Bull. Malays. Math. Sci. Soc. (2024) 47:153
https://doi.org/10.1007/s40840-024-01751-y

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Solution of Linear Damped Wave Equation on Triebel–Lizorkin Spaces

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Received: 16 February 2024 / Revised: 18 July 2024 / Accepted: 23 July 2024 /
Published online: 5 August 2024

Abstract
In this article we study the solution $u(t,x)$ of the Cauchy problem of linear damped wave equation. We obtain the sufficient and necessary condition of the boundedness of the solution $u(t,x)$ on the Triebel–Lizorkin space at a fixed time t.

Keywords Damped wave equation · Triebel–Lizorkin space · Sharp estimate · Multiplier theorem · Wave operator

Mathematics Subject Classification 42B15 · 35L05 · 42B35 · 42B37

1 Introduction

The damped wave equation is the equation describing the fluctuation phenomenon with the damping effect. It is usually used to describe the loss of energy due to friction, air resistance and other factors when waves propagate in the medium. We focus on the

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Communicated by Rosihan M. Ali.

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Cauchy problem of linear damped wave equation

$$(DWE)\begin{cases}\partial_{tt}u + 2u_t - \Delta u = 0,\\ u(0,x) = f(x),\ u_t(0,x) = g(x),\end{cases}$$

where $(t,x) \in (0,\infty) \times \mathbb{R}^n$ $(n \geq 2)$ and Δ is the Laplacian defined by

$$(-\Delta)f(x) = \mathcal{F}^{-1}\left(|\xi|^2 \hat{f}(\xi)\right)(x),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\hat{f}$ denotes the Fourier transform of f. The solution to the Cauchy problem is formally given by

$$u(t,x) = e^{-t}\cosh\left(t\sqrt{L}\right) f(x) + e^{-t}\frac{\sinh\left(t\sqrt{L}\right)}{\sqrt{L}}(f(x) + g(x)),$$

where L is the Fourier multiplier with symbol $1 - |\xi|^2$.

The damped wave equation is a common physical equation used to describe fluctuating phenomena with dissipation, such as the propagation of electromagnetic waves in transmission lines. The study of the solutions to the damped wave equations can provide useful information about fluctuating behavior, which has important implications for many applications in science and engineering. A large number of researchers have studied the behaviors of solutions of $u(t,x)$, such as the decaying nature of solutions, the propagation speed of solutions and the stability of solutions. The property of solutions is an important study. By analyzing the solution of an equation, the researcher can obtain the decay rate of the solution by parameter adjustment. The results of the decay rate have important implications for applications such as electrical signal transmission and energy transmission (see [1–6]). In communication systems, the decay properties and propagation speed of solutions can be used to optimize the reliability and speed of signal transmission. In materials science, the study of the stability of solutions can help predicting the reliability and lifetime of materials. In addition, the study of these properties can also be applied to medical imaging, acoustics and earthquakes. Therefore, there are still many properties worthy of our study of the solution for damped wave equation. In the study of harmonic analysis, the boundedness of $u(t,x)$ on different function spaces is a hot topic in the research on the damped wave equation. See [7–16] and references therein to find researchers on the space–time estimates and asymptotic estimates, the local and global well-posedness of the Cauchy problem of the damped wave equation. Expressly, the result in [17] involves the estimates of damped wave equation in Lebesgue space L^p and the Hardy space H^p. Particularly, the authors in [18] prove the following result. (The case of $\alpha = 1$ denotes the damped wave equation).

Theorem A [18] *Let* $\alpha > 0$, $1 \le r \le p \le \infty$ *and* $\beta \ge n\alpha \left|\frac{1}{2} - \frac{1}{p}\right|$ *for* $\alpha \ne 1$ *or* $\beta > (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|$ *for* $\alpha = 1$. *Then we have, for* $t > 1$, *that*

$$\|u(t,\cdot)\|_{L^p} \preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}(\|f\|_{L^r} + \|g\|_{L^r}) \\ +e^{-t}(1+t)^{\beta}(\|f\|_{\dot{L}^p_\beta} + \|f\|_{\dot{L}^p_{\beta-\alpha}} + \|g\|_{\dot{L}^p_{\beta-\alpha}})$$

where $\dot{L}^p_\beta$ *and* $\dot{L}^p_{\beta-\alpha}$ *are homogeneous Sobolev spaces.*

More elegant and general estimates involving Lebesgue spaces L^p and Hardy spaces H^p, $0 < p \le 1$ are further studied by D'Abbicco. Ebert and Picon in [17]. The long time decay estimates of the inequality in Theorem A describes the diffusion phenomenon of the solution. We notice that if we choose the same space L^p then the inequality is reduced to

$$\|u(t,\cdot)\|_{L^p} \preceq \|f\|_{L^p} + \|g\|_{L^p} + e^{-t}(1+t)^{\beta}(\|f\|_{\dot{L}^p_\beta} + \|f\|_{\dot{L}^p_{\beta-\alpha}} + \|g\|_{\dot{L}^p_{\beta-\alpha}}).$$

Since the Trieble–Lizorkin space $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$ is a space with wider frame, which unites many well-known spaces such as $\dot{L}^p_\beta$, L^p, H^p, BMO, among many others, it naturally inspires us to study the inequality in Theorem A on the Trieble–Lizorkin spaces. In this paper, we mainly examine the boundedness of $u(t,x)$ on the Triebel–Lizorkin space $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$. Specifically, for a fixed t, we want to find the sharp range of p for which the solution $u(t,x)$ is bounded on $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$. The study for the long time decay estimates involved two spaces $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$ will appear in [19].

By focusing on the two fundamental operators $T(t)$ and $S(t)$, we can effectively analyze the solution $u(t,x)$, where

$$T(t) = e^{-t}\cosh(t\sqrt{L}), \quad S(t) = e^{-t}\frac{\sinh(t\sqrt{L})}{\sqrt{L}}.$$

To this end, we need to decompose these operators in their frequency domains, since their performances are quite different in these domains. Let $\Psi(\xi)$ be a C^∞ radial function that takes values in the interval [0, 1] and has support in the set $\{\xi : |\xi| \ge 100\}$. Additionaly, this function satisfies the condition $\Psi(\xi) \equiv 1$ in the set $\{\xi : |\xi| \ge 150\}$. Consider $\Phi(\xi)$ to be a smooth radial function, which takes values in the interval [0, 1]. This function has a support in the set $\{\xi : |\xi| \le 3/4\}$ and satisfies the condition $\Phi(\xi) \equiv 1$ in the set $\{\xi : |\xi| \le 1/2\}$. We further modify the function $\Theta(\xi) = 1 - \Psi(\xi) - \Phi(\xi)$. Suppose that $\Omega(\xi)$ is a smooth radial function that is valued on the interval [0, 1] and has support in the set $\{\xi : 1/2 \le |\xi| \le 2\}$. Additionally, $\Omega(\xi)$ is equal to 1 in the range $\{\xi : 3/4 \le |\xi| \le 4/3\}$ and write

$$\Lambda(\xi) = \Theta(\xi) - \Omega(\xi).$$

For all $\xi \in \mathbb{R}^n$, it is clear that

$$1 = \Psi(\xi) + \Lambda(\xi) + \Omega(\xi) + \Phi(\xi),$$

which allows that we can write

$$T(t) = \Psi(D)T(t) + \Phi(D)T(t) + \Lambda(D)T(t) + \Omega(D)T(t),$$
$$S(t) = \Psi(D)S(t) + \Phi(D)S(t) + \Lambda(D)S(t) + \Omega(D)S(t).$$

The low frequency part of $T(t)$, denoted as $\Phi(D)T(t)$, can be written as

$$\Phi(D)T(t)f(x) = e^{-t}\int_{\mathbb{R}^n} \cosh(t\sqrt{1-|\xi|^2})\Phi(\xi)\widehat{f}(\xi)e^{i<x,\xi>}d\xi.$$

$\Phi(D)S(t)$ is the low frequency part of $S(t)$ defined by

$$\Phi(D)S(t)f(x) = e^{-t}\int_{\mathbb{R}^n} \Phi(\xi)\frac{\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}\widehat{f}(\xi)e^{i<x,\xi>}d\xi,$$

and the other six operators are defined similarly.

The high frequency part of the wave operator (not damped) is defined by

$$T_\alpha f(x) = K_\alpha * f(x),$$

where

$$\widehat{K_\alpha}(\xi) = \frac{e^{\pm i|\xi|}\Psi(|\xi|)}{|\xi|^\alpha}.$$

Cao and Jia [20] have proved the following result.

Theorem B [20] *Suppose $\gamma \in \mathbb{R}$. Then we have the following conclusions.*

1. *Let $1 < p \le q \le 2$ or $2 \le q \le p < \infty$. If $\alpha \ge (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, then T_α is bounded on $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$.*
2. *Let $1 < p, q < \infty$ or $0 < q \le p \le 1$. If $\alpha > (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, then T_α is bounded on $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$.*
3. *Let $0 < p \le 1 < q < \infty$. T_α is bounded on $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$ if $\alpha \ge (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$.*

As we mentioned before, the aim of this paper is to investigate the boundedness of damped wave operators on Triebel–Lizorkin spaces. To achieve our target, estimates on the Triebel–Lizorkin spaces for $T(t)$ and $S(t)$ will be established. We will invoke partially the result from Theorem B in our research. However, Theorem B does not provide any clue if the conditions in it are sharp.

Our intention is to prove the following theorems.

Theorem 1 *Suppose $(n-1)\,|1/p-1/2| \le \beta$ and $\gamma \in \mathbb{R}$. Let $\widetilde{\delta} > 0$ and*

$$c_t = e^{-t}(1+t)^{\widetilde{\delta}}.$$

We have the following conclusions.

