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ORIGINAL ARTICLE

Kowalski–Słodkowski Theorem for Reproducing Kernel Hilbert Spaces

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Abstract

On reproducing kernel Hilbert spaces with normalized complete Pick kernel, we establish an equivalent result to the Gleason–Kahane–Żelazko theorem without assuming linearity. On the way of establishing this, we observe that linearity on a multiplier algebra is enough to conclude linearity on the whole Hilbert space. By constructing a counter-example, we show that the condition of complete Pick kernel can not be removed. Also, we demonstrate the automatic continuity of such functionals. Leveraging these findings, we extend the Kowalski–Słodkowski theorem in this setup.

Keywords Reproducing kernel Hilbert space · Complete Pick kernel · Multiplier algebra · Cyclic function · Gleason–Kahane–Żelazko theorem · Kowalski–Słodkowski theorem · Automatic continuity.

Mathematics Subject Classification Primary 46E22; Secondary 46H40 · 30H20 · 30H15 · 30H10

1 Introduction

Theorem 1.1 *[1–3] Let A be a complex unital Banach algebra, and $\Lambda : A \to \mathbb{C}$ be a linear functional such that $\Lambda \not\equiv 0$. Then, the following statements are equivalent.*

1. $\Lambda(1) = 1$ *and* $\Lambda(a) \neq 0$ *for all invertible elements* $a \in A$.
2. $\Lambda(ab) = \Lambda(a)\Lambda(b)$ *for all* $a, b \in A$.

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The above theorem is known as the Gleason–Kahane–Żelazko (GKZ) theorem, and it is one of the interesting theorems in Banach algebra that characterizes the multiplicativity of a linear functional. Unlike Banach algebra, there is no multiplication operation in Hilbert space. Whereas, in reproducing kernel Hilbert space, multiplication can be defined pointwise. Can we generalize this theorem to Hilbert spaces? Cheng Chu, Michael Hartz, Javad Mashreghi and Thomas Ransford gave a generalization in a special class of Hilbert spaces, namely reproducing kernel Hilbert space (RKHS) with complete normalized Pick property, as follows:

Theorem 1.2 *[4] Let $\mathcal{H}$ be a RKHS with normalized complete Pick kernel, and let $\Lambda : \mathcal{H} \to \mathbb{C}$ be a linear functional such that $\Lambda \neq 0$. Then the following statements are equivalent*

1. $\Lambda(1) = 1$ *and* $\Lambda(f) \neq 0$ *for all cyclic elements* $f \in \mathcal{H}$.
2. $\Lambda(fg) = \Lambda(f)\Lambda(g)$ *for all* $f, g \in \mathcal{H}$ *such that* $fg \in \mathcal{H}$.

Note that a functional Λ satisfying (2) in Theorem 1.2 is considered multiplicative functional in Hilbert space. There is another version of the GKZ theorem for Dirichlet space [5]. This paper attempts to prove the implications (2) $\implies$ (1) in Sect. 2 and (1) $\implies$ (2) in Sect. 3 by weakening the linearity assumption in the above theorem. We deduced the linearity in the hypothesis of theorem 1.2 using some conditions on the functionals.

In Sect. 2, we prove a functional linear in multiplier algebra will be linear in whole RKHS with normalized complete Pick kernel. To underscore the importance of a complete Pick kernel, we construct a concrete example to show that this condition can not be removed. Using this, we prove the continuity of such functionals. Also, we introduce additional conditions from the work of Kowalski and Słodkowski [6] to demonstrate a weaker version of (2) implies (1) of Theorem 1.2. Additionally, our discussion extends to the prospect of generalizing the Kowalski-Słodkowski theorem to RKHS with normalized complete Pick kernel.

In Sect. 3, we generalize (linearity) the reverse implication by incorporating conditions from the paper [7] by one of the authors of this paper, whereas to prove the reverse implication, authors used their GKZ theorem for modules [8]. Also, we add some additional conditions to deduce the linearity of such functionals. And we construct a counter example to show that the condition of complete Pick kernel can not be removed from the statement.

1.1 Preliminaries

The subsection provides foundational definitions of reproducing kernel Hilbert spaces, kernels, multiplier algebra, and cyclic functions. Those with a solid understanding of these concepts can proceed directly to the following section. For a more in-depth exploration of these topics, we recommend consulting the book by Paulsen and Ragupathi [9] and the book authored by Agler and McCarthy [10].

