Bull. Malays. Math. Sci. Soc. (2024) 47:150
https://doi.org/10.1007/s40840-024-01749-6

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Sharp Bounds on Coefficients Functionals of Hankel Determinants for Ozaki Close-to-Convex Functions

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Received: 2 March 2024 / Revised: 28 June 2024 / Accepted: 15 July 2024 /
Published online: 5 August 2024

Abstract
We determine the sharp bounds on the second Hankel determinants of logarithmic coefficients and the third Hankel determinants for Ozaki close-to-convex functions.

Keywords Close-to-convex function · Hankel determinant · Logarithmic coefficient · Sharp bound

Mathematics Subject Classification 30C55 · 30C45

1 Introduction

Let $\mathcal{A}$ be the class of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

We denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of univalent functions.

In 1941, Ozaki [33] introduced the classes of functions $\mathcal{F}$ and $\mathcal{G}$, and defined as

$$\mathcal{F} = \left\{ f \in \mathcal{A} : \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \qquad z \in \mathbb{D} \right\}, \tag{1.2}$$

Communicated by V. Ravichandran.

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and

$$\mathcal{G} = \left\{ f \in \mathcal{A} : \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \qquad z \in \mathbb{D} \right\}, \tag{1.3}$$

respectively. The author proved the inclusion relation $\mathcal{G} \subset \mathcal{S}$. We also note that $\mathcal{F}$ follows from the original definition of Kaplan [15], and that Umezawa [39] subsequently proved that functions in $\mathcal{F}$ are not necessarily starlike, but are convex in one direction. The functions in the classes $\mathcal{F}$ and $\mathcal{G}$ are known as Ozaki close-to-convex functions, which have nice geometric properties and are used to understand the shape and behavior of various subclasses of univalent functions.

Given $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1.1) is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

In recent years, many papers have been devoted to finding bounds of determinants, whose elements are coefficients of functions in $\mathcal{A}$, or its subclasses. The sharp bounds on the second Hankel determinants $|H_{2,1}(f)|$ and $|H_{2,2}(f)|$ were obtained by [6, 9, 13, 14, 26, 32], for various classes of analytic functions. We refer to [4, 7, 8, 34, 36, 37, 43] for discussions on the upper bounds of the third Hankel determinants $|H_{3,1}(f)|$ for various classes of univalent functions. However, these results are far from sharpness. In a recent paper, Kwon et al. [23] found such a formula of expressing c_4 for Carathéodory functions, the sharp results of the third Hankel determinants are found for some classes of univalent functions (see e.g., [5, 19–21, 24, 25, 35, 40–42]).

Note that for $f \in \mathcal{A}$, $a_1 = 1$, $H_{3,1}(f)$ reduces to

$$H_{3,1}(f) = a_3\big(a_2a_4 - a_3^2\big) - a_4\big(a_4 - a_2a_3\big) + a_5\big(a_3 - a_2^2\big). \tag{1.4}$$

Recently, Kowalczyk et al. [21] proved the sharp inequality $|H_{3,1}(f)| \leq 1/16$ for $f \in \mathcal{F}$. In this paper, we prove that $|H_{3,1}(f)| \leq 19/2160$ for $f \in \mathcal{G}$, and so giving the sharp bound for $|H_{3,1}(f)|$ for a significant subclass of $\mathcal{G}$.

For $f \in \mathcal{S}$, let

$$F_f(z) := \log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \qquad z \in \mathbb{D}. \tag{1.5}$$

The numbers $\gamma_n := \gamma_n(f)$ are called logarithmic coefficients of f. It is well known that the logarithmic coefficients play a crucial role in Milin conjecture [29]. Sharp logarithmic coefficient estimates for the class $\mathcal{S}$ are already known for $n = 1$ and $n = 2$, given by $|\gamma_1| \leq 1$ and $|\gamma_2| \leq 1/2 + 1/e^2$, respectively. However, the bound of

γ_n for $n \geq 3$, is still an open problem. We refer to [1, 2, 10, 12, 22, 38] for discussions on the logarithmic coefficient for various classes of univalent functions.

Given $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(F_f/2)$ which entries are logarithmic coefficients of $f \in \mathcal{A}$ of the form (1.1) is defined by

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

A study of Hankel determinant with entries as logarithmic coefficients was initiated by Kowalczyk and Lecko [16]. Due to the great importance of logarithmic coefficients, the proposed topic seems reasonable and interesting.