1. *Let* $1 < p \le q \le 2$ *or* $2 \le q \le p < \infty$. *When* $\beta \ge (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|$, *there exists a positive number* $\widetilde{\delta}$ *for which*

$$\|\Psi(D)T(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq c_t \, \|f\|_{\dot{F}^{\gamma+\beta}_{p,q}}$$

and

$$\|\Psi(D)S(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq c_t \, \|f\|_{\dot{F}^{\gamma+\beta-1}_{p,q}}.$$

2. *Let* $0 < q \le p \le 1$ *or* $1 < p, q < \infty$. *When* $\beta > (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|$, *then*

$$\|\Psi(D)T(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq c_t \, \|f\|_{\dot{F}^{\gamma+\beta}_{p,q}}$$

and

$$\|\Psi(D)S(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq c_t \, \|f\|_{\dot{F}^{\gamma+\beta-1}_{p,q}}.$$

3. *Let* $0 < p \le 1 < q < \infty$. *When* $\beta \ge (n-1)|\frac{1}{2} - \frac{1}{p}|$, *then*

$$\|\Psi(D)T(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq c_t \, \|f\|_{\dot{F}^{\gamma+\beta}_{p,q}}$$

and

$$\|\Psi(D)S(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq c_t \, \|f\|_{\dot{F}^{\gamma+\beta-1}_{p,q}}.$$

Theorem 2 *Let* $\gamma \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \le \infty$. *All operators* $\Phi(D)T(t)$, $\Lambda(D)T(t)$, $\Omega(D)T(t)$, $\Phi(D)S(t)$, $\Lambda(D)S(t)$, $\Omega(D)S(t)$ *are bounded on* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ *and their bounds are independent of* $t > 0$.

Theorem 2 easily yields the following corollary.

Corollary 3 $S(t)$ *is bounded on* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ *if and only if* $\Psi(D)S(t)$ *is bounded on* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$. $T(t)$ *is bounded on* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ *if and only if* $\Psi(D)T(t)$ *is bounded on* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$.

In this paper, we want to obtain the sufficient and necessary conditions on the boundedness of $\Psi(D)S(t)$ and $\Psi(D)T(t)$ on Triebel–Lizorkin spaces. The sharpness of the condition in Theorem 1 is demonstrated by the following result, except at the endpoint for some indices p, q (see Corollary 5 below).

Theorem 4 *Let* $q > 1$, *for a fix* t, *the inequality*

$$\|\Psi(D)T(t)(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|f\|_{\dot{F}^{\gamma+\beta}_{p,q}}$$

for all $f \in \dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ *fails if*

$$\beta < (n-1)\,|1/p - 1/2|\,.$$

Also the inequality

$$\|\Psi(D)S(t)(g)\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|g\|_{\dot{F}^{\gamma+\beta-1}_{p,q}}$$

for all $g \in \dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ *fails if*

$$\beta < (n-1)\,|1/2 - 1/p|\,.$$

From Theorems 1 and 4, we easily see the following corollary.

Corollary 5 $\beta \geq (n-1)|\frac{1}{2} - \frac{1}{p}|$ *is the sufficient and necessary condition of (1) and (3) in Theorem 1 (for a fixed t).*

From Corollary 5, we know that (2) of Theorem 1 remains open at endpoint $\beta = (n-1)|\frac{1}{2} - \frac{1}{p}|$.

Theorem 4 easily yields the following sharp result.

Corollary 6 *Suppose* $\gamma \in \mathbb{R}$, $n \geq 4$ *and* $1 < p \leq q \leq 2$, *or* $2 \leq q \leq p < \infty$. *Then*

$$\|S(t)g\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|g\|_{\dot{F}^{\gamma}_{p,q}}$$

if and only if

$$\frac{2(n-1)}{n+1} \leq p \leq \frac{2(n-1)}{n-3}.$$

If $n = 2, 3$, *then*

$$\|S(t)g\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|g\|_{\dot{F}^{\gamma}_{p,q}}$$

for all $1 < p < \infty$.

We let $f(x) = 0$ in (DWE), Corollary 6 says that when $n \geq 4$, for $2 \leq q \leq p \leq \infty$ or $1 \leq p \leq q \leq 2$, we have

$$\|u(t,\cdot)\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|g\|_{\dot{F}^{\gamma}_{p,q}}$$

if and only if

$$\frac{2(n-1)}{n+1} \leq p \leq \frac{2(n-1)}{n-3},$$

for a fixed t. Also, when $n = 2, 3$, for $2 \le q \le p \le \infty$ or $1 \le p \le q \le 2$, we have

$$\|u(t, \cdot)\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|g\|_{\dot{F}^{\gamma}_{p,q}}$$

for all $p \in (1, \infty)$.

We obtain the sufficient condition of the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness for the fundamental operators $T(t)$ and $S(t)$ in Theorems 1 and 2. The proof of this part is relatively easy by invoking the known results in Theorem B and the multiplier theorem in Lemma 9. Specifically, we carefully decompose operators $T(t)$ and $S(t)$ on their frequency domains into several operators according to their low, middle and high frequencies. Then, we either write an operator as a sum of several operators, or write an operator as a composition of simpler operators. We then further use the multiplier theorem (Lemma 9) to verify that each operator is an $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ multiplier. This execution process is routine and only some easy computations involved.

The main difficulty comes from the proof for the necessary conditions of the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness for $T(t)$ and $S(t)$. This process is executed in Theorem 4. In the proof, in order to give a unified proof on $T(t)$ and $S(t)$, we consider two more general convolution operators $\Omega^{+}_{\beta,\lambda} * f$ and $\Omega^{-}_{\beta+\lambda} * f$, whose kernels are given by

$$\Omega^{+}_{\beta,\lambda}(x) = \int_{\mathbb{R}^n} \frac{e^{it\sqrt{|\xi|^2-1}}}{|\xi|^{\beta}\left(|\xi|^2 - 1\right)^{\lambda/2}} \Psi(|\xi|)\, e^{i<x,\xi>} d\xi$$

and

$$\Omega^{-}_{\beta,\lambda}(x) = \int_{\mathbb{R}^n} \frac{e^{-it\sqrt{|\xi|^2-1}}}{|\xi|^{\beta}\left(|\xi|^2 - 1\right)^{\lambda/2}} \Psi(|\xi|)\, e^{i<x,\xi>} d\xi,$$

separately, and we estimate them in a similar way. After using the Taylor expansion and a careful calculation, we may conclude that the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness of $\Omega^{+}_{\beta,\lambda} * f$ is equivalent to the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness of the wave operator

$$W_{\widetilde{\beta}}(f)(x) = \int_{\mathbb{R}^n} \frac{e^{-it|\xi|}}{|\xi|^{\widetilde{\beta}}} \Psi_1(|\xi|)\, \widehat{f}(\xi)\, e^{i<x,\xi>} d\xi,$$

with some cutoff function Ψ_1 off the origin. Then, in Proposition 13 we prove the necessary condition for the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness of $W_{\widetilde{\beta}}$, which is also optimal by comparing to the sufficient condition. It should be noted that the proof of Proposition 13 is tricky. We need to carefully study the kernel $K_{\widetilde{\beta}}$ of $W_{\widetilde{\beta}}$. To this end, we separate the leading term of $K_{\widetilde{\beta}}$ by striping away all non-essential items, and we further invoke the Gel'fand–Shilov Lemma (Lemma 11) to obtain the precise asymptotic estimate of the leading term. With this estimate we are able to construct a family of test functions $\{f_{\varepsilon}\}$ to achieve the necessary condition when $1 \le p, q \le 2$, which also implies the necessary condition for $2 \le p, q < \infty$ by an easy duality argument. Finally, we use a contradiction argument with the help of the Stein analytic interpolation to obtain the necessary condition for whole range of p, q via a boosting procedure.

The paper is organized as following. In the second section we will review some known results of the Triebel–Lizorkin spaces. We give the proof of Theorem 2 in the third section. The proof of Theorem 1 is finished in Sect. 4 and we prove Theorem 4 in the fifth section. Potential applications are indicated in Sect. 6.

Throughout the article, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \leq CB$. The notation $A \simeq B$ denotes $A \preceq B$ and $A \succeq B$, and the notation $A \approx B$ means that there exists a constant C such that $A = CB$.

2 Preliminary Knowledge

Let $\varphi \in C^{\infty}(\mathbb{R}^n)$ be a fixed function and satisfy $\mathrm{supp}(\varphi) \subset \{\xi : 3 < |\xi| \leq 5\}$, $0 \leq \varphi(\xi) \leq 1$. Also, $\varphi(\xi) > c > 0$ when $7/2 \leq |\xi| \leq 9/2$. Let $\varphi_j(\xi)$ be defined as $\varphi(2^{-j}\xi)$. We may also impose a simple normalization condition on φ, for all $\xi \in \mathbb{R}^n \backslash \{0\}$,

$$\sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1.$$

For $j \in \mathbb{Z}$, we can represent the functions $\phi_j(x)$ as $\widehat{\phi_j}(\xi) = \varphi_j(\xi)$. The homogeneous Triebel–Lizorkin space $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$, for $\gamma \in \mathbb{R}$, $0 < p, q < \infty$, is defined as the collection of all distributions f that satisfy the following condition:

$$\|f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)} = \left\| \left(\sum_{j \in \mathbb{Z}} \left| 2^{\gamma j} (\phi_j * f) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty. \tag{1}$$

Choose a suitable Schwartz function ψ. To define the norm of the inhomogeneous Triebel–Lizorkin space $F^{\gamma}_{p,q}$, we can modify Eq. (1) by including an additional term $\|\psi * f\|_{L^p}$ in the sum and replacing $\sum_{j \in \mathbb{Z}}$ with $\sum_{j \geq 0}$. It is commonly understood that $\dot{F}^{\gamma}_{p,2} = \dot{L}^p_{\gamma}$ if $1 < p < \infty$ and $\dot{F}^{\gamma}_{p,2} = H^p_{\gamma}$ if $0 < p \leq 1$, where $\dot{L}^p_{\gamma}$ is the classical L^p-Sobolev space and the space of Hardy–Sobolev functions denoted by H^p_{γ} has been extensively studied by Strichartz (refer to [21]). In particular, we use the notation $H^p = H^p_0$, which refers to the real Hardy space. The space $\dot{F}^{\gamma}_{\infty,\infty}$ is equivalent to the Lipshitz space Lip_{γ} if $\gamma > 0$. The space $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ is important in the study of harmonic analysis, so that it has attracted many attentions [22–24].