Definition 1.1 (*Reproducing kernel Hilbert space*) [9] Let X be a non-empty set and $\mathcal{F}(X, \mathbb{C})$ be the collection of all functions from X to $\mathbb{C}$. We call $\mathcal{H} \subset \mathcal{F}(X, \mathbb{C})$ a reproducing kernel Hilbert space (RKHS) on X if

1. $\mathcal{H} \subseteq \mathcal{F}(X, \mathbb{C})$ be a Hilbert space.
2. for every $x \in X$, the evaluation functional $E_x : \mathcal{H} \to \mathbb{C}$ defined by $E_x(f) = f(x)$ is bounded.

Definition 1.2 (*Kernel function*) Let X be a non-empty set. A function $K : X \times X \to \mathbb{C}$ is said be kernel function if

1. K is Hermitian, that is $K(x, y) = \overline{K(y, x)}$ for all $x, y \in X$,
2. K is a positive semi-definite function.

Since evaluation functionals E_x are continuous in RKHS $\mathcal{H}$, by Riesz representation theorem, there exists a unique function k_x satisfying $E_x(f) = f(x) = \langle f, k_x \rangle$ for all $f \in \mathcal{H}$. The reproducing kernel associated to $\mathcal{H}$ is defined as $K : X \times X \to \mathbb{C}$, such that $K(x, y) := k_y(x) = \langle k_y, k_x \rangle$ for all $x, y \in X$. This K satisfies the conditions of the kernel function. Also, every kernel function $K : X \times X \to \mathbb{C}$ gives a RKHS on X with K as its reproducing kernel.

Definition 1.3 (*Normalized Kernel*) A reproducing kernel K of a RKHS $\mathcal{H}$ on X is said to be normalized if there exists $x_0 \in X$ such that $K(x, x_0) = 1$ for all $x \in X$. The function $K(\cdot, x_0) \in \mathcal{H}$ acts as the constant function $1_{\mathcal{H}}$ in the Hilbert space $\mathcal{H}$.

Definition 1.4 (*Complete Pick kernel*) [10] A reproducing kernel K of a RKHS $\mathcal{H}$ on X is said to be complete Pick kernel if

1. $K(x, y) \neq 0$ for all $x, y \in X$,
2. There exists $x_0 \in X$ such that $F(x, y) = 1 - \frac{K(x,x_0)K(x_0,y)}{K(x,y)K(x_0,x_0)}$ is a positive semi-definite function on $X \times X$.

The concept of the complete Pick kernel establishes a significant link with the Pick interpolation problem, with further insights provided in [10]. From now on, for easier readability, we will abbreviate 'Reproducing Kernel Hilbert Space with Complete normalized Pick kernel' as 'RKHS with CNP kernel'. Examples include the Hardy Hilbert space H^2, the Dirichlet space $\mathcal{D}$, and the Drury-Arveson space H^2_d. Notably, the Drury-Arveson space is recognized as a universal RKHS with CNP kernel [11].

Definition 1.5 (*Multiplier algebra*) Let $\mathcal{H}$ be a RKHS on non-empty set X. A function $h : X \to \mathbb{C}$ is said to be a multiplier if for all $f \in \mathcal{H}$, $hf \in \mathcal{H}$.

Let $\mathcal{M}$ denote the collection of all multipliers, and M_h represent the multiplication operator by h. With the norm $||h||_{\mathcal{M}} = ||M_h||_{op}$, the collection $\mathcal{M}$ becomes Banach algebra. If the corresponding kernel function is also normalized, then constant function $1 \in \mathcal{M}$, and $\mathcal{M}$ becomes unital Banach algebra. It is essential to observe that the norm on $\mathcal{M}$, which makes it a Banach algebra, differs from the norm from the Hilbert space $\mathcal{H}$. Note that the norm associated with $\mathcal{M}$ and $\mathcal{H}$ gives rise to distinct separate algebraic and metric structures associated with $\mathcal{M}$ and $\mathcal{H}$.