By differentiating (1.5) and using (1.1) we get

$$\gamma_1 = \frac{1}{2}a_2, \qquad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right),$$
$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \qquad \gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \tag{1.6}$$

Therefore,

$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2 = \frac{1}{4}\Bigg[\left(a_3 - \frac{1}{2}a_2^2\right)a_5 + \left(a_2a_3 - a_4 - \frac{1}{6}a_2^3\right)a_4 + \left(\frac{1}{4}a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{12}a_2^4\right)a_3 + \frac{1}{72}a_2^6\Bigg]. \tag{1.7}$$

By observing that when $f \in \mathcal{S}$, for $f_\theta(z) := e^{-i\theta}f(e^{i\theta}z)$ with $\theta \in \mathbb{R}$, we have

$$H_{2,2}(F_{f_\theta}/2) = e^{6i\theta}H_{2,2}(F_f/2).$$

Thus, the coefficients functional $|H_{2,2}(F_f/2)|$ is a rotationally invariant.

Recently, Kowalczyk and Lecko [16] obtained the sharp bounds of $|H_{2,1}(F_f/2)|$ for the classes of starlike and convex functions. Moreover, the sharp bounds of $|H_{2,1}(F_f/2)|$ for the classes of starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$ were found in [17]. Very recently, Kowalczyk and Lecko [18] examined the sharp bounds of $|H_{2,1}(F_f/2)|$ for the classes of strongly starlike and strongly convex functions. Allu et al. [3] (see also [31]) examined the sharp bounds of $|H_{2,1}(F_f/2)|$ for the classes of starlike and convex functions with respect to symmetric points. Moreover, the sharp bounds of $|H_{2,1}(F_f/2)|$ for the class of starlike functions of order α $(0 \leq \alpha < 1)$ with respect to symmetric points were investigated in [30]. Very recently, Eker et al. [11] obtained the sharp bounds of $|H_{2,1}(F_f/2)|$ and $|H_{2,1}(F_{f^{-1}}/2)|$ for the classes of strongly Ozaki close-to-convex functions and inverse functions, respectively.

The problem of finding sharp bounds of $|H_{2,2}(F_f/2)|$ is technically much more difficult. The purpose of this paper is to prove the sharp bounds of $|H_{2,2}(F_f/2)|$ for the classes $\mathcal{F}$ and $\mathcal{G}$, respectively.

Denote $\mathcal{P}$ by the class of Carathéodory functions p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad z \in \mathbb{D}, \tag{1.8}$$

and satisfy the condition $\Re\big(p(z)\big) > 0$.

The following results are known for functions belonging to the class $\mathcal{P}$, which will be required in the proof of our main results.

Lemma 1.1 (See [23, 27, 28]) *If $p \in \mathcal{P}$ and is given by (1.8) with $c_1 \geq 0$, then*

$$c_1 = 2\zeta_1, \tag{1.9}$$

$$c_2 = 2\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2, \tag{1.10}$$

$$c_3 = 2\zeta_1^3 + 4(1-\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2 + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3 \tag{1.11}$$

and

$$\begin{aligned} c_4 =& 2\zeta_1^4 + 2(1-\zeta_1^2)(3\zeta_1^2 + \zeta_2 - 3\zeta_1^2\zeta_2 + \zeta_1^2\zeta_2^2)\zeta_2 \\ &+ 2(1-\zeta_1^2)(1-|\zeta_2|^2)\big(2\zeta_1 - 2\zeta_1\zeta_2 - \overline{\zeta_2}\zeta_3\big)\zeta_3 \\ &+ 2(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4, \end{aligned} \tag{1.12}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

2 Main Results

In this section, we will prove the sharp bounds on the second Hankel determinants of logarithmic coefficients for the classes $\mathcal{F}$ and $\mathcal{G}$, and the sharp bounds on the third Hankel determinants for the class $\mathcal{G}$, respectively.

We begin by deriving the sharp bounds of $|H_{2,2}(F_f/2)|$ for the class $\mathcal{F}$.

Theorem 2.1 *If $f \in \mathcal{F}$ be of the form (1.1), then*

$$\big|H_{2,2}(F_f/2)\big| \leq \frac{1}{32}. \tag{2.1}$$

The result is sharp for

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+2z^2}{1-z^2}, \qquad z \in \mathbb{D}, \tag{2.2}$$

that is, $f(z) = z + z^3/2 + 3z^5/8 + \cdots$.

