal{T}\dot\Delta_j\partial_x a\,dx\\&\quad\lesssim\|(\dot\Delta_j F,\dot\Delta_j G)\|_{L^2}\|(\dot\Delta_j\partial_x a,\dot\Delta_j\partial_x m)\|_{L^2}+C(\tilde\epsilon)\|\dot\Delta_j\mathcal{T}_x\|_{L^2}^2+\tilde\epsilon\|\dot\Delta_j a_x\|_{L^2}^2.\end{aligned}$$

Choosing $\tilde\epsilon>0$ small enough, we can obtain by adding above equation together with (3.14) for $j_0>>1$

$$\begin{aligned}&\frac{d}{dt}\mathcal{H}_j^2+\|(\dot\Delta_j\partial_{xx}a,\dot\Delta_j\partial_x m,\dot\Delta_j\mathcal{T})\|_{L^2}^2\\&\quad\lesssim\|(\dot\Delta_j\partial_{xx}a,\dot\Delta_j\partial_x m,\dot\Delta_j\mathcal{T})\|_{L^2}\|(\dot\Delta_j F,\dot\Delta_j G,\dot\Delta_j H)\|_{L^2}\\&\qquad+\int\dot\Delta_j\partial_x G\dot\Delta_j\partial_x m+\dot\Delta_j\partial_{xx}F\dot\Delta_j\partial_{xx}a\,dx-\int\dot\Delta_j\partial_x G\dot\Delta_j\mathcal{T}+\dot\Delta_j F\dot\Delta_j\partial_{xx}\mathcal{T}\,dx.\end{aligned}\tag{3.15}$$

Here $\mathcal{H}_j^2:=\mathcal{L}_j^2+\frac{1}{2}\int\dot\Delta_j m\dot\Delta_j\partial_x a\,dx$, and the Cauchy inequality implies

$$\mathcal{H}_j^2\approx\|(\dot\Delta_j\partial_{xx}a,\dot\Delta_j\partial_x m,\dot\Delta_j\mathcal{T})\|_{L^2}^2.$$

Next we bound the regular loss part by the symmetry of system and commutator estimates. More precisely, we bound I_1 with

$$I_1:=\int\dot\Delta_j\partial_x G\dot\Delta_j\partial_x m+\dot\Delta_j\partial_{xx}F\dot\Delta_j\partial_{xx}a\,dx.$$

For $\partial_{xx}F_1$, we can rewrite it as following form

$$\int\partial_{xx}\dot\Delta_j(\partial_x am)\dot\Delta_j\partial_{xx}a\,dx=\int\partial_x\dot\Delta_j(\partial_{xx}am)\dot\Delta_j\partial_{xx}a+\partial_x\dot\Delta_j(\partial_x a\partial_x m)\dot\Delta_j\partial_{xx}a\,dx.\tag{3.16}$$

Defining the commutator as $[f,g]:=fg-gf$, then we have

$$\int\partial_x\dot\Delta_j(\partial_{xx}am)\dot\Delta_j\partial_{xx}a\,dx=\int\partial_x\Big([\dot\Delta_j,m]\partial_{xx}a\Big)\dot\Delta_j\partial_{xx}a+\partial_x\Big(m\dot\Delta_j\partial_{xx}a\Big)\dot\Delta_j\partial_{xx}a\,dx.$$

By using the commutator estimates (5.2), we can easily get

$$\begin{aligned}\int \partial_x\Big([\dot{\Delta}_j, m]\partial_{xx}a\Big)\dot{\Delta}_j\partial_{xx}a\,dx &\lesssim \|\partial_x\Big([\dot{\Delta}_j, m]\partial_{xx}a\Big)\|_{L^2}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}\\ &\lesssim c_j\|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}.\end{aligned}$$

Making use of integration by parts, we can deduce that

$$\begin{aligned}\int \partial_x\Big(m\dot{\Delta}_j\partial_{xx}a\Big)\dot{\Delta}_j\partial_{xx}a\,dx &= \int \partial_x m(\dot{\Delta}_j\partial_{xx}a\Big)^2+\frac{1}{2}m\partial_x\Big(\dot{\Delta}_j\partial_{xx}a)^2\,dx\\ &\lesssim \|\partial_x m\|_{L^\infty}\|\dot{\Delta}_j\partial_{xx}a\|^2_{L^2}.\end{aligned}$$

Hence, we have by the embedding $\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$

$$\int \partial_x\dot{\Delta}_j(\partial_{xx}am)\dot{\Delta}_j\partial_{xx}a\,dx \lesssim c_j\|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}+\|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\dot{\Delta}_j\partial_{xx}a\|^2_{L^2}. \tag{3.17}$$

And similarly, we can deduce by the commutator estimates (5.2) that

$$\begin{aligned}&\int \partial_x\dot{\Delta}_j(\partial_x a\partial_x m)\dot{\Delta}_j\partial_{xx}a-\partial_x a\dot{\Delta}_j\partial_{xx}m\dot{\Delta}_j\partial_{xx}a\,dx\\ &\lesssim \|\partial_x([\dot{\Delta}_j,\partial_x a]\partial_x m)\|_{L^2}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}+\|\partial_{xx}a\|_{L^\infty}\|\dot{\Delta}_j\partial_x m\|_{L^2}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}\\ &\lesssim 2^{-\frac{1}{2}j}c_j\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}+\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\dot{\Delta}_j\partial_x m\|_{L^2}\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}.\end{aligned} \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16), we have

$$\begin{aligned}&\int \partial_{xx}\dot{\Delta}_j(\partial_x am)\dot{\Delta}_j\partial_{xx}a-\partial_x a\dot{\Delta}_j\partial_{xx}m\dot{\Delta}_j\partial_{xx}a\,dx\\ &\lesssim\Big(2^{-\frac{1}{2}j}c_j\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\dot{\Delta}_j\partial_x m\|_{L^2}\Big)\|\dot{\Delta}_j\partial_{xx}a\|_{L^2}.\end{aligned}$$

For $\partial_x G_1$, we can rewrite it as following form

$$\begin{aligned}&\int \partial_x\dot{\Delta}_j(\partial_x a\partial_{xx}a)\dot{\Delta}_j\partial_x m\,dx\\ &=\int\Big(\partial_x\Big([\dot{\Delta}_j,\partial_x a]\partial_{xx}a\Big)+\partial_x\Big(\partial_x a\dot{\Delta}_j\partial_{xx}a\Big)\Big)\dot{\Delta}_j\partial_x m\,dx,\end{aligned}$$

and then it follows the commutator estimates (5.2)

$$\int \partial_x \dot{\Delta}_j(\partial_x a \partial_{xx} a)\dot{\Delta}_j \partial_x m + \partial_x a \dot{\Delta}_j \partial_{xx} m \dot{\Delta}_j \partial_{xx} a \, dx \lesssim 2^{-\frac{1}{2}j} c_j \|a\|^2_{\dot{B}^{\frac{5}{2}}_{2,1}} \|\dot{\Delta}_j \partial_x m\|_{L^2}. \tag{3.19}$$

Adding (3.1) and (3.19) together, we have

$$\begin{aligned}&\int \dot{\Delta}_j \partial_{xx} F_1 \dot{\Delta}_j \partial_{xx} a + \dot{\Delta}_j \partial_x G_1 \dot{\Delta}_j \partial_x m \, dx \\ &\lesssim \left(2^{-\frac{1}{2}j} c_j E(T)\left(\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\right) + E(T)\|\dot{\Delta}_j \partial_x m\|_{L^2}\right) \|(\dot{\Delta}_j \partial_{xx} a, \dot{\Delta}_j \partial_x m)\|_{L^2}.\end{aligned}$$

Very similarly, we can also establish the following estimates by (5.3)

$$\begin{aligned}&\int \dot{\Delta}_j \partial_{xx} F_2 \dot{\Delta}_j \partial_{xx} a + \dot{\Delta}_j \partial_x G_2 \dot{\Delta}_j \partial_x m \, dx \\ &\lesssim \left(2^{-\frac{1}{2}j} c_j E(T)\left(\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\right) + E(T)\|\dot{\Delta}_j \partial_x m\|_{L^2}\right) \|(\dot{\Delta}_j \partial_{xx} a, \dot{\Delta}_j \partial_x m)\|_{L^2}.\end{aligned} \tag{3.20}$$

For $\partial_x G_3$ to $\partial_x G_8$, we can rewrite it as following form

$$\sum_{i=3}^{8} \int \dot{\Delta}_j \partial_x G_i \dot{\Delta}_j \partial_x m \, dx \lesssim \sum_{k=1}^{6} \int \partial_x \left([\dot{\Delta}_j, b_k] d_k\right) \dot{\Delta}_j \partial_x m + \partial_x \left(b_k \dot{\Delta}_j d_k\right) \dot{\Delta}_j \partial_x m \, dx.$$

Here b_k and d_k are sequences of functions

$$\begin{aligned} b_k &:= \{m, a, m, \mathcal{T}, m^2, K_1(a)\}, \\ d_k &:= \{\partial_x m, \partial_x \mathcal{T}, \partial_x a, \partial_x a, \partial_x \widetilde{K}_2(a), \partial_x m^2\}. \end{aligned}$$

