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Local-Nonlocal Schrödinger Equation with Critical Exponent: The Zero Mass Case

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Abstract

In this paper, we consider the critical problem involving local and nonlocal operator with critical exponent under the zero mass case. First, we establish the continuous and compactness Sobolev embedding results. Second, we establish the non-existence result by Pohožaev identity. Finally, we prove the existence results for upper-crtical and lower-crtical cases via Sobolev embedding theorem, Mountain-pass theorem and Nehari manifold.

Keywords Schrödinger equation · Local and nonlocal operator · Hardy Sobolev critical exponent · Henon Sobolev critical exponent · Existence

Mathematics Subject Classification 35J20 · 35J35

1 Introduction

In this study, we are specifically focusing on analyzing Schrödinger equation that incorporates both local and nonlocal operator under the zero mass case, as follows

$$-\lambda \Delta u + \mu(-\Delta)^s u = f(x, u), \quad x \in \Omega. \qquad (S_{\lambda,\mu})$$

Here $0 < s < 1$ and Ω is a domain in $\mathbb{R}^N$ with $N \geqslant 3$. The operator $(-\Delta)^s$ is the fractional Laplacian, which is defined by the Fourier transform as follows

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

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the details of this definition can be found in references such as [9, 30]. Equation ($S_{\lambda,\mu}$) with $\lambda = \mu = 1$ and $\Omega \subset \mathbb{R}^N$ being a bounded open set with C^1 boundary arises in population dynamics models incorporating both classical and nonlocal diffusion, as discussed by Dipierro-Lippi-Valdinocci [18]. Biagi-Dipierro-Valdinoci-Vecchi [4] have highlighted the applicability of equation ($S_{\lambda,\mu}$) in studying different types of “regional" or “global" restrictions that may mitigate the spread of a pandemic disease. Furthermore, Dipierro-Valdinocci [17] introduced equation ($S_{\lambda,\mu}$) as a description of an ecological niche for mixed local and nonlocal dispersal.

Equation ($S_{\lambda,\mu}$) with $\lambda = 1$, $\mu = 0$ and $f(x,u) = \frac{|u|^{p-2}u}{|x|^\alpha}$ corresponds to the nonlinear Schrödinger equation

$$-\Delta u = \frac{|u|^{p-2}u}{|x|^\alpha}, \quad x \in \Omega, \tag{S}$$

where $\alpha \in (-\infty, 2)$, $p \in (2, 2^*_{1,\alpha})$ and $2^*_{1,\alpha} = \frac{2(N-\alpha)}{N-2}$. This equation has a rich history in quantum mechanics and quantum field theory [8, 10]. We point out that

$$\begin{cases} 2^*_{1,\alpha} \text{ is the Hardy Sobolev critical exponent for } \alpha \in (0,2), \\ 2^*_{1,\alpha} \text{ is the Sobolev critical exponent for } \alpha = 0, \\ 2^*_{1,\alpha} \text{ is the Henon Sobolev critical exponent for } \alpha \in (-\infty, 0). \end{cases}$$

For $\alpha = 0$ and $\Omega = \mathbb{R}^N$, Anbin [1] and Talenti [33] established the existence of solutions for equation (S) with the Sobolev critical exponent. For $\alpha \in (0, 2)$ and $\Omega = \mathbb{R}^N$, Lieb [25] and Ghoussoub-Yuan [22] explored the existence resutls for equation (S) with Hardy Sobolev critical exponent. For $\alpha \in (-\infty, 0)$, Ni [31] considered the existence results for equation (S) with $p \in (1, 2^*_{1,\alpha})$ and Ω is a ball, and investigated the existence of radial solution for equation (S) with Henon Sobolev critical exponent and $\Omega = \mathbb{R}^N$.

Equation ($S_{\lambda,\mu}$) with $\lambda = 0$, $\mu = 0$, $\Omega = \mathbb{R}^N$ and $f(x,u) = \frac{|u|^{p-2}u}{|x|^\alpha}$ transforms into the fractional Schrödinger equation

$$(-\Delta)^s u = \frac{|u|^{p-2}u}{|x|^\alpha}, \quad x \in \mathbb{R}^N. \tag{FS}$$

For $\alpha = 0$, Lieb [25] and Cotsiolis-Tavoularis [16] investigated the existence results for equation (FS) with Sobolev critical exponent. For $\alpha \in (0, 2s)$, Ghoussoub-Shakerian [21] studied the existence ground state for equation (FS) with Hardy Sobolev critical exponent. Moreover, Chen [12] considered the existence ground state for fractional Schrödinger equation with two kinds of Hardy-Sobolev critical exponents, Ghoussoub-Shakerian [21] and Yang-Yu [34] established existence results of fractional Schrödinger equation with Sobolev and Hardy Sobolev critical cases.

For the following more generalized operator cases: fractional t-Laplacian equation

$$(-\Delta)_t^s u = \frac{|u|^{p-2}u}{|x|^\alpha}, \quad x \in \Omega. \tag{FPS}$$

where $(-\Delta)_t^s$ is a fractional t-Laplacian, see [11, 24]. For $\alpha = 0$ and $\Omega = \mathbb{R}^N$, Brasco-Mosconi-Squassina [6] obtained the existence and sharp asymptotic behavior of solution for equation (FPS) with Sobolev critical exponent. For $\alpha \in (0, ps)$ and $\Omega = \mathbb{R}^N$, Marano-Mosconi [27] established the existence and sharp asymptotic behavior of solution for equation (FPS) with Hardy Sobolev exponent. Assuncao-Silva-Miyagaki [2] studied the existence of weak solution to fractional p-Laplacian equation involving the Hardy potential and multiple critical Sobolev nonlinearities with singularities. Fiscella-Mirzaee [19] established the existence of innitely many solutions involving a Hardy potential and Hardy Sobolev terms. Mirzaee [28] proved the existence of infinitely many solutions by using variational methods.

For the case where $\lambda = \mu = 1$, significant research efforts have been dedicated to exploring various aspects of equation ($S_{\lambda,\mu}$). Chergui-Gou-Hajaiej [15] delved into the existence and multiplicity of solutions, shedding light on the behavior of the equation in this setting. Luo-Hajaiej [26] focused on the existence of normalized solutions, providing valuable insights into the nature of solutions under these conditions. Meanwhile, Chergui's work [14] centered on the exploration of normalized solutions for equation ($S_{\lambda,\mu}$) with Hartree type nonlinearity, contributing to a deeper understanding of the equation's properties. For a comprehensive overview of related research, we also recommend [13, 23].

The prior research naturally leads to an important inquiry: **What are the existence results for equation ($S_{\lambda,\mu}$) with critical exponents**? This paper aims to address this fundamental question and provide a comprehensive understanding of the equation's behavior under critical exponents.

We consider $\lambda = \mu = 1$ and $\Omega = \mathbb{R}^N$. Moreover, if $f(x, u) = \frac{|u|^{p-2}u}{|x|^\alpha}$, then equation ($S_{\lambda,\mu}$) is

$$-\Delta u + (-\Delta)^s u = \frac{|u|^{p-2}u}{|x|^\alpha}, \quad x \in \mathbb{R}^N. \tag{P}$$

If $f(u) = \frac{|u|^{2^*_{1,\alpha}-2}u}{|x|^\alpha} + \beta\frac{|u|^{p-2}u}{|x|^\alpha}$, then equation ($S_{\lambda,\mu}$) is

$$-\Delta u + (-\Delta)^s u = \frac{|u|^{2^*_{1,\alpha}-2}u}{|x|^\alpha} + \beta\frac{|u|^{p-2}u}{|x|^\alpha}, \quad x \in \mathbb{R}^N. \tag{U}$$

If $f(u) = \frac{|u|^{2^*_{s,\alpha}-2}u}{|x|^\alpha} + \beta\frac{|u|^{p-2}u}{|x|^\alpha}$, then equation ($S_{\lambda,\mu}$) is

$$-\Delta u + (-\Delta)^s u = \frac{|u|^{2^*_{s,\alpha}-2}u}{|x|^\alpha} + \beta\frac{|u|^{p-2}u}{|x|^\alpha}, \quad x \in \mathbb{R}^N, \tag{L}$$

where $2^*_{1,\alpha} = \frac{2(N-\alpha)}{N-2}$ and $2^*_{s,\alpha} = \frac{2(N-\alpha)}{N-2s}$.

Initially, we will demonstrate the non-existence of solutions for equation (P) with critical exponents via the Pohožaev identity.

Theorem 1.1 *Let $N \geqslant 3$, $0 < s < 1$ and $0 \leqslant \alpha < 2s$. If $p = 2^*_{s,\alpha}$ or $p = 2^*_{1,\alpha}$, then equation (P) has no non-trivial solution.*

Remark 1.1 From Theorem 1.1, we know that $p \in (2^*_{s,\alpha}, 2^*_{1,\alpha})$ is the potential case for the existence result.