For extending the GKZ theorem to RKHS with CNP kernel, the conditions on invertible elements are replaced with conditions on cyclic elements. The definition of cyclic function in this context is as follows.

Definition 1.6 (*Cyclic function*) Let $\mathcal{H}$ be a RKHS on X with multiplier algebra $\mathcal{M}$. For an element $f \in \mathcal{H}$, the closed $\mathcal{M}$ invariant subspace generated by f is denoted by $[f] = \overline{\mathcal{M}f}$. We say f is cyclic if $[f] = \mathcal{H}$, meaning that for every $g \in \mathcal{H}$, there exists a sequence $(h_n) \in \mathcal{M}$, such that $h_n f \to g$. It is important to note that every cyclic function is non-vanishing.

Multiplier algebra of Hardy Hilbert space $H^2(\mathbb{D})$ is the set of all bounded analytic functions $H^\infty(\mathbb{D})$. By Beurling's theorem, the set of all cyclic functions is the collection of all outer functions. Please refer to [12] for the definition of $H^2(\mathbb{D})$ and outer functions.

2 Generalized GKZ Theorem-I

In this section, without assuming linearity, we will prove the backward implication of Theorem 1.2. In fact, we deduce the linearity. Additionally, we will discuss the continuity of such multiplicative functional. Furthermore, we embark on proving the Kowalski–Słodkowski theorem for RKHS with CNP kernel.

To substantiate our results, we leverage a factorization theorem articulated in the paper [13] authored by Aleman, McCarthy, Richer, and Michael Hartz. This theorem serves as a foundational element in our proof methodology.

Theorem 2.1 *[13] Let $\mathcal{H}$ be a RKHS on X with CNP kernel, and $\mathcal{M}$ be its multiplier algebra. If $f \in \mathcal{H}$, then there exists $h_1, h_2 \in \mathcal{M}$, where h_2 is cyclic in $\mathcal{H}$, such that $f = \frac{h_1}{h_2}$.*

Another factorization, elucidated in [14], aligns with Theorem 2.1. Additionally, this factorization includes the noteworthy property that $\frac{1}{h_2} \in \mathcal{H}$.

Theorem 2.2 *[6] Let A be complex unital Banach algebra, and A^{-1} be the set of all invertible elements of A. If $\Lambda : A \to \mathbb{C}$ be functional such that $\Lambda(0) = 0$ and*

$$((\Lambda(a) - \Lambda(b))1 - a - b) \notin A^{-1}, \textit{for all } a, b \in A,$$

then Λ is linear and multiplicative on A.

The above theorem is known as the Kowalski–Słodkowski theorem. This theorem holds significance as it provides insights into the linearity and multiplicativity of functionals within the context of Banach algebra.

Our enquiry started with the question, does there exist a multiplicative functional that is linear only in Multiplier algebra? The following theorem asserts this question in a special space.

Theorem 2.3 *Let $\mathcal{H}$ be a RKHS with CNP kernel. If Λ is multiplicative functional and Λ is linear in multiplier algebra $\mathcal{M}$, then Λ is linear in the entire $\mathcal{H}$.*

Proof Let Λ be a multiplicative functional, which is linear in $\mathcal{M}$. Let $f_1, f_2 \in \mathcal{H}$. Then by factorization Theorem 2.1, $f_1 = \frac{h_1}{g_1}$ and $f_2 = \frac{h_2}{g_2}$. where $h_1, h_2, g_1, g_2 \in \mathcal{M}(\mathcal{H})$ and g_1, g_2 are cyclic. From this

$$\begin{aligned}\Lambda(f_1 + f_2) &= \Lambda\Big(\frac{h_1}{g_1} + \frac{h_2}{g_2}\Big)\\ &= \Lambda\Big(\frac{h_1 g_2 + g_1 h_2}{g_1 g_2}\Big).\end{aligned}$$

Given that $g_1, g_2 \in \mathcal{M}$, it follows that the product $(f_1 + f_2).g_1.g_2 \in \mathcal{H}$. Moreover, the multiplicative property of Λ implies

$$\begin{aligned}\Lambda(f_1 + f_2) &= \frac{\Lambda(h_1 g_2 + g_1 h_2)}{\Lambda(g_1 g_2)}\\ &= \frac{\Lambda(h_1)\Lambda(g_2) + \Lambda(g_1)(h_2)}{\Lambda(g_1)\Lambda(g_2)}\\ &= \frac{\Lambda(h_1)}{\Lambda(g_1)} + \frac{\Lambda(h_2)}{\Lambda(g_2)}\\ &= \Lambda(f_1) + \Lambda(f_2).\end{aligned}$$