Making use of the commutator estimates (5.2), we have

$$\int \partial_x \left([\dot{\Delta}_j, b_k] d_k\right) \dot{\Delta}_j \partial_x m \, dx \lesssim 2^{-\frac{1}{2}j} c_j \|(a, m, \mathcal{T})\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\dot{\Delta}_j \partial_x m\|_{L^2}.$$

And the Hölder inequality and Bernstein's inequalities implies

$$\begin{aligned}&\int \partial_x \left(b_k \dot{\Delta}_j d_k\right) \dot{\Delta}_j \partial_x m \, dx \\ &\lesssim \|\partial_x b_k\|_{L^\infty} \|\dot{\Delta}_j d_k\|_{L^2} \|\dot{\Delta}_j \partial_x m\|_{L^2} + \int b_k \dot{\Delta}_j \partial_x d_k \dot{\Delta}_j \partial_x m \, dx.\end{aligned}$$

It not difficult to obtain that for $1 \le k \le 6$

$$\|\partial_x b_k\|_{L^\infty} \|\dot{\Delta}_j d_k\|_{L^2} \|\dot{\Delta}_j \partial_x m\|_{L^2}$$

$$\lesssim E(T)2^j\|(\dot\Delta_j a,\ \dot\Delta_j\widetilde{K}_2(a),\ \dot\Delta_j m,\ \dot\Delta_j\mathcal{T})\|_{L^2}\|\dot\Delta_j\partial_x m\|_{L^2}.$$

By the integration by parts, we can obtain that

$$\int b_1\dot\Delta_j\partial_x d_1\dot\Delta_j\partial_x m\,dx=-\frac{1}{2}\int\partial_x m(\partial_x\dot\Delta_j m)^2\,dx\lesssim\|\partial_x m\|_{L^\infty}\|\dot\Delta_j\partial_x m\|_{L^2}^2.$$

And similarly

$$\begin{aligned}&\int b_6\dot\Delta_j\partial_x d_6\dot\Delta_j\partial_x m\,dx\\&\lesssim\|K_1(a)\|_{L^\infty}\|[\dot\Delta_j,m]\partial_x m\|_{L^2}\|\partial_x\dot\Delta_j m\|_{L^2}+E(T)\|\dot\Delta_j\partial_x m\|_{L^2}^2\\&\lesssim E(T)\left(2^{-\frac{1}{2}j}c_j\|m\|^2_{\dot B^{\frac{3}{2}}_{2,1}}\|\dot\Delta_j\partial_x m\|_{L^2}+\|\dot\Delta_j\partial_x m\|_{L^2}^2\right).\end{aligned}$$

Finally, we can deduce by the Hölder inequalities that

$$\sum_{k=2}^{5}\int b_k\dot\Delta_j\partial_x d_k\dot\Delta_j\partial_x m\,dx\lesssim E(T)\|\dot\Delta_j\partial_{xx}a,\ \dot\Delta_j\partial_{xx}\mathcal{T}\|_{L^2}\|\dot\Delta_j\partial_x m\|_{L^2}.$$

Hence, we have for $E(T)\lesssim\delta_1$

$$\begin{aligned}\sum_{i=3}^{8}\int\dot\Delta_j\partial_x G_i\dot\Delta_j\partial_x m\,dx\lesssim{}&2^{-\frac{1}{2}j}c_j\|(a,m,\mathcal{T})\|^2_{\dot B^{\frac{3}{2}}_{2,1}}\|\dot\Delta_j\partial_x m\|_{L^2}\\&+2^j\|(\dot\Delta_j\partial_x a,\ \dot\Delta_j\partial_x\widetilde{K}_2(a),\ \dot\Delta_j m,\ \dot\Delta_j\partial_x\mathcal{T})\|_{L^2}\|\dot\Delta_j\partial_x m\|_{L^2}.\end{aligned}\tag{3.21}$$

Adding (3.1) to (3.21) together, we can obtain that

$$\begin{aligned}&\int\dot\Delta_j\partial_x G\dot\Delta_j\partial_x m+\dot\Delta_j\partial_{xx}F\dot\Delta_j\partial_{xx}a\,dx\\&\lesssim 2^{-\frac{1}{2}j}c_jE(T)\|(\partial_x a,m,\partial_x\mathcal{T})\|_{\dot B^{\frac{3}{2}}_{2,1}}\|(\dot\Delta_j\partial_{xx}a,\ \dot\Delta_j\partial_x m)\|_{L^2}\\&\quad+\|(\dot\Delta_j\partial_{xx}a,\ \dot\Delta_j\partial_{xx}\widetilde{K}_2(a),\ \dot\Delta_j\partial_x m,\ \dot\Delta_j\partial_{xx}\mathcal{T})\|_{L^2}\|\dot\Delta_j\partial_x m\|_{L^2}.\end{aligned}\tag{3.22}$$

Next, we bound $\int\dot\Delta_j\partial_x G\dot\Delta_j\mathcal{T}dx$, and we rewrite it as following commutator form for $k\in\mathbb{Z}$ and $1\le k\le 8$

$$\sum_{i=1}^{8}\int\dot\Delta_j\partial_x G_i\dot\Delta_j\mathcal{T}\,dx\lesssim\sum_{k=1}^{5}\int\partial_x\Big([\dot\Delta_j,h_k]z_k\Big)\dot\Delta_j\mathcal{T}+\partial_x\Big(h_k\dot\Delta_j z_k\Big)\dot\Delta_j\mathcal{T}\,dx.$$

Here h_k and z_k are sequences of functions

$$h_k := \{\partial_x a, \partial_x \widetilde{K}_1(a), m, a, m^2, \mathcal{T}, m^2, K_1(a)\},$$
$$z_k := \{\partial_{xx} a, \partial_{xx} a, \partial_x m, \partial_x \mathcal{T}, \partial_x a, \partial_x a, \partial_x \widetilde{K}_2(a), \partial_x m^2\}.$$

Making use of the commutator estimates (5.2), we have

$$\int \partial_x \Big([\dot{\Delta}_j, h_k] z_k \Big) \dot{\Delta}_j \mathcal{T}\, dx \lesssim 2^{-\frac{1}{2}j} c_j E(T) \|\partial_x a, m, \mathcal{T}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\dot{\Delta}_j \mathcal{T}\|_{L^2}.$$

And the Hölder inequality and Bernstein's inequalities implies

$$\int \partial_x \Big(h_k \dot{\Delta}_j z_k \Big) \dot{\Delta}_j \mathcal{T}\, dx \lesssim \|\partial_x h_k\|_{L^\infty} \|\dot{\Delta}_j z_k\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + \int h_k \partial_x \dot{\Delta}_j z_k \dot{\Delta}_j \mathcal{T}\, dx.$$

It is not difficult to obtain that

$$\begin{aligned} &\|\partial_x h_k\|_{L^\infty} \|\dot{\Delta}_j z_k\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \\ &\quad \lesssim 2^j E(T) \|(\dot{\Delta}_j \partial_x a, \dot{\Delta}_j m, \dot{\Delta}_j \widetilde{K}_2(a), \dot{\Delta}_j \mathcal{T}, \dot{\Delta}_j m^2)\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2}. \end{aligned}$$

By the Hölder inequalities, we can deduce that

$$\sum_{k=1}^{6} \int h_k \partial_x \dot{\Delta}_j z_k \dot{\Delta}_j \mathcal{T}\, dx \lesssim 2^j E(T) \Big(\|(\dot{\Delta}_j \partial_{xx} a, \dot{\Delta}_j \partial_x m)\|_{L^2} + 2^j \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \Big) \|\dot{\Delta}_j \mathcal{T}\|_{L^2}.$$

Be similar to the previous, we can obtain that

$$\begin{aligned} \int h_7 \partial_x \dot{\Delta}_j z_7 \dot{\Delta}_j \mathcal{T}\, dx &\lesssim \|m^2\|_{L^\infty} \|\partial_x([\dot{\Delta}_j, K_2(a)] \partial_x a)\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \\ &\quad + E(T) \left(\|\partial_x \dot{\Delta}_j a\|_{L^2} + \|\dot{\Delta}_j a\|_{L^2} \right) \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \\ &\lesssim E(T) \left(2^{-\frac{1}{2}j} c_j \|a\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\partial_x \dot{\Delta}_j a\|_{L^2} \right) \|\dot{\Delta}_j \mathcal{T}\|_{L^2}. \end{aligned}$$

And also, we have

$$\begin{aligned} \int h_8 \partial_x \dot{\Delta}_j z_8 \dot{\Delta}_j \mathcal{T}\, dx &\lesssim \|K_1(a)\|_{L^\infty} \|\partial_x([\dot{\Delta}_j, m] \partial_x m)\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \\ &\quad + E(T) \left(\|\dot{\Delta}_j m\|_{L^2} + \|\dot{\Delta}_j \partial_x m\|_{L^2} \right) \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \\ &\lesssim E(T) \left(2^{-\frac{1}{2}j} c_j \|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\partial_x \dot{\Delta}_j m\|_{L^2} \right) \|\dot{\Delta}_j \mathcal{T}\|_{L^2}. \end{aligned}$$

Therefor, we finally establish that

$$\int \dot{\Delta}_j \partial_x G \dot{\Delta}_j \mathcal{T}\, dx \lesssim 2^{-\frac{1}{2}j} c_j E(T) \|a, \partial_x a, m, \mathcal{T}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + E(T) \tilde{D}_j \|\dot{\Delta}_j \mathcal{T}\|_{L^2}. \tag{3.23}$$

Here $\tilde{D}_j := \|(\dot{\Delta}_j \partial_{xx} a, \dot{\Delta}_j \partial_x m, \dot{\Delta}_j \partial_{xx} \widetilde{K}_2(a), \dot{\Delta}_j \partial_{xx} \mathcal{T}, \dot{\Delta}_j \partial_x m^2)\|_{L^2}$.