Proof For the function $f \in \mathcal{F}$ given by (1.1), there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re\big(p(z)\big) > 0$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2}p(z) - \frac{1}{2}, \qquad z \in \mathbb{D}. \tag{2.3}$$

By elementary calculations, we have

$$\begin{aligned} a_2 &= \frac{3}{4}c_1, \qquad a_3 = \frac{1}{8}\big(2c_2 + 3c_1^2\big), \qquad a_4 = \frac{1}{64}\big(8c_3 + 18c_1c_2 + 9c_1^3\big), \\ a_5 &= \frac{3}{640}\big(16c_4 + 32c_1c_3 + 36c_1^2c_2 + 12c_2^2 + 9c_1^4\big). \end{aligned} \tag{2.4}$$

Thus, (1.7) and (2.4) give

$$\begin{aligned} 655360 \cdot H_{2,2}(F_f/2) = &\, 384\big(8c_2 + 3c_1^2\big)c_4 - 1536c_1c_2c_3 - 2560c_3^2 \\ &+ 864c_1^3c_3 - 864c_1^2c_2^2 + 1024c_2^3 - 27c_1^6. \end{aligned} \tag{2.5}$$

Since the class $\mathcal{F}$ and $\big|H_{2,2}(F_f/2)\big|$ are rotationally invariant, we may assume that $c_1 \in [0, 2]$. Thus, in view of (1.9) we assume that $\zeta_1 \in [0, 1]$. Using (2.5) and (1.9)-(1.12), we obtain

$$\begin{aligned} &655360 \cdot H_{2,2}(F_f/2) \\ &= \big[5440\zeta_1^6 + 23552(1-\zeta_1^2)\zeta_1^4\zeta_2 + 512(1-\zeta_1^2)(7-54\zeta_1^2)\zeta_1^2\zeta_2^2 \\ &\quad + 1024(1-\zeta_1^2)(20-12\zeta_1^2+13\zeta_1^4)\zeta_2^3 + 2048(1-\zeta_1^2)^2\zeta_1^2\zeta_2^4\big] \\ &\quad + 512(1-\zeta_1^2)(1-|\zeta_2|^2)\big[47\zeta_1^3 - 28(2+\zeta_1^2)\zeta_1\zeta_2 - 8(1-\zeta_1^2)\zeta_1\zeta_2^2\big]\zeta_3 \\ &\quad - 1024(1-\zeta_1^2)(1-|\zeta_2|^2)\big[(1-\zeta_1^2)(10+2|\zeta_2|^2) + 21\zeta_1^2\overline{\zeta_2}\big]\zeta_3^2 \\ &\quad + 3072(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\big[7\zeta_1^2 + 4(1-\zeta_1^2)\zeta_2\big]\zeta_4, \end{aligned}$$

for some $\zeta_1 \in [0, 1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$. Since $|\zeta_4| \le 1$, we have

$$\begin{aligned} &10240 \cdot \big|H_{2,2}(F_f/2)\big| \\ &\le \big|85\zeta_1^6 + 368(1-\zeta_1^2)\zeta_1^4\zeta_2 + 8(1-\zeta_1^2)(7-54\zeta_1^2)\zeta_1^2\zeta_2^2 \\ &\quad + 16(1-\zeta_1^2)(20-12\zeta_1^2+13\zeta_1^4)\zeta_2^3 + 32(1-\zeta_1^2)^2\zeta_1^2\zeta_2^4\big| \\ &\quad + 8(1-\zeta_1^2)(1-|\zeta_2|^2)\big|47\zeta_1^3 - 28(2+\zeta_1^2)\zeta_1\zeta_2 - 8(1-\zeta_1^2)\zeta_1\zeta_2^2\big| \cdot |\zeta_3| \\ &\quad + 16(1-\zeta_1^2)(1-|\zeta_2|^2)\Big[\big|(1-\zeta_1^2)(10+2|\zeta_2|^2) + 21\zeta_1^2\overline{\zeta_2}\big| \\ &\quad - 3\big|7\zeta_1^2 + 4(1-\zeta_1^2)\zeta_2\big|\Big] \cdot |\zeta_3|^2 \\ &\quad + 48(1-\zeta_1^2)(1-|\zeta_2|^2)\big|7\zeta_1^2 + 4(1-\zeta_1^2)\zeta_2\big|. \end{aligned}$$

