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On Approximation Operators Involving Tricomi Function

Nusrat Raza[1] · Manoj Kumar[2] · M. Mursaleen[2,3,4]

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Abstract

The primary objective of this research article is to introduce and study an approximation operator involving the Tricomi function by using Korovkin's theorem and a conventional method based on the modulus of continuity. In Lipschitz-type spaces, we demonstrate the rate of convergence, and we are also able to determine the convergence properties of our operators. In addition, we illustrate the convergence of our proposed operators using various graphs and error-estimating tables for numerical instances.

Keywords Szász operators · Tricomi function · Order of convergence · Modulus of continuity

Mathematics Subject Classification 33C47 · 33E20 · 41A25 · 41A30 · 41A36

1 Introduction

As a bridge between theoretical and applied mathematics, approximation theory has been significant in the advancement of several computational approaches in recent years. It focuses on approximating the functions in the most effective way possible by employing much simpler or accessible functions and procedures that depend on the

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✉ M. Mursaleen
mursaleenm@gmail.com

Nusrat Raza
nraza.maths@gmail.com

Manoj Kumar
mkumar5@myamu.ac.in

1 Mathematics Section, Women's College, Aligarh Muslim University, Aligarh, India

2 Department of Mathematics, Aligarh Muslim University, Aligarh, India

3 Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan

4 Faculty of Mathematics and Natural Sciences, Universitas Sumatera Utara, Medan 20155, Indonesia

application of modern approximation techniques. In this theory, positive approximation techniques play a crucial role and appear naturally in various problems involving the approximation of continuous functions, particularly when additional qualitative properties are required, such as monotonicity, convexity, shape preservation, symmetry, and so on.

In order to decide a number of practical and symphonic studies, measure hypotheses, PDEs, and probability hypotheses, the positive estimate methods presented by Korovkin [16] play out as an effective component. In 1953, Korovkin [16] found what is likely the most effective and yet surprisingly straightforward criterion for determining whether or not a sequence of positive linear operators $\langle \kappa_\nu \rangle_{\nu \in \mathbb{N}}$ on the space $C[0, 1]$ is an approximation process, i.e., $\kappa_\nu(\tilde{f}) \xrightarrow{\text{uniformly}} \tilde{f}$ on $[0, 1]$ for every $\tilde{f} \in C[0, 1]$. This proof inspired other mathematicians to generalize Korovkin's theorem to more general settings, including all function spaces and general abstract spaces such as Banach spaces, Banach lattices, Banach algebras, and so on. In actuality, Korovkin's work outlined a novel theory that may be referred to as Korovkin's type approximation theory. Numerous researchers have investigated the convergence rates and properties of the Korovokin-type approximation; for example, see [16, 17, 21, 28, 29, 31]. Intriguing improvements to approximation theories can really be traced back to the work in [1, 4, 11–14, 19–22, 25, 27, 30, 32].

The discipline of approximation theory greatly benefits from the use of Szász operators [29], which are extended Bernstein operators to infinite intervals. Szász suggested the following set of positive linear operators:

$$\tilde{\mathcal{S}}_\eta(\hat{f}, \nu) = e^{-\eta\nu} \sum_{r=0}^{\infty} \frac{(\eta\nu)^r}{r!} \hat{f}\left(\frac{r}{\eta}\right), \tag{1.1}$$

where $\nu \in [0, \infty)$ and $\hat{f} \in C[0, \infty)$ the series in Eq. (1.1) converges. Current research has extensively examined the generalizations of Szász operators by employing specialized polynomials. These generalizations expand the scope of approximation theory by introducing a diverse set of new operator sequences.

Within this particular framework, it is advantageous to provide clear definitions for certain terminology and emphasize particular outcomes.

Definition 1.1 The modulus of continuity, denoted as $\omega(f; \sigma)$, is defined for any uniformly continuous function f on the interval $[0, \infty)$ and for any positive value of σ

$$\omega(f; \sigma) := \sup_{\substack{s,t \in [0,\infty) \\ |s-t| \le \sigma}} |f(s) - f(t)|. \tag{1.2}$$

Note that for any $\sigma > 0$ and for each $s, t \in [0, \infty)$, we can write

$$|f(s) - f(t)| \le \omega(f; \sigma) \left(\frac{|s-t|}{\sigma} + 1\right). \tag{1.3}$$

Definition 1.2 Let f be any function in the space of real-valued bounded and uniformly continuous functions $C_B[0, \infty)$, the second modulus of continuity is given by

$$\omega_2(f; \sigma) := \sup_{0<x\leq\sigma} \|f(.+2x) - 2f(.+x) + f(.)\|, \tag{1.4}$$

with the associated norm

$$\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|. \tag{1.5}$$

Let us recall Rasa's result and the second-order Steklov function, which are used to establish some results.

Lemma 1.1 ([10]) *Let $\{\mathfrak{L}_n\}_{n\geq 0}$ be a sequence of linear positive operators, with the property $\mathfrak{L}_n(1; x) = 1$ and $\tilde{f} \in C^2[0, a]$. Then:*

$$|\mathfrak{L}_n(\tilde{f}; x) - \tilde{f}(x)| \leq \|\tilde{f}'\| \sqrt{\mathfrak{L}_n\left((\xi - x)^2; x\right)} + \frac{1}{2}\|\tilde{f}''\|\mathfrak{L}_n\left((\xi - x)^2; x\right). \tag{1.6}$$

Definition 1.3 ([33]) For $f \in C[a, b]$, the second-order Steklov function of f is defined by

$$f_h(x) := \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right) f(h; x+t)dt, \quad x \in [a, b], \tag{1.7}$$

where $f(h; .) : [a-h, b+h] \to \mathbb{R}$, $h > 0$ is given by

$$f(h; x) = \begin{cases} L_-(x), & a-h \leq x \leq a \\ f(x), & a \leq x \leq b \\ L_+(x), & b \leq x \leq b+h \end{cases} \tag{1.8}$$

and L_-, L_+ are the linear best approximations to f on the mentioned intervals.

Lemma 1.2 [33] *Let f_h be the second-order Steklov function of f, where $f \in C[a, b]$ and $h \in \left(0, \frac{a-b}{2}\right)$. Then the following inequalities hold true:*

$$\|f_h - f\| \leq \frac{3}{4}\omega_2(f; h), \tag{1.9}$$

$$\|f_h''\| \leq \frac{3}{2h^2}\omega_2(f; h). \tag{1.10}$$

The Landau inequality given by

$$\|f_h'\| \leq \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\|. \tag{1.11}$$

In applied mathematics, the special functions are of paramount significance. Hypergeometric and confluent hypergeometric functions offer a convenient notational framework for expressing a rich variety of special functions. When it comes to the institutionalization of mathematical physics, the theory of special functions is paramount. In physics, engineering, and mathematical analysis, Bessel functions are among the most useful special functions. Several scholars, both in purely mathematical and practical contexts (for example, see [6–8]), have introduced various generalizations of Bessel functions. Much progress has been made in the analysis of radiation phenomena related to charge motion in magnetic devices attributable to the theory of generalized Bessel functions.