□

Can we generalize the previous theorem to any RKHS that does not have complete Pick property? That is, if Λ is multiplicative functional on a general RKHS and Λ is linear in its multiplier algebra $\mathcal{M}$, then is Λ linear on whole $\mathcal{H}$? The answer to this question is negative, as demonstrated by the following example.

Example 1 Consider Segal-Bargmann space $\mathfrak{F}^2(\mathbb{C})$ defined on $\mathbb{C}$, with the reproducing kernel $k(z, w) = e^{z\overline{w}}$, defined by

$$\mathfrak{F}^2(\mathbb{C}) = \Big\{ f \text{ is entire } : \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dz < \infty \Big\}.$$

Since every element of $\mathfrak{F}^2(\mathbb{C})$ is an entire function and $\phi(x)$ is an eigenvalue of M_ϕ^*, it follows that the multiplier algebra contains only constants. For fixed $x, y \in X$, let's define the multiplicative functional as follows:

$$\Lambda(f) = \sqrt{f(x)f(y)}.$$

It is not linear on $\mathcal{H}$. But for any constant function $a \in \mathcal{M}$,

$$\Lambda(a) = a.$$

This function is linear in whole multiplier algebra. So, this example highlights the necessity of the complete Pick property in the Theorem 2.3. That is, there is a multiplicative functional on some normalized RKHS, which is linear on its multiplier algebra and not on the entire space.

Theorem 2.4 *[4] Let $\mathcal{H}$ be a RKHS with CNP kernel, and let $\mathcal{M}$ be its multiplier algebra. Let $\Lambda : \mathcal{H} \to \mathbb{C}$ be a linear functional, such that $\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f \in \mathcal{M}$ and $g \in \mathcal{H}$. Then Λ is continuous on $\mathcal{H}$.*

The fact that every multiplicative linear functional is continuous in a Banach algebra finds its parallel in RKHS with a CNP kernel, as indicated by the theorem mentioned above from [4]. The subsequent theorem delves into the automatic continuity of a multiplicative functional, which is linear only on the multiplier algebra of a RKHS with CNP kernel.

Theorem 2.5 *Let $\mathcal{H}$ be a RKHS with CNP kernel, and let $\mathcal{M}$ be its multiplier algebra. Let $\Lambda : \mathcal{H} \to \mathbb{C}$ be functional, which is linear only in multiplier algebra $\mathcal{M}$, satisfying $\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f \in \mathcal{M}$ and $g \in \mathcal{H}$. Then Λ is continuous on whole Hilbert space $\mathcal{H}$.*

Proof According to Theorem 2.3, the map Λ will be linear in whole Hilbert space $\mathcal{H}$. Furthermore, as per Theorem 2.4, the map Λ is continuous on the entire Hilbert space $\mathcal{H}$. □

Theorem 2.6 *Let $\mathcal{H}$ be a RKHS with a CNP kernel, and let $\Lambda : \mathcal{H} \to \mathbb{C}$ be a function such that $\Lambda \neq 0$. If*

1. *$\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f, g \in \mathcal{H}$, such that $fg \in \mathcal{H}$,*
2. *$((\Lambda(m_1) - \Lambda(m_2))1 - m_1 - m_2) \notin \mathcal{M}^{-1}$ for all $m_1, m_2 \in \mathcal{M}$,*

then $\Lambda(1) = 1$ and $\Lambda(f) \neq 0$ for all cyclic function $f \in \mathcal{H}$.

Proof $\Lambda(1) = \Lambda(1.1) = \Lambda(1)^2$, so $\Lambda(1) = 0$ or 1. But $\Lambda(f) = \Lambda(1.f) = \Lambda(1)\Lambda(f)$, if $\Lambda(1) = 0$, then $\Lambda(f) = 0$ for all $f \in \mathcal{H}$ contrary to hypothesis $\Lambda \neq 0$. so $\Lambda(1) = 1$.