We recall two important properties on the space $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$.

Lemma 7 (Imbedding) [25] *The imbedding relationship on the space* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$*:*

1. *For any real number γ and positive number p less than ∞. If $q_1 \leq q_2$, then*

$$\dot{F}^{\gamma}_{p,q_1} \subset \dot{F}^{\gamma}_{p,q_2}.$$

2. *Given reals* $-\infty < \gamma_2 < \gamma_1 < \infty$ *and* $0 < p_1 < \infty$, $0 < q_1, q_2 \leq \infty$, *let* $0 < p_2 \leq \infty$ *be determined by* $\gamma_1 - \frac{n}{p_1} = \gamma_2 - \frac{n}{p_2}$. *Then*

$$\dot{F}^{\gamma_1}_{p_1,q_1} \subset \dot{F}^{\gamma_2}_{p_2,q_2}.$$

Lemma 8 (Lifting) (see [20, p. 2073]) *There is the lifting property on the space* $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$*:*

$$\|f\|_{\dot{F}^{\gamma}_{p,q}} \simeq \|R_\gamma f\|_{\dot{F}^{0}_{p,q}},$$

where R_γ *represents the Riesz potential, which is defined as*

$$\widehat{R_\gamma f}(\xi) = |\xi|^\gamma \widehat{f}(\xi).$$

According to the lifting property mentioned in [20], to establish the boundedness of a convolution operator on the $\dot{F}^{\gamma}_{p,q}$ space, it is enough to demonstrate its boundedness on $\dot{F}^{0}_{p,q}$. Define the convolution operator C_m as

$$C_m(f)(x) = \int_{\mathbb{R}^n} m(\xi)\, \widehat{f}(\xi)\, e^{i<\xi,x>} d\xi,$$

and define m to be an $\dot{F}^{\gamma}_{p,q}$ multiplier if C_m is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$. We denote the operator norm of C_m on the space $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ as

$$\|m(\cdot)\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n) \to \dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}.$$

Using a scaling argument, it can be easily demonstrated that for every positive t,

$$\|m(\cdot)\|_{\dot{F}^{\gamma}_{p,q} \to \dot{F}^{\gamma}_{p,q}} = \|m(t\cdot)\|_{\dot{F}^{\gamma}_{p,q} \to \dot{F}^{\gamma}_{p,q}}.$$

This article will frequently make use of the following multiplier theorem for $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$.

Lemma 9 [26, Theorem 5.1, pp. 851] *If* s *and* γ *are real numbers,* $0 < q \leq \infty$ *and* $0 < p < \infty$. *Moreover,* $\ell \in N$ *and* $\mu \in C^\ell(\mathbb{R}^n \backslash \{0\})$. *If the following condition is satisfied:*

$$\sup_{R>0} \left(R^{-n+2s+2|\sigma|} \int_{R<|\xi|<2R} \left| \partial^\sigma_\xi \mu(\xi) \right|^2 d\xi \right) \leq A_\sigma, \quad |\sigma| \leq \ell \tag{2}$$

with $\ell > \max\{n/p, n/q\} + n/2$*, then*

$$\|T_\mu(f)\|_{\dot{F}^{s+\gamma}_{p,q}} \leq C\|f\|_{\dot{F}^{\gamma}_{p,q}}.$$

When $s=0$, this condition can be replaced by a simple condition

$$\left|\partial^{\sigma} m(\xi)\right| \preceq |\xi|^{-|\sigma|}, \tag{3}$$

which is called the Mihlin condition.

The asymptotic expansion of the Bessel function $J_\nu(r)$ as $r \to \infty$ is widely used throughout the paper.

Lemma 10 (see [27, Proposition 5.1, p. 93]) *For any positive integer L and $r>0$, $\nu>-1/2$. The following inequality holds on the interval $[1,\infty)$,*

$$J_\nu(r)=\sqrt{\frac{2}{\pi r}}\cos\left(r-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+\sum_{j=1}^{L} a_j e^{ir} r^{-\frac{1}{2}-j}+\sum_{j=1}^{L} b_j e^{-ir} r^{-\frac{1}{2}-j}+E_L(r),$$

where a_j and b_j are constants for $j \in N$. For every nonnegative integer k and the function $E_L(r)$ satisfies

$$E_L^{(k)}(r)=O\left(r^{-\frac{1}{2}-L-1}\right), \quad as \quad r\to\infty.$$

Lemma 11 (Gel'fand–Shilov, see page 171 in [28]) *Let $\sigma>0$ and $\gamma\neq -1,-2,\ldots$. Then*

$$\int_0^\infty e^{-\sigma r} r^{\gamma} e^{isr}\, dr = i e^{i\pi\gamma/2}\Gamma(\gamma+1)(s+i\sigma)^{-\gamma-1},$$

where $\Gamma(\gamma+1)$ is the Gamma function.

3 Proof of Theorem 2

We will use Lemma 9 to verify the boundedness of all operators stated in Theorem 2 on the Triebel–Lizorkin space $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$. The estimates are similar between $\Phi(D)S(t)$, $\Omega(D)S(t)$, $\Lambda(D)S(t)$ and $\Phi(D)T(t)$, $\Omega(D)T(t)$, $\Lambda(D)T(t)$. We will verify the frontal estimates only. Note that the multiplier of $\Phi(D)S(t)$ is

$$m_0(\xi)=\frac{e^{-t}\Phi(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}$$

and the multipliers of $\Omega(D)S(t)$ and $\Lambda(D)S(t)$ are

$$m_1(\xi)=\frac{e^{-t}\Omega(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}$$

and

$$m_2(\xi) = \frac{e^{-t}\Lambda(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}$$

respectively. It suffices to check that all $m_0(\xi), m_1(\xi)$ and $m_2(\xi)$ satisfy condition (3).

We first work on $m_0(\xi)$. Recall that $\Phi(\xi)$ is supported in the set $\{\xi : |\xi| \leq 3/4\}$ and $\Phi(\xi)$ equals 1 when $|\xi| \leq 1/2$. We choose a different radial Schwartz function $\Phi_1(\xi)$ that is supported in the set $\{\xi : |\xi| \leq 13/16\}$ and $\Phi_1(\xi)$ equals 1 when $|\xi| \leq 3/4$. By this choice we can write

$$\Phi(\xi) = \Phi_1(\xi)\Phi(\xi).$$

Now, we can write $m_0(\xi)$ as the product of two multipliers

$$m_0(\xi) = \mu_1(\xi)\,\mu_2(\xi),$$

where

$$\mu_1(\xi) = e^{-t}\Phi_1(\xi)\sinh(t\sqrt{1-|\xi|^2})$$

and

$$\mu_2(\xi) = \frac{\Phi(\xi)}{\sqrt{1-|\xi|^2}}.$$

This means that $\Phi(D)S(t)$ is the composition of two operators with multipliers $\mu_1(\xi)$ and $\mu_2(\xi)$ respectively. By the support condition on $\Phi(\xi)$, it shows that $\mu_2(\xi)$ is a multiplier on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ space trivially. On the other hand, we have

$$\frac{\partial}{\partial\xi_j}\mu_1(\xi) \approx e^{-t}\left(\frac{\partial}{\partial\xi_j}\Phi_1(\xi)\right)\sinh(t\sqrt{1-|\xi|^2}) + \frac{e^{-t}t\Phi_1(\xi)\cosh(t\sqrt{1-|\xi|^2})\xi_j}{\sqrt{1-|\xi|^2}}.$$

It is easy to check that in the support of Φ_1 we have

$$\left|e^{-t}\Phi_1(\xi)\sinh(t\sqrt{1-|\xi|^2})\right| = \left|e^{-t}\Phi_1(\xi)\frac{e^{t\sqrt{1-|\xi|^2}} - e^{-t\sqrt{1-|\xi|^2}}}{2}\right| \preceq \Phi_1(\xi)e^{\frac{-t|\xi|^2}{2}}$$

and

$$\left|e^{-t}\Phi_1(\xi)\cosh(t\sqrt{1-|\xi|^2})\right| \preceq \Phi_1(\xi)e^{\frac{-t|\xi|^2}{2}}.$$

By noting that $\frac{\partial}{\partial \xi_j} \Phi_1(\xi)$ has the same property as that of $\Phi_1(\xi)$ and

$$e^{\frac{-t|\xi|^2}{2}} t\,|\xi|^2 \preceq 1,$$

we easily obtain that

$$\left|\frac{\partial}{\partial \xi_j} \mu_1(\xi)\right| \preceq \left|\frac{\partial}{\partial \xi_j} \Phi_1(\xi) e^{\frac{-t|\xi|^2}{2}}\right| + \Phi_1(\xi) e^{\frac{-t|\xi|^2}{2}} t \left|\xi_j\right| \preceq |\xi|^{-1}$$

uniformly on $t > 0$, for all $j = 1, 2, \cdots, n$. Similarly, we can show that for any multi-index σ

$$\left|\partial^\sigma \mu_1(\xi)\right| \preceq |\xi|^{-|\sigma|}.$$

Thus, $\mu_1(\xi)$ is an $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$ multiplier, and consequently $m_0(\xi) = \mu_1(\xi)\,\mu_2(\xi)$ is also an $\dot{F}^\gamma_{p,q}(\mathbb{R}^n)$ multiplier.