Now, we devote oneself to deal with $\int \dot{\Delta}_j F \dot{\Delta}_j \partial_{xx} \mathcal{T}\, dx$. By integration by parts, we have

$$\int \dot{\Delta}_j F_1 \dot{\Delta}_j \partial_{xx} \mathcal{T}\, dx = \int \partial_x \dot{\Delta}_j (\partial_{xx} a m) \dot{\Delta}_j \mathcal{T} + \partial_x \dot{\Delta}_j (\partial_x a \partial_x m) \dot{\Delta}_j \mathcal{T}\, dx$$

Making use of the commutator estimates, we can obtain

$$\begin{aligned}
&\int \partial_x \dot{\Delta}_j (\partial_{xx} a m) \dot{\Delta}_j \mathcal{T}\, dx \\
&\lesssim \|\partial_x ([\dot{\Delta}_j, m] \partial_{xx} a)\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + E(T) \|\dot{\Delta}_j \partial_{xx} a\|_{L^2} \|\dot{\Delta}_j \partial_x \mathcal{T}\|_{L^2} \\
&\lesssim 2^{-\frac{1}{2}j} c_j \|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + E(T) \|\dot{\Delta}_j \partial_x a\|_{L^2} \|\dot{\Delta}_j \partial_{xx} \mathcal{T}\|_{L^2}.
\end{aligned}$$

And similarly,

$$\begin{aligned}
&\int \partial_x \dot{\Delta}_j (\partial_x a \partial_x m) \dot{\Delta}_j \mathcal{T}\, dx \\
&\lesssim \|\partial_x ([\dot{\Delta}_j, \partial_x a] \partial_x m)\|_{L^2} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + E(T) \|\dot{\Delta}_j \partial_x m\|_{L^2} \|\dot{\Delta}_j \partial_x \mathcal{T}\|_{L^2} \\
&\lesssim 2^{-\frac{1}{2}j} c_j \|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + E(T) \|\dot{\Delta}_j \partial_x m\|_{L^2} \|\dot{\Delta}_j \partial_{xx} \mathcal{T}\|_{L^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \dot{\Delta}_j F_1 \dot{\Delta}_j \partial_{xx} \mathcal{T}\, dx \\
&\lesssim 2^{-\frac{1}{2}j} c_j \|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|m\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\dot{\Delta}_j \mathcal{T}\|_{L^2} + E(T) \|(\dot{\Delta}_j \partial_x a, \dot{\Delta}_j \partial_x m)\|_{L^2} \|\dot{\Delta}_j \partial_{xx} \mathcal{T}\|_{L^2}.
\end{aligned} \tag{3.24}$$

And very similarly, we have

$$\begin{aligned}
&\int \dot{\Delta}_j F_2 \dot{\Delta}_j \partial_{xx} \mathcal{T}\, dx \\
&\lesssim 2^{-\frac{1}{2}j} c_j \left(\|a\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|(a, m)\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|a\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} \right) \|\dot{\Delta}_j \mathcal{T}\|_{L^2} \\
&\quad + E(T) \|(\dot{\Delta}_j \partial_{xx} a, \dot{\Delta}_j \partial_x m)\|_{L^2} \|\dot{\Delta}_j \partial_{xx} \mathcal{T}\|_{L^2}.
\end{aligned} \tag{3.25}$$

Substituting (3.22) to (3.25) into (3.15), we can deduce by Lemma 5.3

$$\begin{aligned}&\|(\dot{\Delta}_j\partial_{xx}a,\dot{\Delta}_j\partial_x m,\dot{\Delta}_j\mathcal{T})\|_{L^\infty_T(L^2)}+\|(\dot{\Delta}_j\partial_{xx}a,\dot{\Delta}_j\partial_x m,\dot{\Delta}_j\mathcal{T})\|_{L^1_T(L^2)}\\&\quad\lesssim\|(\dot{\Delta}_jF,\dot{\Delta}_jG,\dot{\Delta}_jH)\|_{L^2}+E(T)\|\dot{\Delta}_j\partial_{xx}\mathcal{T}\|_{L^2}+2^{-\frac12 j}c_jE(T)\|(\partial_x a,m,\partial_x\mathcal{T})\|_{\dot{B}^{\frac32}_{2,1}}\\&\qquad+2^{-\frac12 j}c_j\|a\|^2_{\dot{B}^{\frac32}_{2,1}}+E(T)\tilde{D}_j+\|(\dot{\Delta}_j\partial_{xx}a_0,\dot{\Delta}_j\partial_x m_0,\dot{\Delta}_j\mathcal{T}_0)\|_{L^2}.\end{aligned}$$

Multiplying by $2^{-\frac12 j}$ then summing up for $j\geq j_0+1$, we can obtain

$$\begin{aligned}&\|(\partial_{xx}a,\partial_x m,\mathcal{T})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac12}_{2,1})}+\|(\partial_{xx}a,\partial_x m,\mathcal{T})\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}\\&\quad\lesssim\|(F,G,H)\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}+E(T)D(T)+\mathcal{X}_0.\end{aligned}$$

In the Lemma 3.1, we have bounded $\|H\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}$ and obtained (3.6). And next we only need to deal with $\|(F,G)\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}$. Making use of the product law (5.6), paralinearization theorem (5.3) and interpolation inequality, we can deduce

$$\|F\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}\lesssim\|m\|_{L^2_T(\dot{B}^{\frac12}_{2,1})}\|(\partial_x a,\partial_x(\widetilde{K}_1(a)))\|_{\widetilde{L}^2_T(\dot{B}^{\frac12}_{2,1})}\lesssim E(T)D(T).\tag{3.26}$$

And similarly, we have

$$\begin{aligned}\|G\|_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}\lesssim{}&\|\partial_x a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac12}_{2,1})}\|\partial_{xx}a\|_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}+\int\|m\|_{\dot{B}^{\frac12}_{2,1}}\|m\|_{\dot{B}^{\frac32}_{2,1}}\,dt\\&+\|(\mathcal{T},\partial_x\mathcal{T})\|_{\widetilde{L}^2_T(\dot{B}^{\frac12}_{2,1})}\|(a,\partial_x a)\|_{\widetilde{L}^2_T(\dot{B}^{\frac12}_{2,1})}.\end{aligned}\tag{3.27}$$

By using (5.5), we can get

$$\begin{aligned}&\|\partial_x a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac12}_{2,1})}\|\partial_{xx}a\|_{\widetilde{L}^1_T(\dot{B}^{\frac12}_{2,1})}\\&\quad\lesssim\left(\|a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac52}_{2,1})}+\|a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{-\frac12}_{2,\infty})}\right)\left(\|a\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac52}_{2,1})}+\|a\|^\ell_{\widetilde{L}^1_T(\dot{B}^{\frac32}_{2,\infty})}\right)\lesssim E(T)D(T).\end{aligned}\tag{3.28}$$

It is easy to obtain the following fact by Hölder inequality and (5.5)

$$\begin{aligned}\int_0^T \|m\|_{\dot B^{\frac12}_{2,1}} \|m\|_{\dot B^{\frac32}_{2,1}}\, dt &\lesssim \int_0^T \|m\|_{\dot B^{\frac12}_{2,1}} \left(\|m\|^h_{\dot B^{\frac32}_{2,1}} + \|m\|^\ell_{\dot B^{\frac12}_{2,1}} \right) dt \\ &\lesssim \|m\|_{\widetilde L^\infty_T(\dot B^{\frac12}_{2,1})} \|m\|^h_{\widetilde L^1_T(\dot B^{\frac32}_{2,1})} + \|m\|^2_{L^2_T(\dot B^{\frac12}_{2,1})} \\ &\lesssim E(T)D(T) + \|m\|^2_{\widetilde L^2_T(\dot B^{\frac12}_{2,1})}.\end{aligned} \tag{3.29}$$