The following generating function comes in handy to define the n^{th}-order Tricomi functions $\mathcal{C}_n(x)$ [5]:

$$\sum_{n=-\infty}^{\infty} \mathcal{C}_n(x) = \exp\left(t - \frac{x}{t}\right), \qquad t \neq 0. \tag{1.12}$$

The series definition of the Tricomi functions $\mathcal{C}_n(x)$ is given by [26]:

$$\mathcal{C}_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(n+k)!}. \tag{1.13}$$

The Bessel functions $\mathcal{J}_n(x)$ are defined by the following generating function [26]:

$$\sum_{n=-\infty}^{\infty} \mathcal{J}_n(x) = \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right), \qquad t \neq 0;\ |x| < \infty, \tag{1.14}$$

and the series definition of the Bessel functions is given by [26]:

$$\mathcal{J}_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{n+2k}}{k!(n+2k)!}. \tag{1.15}$$

Tricomi functions are also characterized by the connection that links them to Bessel functions $\mathcal{J}_n(x)$ given by:

$$\mathcal{C}_n(x) = x^{-\frac{n}{2}} \mathcal{J}_n(2\sqrt{x}). \tag{1.16}$$

In particular, the 0^{th}-order Tricomi function is defined by the following series:

$$\mathcal{C}_0(x) = \mathcal{J}_0(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(k!)^2}, \tag{1.17}$$

where $\mathcal{J}_0(x)$ denotes 0^{th}-order Bessel function. Also, we note that

$$\mathcal{C}_n(x) = \frac{1}{\Gamma(1+n)} {}_0F_1[-; n+1; -x]. \tag{1.18}$$

In 2013, The Mittag-Leffler function was used by Özerslan [23] to introduce an approximation operator. Assume $\beta > 0$ is fixed and let (v_η) be any positive real number sequence. The Mittag-Leffler operators are thus defined for every $\eta \in \mathbb{N}$ as follows:

$$L_\eta^{(\beta)}(\hat{f}; x) = \frac{1}{E_{1,\beta}\left(\frac{\eta x}{v_\eta}\right)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{v_\eta^k \Gamma(k+\beta)} \hat{f}\left(\frac{k}{\eta} v_\eta\right), \tag{1.19}$$

where $E_{\alpha,\beta}(x)$ be the well known 2-parameter Mittag-Leffler function [18].

In [2], Ansari *et al.* introduced a Stancu variant of λ-Schurer operators and studied the pointwise and weighted approximation properties of these operators. Assume that p and r are non-negative parameters with the condition $0 \leq p \leq r$, and γ is a non-negative integer. Then, authors construct λ-Schurer-Stancu operators $\hat{\mathcal{S}}_{\eta,\gamma}^{p,r} : C[0, \gamma + 1] \to C[0, 1]$ as

$$\hat{\mathcal{S}}_{\eta,\gamma}^{p,r}(\hat{f}; x; \lambda) = \sum_{\nu=0}^{\eta+\gamma} \tilde{s}_{\eta,\nu}(\lambda; x) \hat{f}\left(\frac{\nu + p}{\eta + r}\right), \tag{1.20}$$

where $\tilde{s}_{\eta,\nu}$ are defined in [24].

In this article, we construct positive linear approximation operators using the Tricomi function of 0^{th}-order on the interval $\mathbb{R}^+ \cup \{0\}$. In Sect. 2, we give the definition of the operators and find their central moments. In Sect. 3, we establish the transformation properties and global Koronvich's theorem for our operators. Next, we obtain the rate of convergence by using the modulus of continuity and convergence in the different Lipschitz-type spaces. In Sect. 4, we approximate two functions using our operators. We also give a graphical depiction of the approximated functions and compare them with the actual functions. Also, we compute the absolute error for the different values of η on the interval [0, 1] and present the tables and graphical depiction of the error. We run our programming codes in WOLFRAM MATHEMATICA v13.3.1 on the processor MacOS X 13.2.1 x86(64-bit). In Sect. 5, we give some concluding remarks and applications.

2 Construction of Operators

Motivated by the aforementioned applications, in this section, initially, we construct positive linear operators then we establish some equalities. We use these equalities further to determine the convergence characteristics of these operators. In order to construct positive linear operators, we modify $\mathcal{C}_0(x)$ by replacing x by $-x$, we get:

$$\mathcal{T}_0(xt) := \mathcal{C}_0(-xt) = \sum_{k=0}^{\infty} \frac{(xt)^k}{(k!)^2}. \tag{2.1}$$

Now, in view of the above equation, we manifest the positive linear operators for $\eta \in \mathbb{N}$ as follows:

$$\mathscr{T}_\eta(\hat{f}; x) = \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \hat{f}\left(\frac{k^2}{\eta}\right), \tag{2.2}$$

where $\hat{f} \in E := \left\{\hat{f} \in C[0,\infty) : \lim_{x\to\infty} \frac{\hat{f}(x)}{1+x^2} \text{is finite}\right\}$ and $C[0,\infty)$ denotes the space of continuous functions defined on $[0,\infty)$. It is worth noting that the Banach lattice E is endowed with the norm

$$\|\hat{f}\|_* := \sup_{x\in[0,\infty)} \frac{|\hat{f}(x)|}{1+x^2}. \tag{2.3}$$

We prove the following results, which are required to establish our main result:

Lemma 2.1 *The sequence of operators defined by* (2.2) *are linear and positive.*

Proof The lemma can be proved using some direct computations. □

Lemma 2.2 *The following properties are being satisfied by the sequence of operators $\mathscr{T}_\eta(\hat{f}; x)$, for $x \in [0,\infty)$:*

$$\mathscr{T}_\eta(1; x) = 1, \tag{2.4}$$

$$\mathscr{T}_\eta(\nu; x) = x, \tag{2.5}$$

$$\mathscr{T}_\eta(\nu^2; x) = x^2 + 2x^2 \frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)} + \frac{x}{\eta}, \qquad \forall \eta \in \mathbb{N}. \tag{2.6}$$

Proof Since, in view of Eq. (2.1), for $\hat{f}(\nu) = 1$, Eq. (2.2) gives assertion (2.4).

Now, for $\hat{f}(\nu) = \nu$, Eq. (2.2) gives

$$\mathscr{T}_\eta(\nu; x) = \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \frac{k^2}{\eta} \tag{2.7}$$

which is equivalent to

$$\mathscr{T}_\eta(\nu; x) = \frac{x}{\mathcal{T}_0(\eta x)} \sum_{k=1}^{\infty} \frac{(\eta x)^{k-1}}{((k-1)!)^2}, \tag{2.8}$$

Using Eq. (2.1), we get $\mathscr{T}_\eta(\nu; x) = x$.