Furthermore, in this paper, we will establish the existence of solutions for equation ($S_{\lambda,\mu}$) with critical exponents. Our approach will involve novel techniques that extend beyond the existing methods used to study the equation with critical exponents.

Theorem 1.2 *Let $N \geqslant 3$, $0 < s < 1$, $0 \leqslant \alpha < 2s$ and $p \in (2^*_{s,\alpha}, 2^*_{1,\alpha})$. Then we have the following results:*

(i) equation (P) has a radial ground state solution;
(ii) there exists $\beta_1 \in (0, +\infty)$ such that for any $\beta > \beta_1$, equation (U) has a radial ground state solution;
(iii) there exists $\beta_2 \in (0, +\infty)$ such that for any $\beta > \beta_2$, equation (L) has a radial ground state solution.

We also study the case $-\infty < \alpha < 0$, which is called Henon Sobolev case.

Theorem 1.3 *Let $N \geqslant 3$, $\frac{1}{2} < s < 1$, $-\infty < \alpha < 0$ and $p \in (2^*_{s,\alpha}, 2^*_{1,\alpha})$. Then we have the following results:*

(i) equation (P) has a radial solution;
(ii) there exists $\beta_3 \in (0, +\infty)$ such that for any $\beta > \beta_3$, equation (U) has a radial solution;
(iii) there exists $\beta_4 \in (0, +\infty)$such that for any $\beta > \beta_4$, equation (L) has a radial ground state solution.

Motivated by all of the quoted papers above, it is quite natural to present some essential difficulties. For example

Question 1. For the zero mass case, we loss the term of $L^2(\mathbb{R}^N)$ in equation ($S_{\lambda,\mu}$). Hence, the working space is not $H^1(\mathbb{R}^N)$. We set the working space as

$$E{:=}D^{1,2}(\mathbb{R}^N) \cap D^{s,2}(\mathbb{R}^N).$$

But, we do not have the continuous and compact embedding from E to $L^t(\mathbb{R}^N, |x|^\alpha)$ at hand.

Answer 1. For $0 \leqslant \alpha < 2s$, we establish the following embedding results, see Lemmas 2.3 and 2.6 for $\alpha = 0$, and Lemmas 3.1 and 3.2 for $0 < \alpha < 2s$,

$$E \hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), t \in [2^*_{s,\alpha}, 2^*_{1,\alpha}],$$
$$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), t \in (2^*_{s,\alpha}, 2^*_{1,\alpha}).$$

Question 2. Particularly, for $-\infty < \alpha < 0$, we can not establish the continuous embedding from E to $L^t(\mathbb{R}^N, |x|^\alpha)$.

Answer 2. For $s \in (\frac{1}{2}, 1)$, by using the radial inequalities in Lemmas 2.4 and 4.1, we establish the following embedding results, see Lemmas 4.2 and 4.5 for $-\infty < \alpha < 0$,

$$E_{rad} \hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), t \in [2^*_{s,\alpha}, 2^*_{1,\alpha}],$$
$$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), t \in (2^*_{s,\alpha}, 2^*_{1,\alpha}).$$

Remark 1.2 In Answer 2, we just consider the case $s \in (\frac{1}{2}, 1)$. Due to the absense of radial inequality for $D^{s,2}(\mathbb{R}^N)$ for $s \in (0, \frac{1}{2}]$, this remainder case is open.

2 Sobolev Embedding for $\alpha = 0$

Define the following space

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) | |\nabla u| \in L^2(\mathbb{R}^N)\},$$

its norm is taken as

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x.$$

Let $C_0^\infty(\mathbb{R}^N)$ be the collection of smooth functions with compact support. For $N \geqslant 3$ and $s \in (0, 1)$, let the homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with the semi-norm

$$\|u\|^2_{D^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

The mixed Sobolev space E defined by the completion of $C_0^\infty(\mathbb{R}^N)$ under the semi-norm

$$\|u\|^2_E := \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Lemma 2.1 *$E \hookrightarrow D^{1,2}(\mathbb{R}^N)$ and $E \hookrightarrow D^{s,2}(\mathbb{R}^N)$.*

Proof It is easy to see that

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \leqslant \|u\|^2_E,$$

and

$$\|u\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant \|u\|^2_E.$$

These show $E \hookrightarrow D^{1,2}(\mathbb{R}^N)$ and $E \hookrightarrow D^{s,2}(\mathbb{R}^N)$. □

Lemma 2.2 *[21] Let $s \in (0,1)$, $\alpha \in (0,2s)$ and $N > 2s$. Then there exists a constant $S_s > 0$ such that for any $u \in D^{s,2}(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x\right)^{\frac{2}{2^*_{s,\alpha}}} \leqslant S_s^{-1} \|u\|^2_{D^{s,2}(\mathbb{R}^N)},$$

where $2^*_{s,\alpha} := \frac{2(N-\alpha)}{N-2s}$ is the so-called the critical fractional Hardy-Sobolev exponent. In particular [8], when $s = 1$ and $N \geqslant 3$, then there is a constant $S > 0$ such that

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{1,\alpha}}}{|x|^\alpha} \mathrm{d}x\right)^{\frac{2}{2^*_{1,\alpha}}} \leqslant S^{-1} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x,$$

where $2^*_{1,\alpha} := \frac{2(N-\alpha)}{N-2}$ is the so-called the critical Hardy-Sobolev exponent.

Lemma 2.3 *$E \hookrightarrow L^t(\mathbb{R}^N)$, $t \in [2^*_s, 2^*]$.*

Proof Using Hölder's inequality, we have

$$\int_{\mathbb{R}^N} |u|^t \mathrm{d}x \leqslant \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} \mathrm{d}x\right)^{\frac{(tN-2N-2t)(N-2s)}{4N(s-1)}} \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \mathrm{d}x\right)^{\frac{(tN-2N-2ts)(N-2)}{4N(1-s)}}.$$

From Lemma 2.1, we know

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} \mathrm{d}x\right)^{\frac{2}{2^*_s}} \leqslant \|u\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant \|u\|^2_E,$$

and

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant \|u\|^2_{D^{1,2}(\mathbb{R}^N)} \leqslant \|u\|^2_E.$$

Then we get

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^t \mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} \mathrm{d}x\right)^{\frac{(tN-2N-2t)(N-2s)}{4N(s-1)}} \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \mathrm{d}x\right)^{\frac{(tN-2N-2ts)(N-2)}{4N(1-s)}} \\ &\leqslant \|u\|^t_E < \infty.\end{aligned}$$

The proof is completed. □

Lemma 2.4 *[3, 31] For $u \in D^{1,2}(\mathbb{R}^N)$ and $N \geqslant 3$, we have*

$$|u(x)| \leqslant C|x|^{-\frac{N-2}{2}} \|u\|^{\frac{1}{2}}_{D^{1,2}(\mathbb{R}^N)},$$

where $C > 0$ is independent of u.

Lemma 2.5 *[3, Theorem A.I.] Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$P(s)/Q(s) \to 0, \text{ as } |s| \to +\infty.$$

Let $\{u_n\}$ be a sequence of measurable functions: $\mathbb{R}^N \to \mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n)| \mathrm{d}x < +\infty,$$

and

$$P(u_n) \to v \text{ a.e. in } \mathbb{R}^N, \text{ as } n \to \infty.$$

Then for any bounded Borel set B one has $\int_B |P(u_n) - v| \mathrm{d}x \to 0$, as $n \to \infty$.

If one further assumes that

$$P(s)/Q(s) \to 0, \text{ as } |s| \to 0,$$

and

$$u_n(x) \to 0, \text{ as } |x| \to \infty, \text{ uniformly with respect to } n,$$

Then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 2.6 *$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N)$, $t \in (2_s^*, 2^*)$, where $2^* = \frac{2N}{N-2}$, $2_s^* = \frac{2N}{N-2s}$ and E_{rad} is the set of radial functions of E.*

Proof Let $\{u_n\} \subset E_{rad}$ be a sequence such that $\|u_n\|_E$ is bounded. From Lemma 2.4, we have

$$\lim_{|x| \to +\infty} |u_n(x)| = 0,$$

with respect to n. We can extract a subsequence $\{u_{n_k}\}$ which converges almost everywhere in $\mathbb{R}^N$, and weakly in E_{rad} to a radial u. Appling Lemma 2.5 with $P(s) = s^t$ and $Q(s) = s^{2_s^*} + s^{2^*}$, $t \in (2_s^*, 2^*)$, we know that $\{u_{n_k}\}$ converges strongly to u in $L^t(\mathbb{R}^N)$. □

3 Sobolev Embedding for $\alpha \in (0, 2s)$

In this section, we present the continuous and compact embedding results for $\alpha \in (0, 2s)$.