A. Suppose that

$$\left|(1-\zeta_1^2)(10+2|\zeta_2|^2)+21\zeta_1^2\overline{\zeta_2}\right|-3\left|7\zeta_1^2+4(1-\zeta_1^2)\zeta_2\right|\geq 0.$$

Then

$$10240\cdot\left|H_{2,2}(F_f/2)\right|\leq u(\zeta_1,|\zeta_2|),$$

where $u:\ \mathbb{R}^2\to\mathbb{R}$ is defined by

$$\begin{aligned}u(x,y):=&85x^6+8(1-x^2)(20-20x^2+47x^3)\\&+8(1-x^2)(56x+42x^2+28x^3+46x^4)y\\&+8(1-x^2)(-16+8x+16x^2-55x^3+x^2\cdot|7-54x^2|)y^2\\&+8(1-x^2)(40-56x-66x^2-28x^3+26x^4)y^3\\&+32(1-x^2)^2(-1-2x+x^2)y^4.\end{aligned}$$

We show that $u(x,y)\leq 320$ for $(x,y)\in[0,1]\times[0,1]$.
I. On the vertices of $[0,1]\times[0,1]$, we have

$$u(0,0)=160,\quad u(0,1)=320,\quad u(1,0)=u(1,1)=85.$$

II. On the sides of $[0,1]\times[0,1]$, we get

$$\begin{aligned}u(0,y)&=32(5-4y^2+10y^3-y^4)\leq u(0,1)=320,\quad y\in(0,1),\\u(x,0)&=85x^6+8(1-x^2)(20-20x^2+47x^3)\leq u(0,0)=160,\quad x\in(0,1),\\u(1,y)&=85,\quad y\in(0,1),\\u(x,1)&=85x^6+8(1-x^2)(40-20x^2+68x^4+|7x^2-54x^4|)\\&\leq u(0,1)=320,\ x\in(0,1).\end{aligned}$$

III. It remains to consider the set $(0,1)\times(0,1)$.
If $7-54x^2\geq 0$. Then all the real solutions ($x\neq 0,\pm1$) of the system of equations

$$\begin{aligned}\frac{\partial u}{\partial x}=&(-640x+1128x^2+640x^3-1880x^4+510x^5)\\&+8(56+84x-84x^2+16x^3-140x^4-276x^5)y\\&+8(8+78x-189x^2-308x^3+275x^4+324x^5)y^2\\&+8(-56-212x+84x^2+368x^3+140x^4-156x^5)y^3\\&+64(-1+3x+6x^2-6x^3-5x^4+3x^5)y^4=0,\end{aligned}$$

and

$$\frac{\partial u}{\partial y} = 8(1-x^2)\big[(56x+42x^2+28x^3+46x^4) + 2(-16+8x+23x^2-55x^3-54x^4)y + 3(40-56x-66x^2-28x^3+26x^4)y^2 + 16(-1-2x+2x^2+2x^3-x^4)y^3\big] = 0,$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -0.751907, \\ y_1 \approx -26.2555, \end{cases} \quad \begin{cases} x_2 \approx 1.38642, \\ y_2 \approx 16.9888, \end{cases} \quad \begin{cases} x_3 \approx -1.62503, \\ y_3 \approx 5.7912, \end{cases}$$

$$\begin{cases} x_4 \approx -0.808447, \\ y_4 \approx -0.246107, \end{cases} \quad \begin{cases} x_5 \approx -0.764429, \\ y_5 \approx 0.288228, \end{cases} \quad \begin{cases} x_6 \approx -0.0197963, \\ y_6 \approx -0.0303034. \end{cases}$$

Thus the function u has no critical point in $(0, \sqrt{7}/\sqrt{54}] \times (0, 1)$.

If $7 - 54x^2 < 0$. Then all the real solutions ($x \neq 0, \pm 1$) of the system of equations

$$\begin{aligned} \frac{\partial u}{\partial x} = \; & (-640x+1128x^2+640x^3-1880x^4+510x^5) \\ & +8(56+84x-84x^2+16x^3-140x^4-276x^5)y \\ & +8(8+50x-189x^2+180x^3+275x^4-324x^5)y^2 \\ & +8(-56-212x+84x^2+368x^3+140x^4-156x^5)y^3 \\ & +64(-1+3x+6x^2-6x^3-5x^4+3x^5)y^4 = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial y} = \; & 8(1-x^2)\big[(56x+42x^2+28x^3+46x^4) \\ & +2(-16+8x+9x^2-55x^3+54x^4)y \\ & +3(40-56x-66x^2-28x^3+26x^4)y^2 \\ & +16(-1-2x+2x^2+2x^3-x^4)y^3\big] = 0, \end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -0.756068, \\ y_1 \approx -26.0806, \end{cases} \quad \begin{cases} x_2 \approx 2.57483, \\ y_2 \approx 13.8819, \end{cases} \quad \begin{cases} x_3 \approx -1.40321, \\ y_3 \approx 9.79431, \end{cases}$$