Making use of the interpolation inequality, we can deduce that

$$\begin{aligned}\|(m, \mathcal{T}, \partial_x \mathcal{T})\|_{\widetilde L^2_T(\dot B^{\frac12}_{2,1})} &\lesssim \|(m, \mathcal{T}, \partial_x \mathcal{T})\|^{\frac12}_{\widetilde L^\infty_T(\dot B^{-\frac12}_{2,\infty})} \|(m, \mathcal{T}, \partial_x \mathcal{T})\|^{\frac12}_{\widetilde L^1_T(\dot B^{\frac32}_{2,\infty})} \\ &\lesssim \sqrt{E(T)D(T)}.\end{aligned} \tag{3.30}$$

Similarly,

$$\|(a, \partial_x a)\|_{L^2_T(\dot B^{\frac12}_{2,1})} \lesssim \|(a, \partial_x a)\|^{\frac12}_{\widetilde L^\infty_T(\dot B^{-\frac12}_{2,\infty})} \|(a, \partial_x a)\|^{\frac12}_{\widetilde L^1_T(\dot B^{\frac32}_{2,\infty})} \lesssim \sqrt{E(T)D(T)}. \tag{3.31}$$

Substituting (3.28)–(3.31) into (3.27), we have

$$\|G\|_{\widetilde L^1_T(\dot B^{\frac12}_{2,1})} \lesssim \|G\|_{\widetilde L^1_T(\dot B^{\frac12}_{2,1})} \lesssim E(T)D(T). \tag{3.32}$$

By substituting (3.6), (3.26) and (3.32) into (3.1), we can finally obtain

$$\|(\partial_{xx} a, \partial_x m, \mathcal{T})\|^h_{\widetilde L^\infty_T(\dot B^{\frac12}_{2,1})} + \|(\partial_{xx} a, \partial_x m, \mathcal{T})\|^h_{\widetilde L^1_T(\dot B^{\frac12}_{2,1})} \lesssim E(T)D(T) + \mathcal{X}_0.$$

The proof of Lemma 3.2 is finished. □

Based on the Lemma 3.1 and Lemma 3.2, we can prove the Proposition 3.2. Multiplying (3.4) by small constant $\varepsilon > 0$ and adding together with (3.7), we can directly obtain (3.33).

3.2 Low-frequencies estimates

In this subsection, we establish a prior estimates in low-frequencies region ($j \le j_0$) and establish the following Proposition.

Proposition 3.3 *Assume* $(a, m, \mathcal{T})$ *is a solution of* (2.1) *and* (2.2)*, the initial data satisfying* (2.8)*, then*

$$\|(a, m, \mathcal{T})\|^\ell_{\widetilde L^\infty_T(\dot B^{-\frac12}_{2,\infty})} + \|(a, m, \mathcal{T})\|^\ell_{\widetilde L^1_T(\dot B^{\frac32}_{2,\infty})} \lesssim E(T)D(T) + \mathcal{X}_0. \tag{3.33}$$

Proof Be similar to the before, standard energy method implies

$$\frac{1}{2}\frac{d}{dt}\|(\dot{\Delta}_j a, \dot{\Delta}_j m, \dot{\Delta}_j \mathcal{T})\|_{L^2}^2 + \|\dot{\Delta}_j \partial_x \mathcal{T}\|_{L^2}^2 - \int \dot{\Delta}_j \partial_x^3 a \dot{\Delta}_j m \, dx$$
$$= \int \dot{\Delta}_j a \dot{\Delta}_j F + \dot{\Delta}_j m \dot{\Delta}_j G + \dot{\Delta}_j \mathcal{T} \dot{\Delta}_j H \, dx.$$

Applying the operator $\dot{\Delta}_j$ to (2.4) and then multiplying by $\dot{\Delta}_j \partial_x a$, we have

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j \partial_x a\|_{L^2}^2 + \int \dot{\Delta}_j \partial_{xx} m \dot{\Delta}_j \partial_x a \, dx = \int \dot{\Delta}_j \partial_x F \dot{\Delta}_j \partial_x a \, dx.$$

Adding the above two equation together, we have by Hölder's inequality and Bernstein's inequality for $j \le j_0$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(\dot{\Delta}_j a, \dot{\Delta}_j \partial_x a, \dot{\Delta}_j m, \dot{\Delta}_j \mathcal{T})\|_{L^2}^2 + \|\dot{\Delta}_j \partial_x \mathcal{T}\|_{L^2}^2 \\ &= \int \dot{\Delta}_j \partial_x F \dot{\Delta}_j \partial_x a + \dot{\Delta}_j a \dot{\Delta}_j F + \dot{\Delta}_j m \dot{\Delta}_j G + \dot{\Delta}_j \mathcal{T} \dot{\Delta}_j H \, dx \\ &\lesssim \|(\dot{\Delta}_j a, \dot{\Delta}_j m, \dot{\Delta}_j \mathcal{T})\|_{L^2} \|(\dot{\Delta}_j F, \dot{\Delta}_j G, \dot{\Delta}_j H)\|_{L^2}. \end{aligned} \tag{3.34}$$

Multiplying the first and the second equations in (3.8) with $-\dot{\Delta}_j \partial_x m$ and $\dot{\Delta}_j \partial_x a$ and then adding them together, we can obtain the dissipation of a

$$\frac{d}{dt}\int \dot{\Delta}_j m \dot{\Delta}_j \partial_x a \, dx + \|\dot{\Delta}_j \partial_x a\|_{L^2}^2 + \|\dot{\Delta}_j \partial_{xx} a\|_{L^2}^2 - \|\dot{\Delta}_j \partial_x m\|_{L^2}^2 + \int \dot{\Delta}_j \partial_x \mathcal{T} \dot{\Delta}_j \partial_x a \, dx$$
$$= \int \dot{\Delta}_j G \dot{\Delta}_j \partial_x a - \dot{\Delta}_j F \dot{\Delta}_j \partial_x m \, dx.$$

To obtain the dissipation of m, we add the second and the third equations of (3.8) together after multiplying by $-\dot{\Delta}_j \partial_x \mathcal{T}$ and $\dot{\Delta}_j \partial_x m$

$$\frac{d}{dt}\int \dot{\Delta}_j \mathcal{T} \dot{\Delta}_j \partial_x m \, dx + \|\dot{\Delta}_j \partial_x m\|_{L^2}^2 - \|\dot{\Delta}_j \partial_x \mathcal{T}\|_{L^2}^2 - \int \dot{\Delta}_j \partial_x a \dot{\Delta}_j \partial_x \mathcal{T} \, dx$$
$$= \int \dot{\Delta}_j \partial_{xx} \mathcal{T} \dot{\Delta}_j \partial_x m + \dot{\Delta}_j \partial_{xx} a \dot{\Delta}_j \partial_{xx} \mathcal{T} + \dot{\Delta}_j H \dot{\Delta}_j \partial_x m - \dot{\Delta}_j G \dot{\Delta}_j \partial_x \mathcal{T} \, dx.$$

Making use of the Hölder's inequality, Young's inequality and Bernstein's inequality, we can deduce by the above two equation

$$\frac{d}{dt}\left(\int \dot{\Delta}_j m \dot{\Delta}_j \partial_x a + 2\dot{\Delta}_j \mathcal{T} \dot{\Delta}_j \partial_x m \, dx\right) + \frac{1}{2}\|(\dot{\Delta}_j \partial_x a, \dot{\Delta}_j \partial_{xx} a, \dot{\Delta}_j \partial_x m)\|_{L^2}^2$$
$$\lesssim \|\dot{\Delta}_j \partial_x \mathcal{T}\|_{L^2}^2 + \|(\dot{\Delta}_j F, \dot{\Delta}_j G, \dot{\Delta}_j H)\|_{L^2} \|(\dot{\Delta}_j a, \dot{\Delta}_j m, \dot{\Delta}_j \mathcal{T})\|_{L^2}.$$

Defining $\mathcal{U}_j^2 := \|(\dot\Delta_j a, \dot\Delta_j \partial_x a, \dot\Delta_j m, \dot\Delta_j \mathcal{T})\|_{L^2}^2 + \varepsilon \int \dot\Delta_j m \dot\Delta_j \partial_x a + 2\dot\Delta_j \mathcal{T} \dot\Delta_j \partial_x m \, dx$, with ε suitably small, and the Cauchy's inequality implies

$$\mathcal{U}_j^2 \approx \|(\dot\Delta_j a, \dot\Delta_j m, \dot\Delta_j \mathcal{T})\|_{L^2}^2.$$

Adding (3.34) and (3.2) together, we can deduce for $j \le j_0$

$$\frac{d}{dt}\mathcal{U}_j^2 + 2^{2j}\mathcal{U}_j^2 \lesssim \|(\dot\Delta_j F, \dot\Delta_j G, \dot\Delta_j H)\|_{L^2}\mathcal{U}_j.$$

Then we have by Lemma 5.3

$$\begin{aligned}&\|(\dot\Delta_j a, \dot\Delta_j m, \dot\Delta_j \mathcal{T})\|_{L_T^\infty(L^2)} + 2^{2j}\|(\dot\Delta_j a, \dot\Delta_j m, \dot\Delta_j \mathcal{T})\|_{L_T^1(L^2)}\\ &\quad\lesssim \|(\dot\Delta_j F, \dot\Delta_j G, \dot\Delta_j H)\|_{L_T^1(L^2)} + \|(\dot\Delta_j a_0, \dot\Delta_j m_0, \dot\Delta_j \mathcal{T}_0)\|_{L^2}.\end{aligned}$$