Lemma 3.1 *$E \hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha)$, $t \in [2_{s,\alpha}^*, 2_{1,\alpha}^*]$.*

Proof It follows from Hölder's inequality that

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{|u|^t}{|x|^\alpha}\mathrm{d}x\\ &\leqslant\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{2(N-\alpha)}{N-2s}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{\frac{2(N-\alpha)}{N-2}-t}{\frac{2(N-\alpha)}{N-2}-\frac{2(N-\alpha)}{N-2s}}}\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{2(N-\alpha)}{N-2}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{t-\frac{2(N-\alpha)}{N-2s}}{\frac{2(N-\alpha)}{N-2}-\frac{2(N-\alpha)}{N-2s}}}.\end{aligned} \tag{3.1}$$

We recall the following Hardy-Sobolev inequality and fractional Hardy-Sobolev inequality in Lemma 2.2

$$C\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{2(N-\alpha)}{N-2}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{2}{\frac{2(N-\alpha)}{N-2}}}\leqslant\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x, \tag{3.2}$$

and

$$C\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{2(N-\alpha)}{N-2s}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{2}{\frac{2(N-\alpha)}{N-2s}}}\leqslant\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

Combining (3.1)-(3.3), we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u|^t}{|x|^\alpha}\mathrm{d}x\leqslant&\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{2(N-\alpha)}{N-2}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{t-\frac{2(N-\alpha)}{N-2s}}{\frac{2(N-\alpha)}{N-2}-\frac{2(N-\alpha)}{N-2s}}}\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{2(N-\alpha)}{N-2s}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{\frac{2(N-\alpha)}{N-2}-t}{\frac{2(N-\alpha)}{N-2}-\frac{2(N-\alpha)}{N-2s}}}\\ \leqslant&C\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^{\frac{\frac{2(N-\alpha)}{N-2}}{2}\cdot\frac{t-\frac{2(N-\alpha)}{N-2s}}{\frac{2(N-\alpha)}{N-2}-\frac{2(N-\alpha)}{N-2s}}}\\ &\cdot\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{\frac{2(N-\alpha)}{N-2s}}{2}\cdot\frac{\frac{2(N-\alpha)}{N-2}-t}{\frac{2(N-\alpha)}{N-2}-\frac{2(N-\alpha)}{N-2s}}}\\ \leqslant&C\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{t}{2}}\\ =&C\|u\|_E^t.\end{aligned}$$

The proof is completed. □

Lemma 3.2 *$E_{rad}\hookrightarrow\hookrightarrow L^t(\mathbb{R}^N,|x|^\alpha)$, $t\in(2^*_{s,\alpha},2^*_{1,\alpha})$, where E_{rad} is the set of radial functions of E.*

Proof Let u_n be a bounded sequence in E_{rad}. Up to a sequence, one has

$$\begin{aligned}&u_n\rightharpoonup u,\ \text{ in }\ E_{rad},\\ &u_n\to u, a.e.\ in\ \mathbb{R}^N.\end{aligned}$$

We will show that there exists $\varpi(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N} \frac{|u_n - u|^t}{|x|^\alpha} \mathrm{d}x < \varepsilon.$$

By using Holder's and Hardy's inequalities [20, Theorem 1.1], we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u_n - u|^t}{|x|^\alpha} \mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|u_n - u|^\alpha}{|x|^\alpha} |u_n - u|^{t-\alpha} \mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{R}^N} \frac{|u_n - u|^2}{|x|^2} \mathrm{d}x \right)^{\frac{\alpha}{2}} \left(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2(t-\alpha)}{2-\alpha}} \mathrm{d}x \right)^{\frac{2-\alpha}{2}} \\ &\leqslant C \left(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2(t-\alpha)}{2-\alpha}} \mathrm{d}x \right)^{\frac{2-\alpha}{2}},\end{aligned}$$

where

$$\frac{2(N - s\alpha)}{N - 2s} < t < \frac{2(N - \alpha)}{N - 2} \Leftrightarrow \begin{cases} 2_s^* < \frac{2(t-\alpha)}{2-\alpha} \\ \frac{2(t-\alpha)}{2-\alpha} < 2^* \end{cases}.$$

It follows from $E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N)$ with $t \in (2_s^*, 2^*)$ and $\frac{2(N-s\alpha)}{N-2s} < t < \frac{2(N-\alpha)}{N-2}$ that

$$\int_{\mathbb{R}^N} \frac{|u_n - u|^t}{|x|^\alpha} \mathrm{d}x < \varepsilon. \tag{3.4}$$

By using Holder's and fractional Hardy's inequalities [20, Theorem 1.1], we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u_n - u|^t}{|x|^\alpha} \mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|u_n - u|^{\frac{\alpha}{s}}}{|x|^\alpha} |u_n - u|^{t-\frac{\alpha}{s}} \mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{R}^N} \frac{|u_n - u|^{\frac{\alpha}{s}\frac{2s}{\alpha}}}{|x|^{\alpha\frac{2s}{\alpha}}} \mathrm{d}x \right)^{\frac{\alpha}{2s}} \left(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2s(t-\frac{\alpha}{s})}{2s-\alpha}} \mathrm{d}x \right)^{\frac{2s-\alpha}{2s}} \\ &= \left(\int_{\mathbb{R}^N} \frac{|u_n - u|^2}{|x|^{2s}} \mathrm{d}x \right)^{\frac{\alpha}{2s}} \left(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2(ts-\alpha)}{2s-\alpha}} \mathrm{d}x \right)^{\frac{2s-\alpha}{2s}} \\ &\leqslant C \left(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2(ts-\alpha)}{2s-\alpha}} \mathrm{d}x \right)^{\frac{2s-\alpha}{2s}},\end{aligned}$$

where

$$\frac{2(N - \alpha)}{N - 2s} < t < \frac{2(N - \frac{\alpha}{s})}{N - 2} \Leftrightarrow \begin{cases} 2_s^* < \frac{2(ts-\alpha)}{2s-\alpha} \\ \frac{2(ts-\alpha)}{2s-\alpha} < 2^* \end{cases}.$$

It follows from $E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N)$ with $t \in (2_s^*, 2^*)$ and $\frac{2(N-\alpha)}{N-2s} < t < \frac{2(N-\frac{\alpha}{s})}{N-2}$ that

$$\int_{\mathbb{R}^N} \frac{|u_n - u|^t}{|x|^\alpha} dx < \varepsilon. \tag{3.5}$$

Clearly,

$$\begin{cases} \frac{2(N-\frac{\alpha}{s})}{N-2} \leqslant \frac{2(N-s\alpha)}{N-2s}, & \alpha \geqslant \frac{N}{N-1}, \\ \frac{2(N-\frac{\alpha}{s})}{N-2} < \frac{2(N-s\alpha)}{N-2s}, & \alpha < \frac{N}{N-1}. \end{cases}$$

For $\alpha \geqslant \frac{N}{N-1}$, to check $\frac{2(N-\frac{\alpha}{s})}{N-2} \leqslant t \leqslant \frac{2(N-s\alpha)}{N-2}$, we set $\frac{2(N-\alpha)}{N-2s} < t_1 < \frac{2(N-\frac{\alpha}{s})}{N-2}$ and $\frac{2(N-s\alpha)}{N-2s} < t_2 < \frac{2(N-\alpha)}{N-2}$. By using Holder's inequality, (3.4) and (3.5), one has

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u_n - u|^t}{|x|^\alpha} dx &\leqslant \left(\int_{\mathbb{R}^N} \frac{|u_n - u|^{t_1}}{|x|^\alpha} dx \right)^{\frac{t_2-t}{t_2-t_1}} \left(\int_{\mathbb{R}^N} \frac{|u_n - u|^{t_2}}{|x|^\alpha} dx \right)^{\frac{t-t_1}{t_2-t_1}} \\ &< \varepsilon^{\frac{t_2-t}{t_2-t_1}} \varepsilon^{\frac{t-t_1}{t_2-t_1}} \\ &= \varepsilon. \end{aligned}$$

The proof is completed. □

4 Sobolev Embedding for $\alpha \in (-\infty, 0)$

In this section, we present the continuous and compact embedding results for $\alpha \in (-\infty, 0)$.

Lemma 4.1 *[29] Let $N \geqslant 2$ and $s \in (\frac{1}{2}, 1)$. For $u \in D^{s,2}(\mathbb{R}^N)$, we have*

$$|u(x)| \leqslant C|x|^{-\frac{N-2s}{2}} \|u\|_{D^{s,2}(\mathbb{R}^N)}^{\frac{1}{2s}},$$

where $C > 0$ is independent of u.