Let f be a cyclic function in $\mathcal{H}$. According to the definition of cyclicity, a sequence of multipliers h_n exists such that $h_n f \to 1$ in $\mathcal{H}$. By Theorem 2.5, Λ satisfying the hypothesis will be continuous in $\mathcal{H}$. Implies $\Lambda(h_n f) = \Lambda(h_n)\Lambda(f) \to \Lambda(1)$. Since $\Lambda(1) = 1$, $\Lambda(f) \neq 0$ for any cyclic elements. □

In [4], the authors proved the same conclusion for linear multiplicative functional Λ. But we proved for multiplicative functional, which is linear only in multiplier algebra $\mathcal{M}$. Using the Theorem 2.3, we get a generalization of the Kowalski–Słodkowski theorem to RKHS with CNP kernel, as follows.

Theorem 2.7 *(*A generalized Kowalski–Słodkowski theorem*) Let $\mathcal{M}$ be the multiplier algebra of a RKHS with CNP kernel $\mathcal{H}$. Let $\Lambda : \mathcal{H} \to \mathbb{C}$ be functional such that*

1. *$\Lambda(0) = 0$,*
2. *$((\Lambda(m_1) - \Lambda(m_2))1 - m_1 - m_2) \notin \mathcal{M}^{-1}$ for all $m_1, m_2 \in \mathcal{M}$,*
3. *$\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f \in \mathcal{M}$ and $g \in \mathcal{H}$.*

Then Λ is linear and continuous on $\mathcal{H}$.

Proof Since $\mathcal{M}$ is complex unital Banach algebra, if Λ satisfies the hypothesis, by Theorem 2.2, Λ is linear in $\mathcal{M}$. By Theorem 2.3, Λ is linear in $\mathcal{H}$, and by Theorem 2.5, Λ is continuous as well. □

Question: In the Theorem 2.5 mentioned above, the condition (3) is about Λ being multiplicative on the entire RKHS. However, according to Theorem 2.2, the conditions (1) and (2) in the hypothesis imply that Λ is multiplicative on multiplier algebra. Notably, the multiplier algebra is dense within the RKHS with CNP kernel. Instead of relying on condition (3), can we assume a weaker condition solely on the multiplier algebra, leveraging the density of the multiplier algebra within the RKHS with CNP kernel?

3 Generalized GKZ Theorem-II

In this section, without assuming linearity, we will generalize (2) implies (1) of the Theorem 1.2 and discuss the linearity of such functionals. To facilitate this, we will employ the following theorem from [7] by one of the authors of this paper, as follows.

Theorem 3.1 *[7] Let A be complex unital Banach algebra, and let A^{-1} be the set of all invertible elements of A. Let U be a left A-module, and S be a non empty subset of U satisfying*

1. $0 \notin S$, *S generates U as A-module,*
2. *If $x \in A^{-1}$ & $s \in S$, then $xs \in S$,*
3. *For all $s_1, s_2 \in S$, there exists $a_1, a_2 \in A$, such that $a_j s_j \in S$ $(j = 1, 2)$ and $a_1 s_1 = a_2 s_2$.*

Let Λ: $U \to \mathbb{C}$ be a non zero map that satisfies

- $\Lambda(0) = 0$,
- $(\Lambda(x) - \Lambda(y))s - (x - y)\Lambda(s) \notin S$ *for all* $x, y \in U$ & $s \in S$.

Then there exist unique character $\chi : A \to \mathbb{C}$, such that

$$\Lambda(au) = \chi(a)\Lambda(u) \qquad (a \in A, u \in U).$$

The Λ satisfying the hypothesis of the above theorem may not be linear. The following theorem from the same paper [7] tells about the linearity of such Λ.