$$\begin{cases} x_4 \approx 0.690931, \\ y_4 \approx 0.735877, \end{cases} \quad \begin{cases} x_5 \approx 0.452718, \\ y_5 \approx 0.871419, \end{cases} \quad \begin{cases} x_6 \approx -0.799775, \\ y_6 \approx -0.478217, \end{cases}$$

$$\begin{cases} x_7 \approx -0.773773, \\ y_7 \approx 0.164858, \end{cases} \quad \begin{cases} x_8 \approx -0.0197346, \\ y_8 \approx -0.0302118. \end{cases}$$

Thus (x_4, y_4) and (x_5, y_5) are the critical points of u in $(\sqrt{7}/\sqrt{54}, 1) \times (0, 1)$ with

$$u(x_4, y_4) \approx 271.2564 < 320, \qquad u(x_5, y_5) \approx 256.9165 < 320.$$

B. Suppose that

$$\left|(1-\zeta_1^2)(10+2|\zeta_2|^2)+21\zeta_1^2\overline{\zeta_2}\right| - 3\left|7\zeta_1^2+4(1-\zeta_1^2)\zeta_2\right| < 0.$$

Then

$$10240 \cdot \left|H_{2,2}(F_f/2)\right| \le v(\zeta_1, |\zeta_2|),$$

where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned} v(x, y) := \ & 85x^6 + 8(1-x^2)(42x^2+47x^3) \\ & +8(1-x^2)(24+56x-24x^2+28x^3+46x^4)y \\ & +8(1-x^2)(8x-42x^2-55x^3+x^2\cdot|7-54x^2|)y^2 \\ & +8(1-x^2)(16-56x-28x^3+26x^4)y^3 \\ & +32(1-x^2)^2(-2x+x^2)y^4. \end{aligned}$$

We show now that $v(x, y) \le 320$ for $(x, y) \in [0, 1] \times [0, 1]$.
I. On the vertices of $[0, 1] \times [0, 1]$, we have

$$v(0,0) = 0, \quad v(0,1) = 320, \quad v(1,0) = v(1,1) = 85.$$

II. On the sides of $[0, 1] \times [0, 1]$, we get

$$\begin{aligned} v(0, y) &= 64(3y+2y^3) \le v(0,1) = 320, \quad y \in (0,1), \\ v(x, 0) &= 85x^6 + 8(1-x^2)(42x^2+47x^3) \le v(x_0, 0) \approx 169.17, \\ & \quad x_0 \approx 0.788876,\ x \in (0,1), \\ v(1, y) &= 85, \quad y \in (0,1), \\ v(x, 1) &= 85x^6 + 8(1-x^2)\left(40-20x^2+68x^4+|7x^2-54x^4|\right) \\ & \quad \le v(0,1) = 320,\ x \in (0,1). \end{aligned}$$

III. It remains to consider the set $(0, 1) \times (0, 1)$.
If $7 - 54x^2 \ge 0$. Then all the real solutions $(x \ne 0, \pm 1)$ of the system of equations

$$\begin{aligned} \frac{\partial v}{\partial x} = \ & (672x+1128x^2-1344x^3-1880x^4+510x^5) \\ & +8(56-96x-84x^2+280x^3-140x^4-276x^5)y \\ & +8(8-70x-189x^2-76x^3+275x^4+324x^5)y^2 \\ & +8(-56-32x+84x^2+104x^3+140x^4-156x^5)y^3 \\ & +8(-8+8x+48x^2-32x^3-40x^4+24x^5)y^4 = 0, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial v}{\partial y} = &\ 8(1-x^2)\big[(24+56x-24x^2+28x^3+46x^4) \\ &+2(8x-35x^2-55x^3-54x^4)y \\ &+3(16-56x-28x^3+26x^4)y^2 \\ &+4(-8x+4x^2+8x^3-4x^4)y^3\big]=0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases}x_1 \approx -2.3661,\\ y_1 \approx -0.367626,\end{cases} \quad \begin{cases}x_2 \approx -0.93242,\\ y_2 \approx 0.473722,\end{cases} \quad \begin{cases}x_3 \approx -0.7798,\\ y_3 \approx -0.329245,\end{cases}$$