We next study the multiplier

$$m_1(\xi) = \frac{e^{-t}\Omega(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}.$$

By the Taylor expansion,

$$\frac{\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}} = \sum_{k=0}^{\infty} \frac{t^{2k+1}\left(1-|\xi|^2\right)^k}{(2k+1)!},$$

we have that

$$\frac{\partial}{\partial \xi_j}\left(\frac{\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}\right) = -2\sum_{k=1}^{\infty} \frac{t^{2k+1}k\left(1-|\xi|^2\right)^{k-1}\xi_j}{(2k+1)!},$$

and for $i \neq j$,

$$\frac{\partial^2}{\partial \xi_j \partial \xi_i}\left(\frac{\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}\right) = 4\sum_{k=2}^{\infty} \frac{t^{2k+1}k\,(k-1)\left(1-|\xi|^2\right)^{k-2}\xi_j\xi_i}{(2k+1)!}.$$

Using an induction, we easily see that in the support of Ω, for any multi-index σ,

$$\left|\partial^\sigma_\xi\left(\frac{\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}\right)\right| \preceq \sum_{k=|\sigma|}^{\infty} \frac{t^{2k+1}k^{|\sigma|}\,(1/2)^k}{(2k+1)!}\,|\xi|^{-|\sigma|}.$$

Since

$$\frac{t^{2k+1}e^{-t}}{(2k+1)!} \leq 1,$$

we obtain that

$$\left|\partial_\xi^\sigma\left(e^{-t}\frac{\Omega(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}\right)\right| \preceq |\xi|^{-|\sigma|}.$$

This shows that $m_1(\xi)$ is an $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ multiplier.

Finally, we estimate $m_2(\xi)$. Write

$$\frac{\Lambda(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}} = \frac{\Lambda(\xi)\sin(t\sqrt{|\xi|^2-1})}{\sqrt{|\xi|^2-1}}$$

and note $\operatorname{supp}\Lambda(\xi) \subset \{4/3 \leq |\xi| \leq 150\}$. Clearly $\Lambda(\xi)\left(|\xi|^2-1\right)^{-1/2}\sin(t\sqrt{|\xi|^2-1})$ is a C^∞ function and

$$\left|\partial_\xi^\sigma\left(\frac{\Lambda(\xi)\sin(t\sqrt{|\xi|^2-1})}{\sqrt{|\xi|^2-1}}\right)\right| \preceq (1+t)^{|\sigma|}.$$

Thus, it is trivial to see that for any multi-index σ,

$$\left|\partial_\xi^\sigma\left(e^{-t}\frac{\Lambda(\xi)\sinh(t\sqrt{1-|\xi|^2})}{\sqrt{1-|\xi|^2}}\right)\right| \preceq |\xi|^{-|\sigma|}$$

uniformly on $t > 0$.

We have shown that multipliers m_0, m_1, m_2 satisfy condition (3) for all multi-indices σ. This completes the proof of Theorem 2.

4 Proof of Theorem 1

Again we will use the known result combining Lemma 9. Let Φ_0 be a C^∞ radial function supported on the ball $B(0, 5)$ and $\Phi_0(\xi) \equiv 1$ on the ball $B(0, 4)$. Set $\Psi_0(\xi) = 1 - \Phi_0(\xi)$. We consider more general multipliers

$$m_3(\xi, t) = e^{-t}\Psi(\xi)\frac{\sin(t\sqrt{|\xi|^2-1})}{|\xi|^\beta\left(|\xi|^2-1\right)^{\lambda/2}} \quad \text{and} \quad m_4(\xi, t) = \Psi(\xi)\frac{e^{-t}\cos(t\sqrt{|\xi|^2-1})}{|\xi|^\beta\left(|\xi|^2-1\right)^{\lambda/2}}.$$

Proposition 12 *$m_3(\xi,t)$ and $m_4(\xi,t)$ are $\dot F^{\gamma}_{p,q}(\mathbb{R}^n)$ multipliers if*

$$\frac{e^{it|\xi|}\Psi_0\left(|\xi|\right)}{|\xi|^{\beta+\lambda}} \quad and \quad \frac{e^{-it|\xi|}\Psi_0(|\xi|)}{|\xi|^{\beta+\lambda}}$$

are $\dot F^{\gamma}_{p,q}(\mathbb{R}^n)$ multipliers for a fixed t.

Proof We will focus our attention to $m_3\left(\xi,t\right)$, since the calculation of $m_4\left(\xi,t\right)$ is essentially similar. We write that

$$\begin{aligned} m_3(\xi,t) &= e^{-t}\Psi(\xi)\frac{\sin(t\sqrt{|\xi|^2-1})}{|\xi|^{\beta}\left(|\xi|^2-1\right)^{\lambda/2}} \\ &= \widetilde{m_2}\left(\xi,t\right) + e^{-t}\Psi(\xi)\frac{\sin(t\sqrt{|\xi|^2-1})}{|\xi|^{\beta}\left(|\xi|^2-1\right)^{\lambda/2}}\Psi_0\left(t\left|\xi\right|\right), \end{aligned}$$

where

$$\widetilde{m_2}\left(\xi,t\right) = e^{-t}\Phi_0\left(t\left|\xi\right|\right)\frac{\sin(t\sqrt{|\xi|^2-1})}{|\xi|^{\beta}\left(|\xi|^2-1\right)^{\lambda/2}}\Psi(\xi).$$

Note that the support of $\Phi_0\left(t\left|\xi\right|\right)\Psi(\xi)$ is contained in the set

$$\left\{\xi : |\xi| \le 5t^{-1}\right\} \cap \left\{\xi : |\xi| \ge 2\right\}.$$

For any $p, q > 0$ and $\gamma \in \mathbb{R}$, it is easy to verify that $\widetilde{m_2}\left(\xi,t\right)$ satisfies the condition in Lemma 9 so that it is an $\dot F^{\gamma,q}_{p}(\mathbb{R}^n)$ multiplier. Next we write that

$$\begin{aligned} & e^{-t}\Psi(\xi)\frac{\sin(t\sqrt{|\xi|^2-1})}{|\xi|^{\beta}\left(|\xi|^2-1\right)^{\lambda/2}}\Psi_0\left(t\left|\xi\right|\right) \\ &= e^{-t}\Psi(\xi)\frac{e^{i(t\sqrt{|\xi|^2-1})} - e^{-i(t\sqrt{|\xi|^2-1})}}{2i|\xi|^{\beta+\lambda}\left(1-|\xi|^{-2}\right)^{\lambda/2}}\Psi_0\left(t\left|\xi\right|\right) \\ &= m_5(t,\xi) + m_6(t,\xi). \end{aligned}$$

The estimates of $m_5(t,\xi)$ and $m_6(t,\xi)$ are similar. For the first multiplier we write

$$m_5(t,\xi) = e^{-t}\Psi(\xi)\frac{e^{i(t\sqrt{|\xi|^2-1})}}{2i|\xi|^{\beta+\lambda}\left(1-|\xi|^{-2}\right)^{\lambda/2}}\Psi_0\left(t\left|\xi\right|\right) = \widetilde{m_3}\left(\xi,t\right) \times \widetilde{m_4}\left(\xi,t\right),$$

where

$$\widetilde{m_4}\left(\xi,t\right) = e^{-t}\frac{e^{it|\xi|}\Psi_0\left(t\left|\xi\right|\right)}{2i|\xi|^{\beta+\lambda}}, \quad \widetilde{m_3}\left(\xi,t\right) = \frac{e^{itg(\frac{1}{|\xi|})}}{\left(\sqrt{1-|\xi|^{-2}}\right)^{\lambda}}\Psi\left(|\xi|\right),$$

and

$$g(\frac{1}{|\xi|}) = -\frac{1}{2|\xi|}\int_0^1 (1-\frac{s}{|\xi|^2})^{-\frac{1}{2}} ds.$$

Recalling the support condition, for any $p, q > 0$ and $\gamma \in \mathbb{R}$, we easily use Lemma 9 to verify that $\widetilde{m_3}(\xi, t)$ is a multiplier of $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ space. If $\widetilde{m_4}(\xi, t)$ is a multiplier of $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ space, then we can conclude that $m_3(\xi, t)$ is also a multiplier of $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ space. Recall that Cao and Jia in Theorem B obtained the $\dot{F}^{\gamma,q}_{p}$ boundedness of the wave operator with multiplier $\widetilde{m_4}(\xi, t)$, which means that $\widetilde{m_4}(\xi, t)$ is an $\dot{F}^{\gamma}_{p,q}$ multiplier. Combing Proposition 12 and Theorem B, we obtain Theorem 1. □

5 Proof of Theorem 4

The sufficiency of the $\dot{F}^{\gamma,q}_{p}(\mathbb{R}^n)$ boundedness of $\Psi(D)T(t)f$ and $\Psi(D)S(t)f$ have been proved in Theorem 1. Here we mainly discuss the necessity of the boundedness of these operators on $\dot{F}^{\gamma,q}_{p}$.