Theorem 3.2 *[7] Let all the assumptions of Theorem 3.1 hold. If*

$$\sum_{j=1}^{n} \Lambda(s_j) = \Lambda(\sum_{j=1}^{n} s_j) \qquad n \in \mathbb{N}, s_j \in S,$$

then Λ is linear and there exist unique character $\chi : A \to \mathbb{C}$, such that

$$\Lambda(au) = \chi(a)\Lambda(u) \qquad (a \in A, u \in U).$$

Theorem 3.3 *Let $\mathcal{H}$ be a RKHS with CNP kernel, let $\Lambda : \mathcal{H} \to \mathbb{C}$ be a non-zero functional, and S be the set of all cyclic functions. If*

1. $\Lambda(1) = 1$, *and* $\Lambda(0) = 0$,

2. $\Lambda(f) \neq 0$, *for* $f \in S$,
3. $(\Lambda(f) - \Lambda(g))s - (f - g)\Lambda(s) \notin S$ *for all* $f, g \in \mathcal{H}$ & $s \in S$,

then $\Lambda(fg) = \Lambda(f)\Lambda(g)$ *for all* $f, g \in \mathcal{H}$ *such that* $fg \in \mathcal{H}$.

Proof To prove this theorem, we will employ Theorem 3.1, taking $\mathcal{M}$ as the complex unital Banach algebra, Hilbert space $\mathcal{H}$ as the left $\mathcal{M}$-module and S as the set of all cyclic functions in $\mathcal{H}$. This satisfies the hypothesis of the Theorem 3.1. By Theorem 3.1 there exists a unique character $\chi : \mathcal{M} \to \mathbb{C}$ such that

$$\Lambda(hf) = \chi(h)\Lambda(f) \quad (h \in \mathcal{M}, f \in \mathcal{H}). \tag{1}$$

For $f \in \mathcal{M}$, $\Lambda(f) = \Lambda(1.f) = \chi(f)\Lambda(1)$. Implies $\chi(f) = \Lambda(f)$ for all $f \in \mathcal{M}$.

To get the result for the whole $\mathcal{H}$. Consider $f, g \in \mathcal{H}$ be such that $fg \in \mathcal{H}$. By Theorem 2.1, we can write $f = \frac{h_1}{h_2}$, where $h_1, h_2 \in \mathcal{M}$ and h_2 cyclic in $\mathcal{H}$ by repeated usage of (1), we have

$$\begin{aligned} \Lambda(h_2)\Lambda(fg) &= \Lambda(h_2 fg) \\ &= \Lambda(h_1 g) \\ &= \Lambda(h_1)\Lambda(g). \\ \Lambda(h_2)\Lambda(f)\Lambda(g) &= \Lambda(h_2 f)\Lambda(g) \\ &= \Lambda(h_1)\Lambda(g). \end{aligned}$$

since h_2 is cyclic, we have $\Lambda(h_2) \neq 0$, so we conclude that $\Lambda(fg) = \Lambda(f)\Lambda(g)$ for $f, g \in \mathcal{H}$. □

The next Theorem tells when such a functional is linear.

Theorem 3.4 *Let* $\mathcal{H}$ *be a RKHS with CNP kernel, let* $\Lambda : \mathcal{H} \to \mathbb{C}$ *be a non-zero functional, and* S *be the set of all cyclic functions. If*

1. $\Lambda(1) = 1$, *and* $\Lambda(0) = 0$,
2. $\Lambda(f) \neq 0$, *for* $f \in S$,
3. $(\Lambda(f) - \Lambda(g))s - (f - g)\Lambda(s) \notin S$ *for all* $f, g \in \mathcal{H}$ & $s \in S$,
4. $\sum_{j=1}^{n} \Lambda(s_j) = \Lambda(\sum_{j=1}^{n} s_j), \quad s_j \in S, n \in \mathbb{N}$,

then Λ *is linear and multiplicative on* $\mathcal{H}$.

Proof By Theorem 3.2, it is established that Λ is linear in multiplier algebra. Moreover, as per Theorem 3.3, Λ is multiplicative in Hilbert space $\mathcal{H}$. Consequently, according to Theorem 2.3, we can conclude that Λ is linear in the entire Hilbert space $\mathcal{H}$. □

The question of whether the complete Pick property can be omitted from Theorem 3.3 is addressed. The affirmative answer is established by providing an example of a functional that satisfies all the hypotheses of Theorem 3.3 but is not multiplicative in some RKHS. The subsequent example illustrates this.