Lemma 4.2 *Let $\alpha \in (-\infty, 0)$ and $s \in (\frac{1}{2}, 1)$. Then $E_{rad} \hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha)$, $t \in [2_{s,\alpha}^*, 2_{1,\alpha}^*]$.*

Proof From Lemma 2.4, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u|^{2_{1,\alpha}^*}}{|x|^\alpha} dx &= \int_{\mathbb{R}^N} |u|^{2_{1,\alpha}^* - 2^*} \frac{1}{|x|^\alpha} |u|^{2^*} dx \\ &\leqslant C \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\frac{N-2}{2}}} \right)^{2_{1,\alpha}^* - 2^*} \frac{1}{|x|^\alpha} |u|^{2^*} dx \\ &= C \int_{\mathbb{R}^N} |u|^{2^*} dx. \end{aligned}$$

It follows from Lemma 4.1 that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x =& \int_{\mathbb{R}^N} |u|^{2^*_{s,\alpha}-2^*_s} \frac{1}{|x|^\alpha}|u|^{2^*_s}\mathrm{d}x \\ \leqslant& C\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\frac{N-2s}{2}}}\right)^{2^*_{s,\alpha}-2^*_s} \frac{1}{|x|^\alpha}|u|^{2^*_s}\mathrm{d}x \\ =& C\int_{\mathbb{R}^N}|u|^{2^*_s}\mathrm{d}x.\end{aligned} \tag{4.1}$$

□

Lemma 4.3 *[31] Let $\alpha \in (-\infty, 0)$ and $N \geqslant 3$. Then for any $u \in D^{1,2}_{rad}(\mathbb{R}^N)$, we have*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{1,\alpha}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_{1,\alpha}}} \leqslant H^{-1}\|u\|^2_{D^{1,2}(\mathbb{R}^N)}.$$

Lemma 4.4 *Let $\alpha \in (-\infty, 0)$ and $s \in (\frac{1}{2}, 1)$. Then $u \in D^{s,2}_{rad}(\mathbb{R}^N)$, we know*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_{s,\alpha}}} \leqslant H_s^{-1}\|u\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

Proof By using (4.1) and the Sobolev inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_{s,\alpha}}} \leqslant& C\int_{\mathbb{R}^N}|u|^{2^*_s}\mathrm{d}x \\ \leqslant& C\|u\|^2_{D^{s,2}(\mathbb{R}^N)}.\end{aligned}$$

□

Lemma 4.5 *Let $\alpha \in (-\infty, 0)$ and $s \in (\frac{1}{2}, 1)$. Then $E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha)$, $t \in (2^*_{s,\alpha}, 2^*_{1,\alpha})$.*

Proof **Step 1.** By using Lemma 2.4, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u|^t}{|x|^\alpha}\mathrm{d}x =& \int_{\mathbb{R}^N} |u|^{2^*_{1,\alpha}-2^*} \frac{1}{|x|^\alpha}|u|^{t+2^*-2^*_{1,\alpha}}\mathrm{d}x \\ \leqslant& C\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\frac{N-2}{2}}}\right)^{2^*_{1,\alpha}-2^*} \frac{1}{|x|^\alpha}|u|^{t+2^*-2^*_{1,\alpha}}\mathrm{d}x \\ =& C\int_{\mathbb{R}^N}|u|^{t+2^*-2^*_{1,\alpha}}\mathrm{d}x.\end{aligned} \tag{4.2}$$

Let $t \in (2_s^* - 2^* + 2_{1,\alpha}^*, 2_{1,\alpha}^*)$. Then we have

$$2_s^* < t + 2^* - 2_{1,\alpha}^* < 2^*.$$

By using Lemma 2.6 and (4.2), we know

$$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), \quad t \in (2_s^* - 2^* + 2_{1,\alpha}^*, 2_{1,\alpha}^*). \tag{4.3}$$

Step 2. By using Lemma 4.1, we know

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u|^t}{|x|^\alpha} \mathrm{d}x =& \int_{\mathbb{R}^N} |u|^{2_{s,\alpha}^* - 2_s^*} \frac{1}{|x|^\alpha} |u|^{t+2_s^* - 2_{s,\alpha}^*} \mathrm{d}x \\ \leqslant& C \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\frac{N-2s}{2}}} \right)^{2_{s,\alpha}^* - 2_s^*} \frac{1}{|x|^\alpha} |u|^{t+2_s^* - 2_{s,\alpha}^*} \mathrm{d}x \\ =& C \int_{\mathbb{R}^N} |u|^{t+2_s^* - 2_{s,\alpha}^*} \mathrm{d}x.\end{aligned} \tag{4.4}$$

Let $t \in (2_{s,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*)$, we have

$$2_s^* < t + 2_s^* - 2_{s,\alpha}^* < 2^*.$$

By using Lemma 2.6 and (4.4), we know

$$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), \quad t \in (2_{s,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*). \tag{4.5}$$

Step 3. For $\alpha \in [-N, 0)$, we have

$$2_s^* - 2^* + 2_{1,\alpha}^* \leqslant 2^* - 2_s^* + 2_{s,\alpha}^*.$$

Then from (4.3) and (4.5), we get

$$\begin{aligned}E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), \quad t \in& (2_{s,\alpha}^*, 2_{1,\alpha}^*) \\ =& (2_{s,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*) \cup (2_s^* - 2^* + 2_{1,\alpha}^*, 2_{1,\alpha}^*).\end{aligned}$$

Step 4. For $\alpha \in (-\infty, -N)$, we have

$$2_s^* - 2^* + 2_{1,\alpha}^* > 2^* - 2_s^* + 2_{s,\alpha}^*.$$

Then from (4.3) and (4.5), we get

$$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), \quad t \in (2_{s,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*) \cup (2_s^* - 2^* + 2_{1,\alpha}^*, 2_{1,\alpha}^*). \tag{4.6}$$

For $t \in [2_s^* - 2^* + 2_{1,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*]$, let $t_1 \in (2_{s,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*)$ and $t_2 \in (2_s^* - 2^* + 2_{1,\alpha}^*, 2_{1,\alpha}^*)$, applying Holder's inequality, one has

$$\int_{\mathbb{R}^N} \frac{|u|^t}{|x|^\alpha} \mathrm{d}x \leqslant \left(\int_{\mathbb{R}^N} \frac{|u|^{t_2}}{|x|^\alpha} \mathrm{d}x \right)^{\frac{t_1 - t}{t_1 - t_2}} \left(\int_{\mathbb{R}^N} \frac{|u|^{t_1}}{|x|^\alpha} \mathrm{d}x \right)^{\frac{t - t_2}{t_1 - t_2}}. \tag{4.7}$$

Combining (4.6) and (4.7), we have

$$E_{rad} \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N, |x|^\alpha), \quad t \in [2_s^* - 2^* + 2_{1,\alpha}^*, 2^* - 2_s^* + 2_{s,\alpha}^*].$$

The proof is completed. □

5 The Proof of Theorem 1.1

Lemma 5.1 *Let $u \in E$ be a weak solution of equation (P). Then u satisfies the following Pohožaev identity*

$$\frac{N-2}{2} \|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N-2s}{2} \|u\|^2_{D^{s,2}(\mathbb{R}^N)} = \frac{N-\alpha}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x. \tag{5.1}$$

Proof Multiply the equation (*P*) by $x \cdot \nabla u$ on both sides and integrate by parts, we get

$$\langle -\Delta u, x \cdot \nabla u \rangle_{L^2(\mathbb{R}^N)} + \langle (-\Delta)^s u, x \cdot \nabla u \rangle_{L^2(\mathbb{R}^N)} = \left\langle \frac{|u|^{p-2}u}{|x|^\alpha}, x \cdot \nabla u \right\rangle_{L^2(\mathbb{R}^N)}.$$

From [3], we have

$$\langle -\Delta u, x \cdot \nabla u \rangle_{L^2(\mathbb{R}^N)} = -\frac{N-2}{2} \|u\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

From [5, Proposition B.1], we have

$$\langle (-\Delta)^s u, x \cdot \nabla u \rangle_{L^2(\mathbb{R}^N)} = -\frac{N-2s}{2} \|u\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

From [22, Theorem 2.1], we get

$$\left\langle \frac{|u|^{p-2}u}{|x|^\alpha}, x \cdot \nabla u \right\rangle_{L^2(\mathbb{R}^N)} = -\frac{N-\alpha}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x.$$

Then we get the Pohožaev identity. □

By applying the Pohožaev identity, we can prove Theorem 1.1.