Finally, for $\hat{f}(\nu) = \nu^2$

$$\mathscr{T}_\eta(\nu^2; x) = \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \frac{k^4}{\eta^2} \tag{2.9}$$

We consider,

$$C(xt) = \sum_{k=0}^{\infty} \frac{(xt)^k}{(k!)^2} \frac{k^4}{\eta^2} = \frac{1}{\eta^2} \sum_{k=0}^{\infty} \frac{(xt)^k}{((k-1)!)^2} k^2 = \frac{xt}{\eta^2} \sum_{k=1}^{\infty} \frac{(xt)^{k-1}}{((k-1)!)^2} k^2 \tag{2.10}$$

which on simplification, gives

$$C(xt) = \frac{xt}{\eta^2} \left\{ \sum_{k=0}^{\infty} \frac{(xt)^k}{(k!)^2} k^2 + 2 \sum_{k=0}^{\infty} \frac{(xt)^k}{(k!)^2} k + \mathcal{T}_0(xt) \right\} \tag{2.11}$$

We assume

$$A(xt) = \sum_{k=1}^{\infty} \frac{(xt)^k}{(k!)^2} k^2, \tag{2.12}$$

which on solving, gives

$$A(xt) = (xt)\mathcal{T}_0(xt). \tag{2.13}$$

Now, we assume

$$B(xt) = 2 \sum_{k=0}^{\infty} \frac{(xt)^k}{(k!)^2} k. \tag{2.14}$$

Differentiating Eq. (2.1) with respect to t, we obtain

$$\mathcal{T}_0'(xt) = \sum_{k=1}^{\infty} k \frac{(xt)^{k-1}}{(k!)^2}, \tag{2.15}$$

which in view of Eq. (2.14), gives

$$2(xt)\mathcal{T}_0'(xt) = 2 \sum_{k=0}^{\infty} k \frac{(xt)^k}{(k!)^2} = B(xt). \tag{2.16}$$

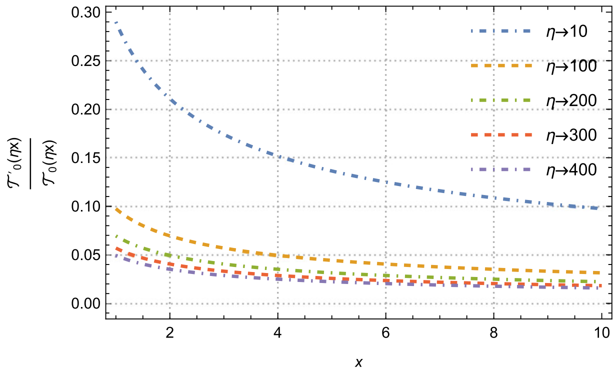

Fig. 1 $\frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)}$ when $x \in [1, 10]$

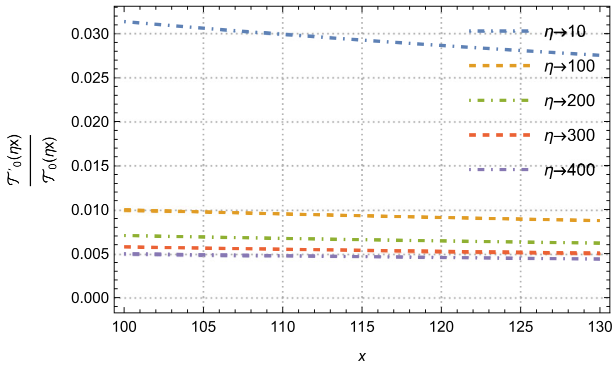

Fig. 2 $\frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)}$ when $x \in [100, 130]$

Using equations (2.13), (2.16) in (2.11), we get

$$C(xt) = \frac{xt}{\eta^2}\left\{(xt)\mathcal{T}_0(xt) + 2(xt)\mathcal{T}_0'(xt) + \mathcal{T}_0(xt)\right\}. \tag{2.17}$$

Taking $t = 1$ and replacing x by ηx in Eq. (2.17), we obtain

$$C(\eta x) = \frac{\eta x}{\eta^2}\left\{(\eta x)\mathcal{T}_0(\eta x) + 2(\eta x)\mathcal{T}_0'(\eta x) + \mathcal{T}_0(\eta x)\right\}, \tag{2.18}$$

which on using in Eq. (2.9), we get assertion (2.6). □

Remark 2.1 From the following figures (Figs. 1, 2, 3, 4), we can observe that

$$\lim_{\eta\to\infty} \frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)} = 0. \tag{2.19}$$

We obtain the central moments of the Tricomi operators as:

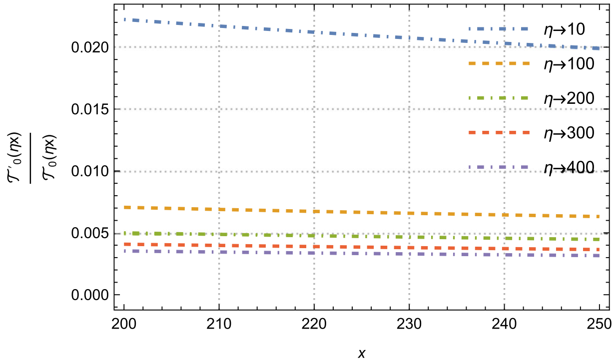

Fig. 3 $\frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)}$ when $x \in [200, 250]$

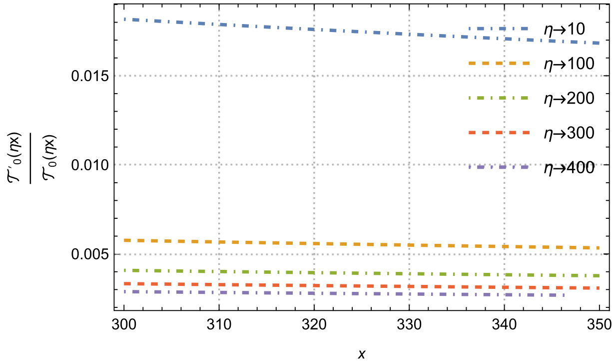

Fig. 4 $\frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)}$ when $x \in [300, 350]$

Lemma 2.3 *If $\mu_{(m,x)}(\nu) = (\nu - x)^m$ denotes the central moments of the operators* (2.2)*, then for $m = 2$, we have*

$$\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x) = 2x^2 \frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)} + \frac{x}{\eta}, \qquad \forall \eta \in \mathbb{N}. \tag{2.20}$$

Proof The left-hand side of the Eq. (2.20) can be rewritten as follows by using the linearity of the operators defined in equation (2.2),

$$\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x) = \mathscr{T}_\eta(\nu^2; x) - 2x\mathscr{T}_\eta(\nu; x) + x^2\mathscr{T}_\eta(1; x) \tag{2.21}$$

on using Eqs. (2.4), (2.5) and (2.6) we get assertion (2.20). □

In the next section, we establish the transformation properties and rate of convergence of the operators $\mathscr{T}_\eta(\hat{f}; x)$ by using the modulus of continuity. Additionally, we compute the rate of convergence in the different Lipsticz-type spaces.