Multiplying $2^{-\frac{1}{2}j}$ and then choosing l^∞ for $j \le j_0$, we can obtain

$$\|(a, m, \mathcal{T})\|^{\ell}_{\widetilde{L}_T^\infty(\dot B^{-\frac{1}{2}}_{2,\infty})} + \|(a, m, \mathcal{T})\|^{\ell}_{\widetilde{L}_T^1(\dot B^{\frac{3}{2}}_{2,\infty})} \lesssim \|(F, G, H)\|^{\ell}_{\widetilde{L}_T^1(\dot B^{-\frac{1}{2}}_{2,\infty})} + \mathcal{X}_0. \tag{3.35}$$

Next, we bound the non-linear part. Taking advantage of the product law (5.7), Proposition 5.3 and interpolation inequality, we can obtain

$$\|F\|_{\widetilde{L}_T^1(\dot B^{-\frac{1}{2}}_{2,\infty})} \lesssim \|a\|_{\widetilde{L}_T^2(\dot B^{\frac{1}{2}}_{2,\infty})}\|m\|_{\widetilde{L}_T^2(\dot B^{\frac{1}{2}}_{2,1})} \lesssim E(T)D(T).$$

Similarly,

$$\|G\|_{\widetilde{L}_T^1(\dot B^{-\frac{1}{2}}_{2,\infty})} \lesssim \|(a, m, \mathcal{T})\|^2_{L_T^2(\dot B^{\frac{1}{2}}_{2,1})}\left(1 + \|(a, m)\|_{L_T^\infty(\dot B^{\frac{1}{2}}_{2,1})}\right) \lesssim E(T)D(T)$$

and

$$\|H\|_{\widetilde{L}_T^1(\dot B^{-\frac{1}{2}}_{2,\infty})} \lesssim \|(a, m, \mathcal{T})\|^2_{\widetilde{L}_T^2(\dot B^{\frac{1}{2}}_{2,1})}\left(1 + \|(a, m, \mathcal{T})\|_{L_T^\infty(\dot B^{\frac{1}{2}}_{2,1})}\right)^2 \lesssim E(T)D(T).$$

Substitute into (3.35), we can obtain

$$\|(a, m, \mathcal{T})\|^{\ell}_{\widetilde{L}_T^\infty(\dot B^{-\frac{1}{2}}_{2,\infty})} + \|(a, m, \mathcal{T})\|^{\ell}_{\widetilde{L}_T^1(\dot B^{\frac{3}{2}}_{2,\infty})} \lesssim E(T)D(T) + \mathcal{X}_0.$$

The proof of Proposition 3.3 is finished. □

4 Global existence and uniqueness

In this subsection, based on Proposition 3.1, we give out the proof of the global existence and uniqueness. Firstly, we introduce the Friedrichs' projector

$$\dot{\mathbb{E}}_n f := \mathcal{F}^{-1}(1_{\mathscr{C}_n}\mathcal{F}f),\ \forall f \in L^p_n,\ n \geq 1,$$

with $1_{\mathscr{C}_n}$ is the characteristic function on the annulus $\mathscr{C}_n$, L^p_n the set of L^p functions spectrally supported in the annulus $\mathscr{C}_n := \{\xi \in \mathbb{R} | \frac{1}{n} \leq |\xi| \leq n\}$ endowed with the standard L^p topology, and consider the approximate scheme as follows

$$\frac{d}{dt}\begin{pmatrix} a^n \\ m^n \\ \vdots \\ \mathcal{T}^n \end{pmatrix} = \dot{\mathbb{E}}_n \begin{pmatrix} -\partial_x m^n + F \\ -\partial_x a^n - \partial_x \mathcal{T}^n + \partial_x^3 a^n + G \\ \vdots \\ -\partial_x m^n + \partial_{xx}\mathcal{T}^n + H \end{pmatrix} \tag{4.1}$$

with the initial data

$$(a^n, m^n, \mathcal{T}^n)|_{t=0} = \dot{\mathbb{E}}_n(a_0, m_0, \mathcal{T}_0). \tag{4.2}$$

It is clear that (4.1) is a system of ordinary differential equations in $L^2_n \times L^2_n$ and locally Lipschitz with respect to the variable $(a^n, m^n, \mathcal{T}^n)$ for every $n \geq 1$. It follows the Cauchy–Lipschitz theorem in [4, Page 124] that there exists a time $T^*_n > 0$ such that the problem (4.1)–(4.2) admits a unique solution $(a^n, m^n, \mathcal{T}^n) \in C([0, T^*_n]; L^2_n)$.

By virtue of Proposition 3.1 and the standard continuity arguments, we can extend the solution $(a^n, u^n, \mathcal{T}^n)$ globally in time and prove that $(a^n, m^n, \mathcal{T}^n)$ satisfies the uniform estimates (2.9) for any $t > 0$ and $n \geq 1$. Actually, because the data satisfies (2.8), there exists a $T^1_n \in (0, T^*_n)$ such that $(a^n, m^n, \mathcal{T}^n)$ satisfies

$$\|(a^n, m^n, \mathcal{T}^n)\|_{\mathcal{X}} \leq 2\delta_0 \tag{4.3}$$

for all $t \in (0, T^1_n)$. Set

$$T^{**}_n = \sup\left\{T \mid (4.3)\text{holds}\right\}, \tag{4.4}$$

we can finally claim $T^{**}_n = \infty$. Otherwise, $T^{**}_n < \infty$, by Proposition 3.1 we can deduce that $(a^n, m^n, \mathcal{T}^n)$ satisfy (3.1) for $T = T^{**}_n$, from which we can deduce

$$\|(a^n, m^n, \mathcal{T}^n)\|_{\mathcal{X}} \leq \frac{3}{2}\delta_0. \tag{4.5}$$

for all $t \in (0, T^1_n)$, due to η_1 small. It implies there exists a $T^{***}_n > T^{**}_n$ such that (4.3) holds. Which contradicts (4.4). Therefore $(a^n, m^n, \mathcal{T}^n)$ is indeed a global solution to the problem (2.1) and satisfies the uniform estimates (2.9).

Be similar to [6], we can prove the strong convergence of the approximate sequence $(a^n, u^n, \mathcal{T}^n)$. More precisely, there exists a limit $(a, m, \mathcal{T})$ such that as $n \to \infty$, the following convergence holds:

$$(a^n, m^n, \mathcal{T}^n) \to (a, m, \mathcal{T}) \quad \text{strongly in} \quad L^\infty(0, T; \dot{B}^{\frac{d}{p}}_{p,1}), \quad \forall\, T > 0. \tag{4.6}$$

Thus, we can prove that the limit $(a, m, \mathcal{T})$ solves (2.1) in the sense of distributions, and thanks to the uniform estimates (2.9) and the Fatou property, $(a, m, \mathcal{T})$ is indeed a global strong solution to the Cauchy problem of System (2.1) subject to the initial data $(a_0, m_0, \mathcal{T}_0)$ and satisfies the estimates (2.8).

Finally, we prove the uniqueness. Let $(a_1, m_1, \mathcal{T}_1)$ and $(a_2, m_2, \mathcal{T}_2)$ are two solutions of System (2.1), and define $(\widetilde{a}, \widetilde{m}, \widetilde{\mathcal{T}}) = (a_1 - a_2, m_1 - m_2, \mathcal{T}_1 - \mathcal{T}_2)$ which satisfies

$$\begin{cases} \partial_t \widetilde{a} + \partial_x \widetilde{m} = F(a_1, m_1) - F(a_2, m_2), \\ \partial_t \widetilde{m} + \partial_x \widetilde{a} + \partial_x \widetilde{\mathcal{T}} - \partial_x^3 \widetilde{a} = G(a_1, m_1, \mathcal{T}_1) - G(a_2, m_2, \mathcal{T}_2), \\ \partial_t \widetilde{\mathcal{T}} + \partial_x \widetilde{m} - \partial_{xx} \widetilde{\mathcal{T}} = H(a_1, m_1, \mathcal{T}_1) - H(a_2, m_2, \mathcal{T}_2). \end{cases} \tag{4.7}$$

Arguing similarly as in Subsections 3.2, for $t \in (0, T]$, one can infer that

$$\|(\widetilde{a}, \widetilde{m}, \widetilde{\mathcal{T}})\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\widetilde{a}, \widetilde{m}, \widetilde{\mathcal{T}})\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})} \lesssim \|(\widetilde{F}, \widetilde{G}, \widetilde{H})\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} \tag{4.8}$$

with

$$\widetilde{F} := F(a_1, m_1) - F(a_2, m_2),\ \widetilde{G} := G(a_1, m_1, \mathcal{T}_1) - G(a_2, m_2, \mathcal{T}_2), \\ \text{and}\ \ \widetilde{H} := H(a_1, m_1, \mathcal{T}_1) - H(a_2, m_2, \mathcal{T}_2).$$