The proof of Theorem 1.1 If $u \in E$ is a weak solution of equation (P), then u satisfies the following Nehari identity

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \|u\|^2_{D^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x. \tag{5.2}$$

Combining (5.1) and (5.2), one has

$$\frac{N-2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N-2s}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} = \frac{N-\alpha}{p}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N-\alpha}{p}\|u\|^2_{D^{s,2}(\mathbb{R}^N)},$$

which gives

$$\left(\frac{N-2}{2} - \frac{N-\alpha}{p}\right)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \left(\frac{N-2s}{2} - \frac{N-\alpha}{p}\right)\|u\|^2_{D^{s,2}(\mathbb{R}^N)} = 0. \tag{5.3}$$

If $p = \frac{2(N-\alpha)}{N-2}$, then $\frac{N-2}{2} - \frac{N-\alpha}{p} = 0$ and $\frac{N-2s}{2} - \frac{N-\alpha}{p} > 0$, and from (5.3), we have

$$\left(\frac{N-2s}{2} - \frac{N-\alpha}{p}\right)\|u\|^2_{D^{s,2}(\mathbb{R}^N)} = 0.$$

This shows $u \equiv 0$, which is a contradiction.

If $p = \frac{2(N-\alpha)}{N-2s}$, then $\frac{N-2}{2} - \frac{N-\alpha}{p} < 0$ and $\frac{N-2s}{2} - \frac{N-\alpha}{p} = 0$, from (5.3) again, we deduce

$$\left(\frac{N-2}{2} - \frac{N-\alpha}{p}\right)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = 0.$$

This implies $u \equiv 0$, which is a contradiction. □

6 Mountain-Pass Geometric Structure and Nehari Manifold

The energy functionals corresponding to the equations (P), (U) and (L) are

$$I_p(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\right) - \frac{1}{p}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x,$$

and

$$\begin{aligned} I_{2^*_{1,\alpha}}(u) =& \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\right) \\ & - \frac{1}{2^*_{1,\alpha}}\int_{\mathbb{R}^N} \frac{|u|^{2^*_{1,\alpha}}}{|x|^\alpha}\mathrm{d}x - \frac{\beta}{p}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x, \end{aligned}$$

and

$$I_{2^*_{s,\alpha}}(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\right) - \frac{1}{2^*_{s,\alpha}}\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x - \frac{\beta}{p}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x.$$

It is worth noting that the mountain-pass geometric structure and Nehari manifold of the three energy functionals mentioned above exhibit remarkable similarities. As a result, we will focus on presenting the case of $I_{2^*_{s,\alpha}}$, which captures the essence of the analysis. In particular, we will consider the Fréchet derivative $I'_{2^*_{s,\alpha}}(u)$ corresponding to $I_{2^*_{s,\alpha}}(u)$, where $\phi \in E$,

$$\langle I'_{2^*_{s,\alpha}}(u), \phi\rangle = \int_{\mathbb{R}^N} \nabla u \nabla \phi \mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}-1}\phi}{|x|^\alpha}\mathrm{d}x - \beta\int_{\mathbb{R}^N} \frac{|u|^{p-1}\phi}{|x|^\alpha}\mathrm{d}x.$$

We set

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_{2^*_{s,\alpha}}(\gamma(t)) > 0 \text{ and } \Gamma = \{\gamma \in C([0,1], E) \,|\, \gamma(0) = 0, I_{2^*_{s,\alpha}}(\gamma(1)) < 0\}.$$

Lemma 6.1 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. Then the functional $I_{2^*_{s,\alpha}}$ has mountain pass geometric structure.*

Proof Using Lemma 2.3, one has

$$I_{2^*_{s,\alpha}}(u) \geqslant \|u\|_E^2 - C\beta\|u\|_E^p - C\|u\|_E^{2^*_{s,\alpha}}.$$

We should keep in mind that the exponent p lies within the range $2^*_{s,\alpha} < p < 2^*_{1,\alpha}$. Under this condition, then there exists a sufficiently small positive number ρ such that

$$\varsigma := \inf_{\|u\|_E=\rho} I_{2^*_{s,\alpha}}(u) > 0 = I(0).$$

For $u \in E \setminus \{0\}$, we have

$$I_{2^*_{s,\alpha}}(tu) = \frac{t^2}{2}\|u\|_E^2 - \beta\frac{t^p}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{t^{2^*_{s,\alpha}}}{2^*_{s,\alpha}}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x.$$

From $2^*_{s,\alpha} < p < 2^*_{1,\alpha}$, it follows that $I_{2^*_{s,\alpha}}(tu) < 0$ for t large enough.

From above, we can choose $t_u > 0$ corresponding to u such that $I_{2^*_{s,\alpha}}(t_u u) < 0$ for $t > t_u$ and $\|t_u u\|_E > \rho$. □

We now set the Nehari manifold as follows

$$\mathcal{N} = \{u \in E \setminus \{0\} | \langle I'_{2^*_{s,\alpha}}(u), u\rangle = 0\}.$$

Lemma 6.2 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. Then for any $u \in E \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $I_{2^*_{s,\alpha}}(t_u u) = \max_{t>0} I_{2^*_{s,\alpha}}(tu)$.*

Proof For any $u \in E\setminus\{0\}$ and $t \in (0, \infty)$, we define

$$f_1(t) = I_{2^*_{s,\alpha}}(tu) = \frac{t^2}{2}\|u\|_E^2 - \beta\frac{t^p}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{t^{2^*_{s,\alpha}}}{2^*_{s,\alpha}}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x.$$

Let's perform the computation

$$f_1'(t) = t\|u\|_E^2 - \beta t^{p-1}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - t^{2^*_{s,\alpha}-1}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x.$$

We know that $f_1'(\cdot) = 0$ iff

$$\|u\|_E^2 = \beta t^{p-2}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x + t^{2^*_{s,\alpha}-2}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x.$$

Let

$$f_2(t) = \beta t^{p-2}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x + t^{2^*_{s,\alpha}-2}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x.$$

Clearly, $\lim_{t\to 0} f_2(t) \to 0$, $\lim_{t\to+\infty} f_2(t) \to +\infty$. Therefore, according to the intermediate value theorem, there must exist a value $0 < t_u < \infty$ such that

$$f_2(t_u) = \|u\|_E^2.$$

Additionally, we can observe that the function $f_2(\cdot)$ is strictly increasing on the interval $(0, \infty)$. This property leads to the conclusion that the value t_u is unique. And then

$$\|u\|_E^2 = \beta t_u^{p-2}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x + t_u^{2^*_{s,\alpha}-2}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x,$$

which gives

$$\|t_u u\|_E^2 = \beta\int_{\mathbb{R}^N}\frac{|t_u u|^p}{|x|^\alpha}\mathrm{d}x + \int_{\mathbb{R}^N}\frac{|t_u u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x.$$

This implies that $t_u u \in \mathcal{N}$. □

Lemma 6.3 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. Then we have $\bar{c} = \inf_{u \in \mathcal{N}} I_{2^*_{s,\alpha}}(u) > 0$.*

Proof By applying $\langle I'_{2^*_{s,\alpha}}(u), u\rangle = 0$, we know

$$0 = \langle I'_{2^*_{s,\alpha}}(u), u\rangle \geqslant \|u\|_E^2 - C\beta\|u\|_E^p - C\|u\|_E^{2^*_{s,\alpha}},$$

which implies

$$C\beta\|u\|_E^{p-2} + C\|u\|_E^{2^*_{s,\alpha}-2} \geqslant 1,$$

and

$$\|u\|_E^2 \geqslant C.$$

Then, for $u \in \mathcal{N}$, we get

$$\begin{aligned} I_{2^*_{s,\alpha}}(u) =& I_{2^*_{s,\alpha}}(u) - \frac{1}{2^*_{s,\alpha}}\langle I'_{2^*_{s,\alpha}}(u), u\rangle \\ =& \frac{1}{2}\|u\|_E^2 - \beta\frac{1}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{1}{2^*_{s,\alpha}}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x \\ & - \frac{1}{2^*_{s,\alpha}}\left(\|u\|_E^2 - \beta\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x\right) \\ =& \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right)\|u\|_E^2 + \beta\left(\frac{1}{2^*_{s,\alpha}} - \frac{1}{p}\right)\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x \\ \geqslant& \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right)\|u\|_E^2 \geqslant C. \end{aligned}$$

Therefore, we can conclude that the functional $I_{2^*_{s,\alpha}}$ is bounded from below on $\mathcal{N}$. And then $\bar{c} > 0$. □

Set

$$\bar{\bar{c}} := \inf_{u \in E\setminus\{0\}} \sup_{t \geqslant 0} I_{2^*_{s,\alpha}}(tu).$$

Lemma 6.4 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. Then we have $c = \bar{c} = \bar{\bar{c}}$.*