3 Transformation Properties and Rate of Convergence

In this section, we establish the transformation properties of the operators $\mathscr{T}_\eta(\hat{f}; x)$ and establish the rate of convergence by using the modulus of continuity. Moreover, we compute the rate of convergence in different Lipschitz-type spaces. We initiate with the following lemma, which proves that $\mathscr{T}_\eta(\hat{f}; x)$ maps E into itself.

Lemma 3.1 *For $\omega(x) = \frac{1}{1+x^2}$, there exists a constant $\mathcal{M}$ such that*

$$\omega(x)\mathscr{T}_\eta\left(\frac{1}{\omega}; x\right) \leq \mathcal{M} \tag{3.1}$$

holds for all $x \in [0, \infty)$ and $\eta \in \mathbb{N}$. Moreover, for all $\hat{f} \in E$, we have

$$\|\mathscr{T}_\eta(\hat{f})\|_* \leq \mathcal{M}\|\hat{f}\|_*. \tag{3.2}$$

Proof Using Lemma 2.1, we can write that

$$\omega(x)\mathscr{T}_\eta\left(\frac{1}{\omega}; x\right) = \frac{1}{1+x^2}\left[\mathscr{T}_\eta(1; x) + \mathscr{T}_\eta(\nu^2; x)\right] \tag{3.3}$$

on using equations (2.4) and (2.6) in equation (3.3), we obtain

$$\omega(x)\mathscr{T}_\eta\left(\frac{1}{\omega}; x\right) = \frac{1}{1+x^2}\left[1 + x^2 + 2x^2\frac{\mathcal{T}_0'(\eta x)}{\mathcal{T}_0(\eta x)} + \frac{x}{\eta}\right] \tag{3.4}$$

which implies

$$\omega(x)\mathscr{T}_\eta\left(\frac{1}{\omega}; x\right) \leq \mathcal{M}. \tag{3.5}$$

Moreover, by using the norm defined in Eq. (2.3), we have

$$\omega(x)\left|\mathscr{T}_\eta(\hat{f}; x)\right| = \omega(x)\left|\mathscr{T}_\eta\left(\omega\frac{\hat{f}}{\omega}; x\right)\right| \leq \|\hat{f}\|_*\omega(x)\mathscr{T}_\eta\left(\frac{1}{\omega}; x\right) \leq \mathcal{M}\|\hat{f}\|_*, \tag{3.6}$$

taking supremum over $x \in [0, \infty)$ in the above inequality, gives the assertion (3.2). □

Now, we obtain the uniform convergence by using universal Korovkin's theorem as:

Theorem 3.1 *Let $\hat{f} \in E := \left\{\hat{f} \in C[0, \infty) : \lim_{x\to\infty} \frac{\hat{f}(x)}{1+x^2} \text{ is finite}\right\}$ and $C[0, \infty)$ denotes the space of continuous functions defined on $[0, \infty)$. Then the sequence of* *operators defined in* (2.2) *converges uniformly on the compact subsets of the interval* $[0, \infty)$*, i.e.,*

$$\lim_{\eta\to\infty} \mathscr{T}_\eta(\hat{f}; x) = \hat{f}(x). \tag{3.7}$$

Proof With the aid of Lemma 2.2 and Remark 2.1, we obtain

$$\lim_{n\to\infty} \mathscr{T}_\eta(\nu^s; x) = x^s, \qquad s = 0, 1, 2, \tag{3.8}$$

uniformly on the compact subsets of the interval $[0, \infty)$. Hence by applying universal Korovkin's theorem [3], we get assertion (3.7). □

From now onward, we present the approximation results.

Theorem 3.2 *Suppose* $\hat{f}$ *is uniformly continuous on the interval* $[0, \infty)$*. Then the following inequality holds for the sequence of operators* $\mathscr{T}_\eta(\hat{f}; x)$

$$\left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right| \leq 2\omega\left(\hat{f}; \sqrt{\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x)}\right) \tag{3.9}$$

where ω *is the modulus of continuity of the function defined by* (1.2).

Proof Consider

$$\left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right| = \left|\frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \left[\hat{f}\left(\frac{k^2}{\eta}\right) - \hat{f}(x)\right]\right| \tag{3.10}$$

using the triangle inequality gives

$$\left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right| \leq \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \left|\hat{f}\left(\frac{k^2}{\eta}\right) - \hat{f}(x)\right| \tag{3.11}$$

using the definition of modulus of continuity and in view of inequality (1.3), we get

$$\left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right| \leq \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \left[1 + \frac{1}{\sigma}\left|\frac{k^2}{\eta} - x\right|\right] \omega(\hat{f}; \sigma) \tag{3.12}$$

$$\left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right| \leq \left[1 + \frac{1}{\sigma} \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \left|\frac{k^2}{\eta} - x\right|\right] \omega(\hat{f}; \sigma). \tag{3.13}$$

Taking into account the Cauchy-Schwartz inequality, we find

$$\sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \left|\frac{k^2}{\eta} - x\right| = \sum_{k=0}^{\infty} \sqrt{\frac{(\eta x)^k}{(k!)^2}} \sqrt{\frac{(\eta x)^k}{(k!)^2}} \left|\frac{k^2}{\eta} - x\right| \tag{3.14}$$

$$\sum_{k=0}^{\infty}\frac{(\eta x)^k}{(k!)^2}\left|\frac{k^2}{\eta}-x\right| \leq \left\{\sum_{k=0}^{\infty}\frac{(\eta x)^k}{(k!)^2}\right\}^{\frac{1}{2}}\left\{\sum_{k=0}^{\infty}\frac{(\eta x)^k}{(k!)^2}\left(\frac{k^2}{\eta}-x\right)^2\right\}^{\frac{1}{2}} \tag{3.15}$$

multiplying and dividing by $\sqrt{\mathcal{T}_0(\eta x)}$ in the right hand side of inequality (3.15) and in the view of Eq. (2.1), we have

$$\sum_{k=0}^{\infty}\frac{(\eta x)^k}{(k!)^2}\left|\frac{k^2}{\eta}-x\right| \leq \mathcal{T}_0(\eta x)\left\{\mathscr{T}_\eta((\nu-x)^2;x)\right\}^{\frac{1}{2}} \tag{3.16}$$

using above inequality in inequality (3.13), we find

$$\left|\mathscr{T}_\eta(\hat{f};x)-\hat{f}(x)\right| \leq \left[1+\frac{1}{\sigma}\sqrt{\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x)}\right]\omega(\hat{f};\sigma) \tag{3.17}$$

by choosing

$$\sigma = \sqrt{\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x)}$$

we get assertion (3.9). □

Now, for $0 < \alpha \leq 1$ and $\nu_1, \nu_2 \in [0,\infty)$, let us introduce the following class of the functions:

$$Lip_{\mathscr{M}}^{(\alpha)} := \{\psi : |\psi(\nu_1)-\psi(\nu_2)| \leq \mathscr{M}|\nu_1-\nu_2|^\alpha\}. \tag{3.18}$$