The kernel of $\Psi(D)S(t)$ is

$$\Omega(t)(x) = e^{-t}\int_{\mathbb{R}^n} \Psi(|\xi|)\frac{\sin(t\sqrt{|\xi|^2-1})}{\sqrt{|\xi|^2-1}} e^{i<x,\xi>} d\xi,$$

and the kernel of $\Psi(D)T(t)$ is

$$K(t)(x) = e^{-t}\int_{\mathbb{R}^n} \Psi(|\xi|)\cos(t\sqrt{|\xi|^2-1}) e^{i<x,\xi>} d\xi.$$

We study more general kernels

$$\Omega_{\beta,\lambda}(x) = \int_{\mathbb{R}^n} \Psi(|\xi|)\frac{\sin(t\sqrt{|\xi|^2-1})}{|\xi|^{\beta}\left(|\xi|^2-1\right)^{\lambda/2}} e^{i<x,\xi>} d\xi$$

and

$$K_{\beta,\lambda}(x) = \int_{\mathbb{R}^n} \Psi(|\xi|)\frac{\cos(t\sqrt{|\xi|^2-1})}{|\xi|^{\beta}\left(|\xi|^2-1\right)^{\lambda/2}} e^{i<x,\xi>} d\xi.$$

We will focus our attention to the kernel $\Omega_{\beta,\lambda}(x)$, where $\beta+\lambda > 0$. The calculation of the kernel $K_{\beta,\lambda}$ is essentially similar. Because of

$$\sin x = (e^{ix} - e^{-ix})/2i,$$

we will study kernels

$$\Omega^+_{\beta,\lambda}(x) = \int_{\mathbb{R}^n} \frac{e^{it\sqrt{|\xi|^2-1}}}{|\xi|^\beta \left(|\xi|^2-1\right)^{\lambda/2}} \Psi(|\xi|) e^{i<x,\xi>} d\xi$$

and

$$\Omega^-_{\beta,\lambda}(x) = \int_{\mathbb{R}^n} \frac{e^{-it\sqrt{|\xi|^2-1}}}{|\xi|^\beta \left(|\xi|^2-1\right)^{\lambda/2}} \Psi(|\xi|) e^{i<x,\xi>} d\xi.$$

Let

$$\Psi_1(|\xi|) = \frac{\Psi(|\xi|)}{(\sqrt{1-|\xi|^{-2}})^\lambda},$$

we notice that $\Psi_1 \in C^\infty$ and $\mathrm{supp}\Psi_1 \subseteq \mathrm{supp}\Psi$. We write

$$\begin{aligned}\mathfrak{T}^+_{\beta+\lambda} f(x) &= \Omega^+_{\beta,\lambda} * f(x) \\ &= \int_{\mathbb{R}^n} \frac{e^{it\sqrt{|\xi|^2-1}}}{|\xi|^{\beta+\lambda}(\sqrt{1-|\xi|^{-2}})^\lambda} \Psi(|\xi|) \widehat{f}(\xi) e^{i<x,\xi>} d\xi \\ &= \int_{\mathbb{R}^n} \frac{e^{it|\xi|\sqrt{1-|\xi|^{-2}}}}{|\xi|^{\beta+\lambda}} \Psi_1(|\xi|) \widehat{f}(\xi) e^{i<x,\xi>} d\xi\end{aligned}$$

and

$$\begin{aligned}\mathfrak{T}^-_{\beta+\lambda} f(x) &= \Omega^-_{\beta,\lambda} * f(x) \\ &= \int_{\mathbb{R}^n} \frac{e^{-it\sqrt{|\xi|^2-1}}}{|\xi|^{\beta+\lambda}(\sqrt{1-|\xi|^{-2}})^\lambda} \Psi(|\xi|) \widehat{f}(\xi) e^{i<x,\xi>} d\xi \\ &= \int_{\mathbb{R}^n} \frac{e^{-it|\xi|\sqrt{1-|\xi|^{-2}}}}{|\xi|^{\beta+\lambda}} \Psi_1(|\xi|) \widehat{f}(\xi) e^{i<x,\xi>} d\xi.\end{aligned}$$

We will only estimate $\mathfrak{T}^+_{\beta+\lambda} f(x)$, the study of $\mathfrak{T}^-_{\beta+\lambda} f(x)$ is similar. Using the Taylor expansion, we have that

$$\sqrt{1-|\xi|^{-2}} = 1 - \frac{1}{2|\xi|^2} + |\xi|^{-4} E(|\xi|),$$

where $E(|\xi|)$ is a C^∞ function whose any derivative is bounded on the support of $\Psi_1(|\xi|)$. We further write

$$e^{it|\xi|\sqrt{1-|\xi|^{-2}}} = e^{it|\xi|} e^{it(-\frac{1}{2|\xi|}+|\xi|^{-3}E(|\xi|))} = e^{it|\xi|}\{e^{it(-\frac{1}{2|\xi|}+|\xi|^{-3}E(|\xi|))} - 1\} + e^{it|\xi|}.$$

The multiplier $\mu_{\beta+\lambda}$ now is decomposed as a sum of two multipliers:

$$\begin{aligned}\mu_{\beta+\lambda}(\xi) &= \frac{e^{it|\xi|\sqrt{1-|\xi|^{-2}}}\Psi_1(|\xi|)}{|\xi|^{\beta+\lambda}} \\ &= \frac{e^{it|\xi|}\Psi_1(|\xi|)}{|\xi|^{\beta+\lambda}} + \frac{e^{it|\xi|}\{e^{it(-\frac{1}{2|\xi|}+|\xi|^{-3}E(|\xi|))} - 1\}\Psi_1(|\xi|)}{|\xi|^{\beta+\lambda}}, \\ &= \mu_{\beta+\lambda,1}(\xi) + \mu_{\beta+\lambda,2}(\xi).\end{aligned}$$

Corresponding to the multiplier $\mu_{\beta+\lambda}$, its convolution operator can be written as

$$\begin{aligned}\mathfrak{T}^{+}_{\beta+\lambda}(f)(x) &= \int_{\mathbb{R}^n} \mu_{\beta+\lambda}(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi \\ &= \int_{\mathbb{R}^n} \mu_{\beta+\lambda,1}(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi + \int_{\mathbb{R}^n} \mu_{\beta+\lambda,2}(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi.\end{aligned}$$

We further write

$$\mu_{\beta+\lambda,2}(\xi) = \frac{e^{it|\xi|}\Psi_2(|\xi|)}{|\xi|^{\beta+\lambda+1}},$$

where

$$\Psi_2(|\xi|) = \{e^{i(-\frac{1}{2|\xi|}+|\xi|^{-3}E(|\xi|))} - 1\}\Psi_1(|\xi|)$$

is $C^\infty(\mathbb{R}^n)$ and $supp(\Psi_2) \subseteq supp(\Psi_1)$. We fix $t = 1$ for simplicity. We can easily observe that $\mathfrak{T}^{+}_{\widetilde{\beta}}(f)$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ if and only if

$$W_{\widetilde{\beta}}(f)(x) = \int_{\mathbb{R}^n} \frac{e^{i|\xi|}}{|\xi|^{\widetilde{\beta}}}\Psi_1(|\xi|)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi = K_{\widetilde{\beta}} * f(x)$$

is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$, where the Fourier transform of $K_{\widetilde{\beta}}$ is $\frac{e^{i|\xi|}}{|\xi|^{\widetilde{\beta}}}\Psi_1(|\xi|)$ and $\widetilde{\beta} = \beta + \lambda$.

Proposition 13 *Let $q > 1$. When $\widetilde{\beta} < (n-1)|\frac{1}{p} - \frac{1}{2}|$, $W_{\widetilde{\beta}}(f)$ is not bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$.*

Proof Without loss of generality, it suffices to discuss the regularity index $\widetilde{\beta}$ lies in the interval

$$\max\left\{0,\ (n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}\right\} < \widetilde{\beta} < (n-1)|\frac{1}{p} - \frac{1}{2}|.$$

Let $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ be radial and C^∞ and satisfy

(a) $\widetilde{\Phi}_1(s) = 1$ if $s \in [0, \frac{1}{2}]$ and $\widetilde{\Phi}_1(s) = 0$ if $s \in [2/3, \infty)$, and $0 \le \widetilde{\Phi}_1(s) \le 1$,

(b) $\widetilde{\Phi}_2(s) = 1$ if $s \in [2, \infty)$ and $\widetilde{\Phi}_2(s) = 0$ if $s \in [0, \frac{4}{3})$, and $0 \le \widetilde{\Phi}_2(s) \le 1$, and

$$\widetilde{\Theta}(s) = 1 - \widetilde{\Phi}_1(s) - \widetilde{\Phi}_2(s),$$

We then write

$$\begin{aligned} K_{\widetilde{\beta}}(x) &= K_{\widetilde{\beta}}(x)\widetilde{\Phi}_1(x) + K_{\widetilde{\beta}}(x)\widetilde{\Theta}(x) + K_{\widetilde{\beta}}(x)\widetilde{\Phi}_2(x) \\ &= K_{loc}(x) + K_{mid}(x) + K_{\infty}(x). \end{aligned}$$

By this definition we have that

$$W_{\widetilde{\beta}}(f) = W_{loc}(f) + W_{mid}(f) + W_{\infty}(f),$$

and $K_{loc}(x)$, $K_{mid}(x)$, $K_{\infty}(x)$ are kernels of W_{loc}, W_{mid}, W_{∞} respectively.

In order to complete the proof of Proposition 13, we also need to invoke the following lemma.

Lemma 14 *For $p, q \ge 1$ and $\widetilde{\beta} > 0$, the boundedness of $W_{\widetilde{\beta}}$ on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ holds if and only if W_{mid} is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$.*

Proof of Lemma 14 It suffiecs to show that for $p, q \ge 1$,

$$\begin{aligned} \|W_{loc}(f)\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)} &\preceq \|f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}, \\ \|W_{\infty}(f)\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)} &\preceq \|f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}. \end{aligned}$$

We consider the case of $t = 1$ for simplicity. We first estimate the kernel and use the polar coordinate (see [29, Theorem 3.3 in pp. 155]) to obtain

$$\begin{aligned} K_{loc}(x) &= \widetilde{\Phi}_1(x) \int_{\mathbb{R}^n} \frac{e^{i|\xi|}}{|\xi|^{\widetilde{\beta}}} \Psi_1(|\xi|) e^{ix\cdot\xi} d\xi \\ &\simeq \widetilde{\Phi}_1(x) \int_0^{\infty} \frac{e^{ir}}{r^{\widetilde{\beta}-n+1}} \Psi_1(r) \frac{J_{\frac{n-2}{2}}(r|x|)}{(r|x|)^{\frac{n-2}{2}}} dr, \end{aligned}$$

where $J_{\frac{n-2}{2}}$ is the Bessel function of order $\frac{n-2}{2}$. Now we let ϕ be a C^{∞} radial function satisfying $\phi = 1 - \Psi_1$. We further write, since $\widetilde{\beta} > 0$,

$$\begin{aligned} K_{loc}(x) &\approx \widetilde{\Phi}_1(x) \int_0^{\infty} \frac{e^{ir}}{r^{\widetilde{\beta}-n+1}} \phi(r|x|) \Psi_1(r) \frac{J_{\frac{n-2}{2}}(r|x|)}{(r|x|)^{\frac{n-2}{2}}} dr \\ &\quad + \widetilde{\Phi}_1(x) \int_0^{\infty} \frac{e^{ir}}{r^{\widetilde{\beta}-n+1}} \Psi_1(r|x|) \Psi_1(r) \frac{J_{\frac{n-2}{2}}(r|x|)}{(r|x|)^{\frac{n-2}{2}}} dr \\ &= K_{loc,1}(x) + K_{loc,2}(x). \end{aligned}$$