Example 2 Consider the Segal-Bargmann space $\mathfrak{F}^2(\mathbb{C})$ as described in Example 1. Notably, there is no cyclic function in $\mathfrak{F}^2(\mathbb{C})$, since only constant functions serve as multipliers. Define $\Lambda : \mathcal{H} \to \mathbb{C}$ by

$$\Lambda\Big(\sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}}\Big) = \begin{cases} a_0, & a_0 \neq 0 \\ a_1, & a_0 = 0 \end{cases}$$

Clearly, $\Lambda(1) = 1$, and given the absence of a cyclic function, Λ satisfies both (2) and (3) conditions in Theorem 3.3. According to the theory of Segal-Bargmann spaces, both z and z^2 belong to $\mathfrak{F}^2(\mathbb{C})$. However, $\Lambda(z) = 1$, while $\Lambda(z^2) = 0$, indicating that $\Lambda(z^2)$ is not equal to $(\Lambda(z))^2$. Consequently, Λ is not a multiplicative functional. Furthermore, Λ is not linear, as $\Lambda(1+z)$ does not equal $\Lambda(1)+\Lambda(z)$. This non-linearity arises from the fact that $\mathfrak{F}^2(\mathbb{C})$ lacks a complete Pick kernel.

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Declarations

Conflict of interest The authors declare that there is no Conflict of interest.

References

1. Gleason, A.M.: A characterization of maximal ideals. J. Analyse Math. **19**, 171–172 (1967). https://doi.org/10.1007/BF02788714
2. Kahane, J.-P., Żelazko, W.: A characterization of maximal ideals in commutative Banach algebras. Studia Math. **29**, 339–343 (1968). https://doi.org/10.4064/sm-29-3-339-343
3. Żelazko, W.: A characterization of multiplicative linear functionals in complex Banach algebras. Studia Math. **30**, 83–85 (1968). https://doi.org/10.4064/sm-30-1-83-85
4. Chu, C., Hartz, M., Mashreghi, J., Ransford, T.: A Gleason-Kahane-Żelazko theorem for reproducing kernel Hilbert spaces. Bull. Lond. Math. Soc. **54**(3), 1120–1130 (2022). https://doi.org/10.1112/blms.12618
5. Mashreghi, J., Ransford, J., Ransford, T.: A Gleason-Kahane-Żelazko theorem for the Dirichlet space. J. Funct. Anal. **274**(11), 3254–3262 (2018). https://doi.org/10.1016/j.jfa.2017.10.013
6. Kowalski, S., Słodkowski, Z.: A characterization of multiplicative linear functionals in Banach algebras. Studia Math. **67**(3), 215–223 (1980). https://doi.org/10.4064/sm-67-3-215-223
7. Sebastian, G., Daniel, S.: A weaker Gleason-Kahane-Żelazko theorem for modules and applications to Hardy spaces. Colloq. Math. **164**(2), 273–282 (2021). https://doi.org/10.4064/cm8015-9-2019
8. Mashreghi, J., Ransford, T.: A Gleason-Kahane-Żelazko theorem for modules and applications to holomorphic function spaces. Bull. Lond. Math. Soc. **47**(6), 1014–1020 (2015). https://doi.org/10.1112/blms/bdv080
9. Raghupathi, M., Paulsen, V.: An introduction to the theory of reproducing kernel Hilbert spaces, vol. 152. Cambridge University Press, Cambridge (2016)
10. Agler, J., McCarthy, J.E.: Pick interpolation and Hilbert function spaces, vol. 44. American Mathematical Society, Providence (2002)

11. Agler, J., MCarthy, J.E.: Complete Nevanlinna-Pick kernels. J. Funct. Anal. **175**(1), 111–124 (2000). https://doi.org/10.1006/jfan.2000.3599
12. Martínez-Avendaño, R., Rosenthal, P.: An introduction to operators on the Hardy-Hilbert space, vol. 237. Springer, New York (2007). https://doi.org/10.1007/978-0-387-48578-2
13. Aleman, A., Hartz, M., McCarthy, J.E., Richter, S.: The Smirnov class for spaces with the complete Pick property. J. Lond. Math. Soc. (2) **96**(1), 228–242 (2017). https://doi.org/10.1112/jlms.12060
14. Jury, M.T., Martin, R.T.W.: The Smirnov classes for the Fock space and complete Pick spaces. Indiana Univ. Math. J. **70**(1), 269–284 (2021). https://doi.org/10.1512/iumj.2021.70.8203