Theorem 3.3 *Assume that $\psi \in Lip_{\mathscr{M}}^{(\alpha)}$. Then*

$$\left|\mathscr{T}_\eta(\psi;x)-\psi(x)\right| \leq \mathscr{M}\left[\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x)\right]^{\frac{\alpha}{2}}. \tag{3.19}$$

Proof Since, $\mathscr{T}_\eta(\hat{f};x)$ is positive linear operator and $\psi \in Lip_{\mathscr{M}}^{(\alpha)}$, therefore in view of Eq. (3.18), we obtain

$$\begin{aligned}\left|\mathscr{T}_\eta(\psi;x)-\psi(x)\right| &= \left|\mathscr{T}_\eta(\psi(\nu)-\psi(x);x)\right| \leq \mathscr{T}_\eta(|\psi(\nu)-\psi(x)|;x)\\ &\leq \mathscr{M}\,\mathscr{T}_\eta(|\nu-x|^\alpha;x)\end{aligned} \tag{3.20}$$

which on using Hölder's inequality on the right-hand side, gives

$$\mathscr{T}_\eta(|\nu-x|^\alpha;x) = \frac{1}{\mathcal{T}_0(\eta x)}\sum_{k=0}^{\infty}\frac{(\eta x)^k}{(k!)^2}\left|\frac{k^2}{\eta}-x\right|^\alpha \tag{3.21}$$

$$\mathscr{T}_\eta(|\nu-x|^\alpha;x) = \frac{1}{\mathcal{T}_0(\eta x)}\sum_{k=0}^{\infty}\left\{\frac{(\eta x)^k}{(k!)^2}\right\}^{\frac{2-\alpha}{2}}\left\{\frac{(\eta x)^k}{(k!)^2}\right\}^{\frac{\alpha}{2}}\left|\frac{k^2}{\eta}-x\right|^\alpha \tag{3.22}$$

$$\mathscr{T}_\eta(|\nu - x|^\alpha; x) \leq \frac{1}{\mathcal{T}_0(\eta x)} \times [\mathcal{T}_0(\eta x)]^{\frac{2-\alpha}{2}} \left\{ \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \right\}^{\frac{2-\alpha}{2}}$$
$$\times [\mathcal{T}_0(\eta x)]^{\frac{\alpha}{2}} \left\{ \frac{1}{\mathcal{T}_0(\eta x)} \sum_{k=0}^{\infty} \frac{(\eta x)^k}{(k!)^2} \left(\frac{k^2}{\eta} - x \right)^2 \right\}^{\frac{\alpha}{2}}. \quad (3.23)$$

In view of equations (2.2) and (2.20), the above inequality gives

$$\mathscr{T}_\eta(|\nu - x|^\alpha; x) \leq [\mathscr{T}_\eta(1; x)]^{\frac{2-\alpha}{2}} \left[\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x)\right]^{\frac{\alpha}{2}}, \quad (3.24)$$

which on using inequalities (3.20) and (3.24), gives assertion (3.19). □

Theorem 3.4 *Suppose that* χ *is a continuous function on* $[0, \infty)$*. Then we have*

$$\left|\mathscr{T}_\eta(\chi; x) - \chi(x)\right| \leq \frac{3}{2}\left(1 + \frac{a}{2} + \frac{h^2}{2}\right)\omega_2(\chi, h) + \frac{2h^2}{a}\|\chi\|, \quad (3.25)$$

where

$$h := \Omega_n(x) = \left\{\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x)\right\}^{\frac{1}{4}}$$

and ω_2 *is the second modulus of the continuity of the function* χ *defined in* (1.4).

Proof Let f_h be the second order Steklov function of the function χ, Then we have

$$\left|\mathscr{T}_\eta(\chi; x) - \chi(x)\right| \leq \left|\mathscr{T}_\eta(|\chi - f_h|; x)\right| + \left|\mathscr{T}_\eta(f_h; x) - f_h(x)\right| + |f_h(x) - \chi(x)| \quad (3.26)$$

using Eq. (2.4), we have

$$\left|\mathscr{T}_\eta(\chi; x) - \chi(x)\right| \leq 2\|\chi - f_h\| + \left|\mathscr{T}_\eta(f_h; x) - f_h(x)\right| \quad (3.27)$$

in view of inequality (1.9), above inequality becomes

$$\left|\mathscr{T}_\eta(\chi; x) - \chi(x)\right| \leq \frac{3}{2}\omega_2(\chi, h) + \left|\mathscr{T}_\eta(f_h; x) - f_h(x)\right|. \quad (3.28)$$

Keeping in view that $f_h \in C^2[0, a]$, from inequality (1.6) it follows that

$$\left|\mathscr{T}_\eta(f_h; x) - f_h(x)\right| \leq \|f_h'\|\sqrt{\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x)} + \frac{1}{2}\|f_h''\|\mathscr{T}_\eta(\mu_{(2,x)}(\nu); x), \quad (3.29)$$

now using inequality (1.10) in above inequality, we find

$$\left|\mathscr{T}_\eta(f_h;x)-f_h(x)\right| \leq \|f_h'\|\sqrt{\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x)}+\frac{3}{4h^2}\omega_2(f,h)\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x). \tag{3.30}$$

Further, in view of inequalities (1.10) and (1.11), it gives

$$\begin{aligned}\left|\mathscr{T}_\eta(f_h;x)-f_h(x)\right| \leq & \left(\frac{2}{a}\|\chi\|+\frac{3a}{4\,h^2}\omega_2(\chi,h)\right)\sqrt{\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x)}\\ &+\frac{3}{4\,h^2}\omega_2(\chi,h)\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x).\end{aligned} \tag{3.31}$$

Therefore, in view of equations (3.28) and (3.31), we get assertion (3.25) by choosing

$$h := \Omega_n(x) = \left\{\mathscr{T}_\eta(\mu_{(2,x)}(\nu);x)\right\}^{\frac{1}{4}}. \tag{3.32}$$

□

Remark 3.1 In Theorem 3.4, $\Omega_n(x) \to 0$, as $n \to \infty$.

In the next section, we observe the rate of convergence through numerical examples. We give a graphical depiction of the approximated function and its absolute error and provide tables for absolute error for the different values of x and η.

4 Numerical Examples

In this section, we explain the convergence of our operators through numerical examples. We give a graphical depiction of approximated expressions and actual functions. Also, we give the graphical depiction of absolute error and tables for the different values of x and η.

Example 4.1 For $\eta = 10, 10^2, 10^3, 10^4, 10^5$ the rate of convergence of the operators $\mathscr{T}_\eta(\hat{f};x)$ to the function $\hat{f}(x) = x^2 + \frac{1}{1+x+x^2}$ is illustrated in Fig. 5. Further, in Table 1, we estimate the absolute error $E_\eta = \left|\mathscr{T}_\eta(\hat{f};x)-\hat{f}(x)\right|$ for different values of η and given the corresponding graph for the error depicting the convergence in Fig. 6.
It can be clearly seen from Figs. 5, 6, and Table 1 that for larger values of η the proposed operator (2.2) converges to $\hat{f}(x)$.