Next, we bound the non-linear part. First, we rewrite the $\widetilde{F}$ as

$$\widetilde{F}_1 = F_1(a_1, m_1) - F_1(a_2, m_2) = \partial_x a_1 m_1 - \partial_x a_2 m_1 + \partial_x a_2 m_1 - \partial_x a_2 m_2,$$

then the product law (5.7) and the interpolation inequality implies

$$\begin{aligned} \|\widetilde{F}_1\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} &\lesssim \|a_1 - a_2\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,\infty})} \|m_1\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|a_2\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,\infty})} \|m_1 - m_2\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \\ &\lesssim \mathcal{X}_0\big(\|(\widetilde{a}, \widetilde{m})\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\widetilde{a}, \widetilde{m})\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})}\big) \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} \|\widetilde{F}_2\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} &\lesssim \|\widetilde{K}_1(a_1) - \widetilde{K}_1(a_2)\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,\infty})} \|m_1\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \|a_2\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,\infty})} \|m_1 - m_2\|_{\widetilde{L}^2_T(\dot{B}^{\frac{1}{2}}_{2,1})} \\ &\lesssim \mathcal{X}_0\big(\|(\widetilde{a}, \widetilde{m})\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\widetilde{a}, \widetilde{m})\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})}\big) \end{aligned}$$

In the last inequality, we use the Corollary 5.1. Adding above two together, we have

$$\|\widetilde{F}\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} \lesssim \mathcal{X}_0\big(\|(\widetilde{a}, \widetilde{m})\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\widetilde{a}, \widetilde{m})\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})}\big) \tag{4.9}$$

By the same argument, we can deduce

$$\|(\widetilde{G},\widetilde{H})\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} \lesssim \mathcal{X}_0\big(\|(\widetilde{a},\widetilde{m},\widetilde{\mathcal{T}})\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\widetilde{a},\widetilde{m},\widetilde{\mathcal{T}})\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})}\big) \tag{4.10}$$

Inserting (4.9) and (4.10) into (4.8), we can finally obtain

$$\|(\widetilde{a},\widetilde{m},\widetilde{\mathcal{T}})\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,\infty})} + \|(\widetilde{a},\widetilde{m},\widetilde{\mathcal{T}})\|_{\widetilde{L}^1_T(\dot{B}^{\frac{3}{2}}_{2,\infty})} \lesssim 0. \tag{4.11}$$

The uniqueness is proved.

5 Appendix

5.1 Appendix A: Fourier analysis

To make the paper self-contained, we briefly recall Littlewood-Paley decomposition, Besov spaces and analysis tools. The reader is referred to Chap. 2 and Chap. 3 of [4] for more details.

Firstly, we introduce the following so-called Bernstein's inequalities.

Lemma 5.1 (Bernstein's inequalities see [17]) *Let $k \in \mathbb{N}$, $1 \le a \le b \le \infty$, C is a constant and f is an any function in L^p, then we have if* $\mathrm{Supp}\,\mathcal{F}f \subset \big\{\xi \in \mathbb{R}^d : |\xi| \le R\lambda\big\}$ *for some $R > 0$*

$$\sup_{|\alpha|=k} \|D^\alpha f\|_{L^b} \le C^{1+k}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}.$$

More generally, If $\mathrm{Supp}\,\mathcal{F}f \subset \big\{\xi \in \mathbb{R}^d : R_1\lambda \le |\xi| \le R_2\lambda\big\}$ *for some $0 < R_1 < R_2$, we have*

$$C_0^{-k-1}\lambda^k\|u\|_{L^a} \le \sup_{|\alpha|=k}\|D^\alpha u\|_{L^a} \le C_0^{k+1}\lambda^k\|u\|_{L^a}$$

and

$$\|A(D)f\|_{L^b} \lesssim \lambda^{m+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a},$$

Now, we define homogeneous Besov space as follows:

Definition 5.1 For $\sigma \in \mathbb{R}$ and $1 \le p, r \le \infty$, the homogeneous Besov spaces $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} \triangleq \left\{ f \in \mathcal{S}'_0 : \|f\|_{\dot{B}^s_{p,r}} < +\infty \right\},$$

where

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{def}{=} \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|^r_{L^p}\right)^{\frac{1}{r}} \quad \text{if} \quad r < \infty \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,\infty}} \stackrel{def}{=} \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^p}. \tag{5.1}$$

We often use the following classical properties:

- *Completeness:* $\dot{B}^s_{p,r}$ is a Banach space whenever $s < \frac{d}{p}$ or $s \le \frac{d}{p}$ and $r = 1$.
- *Action of Fourier multipliers:* If F is a smooth homogeneous of degree m function on $\mathbb{R}^d \backslash \{0\}$ and $F(D)$ maps $\mathcal{S}'_0$ to itself, then

$$F(D) : \dot{B}^s_{p,r} \to \dot{B}^{s-m}_{p,r}.$$

  In particular, the gradient operator maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-1}_{p,r}$, further we have $\|D^k f\|_{\dot{B}^s_{p,r}} \approx \|f\|_{\dot{B}^{s+k}_{p,r}}$ for all $k \in \mathbb{N}$ by the homogeneous Besov space and Bernstein's inequalities.

The mixed space-time Besov spaces are also used, which was initiated by Chemin and ∼Lerner [14].

Definition 5.2 For $T > 0$, $s \in \mathbb{R}$, $1 \le r, \theta \le \infty$, the homogeneous Chemin-Lerner space $\widetilde{L}^\theta_T(\dot{B}^s_{p,r})$ is defined by

$$\widetilde{L}^\theta_T(\dot{B}^s_{p,r}) \triangleq \left\{ f \in L^\theta(0, T; \mathcal{S}'_0) : \|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})} < +\infty \right\},$$

where

$$\|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})} \triangleq \|(2^{ks}\|\dot{\Delta}_k f\|_{L^\theta_T(L^p)})\|_{\ell^r(\mathbb{Z})}. \tag{5.2}$$

For notational simplicity, index T will be omitted if $T = +\infty$. We also use the following functional space:

$$\tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^s_{p,r}) \triangleq \left\{ f \in \mathcal{C}(\mathbb{R}_+; \dot{B}^s_{p,r}) \text{ s.t } \|f\|_{\tilde{L}^\infty(\dot{B}^s_{p,r})} < +\infty \right\}.$$

The above norm (5.2) may be linked with those of the standard spaces $L^\theta_T(\dot{B}^s_{p,r})$ by means of Minkowski's inequality.

Remark 5.1 It holds that

$$\|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})} \le \|f\|_{L^\theta_T(\dot{B}^s_{p,r})} \ \text{if} \ r \ge \theta; \quad \|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})} \ge \|f\|_{L^\theta_T(\dot{B}^s_{p,r})} \ \text{if} \ r \le \theta.$$

Restricting the above norms (5.1) and (5.2) to the low or high frequencies parts of distributions will be fundamental in our approach and we give out the following definition

Definition 5.3 Assume k_0 is some fixed integer (the value of which will follow from the proofs of our main results) and T, p, s is setting as in Definition 5.2 then we can define

$$\|f\|^\ell_{\dot{B}^s_{p,1}} \triangleq \sum_{k \le k_0} 2^{ks}\|\dot{\Delta}_k f\|_{L^p} \text{ and } \|f\|^h_{\dot{B}^s_{p,1}} \triangleq \sum_{k \ge k_0+1} 2^{js}\|\dot{\Delta}_k f\|_{L^p}, \tag{5.3}$$

$$\|f\|^{\ell}_{\tilde{L}^{\infty}_T(\dot{B}^s_{p,1})} \triangleq \sum_{k\le k_0} 2^{ks}\|\dot{\Delta}_k f\|_{L^{\infty}_T(L^p)} \text{ and } \|f\|^{h}_{\tilde{L}^{\infty}_T(\dot{B}^s_{p,1})} \triangleq \sum_{k\ge k_0+1} 2^{js}\|\dot{\Delta}_k f\|_{L^{\infty}_T(L^p)}. \tag{5.4}$$

Remark 5.2 It is obviously that, when $s < \sigma$ (when $s = \sigma, r_2 \le r_1$), the Bernstein's inequality implies

$$\|f\|^{\ell}_{\dot{B}^{\sigma}_{p,r_1}} \lesssim \|f\|^{\ell}_{\dot{B}^{s}_{p,r_2}} \text{ and } \|f\|^{h}_{\dot{B}^{s}_{p,r_1}} \lesssim \|f\|^{h}_{\dot{B}^{\sigma}_{p,r_2}}. \tag{5.5}$$

Next, we states interpolation inequalities for high and low frequencies.