Proof By using Lemma 6.2, We can directly obtain the following result:

$$\bar{c} = \bar{\bar{c}}.$$

For any $u \in E \setminus \{0\}$, there exists some $\tilde{t} > 0$ that is sufficiently large such that $I_{2^*_{s,\alpha}}(\tilde{t}u) < 0$. We can construct a path $\gamma : [0, 1] \to E$ by setting $\gamma(t) = t\tilde{t}u$. It is clear that $\gamma \in \Gamma$ and that

$$c \leqslant \bar{\bar{c}}.$$

Alternatively, for every path $\gamma \in \Gamma$, we can define $g(t) = \langle I'_{2^*_{s,\alpha}}(\gamma(t)), \gamma(t)\rangle$. It is evident that $g(0) = 0$ and $g(t) > 0$ for small values of t. By performing a direct calculation, we obtain the following expression:

$$\begin{aligned} &I_{2^*_{s,\alpha}}(\gamma(1)) - \frac{1}{2^*_{s,\alpha}}\langle I'_{2^*_{s,\alpha}}(\gamma(1)), \gamma(1)\rangle \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right)\|\gamma(1)\|_E^2 + \beta\left(\frac{1}{2^*_{s,\alpha}} - \frac{1}{p}\right)\int_{\mathbb{R}^N}\frac{|\gamma(1)|^p}{|x|^\alpha}\mathrm{d}x \geqslant 0, \end{aligned}$$

which shows

$$\begin{aligned} \langle I'_{2^*_{s,\alpha}}(\gamma(1)), \gamma(1)\rangle \leqslant & 2^*_{s,\alpha} \cdot I_{2^*_{s,\alpha}}(\gamma(1)) \\ = & 2^*_{s,\alpha} \cdot I_{2^*_{s,\alpha}}(\tilde{t}u) < 0. \end{aligned}$$

Thus, there exists $\tilde{\tilde{t}} \in (0, 1)$ such that $g(\tilde{\tilde{t}}) = 0$, i.e. $\gamma(\tilde{\tilde{t}}) \in \mathcal{N}$ and $c \geqslant \bar{c}$. This deduces $c = \bar{c} = \bar{\bar{c}}$.

□

Lemma 6.5 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. For $u \in \mathcal{N}$, we have $\Phi'(u) \neq 0$, where*

$$\Phi(u) = \langle I'_{2^*_{s,\alpha}}(u), u\rangle = \|u\|_E^2 - \beta\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x, \tag{6.1}$$

and

$$\langle \Phi'(u), u\rangle = 2\|u\|_E^2 - p\beta\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - 2^*_{s,\alpha}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x. \tag{6.2}$$

Moreover, if $u \in \mathcal{N}$ and $I_{2^*_{s,\alpha}}(u) = c$, then u is a ground state solution for equation (L).

Proof For $u \in \mathcal{N}$, it follows from (6.1) and (6.2) that

$$\begin{aligned}\langle \Phi'(u), u\rangle =&\langle \Phi'(u), u\rangle - 2^*_{s,\alpha}\Phi(u)\\ =&\left(2\|u\|_E^2 - p\beta\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - 2^*_{s,\alpha}\int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x\right)\\ &-2^*_{s,\alpha}\left(\|u\|_E^2 - \beta\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \int_{\mathbb{R}^N}\frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x\right)\\ =&(2-2^*_{s,\alpha})\|u\|_E^2 + \beta(2^*_{s,\alpha}-p)\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x\\ <&0.\end{aligned}$$

Thus, $\Phi'(u) \neq 0$ for $u \in \mathcal{N}$.

Suppose $u \in \mathcal{N}$ and $I_{2^*_{s,\alpha}}(u) = \bar{c}$, where $\bar{c}$ is the minimum of $I_{2^*_{s,\alpha}}$ on $\mathcal{N}$. By applying the Lagrange multiplier theorem, we can conclude that there exists a scalar $\lambda \in \mathbb{R}$ such that $I'_{2^*_{s,\alpha}}(u) = \lambda\Phi'(u)$.So

$$\langle \lambda\Phi'(u), u\rangle = \langle I'_{2^*_{s,\alpha}}(u), u\rangle = \Phi(u) = 0.$$

This shows $\lambda = 0$ and $I'_{2^*_{s,\alpha}}(u) = 0$. Thus, u is a ground state solution for equation (L). □

7 The Proof of Theorem 1.2

We recall the $(PS)_c$ sequence as follows.

Definition 7.1 If sequence $\{u_n\} \subset E$ satisfies the condition

$$I_{2^*_{s,\alpha}}(u_n) \to c \text{ and } I'_{2^*_{s,\alpha}}(u_n) \to 0 \text{ in } E^{-1}, \text{ as } n \to \infty.$$

Then $\{u_n\}$ is called the Palais-Smale sequence of $I_{2^*_{s,\alpha}}$ with respect to c, short for $(PS)_c$ sequence, where E^{-1} is the dual space of E.

Lemma 7.1 *Let* $N \geqslant 3$, $0 < s < 1$ *and* $0 < \alpha < 2s$. *Then there exists a bounded* $(PS)_c$ *sequence* $\{u_n\} \subset \mathcal{N}$ *such that*

$$I_{2^*_{s,\alpha}}(u_n) \to c \text{ and } \|I'_{2^*_{s,\alpha}}(u_n)\|_{E^{-1}} \to 0, \text{ as } n \to \infty.$$

Proof Based on Lemmas 6.2 and 6.4, we know that $\mathcal{N} \neq \emptyset$ and $\inf\limits_{u\in\mathcal{N}} I_{2^*_{s,\alpha}}(u) = \bar{c} = c$. By applying Ekeland's variational principle, there exist $\{u_n\} \subset \mathcal{N}$ and $\lambda_n \in \mathbb{R}$ such that

$$I_{2^*_{s,\alpha}}(u_n) \to \bar{c} \text{ and } I'_{2^*_{s,\alpha}}(u_n) - \lambda_n\Phi'(u_n) \to 0 \text{ in } E^{-1}, \text{ as } n \to \infty.$$

So

$$\bar{c} = I_{2^*_{s,\alpha}}(u_n) = I_{2^*_{s,\alpha}}(u_n) - \frac{1}{2^*_{s,\alpha}} \langle I'_{2^*_{s,\alpha}}(u_n), u_n \rangle \geqslant \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}} \right) \|u_n\|^2_E + \beta \left(\frac{1}{2^*_{s,\alpha}} - \frac{1}{p} \right) \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^\alpha} \mathrm{d}x,$$

which implies that $\{u_n\}$ is bounded in E.

Taking $n \to \infty$, we have

$$|\langle I'_{2^*_{s,\alpha}}(u_n), u_n \rangle - \langle \lambda_n \Phi'(u_n), u_n \rangle| \leqslant \|I'_{2^*_{s,\alpha}}(u_n) - \lambda_n \Phi'(u_n)\|_{E^{-1}} \|u_n\|_E \to 0,$$

we have

$$\langle I'_{2^*_{s,\alpha}}(u_n), u_n \rangle - \lambda_n \langle \Phi'(u_n), u_n \rangle \to 0, \text{ as } n \to \infty. \tag{7.1}$$

Note that $\{u_n\} \subset \mathcal{N}$. From Lemma 6.5, we obtain

$$\langle I'_{2^*_{s,\alpha}}(u_n), u_n \rangle = 0, \tag{7.2}$$

and

$$\langle \Phi'(u_n), u_n \rangle \neq 0. \tag{7.3}$$

Combining (7.1)–(7.3), we conclude $\lambda_n \to 0$.