Now, we consider another example:

Example 4.2 For $\eta = 10, 10^2, 10^3, 10^4, 10^5$ the rate of convergence of the operators $\mathscr{T}_\eta(\hat{f};x)$ to the function $\hat{f}(x) = \frac{x}{2}+\frac{1}{10+x^2+x^3}+\sqrt{x}$ is illustrated in Fig. 7. Further, in Table 2, we estimate the absolute error $E_\eta = \left|\mathscr{T}_\eta(\hat{f};x)-\hat{f}(x)\right|$ for different values

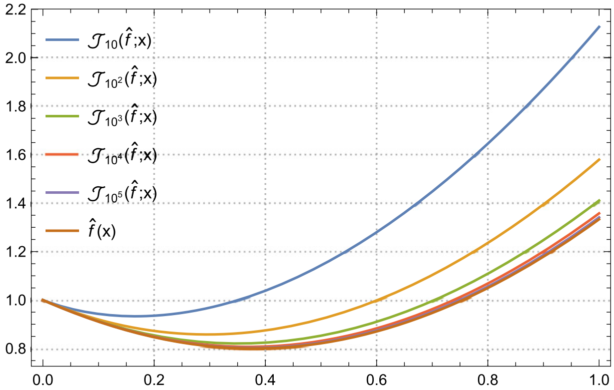

Fig. 5 The Convergence of operators $\mathscr{T}_\eta(\hat{f}; x)$ to $\hat{f}(x) = x^2 + \frac{1}{1+x+x^2}$

Table 1 Error of approximation process for $\hat{f}(x) = x^2 + \frac{1}{1+x+x^2}$

x	$\mathcal{E}_{10}$	$\mathcal{E}_{10^2}$	$\mathcal{E}_{10^3}$	$\mathcal{E}_{10^4}$	$\mathcal{E}_{10^5}$
0.05	0.01152	0.00294664	0.000845209	0.0002581	0.0000806922
0.10	0.0318827	0.00876897	0.0025807	0.000795059	0.000249264
0.15	0.0581969	0.0166527	0.00499648	0.00154961	0.000486867
0.20	0.0888368	0.026128	0.00795773	0.00248145	0.000781033
0.25	0.122745	0.0368263	0.0113459	0.00355346	0.00112008
0.30	0.159184	0.0484544	0.015059	0.00473244	0.00149343
0.35	0.197628	0.0607836	0.0190126	0.0059903	0.00189205
0.40	0.237692	0.0736394	0.0231401	0.00730439	0.0023086
0.45	0.279102	0.0868926	0.027391	0.0086574	0.00273746
0.50	0.321658	0.10045	0.0317287	0.0100367	0.00317449
0.55	0.365222	0.114248	0.0361281	0.0114336	0.00361688
0.60	0.409696	0.128245	0.0405733	0.0128426	0.00406284
0.65	0.455016	0.142416	0.0450552	0.0142607	0.0045114
0.70	0.501143	0.156749	0.0495699	0.0156868	0.00496219
0.75	0.548053	0.171239	0.0541173	0.0171209	0.00541527
0.80	0.595733	0.185889	0.0586997	0.018564	0.00587097
0.85	0.644182	0.200707	0.0633211	0.0200176	0.00632982
0.90	0.693403	0.2157	0.0679865	0.0214837	0.00679245
0.95	0.743401	0.230881	0.072701	0.0229643	0.00725952
1.00	0.794185	0.246259	0.0774703	0.0244612	0.00773168

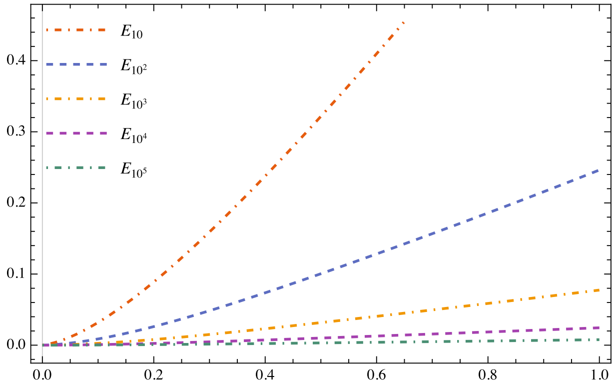

Fig. 6 Graphical depiction of absolute error of operators $\mathscr{T}_\eta(\hat{f};x)$ to $\hat{f}(x) = x^2 + \frac{1}{1+x+x^2}$, for $\eta = 10, 10^2, 10^3, 10^4, 10^5$.

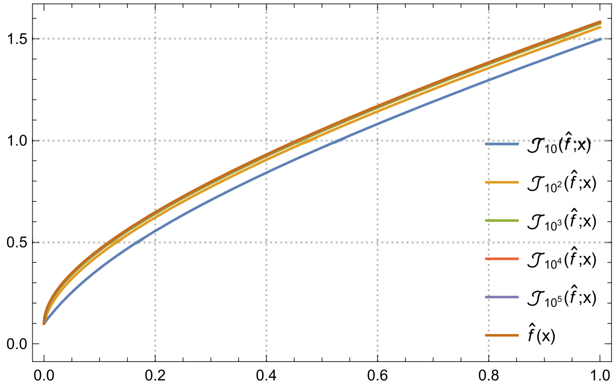

Fig. 7 The Convergence of operators $\mathscr{T}_\eta(\hat{f};x)$ to $\hat{f}(x) = \frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$

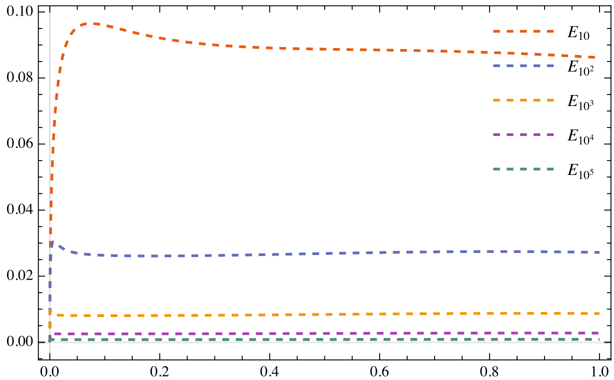

Fig. 8 Graphical depiction of absolute error of operators $\mathscr{T}_\eta(\hat{f};x)$ to $\hat{f}(x) = \frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$, for $\eta = 10, 10^2, 10^3, 10^4, 10^5$.