Invoke the fact

$$\left|\frac{J_{\frac{n-2}{2}}(r|x|)}{(r|x|)^{\frac{n-2}{2}}}\right| \preceq 1$$

on the support of $\phi(r|x|)$, within the support of Ψ_1 we know that

$$|K_{loc,1}(x)| \preceq \widetilde{\Phi}_1(x) \int_1^{\frac{2}{|x|}} \frac{1}{r^{\widetilde{\beta}-n+1}} \Psi_1(r)dr \preceq \widetilde{\Phi}_1(x) \max\{1, |x|^{\widetilde{\beta}-n}\}.$$

Also, by the asymptotic expansion of the Bessel function (Lemma 10), the main term of $K_{loc,2}(x)$ is dominated by

$$\left|\frac{A\widetilde{\Phi}_1(x)}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{e^{ir(1-|x|)}}{r^{\widetilde{\beta}-n+1}} \Psi_1(r|x|)\Psi_1(r)dr\right| + \left|\frac{B\widetilde{\Phi}_1(x)}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{e^{ir(1+|x|)}}{r^{\widetilde{\beta}-n+1}} \Psi_1(r|x|)\Psi_1(r)dr\right|,$$

where $A, B > 0$ are constants. Hence, by performing integration by parts with respect to the variable r, it is straightforward to derive

$$\left|K_{loc,2}(x)\right| \preceq \left|\frac{\widetilde{\Phi}_1(x)}{|x|^{\frac{n-1}{2}}(1+|x|)^N}\right| + \left|\frac{\widetilde{\Phi}_1(x)}{|x|^{\frac{n-1}{2}}(1-|x|)^N}\right|,$$

for any positive integer N. Noting the support condition of $\widetilde{\Phi}_1$, we obtain the estimate

$$|K_{loc}(x)| \preceq \max\left\{\frac{1}{|x|^{\frac{n-1}{2}}}, |x|^{\widetilde{\beta}-n}\right\} \quad if \quad |x| < \frac{2}{3}$$
$$|K_{loc}(x)| = 0 \quad if \quad |x| \geq \frac{2}{3},$$

Thus we show that the kernel $K_{loc}(x)$ is a Lebesgue integrable function. For the kernel $K_\infty(x)$, again, its main term is

$$\frac{A\widetilde{\Phi}_2(x)}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{e^{ir(1-|x|)}}{r^{\widetilde{\beta}-n+1}} \Psi_1(r)dr + \frac{B\widetilde{\Phi}_2(x)}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{e^{ir(1+|x|)}}{r^{\widetilde{\beta}-n+1}} \Psi_1(r)dr.$$

Similar to the estimate for K_{loc}, for any positive value L, an integration by parts reveals that

$$|K_\infty(x)| \preceq \frac{1}{|x|^L} \quad if \quad |x| > \frac{3}{2}$$

and

$$|K_\infty(x)| = 0 \quad if \quad |x| < \frac{3}{2}.$$

Thus, we know

$$\|K_\infty\|_{L^1} < \infty \quad and \quad \|K_{loc}\|_{L^1} < \infty.$$

By the generalized Young inequality on $\dot{F}^{\gamma}_{p,q}$ space,

$$\|W_{loc}(f)\|_{\dot{F}^{\gamma}_{p,q}} = \|K_{loc} * f\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|K_{loc}\|_{L^1} \|f\|_{\dot{F}^{\gamma}_{p,q}}.$$

Similarly, we have

$$\|W_\infty(f)\|_{\dot{F}^{\gamma}_{p,q}} \preceq \|f\|_{\dot{F}^{\gamma}_{p,q}}.$$

Lemma 14 is proved. □

As a consequence of Lemma 14, $W_{\widetilde{\beta}} f(x)$ is bounded on $\dot{F}^{\gamma}_{p,q}$ space if and only if $W_{mid} f(x)$ is bounded on $\dot{F}^{\gamma}_{p,q}$ provided $p, q \geq 1$. In order to proceed with proving Proposition 13, we will focus on the convolution operator W_{mid} with the kernel K_{mid}.

Using the polar coordinates, we obtain that

$$\begin{aligned} K_{mid}(x) &= \widetilde{\Theta}(x) \int_{\mathbb{R}^n} \frac{e^{i|\xi|}}{|\xi|^{\widetilde{\beta}}} \Psi_1(|\xi|) e^{ix\cdot\xi} d\xi \\ &\approx \widetilde{\Theta}(x) \int_0^\infty \frac{\Psi_1(r) e^{ir}}{r^{\widetilde{\beta}-n+1}} \frac{J_{\frac{n-2}{2}}(r|x|)}{(r|x|)^{\frac{n-2}{2}}} dr \\ &\approx \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-2}{2}}} \int_0^\infty \frac{\Psi_1(r) e^{ir}}{r^{\widetilde{\beta}-\frac{n}{2}}} J_{\frac{n-2}{2}}(r|x|) dr. \end{aligned}$$

Hence, the asymptotic formula in Lemma 10 yields that

$$\begin{aligned} K_{mid}(x) \approx & \sum_{j=0}^{N} \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} c_j \int_0^\infty \frac{\Psi_1(r) e^{ir(1+|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}} dr \\ & + \sum_{j=0}^{N} \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} d_j \int_0^\infty \frac{\Psi_1(r) e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}} dr \\ & + \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-2}{2}}} \int_0^\infty \frac{\Psi_1(r) e^{ir}}{r^{\widetilde{\beta}-\frac{n}{2}}} E_N(r|x|) dr, \end{aligned}$$

where all c_j and d_j are constants. Write

$$R_N(x) = \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-2}{2}}} \int_0^\infty \frac{\Psi_1(r) e^{ir}}{r^{\widetilde{\beta}-\frac{n}{2}}} E_N(r|x|) dr.$$

Let j_0 be the integer for which

$$\widetilde{\beta}-\frac{n-1}{2}+j>1$$

when $j>j_0$, and

$$\widetilde{\beta}-\frac{n-1}{2}+j\leq 1$$

if $j\leq j_0$. We now write

$$G(x)=\sum_{j=j_0+1}^{N}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}d_j\int_0^\infty\frac{\Psi_1(r)e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr,$$
$$H(x)=\sum_{j=0}^{N}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}c_j\int_0^\infty\frac{\Psi_1(r)e^{ir(1+|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr,$$
$$\mathfrak{R}(x)=\sum_{j=0}^{j_0}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}d_j\int_0^\infty\frac{\Psi_1(r)e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr.$$

Since the support of $\Psi_1(r)$ is off the origin, we choose a large N for which $R_N(x)$ is a Lebesgue integrable function. Similarly by the choice of j_0, $G(x)$ is a Lebesgue integrable function. For $H(x)$, we take integration by parts N times for a sufficiently large N to obtain

$$|H(x)|\leq\sum_{j=0}^{N}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}\left|c_j\int_0^\infty\frac{\Psi_1(r)e^{ir(1+|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr\right|$$
$$\preceq\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}(1+|x|)^N}.$$

Clearly, $H(x)$ is also a Lebesgue integrable function by noting the support of $\widetilde{\Theta}(x)$. According to the Minkowski inequality, we have that, for $p,q\geq 1$,

$$\|H*f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}+\|G*f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}+\|R_N*f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}\preceq\|f\|_{\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)}.$$

By this inequality and Lemma 14, we know that $W_{\widetilde{\beta}}$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ for $p,q\geq 1$ if and only if the convolution operator $\mathfrak{R}*f$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$.

Finally, we focus on the kernel

$$\mathfrak{R}(x)=\sum_{j=0}^{j_0}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}d_j\int_0^\infty\frac{\Psi_1(r)e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr,$$

in which we notice that each $\widetilde{\beta} - \frac{n-1}{2} + j \le 1$. We will use Lemma 11 to estimate the kernel $\mathfrak{R}(x)$. To this end, we first estimate the case $\widetilde{\beta} - \frac{n-1}{2} + j_0 = 1$ since this case is not included in Lemma 11. We change variables to obtain

$$\int_0^\infty \frac{\Psi_1(r)e^{ir(1-|x|)}}{r}dr = \int_0^\infty \frac{\Psi_1(\frac{r}{1-|x|})e^{ir}}{r}dr.$$

By the choice of Ψ_1 it is easy to see that

$$\left|\int_0^\infty \frac{\Psi_1(\frac{r}{1-|x|})e^{ir}}{r}dr\right| \preceq \int_{|1-|x||}^{100} \frac{1}{r}dr + \left|\int_{100}^\infty \frac{e^{ir}}{r}dr\right| \preceq \log\frac{1}{|1-|x||}$$

in the support of $\widetilde{\Theta}(x)$. Thus

$$\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}\int_0^\infty \frac{\Psi_1(r)e^{ir(1-|x|)}}{r}dr$$

is an L^1 integrable function.