It follows from Hölder's and Sobolev's inequalities that

$$\begin{aligned} &\|I'_{2^*_{s,\alpha}}(u_n)\|_{E^{-1}} \\ &= \sup_{\varphi \in E, \|\varphi\|_E = 1} |\langle \Phi'(u_n), \varphi \rangle| \\ &= \sup_{\varphi \in E, \|\varphi\|_E = 1} \Bigg| 2 \int_{\mathbb{R}^N} \nabla u \nabla \varphi \mathrm{d}x - \beta p \int_{\mathbb{R}^N} \frac{|u_n|^{p-2} u_n \varphi}{|x|^\alpha} \mathrm{d}x - 2^*_{s,\alpha} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,\alpha}-2} u_n \varphi}{|x|^\alpha} \mathrm{d}x \\ &\quad + 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \Bigg| \\ &\leqslant C. \end{aligned}$$

Then we obtain

$$\|I'_{2^*_{s,\alpha}}(u_n)\|_{E^{-1}} \leqslant \|I'_{2^*_{s,\alpha}}(u_n) - \lambda_n \Phi'(u_n)\|_{E^{-1}} + |\lambda_n| \|\Phi'(u_n)\|_{E^{-1}} = o(1).$$

That is, $I'_{2^*_{s,\alpha}}(u_n) \to 0$ in E^{-1}. Hence, $\{u_n\}$ is a $(PS)_c$ sequence of $I_{2^*_{s,\alpha}}$. □

Lemma 7.2 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. Then there exists a bounded nonnegative radial sequence $\{u_n\} \subset \mathcal{N}$ such that*

$$I_{2^*_{s,\alpha}}(u_n) \to c \text{ and } \langle I'_{2^*_{s,\alpha}}(u_n), u_n \rangle = 0.$$

Proof According to Lemma 7.1, we can deduce that there exists a bounded $(PS)_c$ sequence $\{u_n\} \subset \mathcal{N}$. It is easy to see that

$$\begin{aligned}&\int_{u_n(y)\geqslant 0}\int_{u_n(x)<0}\frac{||u_n(x)|-|u_n(y)||^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_{u_n(y)<0}\int_{u_n(x)\geqslant 0}\frac{||u_n(x)|-|u_n(y)||^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\&\leqslant\int_{u_n(y)\geqslant 0}\int_{u_n(x)<0}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\int_{u_n(y)<0}\int_{u_n(x)\geqslant 0}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

which implies

$$\|\,|u_n|\,\|_{D^{s,2}(\mathbb{R}^N)} \leqslant \|u_n\|_{D^{s,2}(\mathbb{R}^N)}.$$

Then,

$$I_{2^*_{s,\alpha}}(t|u_n|) \leqslant I_{2^*_{s,\alpha}}(tu_n), \quad t > 0.$$

Note that $\{u_n\} \subset \mathcal{N}$. Then $|u_n| \not\equiv 0$. And there exists a sequence $t_{1,u_n} > 0$ such that $t_{1,u_n}|u_n| \in \mathcal{N}$ and

$$\|\,|u_n|\,\|^2_{D^{1,2}(\mathbb{R}^N)} + \|\,|u_n|\,\|^2_{D^{s,2}(\mathbb{R}^N)} = t_{1,u_n}^{2^*_{s,\alpha}-2}\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x + t_{1,u_n}^{p-2}\beta\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x|^\alpha}\mathrm{d}x.$$

It follows from $\{u_n\} \subset \mathcal{N}$ that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x+\beta\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x|^\alpha}\mathrm{d}x&=\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}+\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)}\\&\geqslant\|\,|u_n|\,\|^2_{D^{1,2}(\mathbb{R}^N)}+\|\,|u_n|\,\|^2_{D^{s,2}(\mathbb{R}^N)}\\&=t_{1,u_n}^{2^*_{s,\alpha}-2}\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x+t_{1,u_n}^{p-2}\beta\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x|^\alpha}\mathrm{d}x,\end{aligned}$$

which gives

$$t_{1,u_n} \in (0, 1].$$

Furthermore, we have

$$\bar{c} \leqslant I_{2^*_{s,\alpha}}(t_{1,u_n}|u_n|) \leqslant I_{2^*_{s,\alpha}}(t_{1,u_n}u_n) \leqslant \max_{t\geqslant 0} I_{2^*_{s,\alpha}}(tu_n) = I_{2^*_{s,\alpha}}(u_n) = \bar{c}.$$

Then we know $I_{2^*_{s,\alpha}}(t_{1,u_n}|u_n|) = \bar{c} = c$.

Let us define v_n^* as the symmetric decreasing rearrangement of $v_n := t_{1,u_n}|u_n|$. Then

$$\|v_n^*\|_{D^{1,2}(\mathbb{R}^N)} \leqslant \|v_n\|_{D^{1,2}(\mathbb{R}^N)},$$

and

$$\|v_n^*\|_{D^{s,2}(\mathbb{R}^N)} \leqslant \|v_n\|_{D^{s,2}(\mathbb{R}^N)},$$

and

$$\int_{\mathbb{R}^N} \frac{|v_n^*|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x \geqslant \int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x,$$

and

$$\int_{\mathbb{R}^N} \frac{|v_n^*|^p}{|x|^\alpha} \mathrm{d}x \geqslant \int_{\mathbb{R}^N} \frac{|v_n|^p}{|x|^\alpha} \mathrm{d}x.$$

These deduce

$$I_{2^*_{s,\alpha}}(t|v_n^*|) \leqslant I_{2^*_{s,\alpha}}(tv_n), \quad t > 0.$$

Notice that $\{v_n\} \subset \mathcal{N}$. Then $v_n \not\equiv 0$ and there exists $t_{1,v_n^*} > 0$ such that $t_{1,v_n^*} v_n^* \in \mathcal{N}$. And

$$\|v_n^*\|^2_{D^{1,2}(\mathbb{R}^N)} + \|v_n^*\|^2_{D^{s,2}(\mathbb{R}^N)} = t_{1,v_n^*}^{2^*_{s,\alpha}-2} \int_{\mathbb{R}^N} \frac{|v_n^*|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x + t_{1,v_n^*}^{p-2} \beta \int_{\mathbb{R}^N} \frac{|v_n^*|^p}{|x|^\alpha} \mathrm{d}x.$$

It follows from $v_n := t_{1,u_n}|u_n| \in \mathcal{N}$ that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|v_n^*|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x + \beta \int_{\mathbb{R}^N} \frac{|v_n^*|^p}{|x|^\alpha} \mathrm{d}x &\geqslant \int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x + \beta \int_{\mathbb{R}^N} \frac{|v_n|^p}{|x|^\alpha} \mathrm{d}x \\
&= \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \|v_n\|^2_{D^{s,2}(\mathbb{R}^N)} \\
&\geqslant \|v_n^*\|^2_{D^{1,2}(\mathbb{R}^N)} + \|v_n^*\|^2_{D^{s,2}(\mathbb{R}^N)} \\
&= t_{1,v_n^*}^{2^*_{s,\alpha}-2} \int_{\mathbb{R}^N} \frac{|v_n^*|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x + t_{1,v_n^*}^{p-2} \beta \int_{\mathbb{R}^N} \frac{|v_n^*|^p}{|x|^\alpha} \mathrm{d}x,
\end{aligned}$$

which gives

$$t_{1,v_n^*} \in (0, 1].$$

and

$$\bar{c} \leqslant I_{2^*_{s,\alpha}}(t_{1,v_n^*}|v_n^*|) \leqslant I_{2^*_{s,\alpha}}(t_{1,v_n^*} v_n) \leqslant \max_{t \geqslant 0} I_{2^*_{s,\alpha}}(tv_n) = I_{2^*_{s,\alpha}}(v_n) = \bar{c}.$$

Then we know $I_{2^*_{s,\alpha}}(t_{1,v_n^*}|v_n^*|) = \bar{c} = c$. □

Lemma 7.3 *Assume that the assumptions of Theorem 1.2 hold. There exist* $\beta_1 \in (0, +\infty)$ *such that for any* $\beta > \beta_1$*, we have*

$$c \in (0, c^*),$$

where

$$c^* := \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right) S_s^{\frac{2^*_{s,\alpha}}{2^*_{s,\alpha}-2}},$$

where S_s *is the best constant of Sobolev inequality, see Lemma 2.2.*

Proof Let us select $w \in E$ in the following way:

$$\|w\|_E = 1 \text{ and } \int_{\mathbb{R}^N} |w|^p \mathrm{d}x > 0.$$

From the Mountain Pass geometric structure, one can deduce

$$\lim_{t \to +\infty} I_{2^*_{s,\alpha}}(tw) = -\infty,$$

and $t_{w,\beta} > 0$ such that $t_{w,\beta} w \in \mathcal{N}$

$$\sup_{t \geqslant 0} I_{2^*_{s,\alpha}}(tw) = I_{2^*_{s,\alpha}}(t_{w,\beta} w).$$

Thus, $t_{w,\beta}$ satisfies

$$t_{w,\beta}^2 \|w\|_E^2 = t_{w,\beta}^{2^*_{s,\alpha}} \int_{\mathbb{R}^N} \frac{|w|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x + \beta t_{w,\beta}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^\alpha} \mathrm{d}x. \tag{7.4}$$

Furthermore,

$$t_{w,\beta}^2 \|w\|_E^2 \geqslant t_{w,\beta}^{2^*_{s\alpha}} \int_{\mathbb{R}^N} \frac{|w|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x.$$

This gives $\{t_{w,\beta}\}_\beta$ is bounded.