Lemma 5.2 (see [40]) *Let $s_1 \le s_2$, $q, r \in [1, +\infty]$, $\theta \in (0, 1)$ and $1 \le \alpha_1 \le \alpha \le \alpha_2 \le \infty$ satisfying $\frac{1}{\alpha} = \frac{\theta}{\alpha_1} + \frac{1-\theta}{\alpha_2}$, then we have*

$$\|f\|^{\ell}_{\widetilde{L}^{\alpha}_T(\dot{B}^{\theta s_1+(1-\theta)s_2}_{q,r})} \lesssim \left(\|f\|^{\ell}_{\widetilde{L}^{\alpha_1}_T(\dot{B}^{s_1}_{q,r})}\right)^{\theta}\left(\|f\|^{\ell}_{\widetilde{L}^{\alpha_2}_T(\dot{B}^{s_2}_{q,r})}\right)^{1-\theta},$$
$$\|f\|^{h}_{\widetilde{L}^{\alpha}_T(\dot{B}^{\theta s_1+(1-\theta)s_2}_{q,r})} \lesssim \left(\|f\|^{h}_{\widetilde{L}^{\alpha_1}_T(\dot{B}^{s_1}_{q,r})}\right)^{\theta}\left(\|f\|^{h}_{\widetilde{L}^{\alpha_2}_T(\dot{B}^{s_2}_{q,r})}\right)^{1-\theta}.$$

The following product estimates in Besov spaces play a fundamental role in our analysis of the nonlinear terms.

Proposition 5.1 (Product estimates see [17, 36]) *Let $\sigma > 0$ and $1 \le p, r \le \infty$. Then $\dot{B}^{\sigma}_{p,r} \cap L^{\infty}$ is an algebra and*

$$\|fg\|_{\dot{B}^{\sigma}_{p,r}} \lesssim \|f\|_{L^{\infty}}\|g\|_{\dot{B}^{\sigma}_{p,r}} + \|g\|_{L^{\infty}}\|f\|_{\dot{B}^{\sigma}_{p,r}}.$$

Let the real numbers $\sigma_1,\ \sigma_2,\ p_1$ and p_2 be such that

$$\sigma_1 + \sigma_2 > 0,\ \sigma_1 \le \frac{d}{p_1},\ \sigma_2 \le \frac{d}{p_2},\ \sigma_1 \ge \sigma_2,\ \frac{1}{p_1} + \frac{1}{p_2} \le 1.$$

Then we have

$$\|fg\|_{\dot{B}^{\sigma_2}_{q,1}} \lesssim \|f\|_{\dot{B}^{\sigma_1}_{p_1,1}}\|g\|_{\dot{B}^{\sigma_2}_{p_2,1}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma_1}{d}. \tag{5.6}$$

Additionally, for exponents $s > 0$ and $1 \le p_1,\ p_2,\ q \le \infty$ satisfying

$$\frac{d}{p_1} + \frac{d}{p_2} - d \le s \le \min\left(\frac{d}{p_1}, \frac{d}{p_2}\right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s}{d},$$

we have

$$\|fg\|_{\dot{B}^{-s}_{q,\infty}} \lesssim \|f\|_{\dot{B}^{s}_{p_1,1}}\|g\|_{\dot{B}^{-s}_{p_2,\infty}}. \tag{5.7}$$

Next, we states the commutator estimates as follows.

Proposition 5.2 (see [4]) *Assume* $d=1$, $1\le p\le\infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1$. *Then, we have*

- If $s\in\left(-\min\left(\frac{1}{p},\frac{1}{p'}\right),\frac{1}{p}+1\right]$, then

$$2^{sj}\|[\dot{\Delta}_j,f]\partial_x g\|_{L^p}\lesssim c_j\|\partial_x f\|_{\dot{B}^{\frac{1}{p}}_{p,1}}\|g\|_{\dot{B}^s_{p,1}},\quad \textit{with}\quad \sum_{j\in\mathbb{Z}}c_j=1.$$

- If $s\in\left[-\min\left(\frac{1}{p},\frac{1}{p'}\right),\frac{1}{p}+1\right)$, then

$$\sup_{j\in\mathbb{Z}}2^{sj}\|[\dot{\Delta}_j,f]\partial_x g\|_{L^p}\lesssim\|\partial_x f\|_{\dot{B}^{\frac{1}{p}}_{p,1}}\|g\|_{\dot{B}^s_{p,\infty}}.$$

- If $s\in\left(-1-\min\left(\frac{1}{p},\frac{1}{p'}\right),\frac{1}{p}\right]$, then

$$2^{sj}\|\partial_x([\dot{\Delta}_j,f]g)\|_{L^p}\lesssim\|\partial_x f\|_{\dot{B}^{\frac{1}{p}}_{p,1}}\|g\|_{\dot{B}^s_{p,1}},\quad \textit{with}\quad \sum_{j\in\mathbb{Z}}c_j=1.$$

To investigate the effect of composition by smooth function on Besov spaces, we state the following Lemma that called para-linearization theorem

Proposition 5.3 (Para-linearization theorem see [17]) *Let* $F:\mathbb{R}\to\mathbb{R}$ *be smooth with* $F(0)=0$. *For all* $1\le p,r\le\infty$ *and* $\sigma>0$ *we have* $F(f)\in\dot{B}^\sigma_{p,r}\cap L^\infty$ *for* $f\in\dot{B}^\sigma_{p,r}\cap L^\infty$ *and*

$$\|F(f)\|_{\dot{B}^\sigma_{p,r}}\le C\|f\|_{\dot{B}^\sigma_{p,r}}$$

with C *depending only on* $\|f\|_{L^\infty}$, σ, p *and* d.

The following corollary are also used.

Corollary 5.1 (see [7]) *Assume that* $F(m)$ *is a smooth function satisfying* $F'(0)=0$. *Let* $1\le p\le\infty$. *For any couple* (m_1,m_2) *of functions in* $\dot{B}^s_{p,1}\cap L^\infty$, *there exists a constant* $C_{m_1,m_2}>0$ *depending on* F'' *and* $\|(m_1,m_2)\|_{L^\infty}$ *such that*

- *Let* $-\min\{\frac{d}{p},d(1-\frac{1}{p})\}<s\le\frac{d}{p}$ *and* $1\le r\le\infty$. *Then, we have*

$$\|F(m_1)-F(m_2)\|_{\dot{B}^s_{p,r}}\le C\|(m_1,m_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|m_1-m_2\|_{\dot{B}^s_{p,r}}.\tag{5.8}$$

- *In the limiting case* $r=\infty$, *for any* $-\min\{\frac{d}{p},d(1-\frac{1}{p})\}\le s<\frac{d}{p}$, *it holds that*

$$\|F(m_1)-F(m_2)\|_{\dot{B}^s_{p,\infty}}\le C\|(m_1,m_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|m_1-m_2\|_{\dot{B}^s_{p,\infty}}.\tag{5.9}$$

Finally, we state the following Lemma that proofed in [5].

Lemma 5.3 *Let $p \geq 1$ and $X : [0, T] \to \mathbb{R}^+$ be a continuous function such that X^p is differentiable almost everywhere. We assume that there exists a constant $B \geq 0$ and a measurable function $A : [0, T] \to \mathbb{R}^+$such that*

$$\frac{1}{p}\frac{d}{dt}X^p + BX^p \leq AX^{p-1} \quad a.\ e.\ on \quad [0, T]. \tag{5.10}$$

Then, for all $t \in [0, T]$, we have

$$X(t) + B\int_0^t Xd\tau \leq X(0) + \int_0^t Ad\tau. \tag{5.11}$$

5.2 Appendix B: Derivation of model (1.1)

The purpose of this appendix is to give the derivation process of (1.1), and the special case that $\kappa(\rho)$ is a constant has be proved by Hou et al. in [22]. It is clear that

$$\begin{aligned}
&\mathcal{K} = \left(\frac{1}{2}\kappa(\rho)(\partial_x\rho)^2 - \rho\theta + \rho\partial_x(\kappa(\rho)\partial_x\rho)\right) - \kappa(\rho)(\partial_x\rho)^2 = K - P,\\
&\kappa(\rho) = 2\rho\Psi_\phi = \tilde{\kappa}(\rho),
\end{aligned}$$

$\mathscr{E}$ then we can rewrite the conservation of momentum as following by the mass conservation equation

$$\partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x P(\rho, \theta) = \partial_x(\mu\partial_x u) + \partial_x K.$$

And then the energy conservation equation can be written as

$$\partial_t(\rho e) + \partial_x(\rho u e) + P\partial_x u = \mu(\partial_x u)^2 + \widetilde{\alpha}\partial_{xx}\theta + K\partial_x u + \partial_x W.$$

Substituting into the following equations

$$e = \Psi - \theta\Psi_\theta = \theta + \frac{\tilde{\kappa}(\rho)}{2\rho}\partial_x\rho,$$

then the conservation equation of mass implies

$$\begin{aligned}
&\rho\partial_t\theta + \rho u\partial_x\theta + \partial_t\left(\frac{\kappa(\rho)}{2}(\partial_x\rho)^2\right) + \partial_x\left(u\frac{\kappa(\rho)}{2}(\partial_x\rho)^2\right) + P\partial_x u\\
&\quad = \mu(\partial_x u)^2 + \widetilde{\alpha}\partial_{xx}\theta + K\partial_x u + \partial_x W.
\end{aligned}$$

By direct calculation, we can obtain

$$Ku_x = \kappa(\rho)\rho_{xx}\rho u_x + \frac{1}{2}\kappa'(\rho)\rho_x^2\rho u_x - \frac{1}{2}\kappa(\rho)\rho_x^2 u_x,$$

$$W_x = \kappa(\rho)\rho_{xx}(\rho_t + \rho_x u) + \kappa'(\rho)\rho_x^2(\rho_t + \rho_x u) + \kappa(\rho)\rho_x^2 u_x + \kappa(\rho)\rho_x\rho_{xx}u + \kappa(\rho)\rho_x\rho_{xt}.$$

And further, we have

$$Ku_x + W_x = \frac{1}{2}\kappa'(\rho)\rho_x^2(\rho_t + \rho_x u) + \frac{1}{2}\kappa(\rho)\rho_x^2 u_x + \kappa(\rho)\rho_x\rho_{xx}u + \kappa(\rho)\rho_x\rho_{xt} = \left(u\frac{\kappa(\rho)}{2}\rho_x^2\right)_x + \left(\frac{\kappa(\rho)}{2}\rho_x^2\right)_t.$$

The derivation of model (1.1) is finished.