Now, we assume $\widetilde{\beta} - \frac{n-1}{2} + j < 1$ for $j = 1, 2 \cdots, j_0$. By Lemma 11, we have that

$$\begin{aligned}
&\sum_{j=0}^{j_0} \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} d_j \int_0^\infty \frac{\Psi_1(r)e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr \\
&= \lim_{\sigma\to 0^+}\sum_{j=0}^{j_0} \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} d_j \int_0^\infty \frac{e^{-\sigma r}\Psi_1(r)e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr \\
&= \mathfrak{R}_1(x) - \mathfrak{R}_2(x),
\end{aligned}$$

where

$$\mathfrak{R}_1(x) = \lim_{\sigma\to 0^+}\sum_{j=0}^{j_0} \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} d_j \int_0^\infty \frac{e^{-\sigma r}e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr$$

and

$$\mathfrak{R}_2(x) = \lim_{\sigma\to 0^+}\sum_{j=0}^{j_0} \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} d_j \int_0^\infty \frac{e^{-\sigma r}\phi(r)e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr,$$

where

$$\phi(r) = 1 - \Psi_1(r).$$

Clearly by the support of $\phi(r)$,

$$|\mathfrak{R}_2(x)| \preceq \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}} \preceq \widetilde{\Theta}(x),$$

which yields that $\mathfrak{R}_2$ is a Lebesgue integrable function. We see that

$$\begin{aligned}\mathfrak{R}_1(x) &= \lim_{\sigma\to0^+}\sum_{j=0}^{j_0}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}d_j\int_0^\infty\frac{e^{-\sigma r}e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr\\ &\approx \mathfrak{L}_+(x)+\mathfrak{M}(x),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{L}_+(x) &= \lim_{\sigma\to0^+}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}}}d_0\int_0^\infty\frac{e^{-\sigma r}e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}}}dr,\\ \mathfrak{M}(x) &= \lim_{\sigma\to0^+}\sum_{j=1}^{j_0}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}d_j\int_0^\infty\frac{e^{-\sigma r}e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr.\end{aligned}$$

Now by Lemma 11, we further have that the distribution kernel

$$\begin{aligned}\mathfrak{L}_+(x) &\approx \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}}}\int_0^\infty\frac{e^{-\sigma r}e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}}}dr\\ &= \frac{i\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}}}e^{i\left(\frac{n-1}{2}-\widetilde{\beta}\right)\frac{\pi}{2}}\Gamma\left(\frac{n+1}{2}-\widetilde{\beta}\right)\left(\frac{1}{1-|x|+i\sigma}\right)^{\frac{n+1}{2}-\widetilde{\beta}}.\end{aligned}$$

Similarly, we see that

$$\begin{aligned}\mathfrak{M}(x) &= \sum_{j=1}^{j_0}\frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}d_j\int_0^\infty\frac{e^{-\sigma r}e^{ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}+j}}dr\\ &\preceq \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}+j}}\left|\frac{1}{1-|x|+i\sigma}\right|^{\frac{n+1}{2}-\widetilde{\beta}-1}.\end{aligned}$$

Recall that we consider

$$\max\left\{0,\,(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}\right\}<\widetilde{\beta}<(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$$

and $\widetilde{\beta}\geq\beta$. Clearly, for $p,q\geq1$, we obtain that $W_{\widetilde{\beta}}*f$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ if and only if $\mathfrak{L}_+*f$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$. As a conclusion, we obtain that $\mathfrak{T}^+_{\beta+\lambda}$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ if and only if $\mathfrak{L}_+*f$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ provided $p,q\geq1$.

A similar argument demonstrates that $\mathfrak{T}^{-}_{\beta+\lambda}$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ if and only if $\mathfrak{L}_- * f$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$, where

$$\begin{aligned}\mathfrak{L}_-(x) &\approx \frac{\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}}}\int_0^\infty \frac{e^{-\sigma r}e^{-ir(1-|x|)}}{r^{\widetilde{\beta}-\frac{n-1}{2}}} \\ &= \frac{i\widetilde{\Theta}(x)}{|x|^{\frac{n-1}{2}}}e^{i\left(\frac{n-1}{2}-\widetilde{\beta}\right)\frac{\Pi}{2}}\Gamma\left(\frac{n+1}{2}-\widetilde{\beta}\right)\left(\frac{1}{1-|x|-i\sigma}\right)^{\frac{n+1}{2}-\widetilde{\beta}}(e^{i\pi})^{\frac{n+1}{2}-\widetilde{\beta}}.\end{aligned}$$

It is easy to see that there is a $c>0$ such that

$$\lim_{\sigma\to 0}\left|\frac{\mathfrak{L}_+(x)-\mathfrak{L}_-(x)}{2i}\right| \geq c\widetilde{\Theta}(x)\left|\frac{1}{1-|x|}\right|^{\frac{n+1}{2}-\widetilde{\beta}}, \tag{4}$$

$$\lim_{\sigma\to 0}|\mathfrak{L}_-(x)| \geq c\widetilde{\Theta}(x)\left|\frac{1}{1-|x|}\right|^{\frac{n+1}{2}-\widetilde{\beta}}, \tag{5}$$

and

$$\lim_{\sigma\to 0}|\mathfrak{L}_+(x)| \geq c\widetilde{\Theta}(x)\left|\frac{1}{1-|x|}\right|^{\frac{n+1}{2}-\widetilde{\beta}}.$$

By the above discussion, the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness of $\Omega_{\beta,\lambda} * f(x)$ is equivalent to the $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ boundedness of $\frac{(\mathfrak{L}_+-\mathfrak{L}_-)}{2i} * f$.

To complete the proof of Proposition 13, we first consider $1 \leq q \leq 2$. Denote the sets

$$\begin{aligned}B_{\varepsilon,l} &= \{x\in\mathbb{R}^n : 10\varepsilon < |x|-1 \leq 2^{-l}\} \\ A_{\varepsilon,l} &= \{x\in\mathbb{R}^n : 5\varepsilon < |x|-1 \leq 2^{-l+1}\},\end{aligned}$$

where l is sufficiently large and ε is sufficiently small. Since the lift property says that $\mathfrak{L}_+ * f$ is bounded on $\dot{F}^{\gamma}_{p,q}(\mathbb{R}^n)$ if and only if $\mathfrak{L}_+ * f$ is bounded on $\dot{F}^{0}_{p,q}(\mathbb{R}^n)$, we will allow that only the space $\dot{F}^{0}_{p,q}(\mathbb{R}^n)$ involves in the following argument. By the definition of $\dot{F}^{0}_{p,q}(\mathbb{R}^n)$, since $1 \leq q \leq 2$ and $\dot{F}^{0}_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, we have that

$$\begin{aligned}\|\mathfrak{L}_+ * f\|^p_{\dot{F}^0_{p,q}} &\succeq \left\|\left(\sum_{j\in\mathbb{Z}}|\phi_j * \mathfrak{L}_+ * f|^2\right)^{\frac{1}{2}}\right\|^p_{L^p} \\ &\approx \|\mathfrak{L}_+ * f\|^p_{L^p} \geq \int_{B_{\varepsilon,l}}|\mathfrak{L}_+ * f|^p dx.\end{aligned}$$

We take a C^∞ function g satisfying $g(x)\geq 0$ and

$$supp(g) \subset \{x\in\mathbb{R}^n : |x|\leq 1\}.$$

Denote a family of test functions

$$g_\varepsilon(x) = g(\frac{x}{\varepsilon}), \quad \varepsilon > 0$$

It is easy to check that

$$\int_{\mathbb{R}^n} |g_\varepsilon(x)|^p dx \approx \varepsilon^n$$

and

$$\|g_\varepsilon\|^p_{\dot{F}^0_{p,q}} \approx \varepsilon^n.$$

On the other hand, we have that for $x \in B_{\varepsilon,l}$,

$$\begin{aligned}\int_{B_{\varepsilon,l}} (|(\mathfrak{L}_+) * (g_\varepsilon)(x)|^q)^{\frac{p}{q}} dx &\succeq \|g_\varepsilon\|^p_{L^1} \int_{A_{\varepsilon,l}} (|x|-1)^{(\widetilde{\beta}-\frac{n+1}{2})p} dx \\ &\approx \varepsilon^{np} \varepsilon^{-(\frac{n+1}{2}-\widetilde{\beta})p+1}.\end{aligned}$$

We finally obtain that if

$$\|(\mathfrak{L}_+) * g_\varepsilon\|^p_{\dot{F}^0_{p,q}} \preceq \|g_\varepsilon\|^p_{\dot{F}^0_{p,q}} \approx \varepsilon^n$$

holds, then we must have that

$$\varepsilon^n \succeq \varepsilon^{np} \varepsilon^{-\left(\frac{n+1}{2}-\widetilde{\beta}\right)p+1},$$

for any small ε. This implies the inequality

$$n \le np - \left(\frac{n+1}{2} - \widetilde{\beta}\right) p + 1.$$

After a trivial computation, we obtain that, for $0 < \widetilde{\beta} < \frac{n-1}{2}$ and $2 \ge p, q \ge 1$, if $W_{\widetilde{\beta}}(f)$ is bounded on $\dot{F}^\gamma_{p,q}$ then we must have

$$\widetilde{\beta} \ge (n-1)\left(\frac{1}{p} - \frac{1}{2}\right).$$

Next, We show that the above inequality on $\widetilde{\beta}$ is also a necessary condition for any $q \ge 2$. To this end, We use an argument of contradiction by assuming that there exist $p_0 \in (1, 2)$, $q_0 > 2$ and $\delta > 0$ satisfying

$$\beta_0 = (n-1)\left(\frac{1}{p_0} - \frac{1}{2}\right) - \delta$$

such that

$$\left\| W_{\beta_0}(f) \right\|_{\dot{F}^0_{p_0,q_0}} \preceq \|f\|_{\dot{F}^0_{p_0,q_0}}.$$

Let $z_0,\ z_1 \in \mathbb{C}$ satisfying $Re(z_0) = \beta_0$ and for any $\eta > 0$, $\beta_1 = (n-1)(\frac{1}{p_0} - \frac{1}{2}) + \eta$, such that $Re(z_1) = \beta_1$. Also, we let $q_1 \in (1, \frac{3}{2})$. By the known sufficient result, we can show

$$\left\| W_{z_0}(f) \right\|_{\dot{F}^\gamma_{p_0,q_0}} \preceq \|f\|_{\dot{F}^\gamma_{p_0,q_0}}$$

and

$$\left\| W_{z_1}(f) \right\|_{\dot{F}^\gamma_{p_0,q_1}} \preceq \|f\|_{\dot{F}^\gamma_{p_0,q_1}}.$$

According to the analytical interpolation relaton

$$\frac{1}{q_*} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1},$$

we can obtain $q_* \in (1, 2),\ Re(z_*) = \beta_*$ and $\beta_* < (n-1)(\frac{1}{p_0} - \frac{1}{2})$, for which we have

$$\left\| W_{z_*}(f) \right\|_{\dot{F}^\gamma_{p_0,q_*}} \preceq \|f\|_{\dot{F}^\gamma_{p_0,q_*}}.$$

This leads to a contradiction to the previous proved result.