We assert that $t_{w,\beta} \to 0$ as $\beta \to +\infty$. Let us argue by contradiction and assume that there exist $t_0 > 0$ and a sequence $\{\beta_n\}$ with $\beta_n \to \infty$ such that $t_{w,\beta_n} \to t_0$ as $n \to +\infty$. Then, we have the following:

$$\beta_n t_{w,\beta_n}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^\alpha} \mathrm{d}x \to +\infty, \quad \text{as } n \to +\infty.$$

Putting this into (7.4), we know

$$t_0^2\|w\|_E^2 = +\infty.$$

This is a contradiction with $\|w\|_E = 1$.

By applying $t_{w,\beta} \to 0$ as $\beta \to +\infty$, we obtain

$$\lim_{\beta\to+\infty}\sup_{t\geqslant 0} I_{2^*_{s,\alpha}}(tw) = \lim_{\beta\to+\infty} I_{2^*_{s,\alpha}}(t_{w,\beta}w) = 0.$$

Then there exists $\beta_1 \in (0,+\infty)$ such that for any $\beta > \beta_1$ there holds

$$\sup_{t\geqslant 0} I_{2^*_{s,\alpha}}(tw) < c^*.$$

For any $\beta > \beta_1$, we construct a mountain pass path as: taking $e = Tw$ and $\gamma(t) = te$ with T large enough to satisfies $I_{2^*_{s,\alpha}}(e) < 0$, then

$$c \leqslant \max_{t\in[0,1]} I_{2^*_{s,\alpha}}(\gamma(t)).$$

Hence, $c \leqslant \sup_{t\geqslant 0} I_{2^*_{s,\alpha}}(tw) < c^*$. □

Lemma 7.4 *Let $N \geqslant 3$, $0 < s < 1$ and $0 < \alpha < 2s$. Let $\{u_n\} \subset \mathcal{N}$ be a bounded nonnegative radial sequence such that*

$$I_{2^*_{s,\alpha}}(u_n) \to c \text{ and } \langle I'_{2^*_{s,\alpha}}(u_n), u_n\rangle = 0.$$

Then u_n converges strongly to $u \not\equiv 0$ in E. Moreover, we know that $I_{2^*_{s,\alpha}}(u) = c$.

Proof From Lemma 7.2, we know that bounded nonnegative radial sequence $\{u_n\} \subset \mathcal{N}$ with $c \in (0, c^*)$. If $\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x|^\alpha}\mathrm{d}x = 0$, then

$$c = I_{2^*_{s,\alpha}}(u_n) = \frac{1}{2}\|u_n\|_E^2 - \frac{1}{2^*_{s,\alpha}}\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x,$$

and

$$0 = \langle I'_{2^*_{s,\alpha}}(u_n), u_n\rangle = \|u_n\|_E^2 - \int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{s,\alpha}}}{|x|^\alpha}\mathrm{d}x, \tag{7.5}$$

which gives

$$c = \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right)\|u_n\|_E^2. \tag{7.6}$$

From (7.5) and Lemma 2.2

$$\|u_n\|_E^2 = \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x \leqslant S_s^{-\frac{2^*_{s,\alpha}}{2}} \|u_n\|_{D^{s,2}(\mathbb{R}^N)}^{2^*_{s,\alpha}} \leqslant S_s^{-\frac{2^*_{s,\alpha}}{2}} \|u_n\|_E^{2^*_{s,\alpha}},$$

which shows

$$S_s^{\frac{2^*_{s,\alpha}}{2}} \leqslant \|u_n\|_E^{2^*_{s,\alpha}-2} \Rightarrow \|u_n\|_E^2 \geqslant S_s^{\frac{2^*_{s,\alpha}}{2^*_{s,\alpha}-2}}. \tag{7.7}$$

Combining (7.6) and (7.7),

$$c \geqslant \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right) S_s^{\frac{2^*_{s,\alpha}}{2^*_{s,\alpha}-2}}.$$

This contradicts $0 < c < c^* = \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right) S_s^{\frac{2^*_{s,\alpha}}{2^*_{s,\alpha}-2}}$ in Lemma 7.3. Then we get $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^\alpha} \mathrm{d}x > 0$. By using Lemma 3.2, we know that $\{u_n\}$ converges strongly to $u \not\equiv 0$ in $L^p(\mathbb{R}^N, |x|^\alpha)$.

Now, by virtue of the Brezis-Lieb Lemma [7], one deduces

$$\begin{aligned}
\bar{c} \leqslant I_{2^*_{s,\alpha}}(u) =& I_{2^*_{s,\alpha}}(u) - \frac{1}{2^*_{s,\alpha}} \langle I'_{2^*_{s,\alpha}}(u), u\rangle \\
=& \frac{1}{2}\|u\|_E^2 - \beta \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x - \frac{1}{2^*_{s,\alpha}} \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x \\
&- \frac{1}{2^*_{s,\alpha}} \left(\|u\|_E^2 - \beta \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x - \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha} \mathrm{d}x \right) \\
=& \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right) \|u\|_E^2 + \beta \left(\frac{1}{2^*_{s,\alpha}} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x \\
\leqslant& \lim_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right) \|u_n\|_E^2 + \beta \left(\frac{1}{2^*_{s,\alpha}} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x \right] \\
=& \lim_{n\to\infty} I_{2^*_{s,\alpha}}(u_n) - \frac{1}{2^*_{s,\alpha}} \lim_{n\to\infty} \langle I'_{2^*_{s,\alpha}}(u_n), u_n\rangle \\
=& \lim_{n\to\infty} I_{2^*_{s,\alpha}}(u_n) \\
=& c = \bar{c},
\end{aligned}$$

which gives $I_{2^*_{s,\alpha}}(u) = \bar{c}$. □

At this point, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 According to Lemma 7.4, we can conclude that there exists $u \not\equiv 0$ such that $I_{2^*_{s,\alpha}}(u) = c$. Utilizing Lemma 6.5, we can further deduce that u serves as a ground state solution for equation (L). □

8 Proof of Theorem 1.3

Let

$$I_{p,rad} = I_p|_{E_{rad}}, \quad I_{2^*_{1,\alpha},rad} = I_{2^*_{1,\alpha}}|_{E_{rad}}, \quad \text{and} \quad I_{2^*_{s,\alpha},rad} = I_{2^*_{s,\alpha}}|_{E_{rad}}.$$

It is worth noting that the mountain-pass geometric structure and Nehari manifold of the three energy functionals mentioned above exhibit remarkable similarities. As a result, we will focus on presenting the case of $I_{2^*_{s,\alpha},rad}$, which captures the essence of the analysis. We set the Nehari manifold as follows

$$\mathcal{M} = \{u \in E_{rad} \setminus \{0\} | \langle I'_{2^*_{s,\alpha},rad}(u), u\rangle = 0\}.$$

We set

$$\begin{aligned} c_{rad} &= \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_{2^*_{s,\alpha},rad}(\gamma(t)) > 0 \text{ and} \\ \Gamma &= \{\gamma \in C([0,1], E_{rad}) \,|\gamma(0) = 0, I_{2^*_{s,\alpha},rad}(\gamma(1)) < 0\}, \end{aligned}$$

and

$$\bar{c}_{rad} = \inf_{u \in \mathcal{M}} I_{2^*_{s,\alpha},rad}(u) > 0,$$

and

$$\bar{\bar{c}}_{rad} = \inf_{u \in E_{rad} \setminus \{0\}} \sup_{t \geqslant 0} I_{2^*_{s,\alpha},rad}(tu).$$

The proof of Theorem 1.3 We have the similarly results for $I_{2^*_{s,\alpha},rad}$ without the proof of Lemmas 6.1-6.5. Repeatting the proof of Lemma 7.1, we know that there exists a bounded $(PS)_{c_{rad}}$ sequence $\{u_n\} \subset \mathcal{M}$ such that

$$I_{2^*_{s,\alpha},rad}(u_n) \to c_{rad} \text{ and } \|I'_{2^*_{s,\alpha},rad}(u_n)\|_{E^{-1}_{rad}} \to 0, \text{ as } n \to \infty.$$

Arguement as Lemma 7.3, there exists $\beta_3 \in (0, +\infty)$ such that for any $\beta > \beta_3$, we have

$$0 < c_{rad} < c^*_{rad} = \left(\frac{1}{2} - \frac{1}{2^*_{s,\alpha}}\right) H_s^{\frac{2^*_{s,\alpha}}{2^*_{s,\alpha}-2}}.$$

According to Lemma 7.4, we can conclude that there exists $u \not\equiv 0$ such that $I_{2^*_{s,\alpha},rad}(u) = c_{rad}$. Then we have $I'_{2^*_{s,\alpha},rad}(u) = 0$. From the Palais' principle of symmetric criticality [32], we know that the critical point of $I_{2^*_{s,\alpha},rad}$ are also the critical point of $I_{2^*_{s,\alpha}}$. □

Data Availibility There is no data in this paper.

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