Table 2 Error of approximation process for $\hat{f}(x) = \frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$

x	$\mathcal{E}_{10}$	$\mathcal{E}_{10^2}$	$\mathcal{E}_{10^3}$	$\mathcal{E}_{10^4}$	$\mathcal{E}_{10^5}$
0.05	0.0953406	0.0269737	0.00806523	0.00251692	0.000792794
0.10	0.0959458	0.026303	0.00803711	0.00251835	0.00079416
0.15	0.0938896	0.0261318	0.00804496	0.00252504	0.000796656
0.20	0.0921139	0.0261086	0.00807011	0.00253532	0.000800123
0.25	0.0908544	0.0261606	0.00810767	0.00254874	0.000804503
0.30	0.0900103	0.0262594	0.00815519	0.00256494	0.000809732
0.35	0.0894598	0.0263891	0.00821073	0.00258357	0.000815718
0.40	0.0891061	0.0265381	0.00827236	0.00260418	0.000822336
0.45	0.0888775	0.0266966	0.00833802	0.00262623	0.000829427
0.50	0.0887218	0.0268561	0.00840544	0.00264909	0.000836799
0.55	0.0886004	0.0270087	0.00847224	0.00267204	0.000844232
0.60	0.0884849	0.0271472	0.00853595	0.0026943	0.000851485
0.65	0.0883541	0.0272653	0.00859412	0.00271509	0.000858302
0.70	0.0881924	0.0273574	0.00864438	0.00273358	0.000864424
0.75	0.0879888	0.027419	0.00868455	0.002749	0.000869596
0.80	0.0877356	0.0274465	0.00871271	0.00276063	0.00087358
0.85	0.0874279	0.0274373	0.00872726	0.00276785	0.000876165
0.90	0.0870634	0.0273898	0.00872699	0.00277014	0.000877173
0.95	0.0866417	0.0273035	0.00871108	0.00276712	0.000876467
1.00	0.0861639	0.0271786	0.00867915	0.00275855	0.000873958

of η and given the corresponding graph for the error depicting the convergence in Fig. 8.

It can be clearly seen from Figs. 7, 8, and Table 2 that for larger values of η the proposed operator (2.2) converges to $\hat{f}(x)$.

4.1 Comparative Study

This section illustrates the efficacy and efficiency of the 0^{th}-order Tricomi function operators through instructive examples. Our work employs distinct test functions to assess the efficacy of the Schurer-Stancu operators with shape parameter λ, Mittag-Leffler operators, and the recently introduced 0^{th}-order Tricomi function operators. Our results indicate that the newly introduced operators, when applied to the selected test functions and parameters, perform better than the Schurer-Stancu operators with shape parameter λ and Mittag-Leffler operator within the specified range. We need to acknowledge that we do not assert absolute superiority of the newly constructed operator over the Schurer-Stancu operators, which have a shape parameter λ, and the Mittag-Leffler operators. Instead, we propose it as a feasible alternative. Furthermore, the illustrations shown in this section were calculated using Wolfram Mathematica 13.3.1.0 on the MAC OS X (64-bit) operating system.

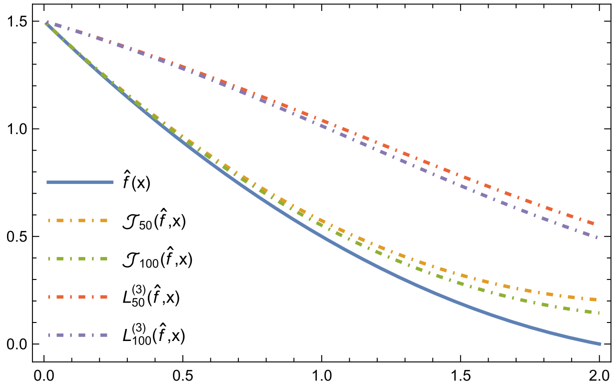

Fig. 9 Comparison between operators $\mathscr{T}_\eta(\hat{f}; x)$ and $L^{\beta}_{\eta}(\hat{f}; x)$ with exact function $\hat{f}(x) = \frac{1}{4}(x-2)(x-3)$

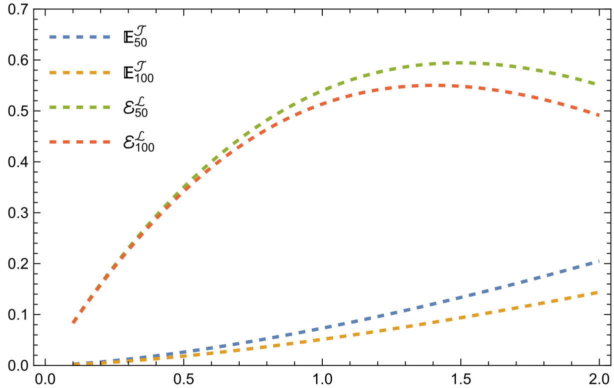

Fig. 10 Absolute error $\mathbb{E}^{\mathscr{T}}_{\eta}$ and $\mathcal{E}^{\mathcal{L}}_{\eta}$ of operators $\mathscr{T}_\eta(\hat{f}; x)$ and $L^{\beta}_{\eta}(\hat{f}; x)$ to the function $\hat{f}(x) = \frac{1}{4}(x-2)(x-3)$

Example 4.3 In this example, we examine a test function $\hat{f}(x) = \frac{1}{4}(x-2)(x-3)$ with parameter $\beta = 3$, and $(v_\eta) = \eta^{\frac{4}{5}}$, where $\eta = 50, 100$.
Using the same interval, Figure 9 compares the performance of the newly defined 0^{th}-order Tricomi function operators in (2.2) with that of the Mittag-Leffler operators presented in (1.19). This graphic clearly shows that the approximation with the new operator fits the test function more smoothly.
Furthermore, we present the error of approximation of both the Mittag-Leffler operator and the newly defined 0^{th}-order Tricomi function operators in Figure 10. Here, $\mathcal{E}^{\mathcal{L}}_{\eta} = \left|L^{(\beta)}_{\eta}(\hat{f}; x) - \hat{f}(x)\right|$ and $\mathbb{E}^{\mathscr{T}}_{\eta} = \left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right|$ denote the error functions of approximations by the Mittag-Leffler operators and newly defined 0^{th}-order Tricomi function operators, respectively.
Table 3 provides a numerical comparison of the approximation error of these operators.

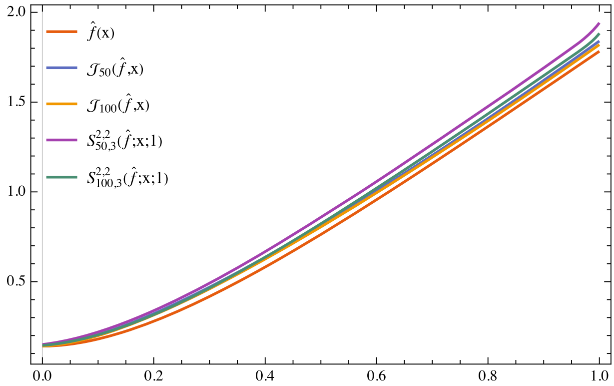

Fig. 11 Comparison between operators $\mathscr{T}_\eta(\hat{f}; x)$ and $\hat{\mathcal{S}}^{p,r}_{\eta,\gamma}(\hat{f}; x; \lambda)$ with exact function $\hat{f}(x) = \frac{(\pi x)^2}{3x+2} + \sin\left(\frac{x^2}{3} + 3\right)$