Use the same way we may prove $\widetilde{\beta} \geq (n-1)(\frac{1}{p} - \frac{1}{2})$ is the necessary condition of the $\dot{F}^{\gamma,q}_p$ boundedness for all $p \geq 1$, $q > 1$.

For the case of $q > 1$ and $0 < p < 1$, again, we will use a contradiction argument based on the obtained result in the previous case. Suppose that we find a p_0 and a $\delta > 0$ satisfying

$$0 < p_0 < 1$$

and $W_{\beta_0}(f)$ is bounded on $\dot{F}^\gamma_{p_0,q_0}$ for some $q_0 > 1$ at

$$\beta_0 = (n-1)\left(\frac{1}{p_0} - \frac{1}{2}\right) - \delta.$$

Clearly we can replace $W_{\beta_0}(f)$ by $W_{z_0}(f)$ for $z_0 \in \mathbb{C}$ satisfying $Re(z_0) = \beta_0$. We pick a $p_1 \in (\frac{3}{2}, 2)$ and $\beta_1 = (n-1)(\frac{1}{p_1} - \frac{1}{2}) + \eta$, from the known sufficient condition we know that $W_{z_1}(f)$ is bounded on $\dot{F}^\gamma_{p_1,q_0}$ for arbitrarily small $\eta > 0$, where $Rez_1 = \beta_1$.

Using the analytical interpolation and letting η be sufficiently small, we should obtain a $p_* \in (1,2)$ and a β_* satisfying

$$\beta_* < (n-1)\left(\frac{1}{p_*} - \frac{1}{2}\right)$$

such that $W_{\beta_*}(f)$ is bounded on $\dot{F}^{\gamma}_{p_*,q_0}$, which leads to a contradiction.

Combining all discussions, we obtain the proof of Proposition 13.

Conclusion 1: Combining Theorem B and Proposition 13, we obtain that $\alpha \geq (n-1)|\frac{1}{p} - \frac{1}{2}|$ is also a necessary condition in Theorem B.

Conclusion 2: Similarly, we can obtain the same estimate (mainly see (5)) for the boundedness of $\mathfrak{T}^{-}_{\beta+\lambda} f(x)$ on $\dot{F}^{\gamma}_{p,q}$. In terms of (4), by the same method, we obtain the necessary condition $\beta + \lambda \geq (n-1)|\frac{1}{p} - \frac{1}{2}|$ of the boundedness on $\dot{F}^{\gamma}_{p,q}$ for $\Omega_{\beta,\lambda} * f$ and $K_{\beta,\lambda} * f$.

These clearly imply the proof of Theorem 4, the interested reader can make up the conclusion easily.

6 Application

We are grateful to an anonymous referee for valuable suggestion on this manuscript. For the Cauchy problem of the nonlinear damped wave equation

$$\begin{cases} \partial_{tt}u + 2u_t - \Delta u = F(u) \\ u(0,x) = 0, \ \ u_t(0,x) = g(x), \end{cases} \tag{6}$$

where

$$F(u) = |u|^{2k}\, u \quad for\ some\ positive\ integer\ k.$$

We have studied the well posedness of the problem with intial data in some α modulation spaces [30]. We will further study the well posedness of the problem by assuming that the initial velocity g is in some Triebel–Lizorkin spaces.

Acknowledgements This work was supported by the National Key Research and Development Program of China (No. 2022YFA1005700); the Natural Science Foundation of Guangdong Province (No. 2023A1515012034) and the National Natural Science Foundation of China (Nos. 12371105, 11971295).

Data Availability This is not applicable in our paper.

Declaration

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Burq, N., Joly, R.: Exponential decay for the damped wave equation in unbounded domains. Commun. Contemp. Math. **18**(6), 1650012–27 (2016). https://doi.org/10.1142/S0219199716500127
2. Gerbi, S., Said-Houari, B.: Exponential decay for solutions to semilinear damped wave equation. Discrete Contin. Dyn. Syst. Ser. S **5**(3), 559–566 (2012). https://doi.org/10.3934/dcdss.2012.5.559
3. Kleinhenz, P.: Decay Rates for the Damped Wave Equation on The Torus, p. 85. ProQuest LLC, Ann Arbor (2020). Thesis (Ph.D.)—Northwestern University. http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:28087953
4. Messaoudi, S.A., Al-Smail, J.H., Talahmeh, A.A.: Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities. Comput. Math. Appl. **76**(8), 1863–1875 (2018). https://doi.org/10.1016/j.camwa.2018.07.035
5. Pan, Y.: The damped wave equation and associated polymer (2023). arXiv:2310.01631
6. Yang, Y., Xu, R.: Nonlinear wave equation with both strongly and weakly damped terms: supercritical initial energy finite time blow up. Commun. Pure Appl. Anal. **18**(3), 1351–1358 (2019). https://doi.org/10.3934/cpaa.2019065
7. Coimbra Charão, R., D'Abbicco, M., Ikehata, R.: Asymptotic profiles for a wave equation with parameter-dependent logarithmic damping. Math. Methods Appl. Sci. **44**(18), 14003–14024 (2021). https://doi.org/10.1002/mma.7671
8. Ikehata, R., Tanizawa, K.: Global existence of solutions for semilinear damped wave equations in R^N with noncompactly supported initial data. Nonlinear Anal. **61**(7), 1189–1208 (2005). https://doi.org/10.1016/j.na.2005.01.097
9. Ikehata, R., Miyaoka, Y., Nakatake, T.: Decay estimates of solutions for dissipative wave equations in $\mathbf{R}^N$ with lower power nonlinearities. J. Math. Soc. Jpn. **56**(2), 365–373 (2004). https://doi.org/10.2969/jmsj/1191418635
10. Marcati, P., Nishihara, K.: The L^p-L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media. J. Differ. Equ. **191**(2), 445–469 (2003). https://doi.org/10.1016/S0022-0396(03)00026-3
11. Narazaki, T.: L^p-L^q estimates for damped wave equations and their applications to semi-linear problem. J. Math. Soc. Jpn. **56**(2), 585–626 (2004). https://doi.org/10.2969/jmsj/1191418647
12. Narazaki, T.: L^p-L^q estimates for damped wave equations with odd initial data. Electron. J. Differ. Equ. **74**, 17 (2005)
13. Narazaki, K.: L^p-L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application. Math. Z. **244**(3), 631–649 (2003). https://doi.org/10.1007/s00209-003-0516-0
14. Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping. J. Differ. Equ. 464–489 (2001). https://doi.org/10.1006/jdeq.2000.3933
15. D'Abbicco, M.: A New Critical Exponent for the Heat and Damped Wave Equations with Nonlinear Memory and not Integrable Data, pp. 191–211. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-61346-4_9
16. D'Abbicco, M., Lucente, S.: A modified test function method for damped wave equations. Adv. Nonlinear Stud. 867–892 (2013). https://doi.org/10.1515/ans-2013-0407
17. D'Abbicco, M., Ebert, M.R., Picon, T.: Long time decay estimates in real Hardy spaces for evolution equations with structural dissipation. J. Pseudo-Differ. Oper. Appl. **7**(2), 261–293 (2016). https://doi.org/10.1007/s11868-015-0141-9
18. Chen, J., Fan, D., Zhang, C.: Space–time estimates on damped fractional wave equation. Abstr. Appl. Anal. 17 : 428909 (2014). https://doi.org/10.1155/2014/428909
19. Wang, M., Fan, D.: Asymptotic estimates of solution to damped fractional wave equation. Submitted to J. Inequal. Appl. (2024)
20. Cao, W., Jia, H.: Boundedness of certain oscillating multiplier operator related to wave equation in the Triebel–Lizorkin space. Chin. Ann. Math. Ser. A. Shuxue Niankan. A Ji 759–768 (2012)
21. Strichartz, R.S.: Multipliers on fractional Sobolev spaces. J. Math. Mech. **16**, 1031–1060 (1967)
22. Frazier, M., Jawerth, B.: A discrete transform and decompositions of distribution spaces. J. Funct. Anal. **93**(1), 34–170 (1990). https://doi.org/10.1016/0022-1236(90)90137-A
23. Frazier, M., Jawerth, B., Weiss, G.: Littlewood–Paley theory and the study of function spaces. In: CBMS Regional Conference Series in Mathematics, vol. 79, p. 132. Conference Board of the Mathematical

Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1991). https://doi.org/10.1090/cbms/079
24. Liu, L., Yang, D.: Boundedness of sublinear operators in Triebel–Lizorkin spaces via atoms. Stud. Math. **190**(2), 163–183 (2009). https://doi.org/10.4064/sm190-2-5
25. Triebel, H.: Theory of function spaces IV. In: Monographs in Mathematics, vol. 107, p. 160. Birkhäuser/Springer, Cham (2020). https://doi.org/10.1007/978-3-030-35891-4
26. Cho, Y.-K.: Continuous characterization of the Triebel–Lizorkin spaces and Fourier multipliers. Bull. Korean Math. Soc. **47**(4), 839–857 (2010). https://doi.org/10.4134/BKMS.2010.47.4.839
27. Fan, D., Zhao, F.: Approximation properties of combination of multivariate averages on Hardy spaces. J. Approx. Theory **223**, 77–95 (2017). https://doi.org/10.1016/j.jat.2017.07.008
28. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. 1, p. 423. Academic Press, New York (1964)
29. Stein, E.M., Weiss, G.: Introduction to Fourier analysis on Euclidean spaces. In: Princeton Mathematical Series, vol. 32, p. 297. Princeton University Press, Princeton, NJ (1971)
30. Wang, M.: Estimates of some damped equations on α modulation space. Submitted (2024)