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References

1. Audiard, C.: Dispersive smoothing for the Euler–Korteweg model. SIAM J. Math. Anal. **44**, 3018–3040 (2012)
2. Audiard, C., Haspot, B.: Global well-posedness of the Euler–Korteweg system for small irrotational data. Commun. Math. Phys. **351**, 201–247 (2017)
3. Audiard, C., Haspot, B.: From the Gross-Pitaevskii equation to the Euler Korteweg system, existence of global strong solutions with small irrotational initial data. Ann. Mat. Pura Appl. **197**, 721–760 (2018)
4. Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, p. xvi+523. Springer, Heidelberg (2011)
5. Barat, T.C., Danchin, R.: Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. Pure Appl. Anal. **4**, 85–125 (2022)
6. Crin-Barat, T., He, Q.Y., Shou, L.Y.: The hyperbolic-parabolic chemotaxis system for vasculogenesis: global dynamics and relaxation limit toward a Keller-Segel model. SIAM J. Math. Anal. **55**, 4445–4492 (2023)
7. Crin-Barat, T., Shou, L.Y., Zhang, J.Z.: Strong relaxation limit and uniform time asymptotics of the Jin-Xin model in the L^p framework (2023). arXiv:2311.04105
8. Berti, M., Maspero, A., Murgante, F.: Local well posedness of the Euler–Korteweg equations on $\mathbb{T}^d$. J. Dyn. Differ. Equ. **33**, 1475–1513 (2021)
9. Bresch, D., Gisclon, M., L. -Violet, I.: On Navier–Stokes–Korteweg and Euler–Korteweg systems: application to quantum fluids models. Arch. Ration. Mech. Anal. **233**, 975–1025 (2019)
10. Chen, Z.Z., Chai, X.J., Dong, B.Q., Zhao, H.J.: Global classical solutions to the one-dimensional compressible fluid models of Korteweg type with large initial data. J. Differ. Equ. **259**, 4376–4411 (2015)
11. Chen, Z.Z., He, L., Zhao, H.J.: Global smooth solutions to the non-isothermal compressible fluid models of Korteweg type with large initial data. Z. Angew. Math. Phys. **68**, 37 (2017)
12. Chen, Z.Z., Li, Y.P.: Asymptotic behavior of solutions to an impermeable wall problem of the compressible fluid models of Korteweg type with density-dependent viscosity and capillarity. SIAM J. Math. Anal. **53**, 1434–1473 (2021)

13. Charve, F., Danchin, R., Xu, J.: Gevrey analyticity and decay for the compressible Navier–Stokes system with capillarity. Indiana Univ. Math. J. **70**, 903–1944 (2021)
14. Chemin, J.-Y., Lerner, N.: Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes. J. Differ. Equ. **121**, 314–328 (1995)
15. Danchin, R., Desjardins, B.: Existence of solutions for compressible fluid models of Korteweg type. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **18**, 97–133 (2001)
16. Dunn, J.E., Serrin, J.: On the thermodynamics of interstitial working. Arch. Ration. Mech. Anal. **88**, 95–133 (1985)
17. Danchin, R., Xu, J.: Optimal time-decay estimates for the compressible Navier–Stokes equations in the critical L^p framework. Arch. Ration. Mech. Anal. **224**, 53–90 (2017)
18. B. -Gavage, S., Danchin, R., Descombes, S.: Well-posedness of one-dimensional Korteweg models. Electron. J. Differ. Equ. **59**, 1–35 (2006)
19. B. -Gavage, S., Danchin, R., Descombes, S.: On the well-posedness for the Euler–Korteweg model in several space dimensions. Indiana Univ. Math. J. **56**, 1499–1579 (2007)
20. Haspot, B.: Existence of global strong solutions in critical spaces for barotropic viscous fluids. Arch. Ration. Mech. Anal. **202**, 427–460 (2011)
21. Hattori, H., Li, D.: The existence of global solutions to a fluid dynamic model for materials for Korteweg type. J. Partial Differ. Equ. **9**, 323–342 (1996)
22. Hou, X.F., Peng, H.Y., Zhu, C.J.: Global well-posedness of the 3 D non-isothermal compressible fluid model of Korteweg type. Nonlinear Anal. Real World Appl. **43**, 18–53 (2018)
23. Korteweg, D.J.: Sur la forme que prennent les équations des mouvement des fluids si l'on tient comple des forces capillaries par des variations de densité. Arch. Neerl. Sci. Exactes Nat. Ser
24. Kawashima, S.: Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics. Doctoral Thesis (1983)
25. Kawashima, S., Shibata, Y., Xu, J.: The L^p energy methods and decay for the compressible Navier–Stokes equations with capillarity. J. Math. Pures Appl. **154**, 146–184 (2021)
26. Kawashima, S., Shibata, Y., Xu, J.: Dissipative structure for symmetric hyperbolic-parabolic systems with Korteweg-type dispersion. Commun. Partial Differ. Equ. **47**, 378–400 (2022)
27. Lions, P.L.: Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models. Oxford University Press, New York (1998)
28. Li, Y.P., Chen, Z.Z.: Large-time behavior of solutions to an inflow problem for the compressible Navier–Stokes–Korteweg equation in the half space. J. Math. Fluid Mech. **24**, 103 (2022)
29. Li, Y.P., Chen, Z.Z., Luo, Z.: Stability of the planar rarefaction wave to two dimensional Navier–Stokes–Korteweg equations of compressible fluids. Math. Methods Appl. Sci. **43**, 3307–3330 (2020)
30. Li, Y.P., Luo, Z.: Stability of the planar rarefaction wave to three-dimensional Navier–Stokes–Korteweg equations of compressible fluids. Nonlinearity **34**, 2689–2714 (2021)
31. Li, Y.P., Tang, J., Yu, S.Q.: Asymptotic stability of rarefaction wave for the compressible Navier–Stokes–Korteweg equations in the half space. Proc. R. Soc. Edinb. A-Math. **152**, 756–779 (2022)
32. Li, Y.P., Zhu, P.C.: Asymptotic stability of the stationary solution to an out-flow problem for the Navier–Stokes–Korteweg equations of compressible fluids. Nonlinear Anal. Real World Appl. **57**, 103193 (2021)
33. Li, J., Xin, Z.P.: Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier–Stokes equations with vacuum. Ann. PDE **5**, 1–37 (2019)
34. Serre, D.: Systèmes de lois de conservation, tome 1. Diderot editeur, Arts et Sciences, Paris, New-York, Amsterdam (1996)
35. Shizuta, S., Kawashima, S.: Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. Hokkaido Math. J. **14**, 249–275 (1985)
36. Shi, W.X., Xu, J.: A sharp time-weighted inequality for the compressible Navier–Stokes–Poisson system in the critical L^p framework. J. Differ. Equ. **266**, 6426–6458 (2019)
37. Shi, W.X., Song, Z.H., Zhang, J.Z.: Large-time behavior of solutions in the critical spaces for the non-isentropic compressible Navier–Stokes equations with capillarity. J. Math. Fluid Mech. **24**, 59 (2022)
38. Tan, Z., Wu, G.C.: Large time behavior of solutions for compressible Euler equations with damping in R^3. J. Differ. Equ. **252**, 1546–1561 (2012)
39. Tan, Z., Wang, H.Q., Xu, J.K.: Global existence and optimal L^2 decay rate for the strong solutions to the compressible fluid models of Korteweg type. J. Math. Anal. Appl. **390**, 181–187 (2012)

40. Xu, J., Zhang, J.Z.: A remark on the multi-dimensional compressible Euler system with damping in the L^p critical Besov spaces. Proc. Am. Math. Soc. **152**, 239–252 (2024)
41. Yong, W.-A.: Entropy and global existence for hyperbolic balance laws. Arch. Ration. Mech. Anal. **172**, 47–266 (2004)
42. Yang, J.W., Ju, Q.C., Yang, Y.F.: Asymptotic limits of Navier–Stokes equations with quantum effects. Z. Angew. Math. Phys. **66**, 2271–2283 (2015)