Example 4.4 In this example, we examine a test function $\hat{f}(x) = \frac{(\pi x)^2}{3x+2} + \sin\left(\frac{x^2}{3} + 3\right)$ with parameters $p = 2$, $r = 2$, $\gamma = 3$ and $\lambda = 1$, where $\eta = 50, 100$.
Using the same interval, Fig. 11 compares the performance of the newly defined 0^{th}-order Tricomi function operators in (2.2) with that of the Schurer-Stancu operators with shape parameter λ presented in (1.20). This graphic clearly shows that the approximation with the new operator fits the test function more smoothly.
Furthermore, we present the error of approximation of both the Schurer-Stancu operators with shape parameter λ and the newly defined 0^{th}-order Tricomi function operators in Fig. 12. Here, $\mathcal{E}^{\mathcal{S}}_\eta = \left|\hat{\mathcal{S}}^{p,r}_{\eta,\gamma}(\hat{f}; x; \lambda) - \hat{f}(x)\right|$ and $\mathbb{E}^{\mathscr{T}}_\eta = \left|\mathscr{T}_\eta(\hat{f}; x) - \hat{f}(x)\right|$ denote the error functions of approximations by the Schurer-Stancu operators with shape parameter λ operators and newly defined 0^{th}-order Tricomi function operators, respectively.
For a numerical comparison of the error of approximation between these operators, refer to Table 4. Similar to the previous example, it's evident that the newly introduced operator performs comparably well.

5 Conclusions and Applications

With the help of the Tricomi function, we present a sequence of new operators in this paper. The approximation properties and convergence qualities of the sequence of a positive linear operator in (2.2) are obtained. The numerical examples are done in Wolfram Mathematica. Also, we analyze the error of the approximation and give the graphical depictions of the approximated function $\hat{f}$ and the error.
In further studies, researchers can look at a new sequence of operators that generalizes the operators in (2.2). For instance, modification or generalization of this operator can be considered for better approximation. Furthermore, the derived results are useful and can be applied in mathematical analysis, mathematical physics, and quantum calcu-

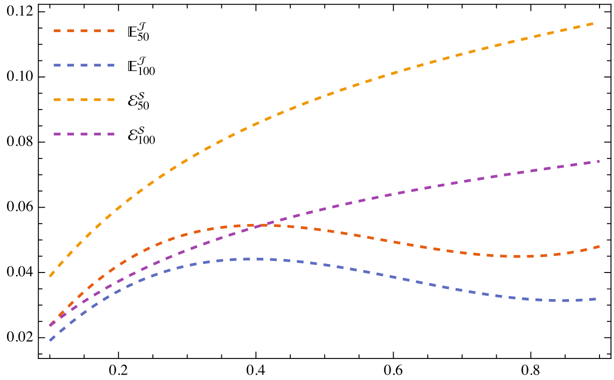

Fig. 12 Absolute error $\mathbb{E}_{\eta}^{\mathcal{T}}$ and $\mathcal{E}_{\eta}^{\mathcal{S}}$ of operators $\mathscr{T}_{\eta}(\hat{f};x)$ and $\hat{\mathcal{S}}_{\eta,\gamma}^{p,r}(\hat{f};x;\lambda)$ to the function $\hat{f}(x) = \frac{(\pi x)^2}{3x+2} + \sin\left(\frac{x^2}{3}+3\right)$

Table 3 Error of approximation process for $\hat{f}(x) = \frac{1}{4}(x-2)(x-3)$

Error at $x =$	$\mathbb{E}_{50}^{\mathcal{T}}$	$\mathcal{E}_{50}^{\mathcal{L}}$	$\mathbb{E}_{100}^{\mathcal{T}}$	$\mathcal{E}_{100}^{\mathcal{L}}$
0.1	0.00247	0.08369	0.00170	0.08296
0.2	0.00680	0.16049	0.00471	0.15858
0.3	0.01234	0.23046	0.00858	0.22689
0.4	0.01886	0.29369	0.01314	0.28801
0.5	0.02621	0.35029	0.01829	0.34205
0.6	0.03433	0.40040	0.02397	0.38920
0.7	0.04312	0.44419	0.03014	0.42966
0.8	0.05255	0.48184	0.03676	0.46367
0.9	0.06258	0.51358	0.04380	0.49153
1.0	0.07316	0.53964	0.05123	0.51355
1.1	0.08428	0.56029	0.05904	0.53005
1.2	0.09590	0.57579	0.06721	0.54141
1.3	0.10801	0.58644	0.07572	0.54798
1.4	0.12058	0.59254	0.08456	0.55015
1.5	0.13360	0.59440	0.09371	0.54829
1.6	0.14705	0.59231	0.10317	0.54278
1.7	0.16092	0.58660	0.11293	0.53399
1.8	0.17520	0.57756	0.12298	0.52225
1.9	0.18987	0.56549	0.13330	0.50791
2.0	0.20493	0.55067	0.14390	0.49127

Table 4 Error of approximation process for $\hat{f}(x) = \frac{(\pi x)^2}{3x+2} + \sin\left(\frac{x^2}{3} + 3\right)$

Error at $x =$	$\mathbb{E}_{50}^{\mathscr{T}}$	$\mathcal{E}_{50}^{\mathcal{S}}$	$\mathbb{E}_{100}^{\mathscr{T}}$	$\mathcal{E}_{100}^{\mathcal{S}}$
0.05	0.01104	0.02466	0.00885	0.01442
0.10	0.02359	0.03876	0.01906	0.02367
0.15	0.03413	0.05029	0.02768	0.03116
0.20	0.04222	0.05981	0.03432	0.03732
0.25	0.04801	0.06779	0.03905	0.04248
0.30	0.05177	0.07459	0.04209	0.04688
0.35	0.05383	0.08046	0.04370	0.05067
0.40	0.05450	0.08559	0.04413	0.05398
0.45	0.05411	0.09013	0.04360	0.05692
0.50	0.05294	0.09418	0.04236	0.05954
0.55	0.05127	0.09783	0.04062	0.06190
0.60	0.04940	0.10116	0.03859	0.06405
0.65	0.04758	0.10421	0.03648	0.06603
0.70	0.04607	0.10703	0.03451	0.06785
0.75	0.04513	0.10966	0.03286	0.06956
0.80	0.04499	0.11214	0.03176	0.07116
0.85	0.04588	0.11449	0.03140	0.07268
0.90	0.04801	0.11685	0.03199	0.07414
0.95	0.05160	0.12123	0.03371	0.07584
1.00	0.05680	0.13231	0.03674	0.07698

lus, particularly with q and (p, q) analogues of the proposed operators. Regarding the applications of the proposed operators, they can be utilized to solve fractional Volterra integral equations of the first and second kinds, providing numerical approximate solutions [9]. Additionally, the proposed operator can be employed to examine and regulate a real-life issue associated with the daily average global surface air temperature [15]. This sequence of operators has the capacity to exert impact over numerous domains of scientific inquiry.

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Availability of data and materials None.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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