$$\begin{cases}x_4 \approx -0.62236,\\ y_4 \approx 0.0596674,\end{cases} \quad \begin{cases}x_5 \approx -0.325247,\\ y_5 \approx 1.55834,\end{cases} \quad \begin{cases}x_6 \approx -0.311599,\\ y_6 \approx -1.54511,\end{cases}$$

$$\begin{cases}x_7 \approx 0.164715,\\ y_7 \approx -0.77519,\end{cases} \quad \begin{cases}x_8 \approx 0.273348,\\ y_8 \approx 0.745703,\end{cases} \quad \begin{cases}x_9 \approx 1.46314,\\ y_9 \approx 5.14397.\end{cases}$$

Thus (x_8, y_8) is the unique critical point of v in $(0, \sqrt{7}/\sqrt{54}] \times (0, 1)$ with

$$v(x_8, y_8) \approx 231.2083 < 320.$$

If $7-54x^2 < 0$. Then all the real solutions $(x \neq 0, \pm 1)$ of the system of equations

$$\begin{aligned}\frac{\partial v}{\partial x} = &\ (672x+1128x^2-1344x^3-1880x^4+510x^5) \\ &+8(56-96x-84x^2+280x^3-140x^4-276x^5)y \\ &+8(8-98x-189x^2+412x^3+275x^4-324x^5)y^2 \\ &+8(-56-32x+84x^2+104x^3+140x^4-156x^5)y^3 \\ &+8(-8+8x+48x^2-32x^3-40x^4+24x^5)y^4=0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial v}{\partial y} = &\ 8(1-x^2)\big[(24+56x-24x^2+28x^3+46x^4) \\ &+2(8x-49x^2-55x^3+54x^4)y \\ &+3(16-56x-28x^3+26x^4)y^2 \\ &+4(-8x+4x^2+8x^3-4x^4)y^3\big]=0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases}x_1 \approx -2.3661,\\ y_1 \approx -0.367626,\end{cases} \quad \begin{cases}x_2 \approx -0.93242,\\ y_2 \approx 0.473722,\end{cases} \quad \begin{cases}x_3 \approx -0.7798,\\ y_3 \approx -0.329245,\end{cases}$$

$$\begin{cases} x_4 \approx -0.62236, \\ y_4 \approx 0.0596674, \end{cases} \quad \begin{cases} x_5 \approx -0.325247, \\ y_5 \approx 1.55834, \end{cases} \quad \begin{cases} x_6 \approx -0.311599, \\ y_6 \approx -1.54511, \end{cases}$$
$$\begin{cases} x_7 \approx 0.164715, \\ y_7 \approx -0.77519, \end{cases} \quad \begin{cases} x_8 \approx 0.273348, \\ y_8 \approx 0.745703, \end{cases} \quad \begin{cases} x_9 \approx 1.46314, \\ y_9 \approx 5.14397. \end{cases}$$

Thus the function v has no critical point in $(\sqrt{7}/\sqrt{54}, 1) \times (0, 1)$.

Summarizing, we see that the bounds obtained in Parts A and B give

$$\left|H_{2,2}(F_f/2)\right| \leq \frac{1}{10240} \cdot 320 = \frac{1}{32}.$$

We finally note that equality in (2.1) holds for the function $f \in \mathcal{F}$ defined by (1.1), and satisfying (2.3) with

$$p(z) := \frac{1+z^2}{1-z^2}, \qquad z \in \mathbb{D},$$

for which $a_2 = a_4 = 0$, $a_3 = 1/2$ and $a_5 = 3/8$. This completes the proof of the theorem. □

We next consider the sharp bounds of $|H_{2,2}(F_f/2)|$ for the class $\mathcal{G}$.

Theorem 2.2 *If $f \in \mathcal{G}$ be of the form (1.1), then*

$$\left|H_{2,2}(F_f/2)\right| \leq \frac{1}{576}. \tag{2.6}$$

The result is sharp for

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1-2z^3}{1-z^3}, \qquad z \in \mathbb{D}, \tag{2.7}$$

that is, $f(z) = z - z^4/12 + \cdots$.

Proof For the function $f \in \mathcal{G}$ given by (1.1), there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re\big(p(z)\big) > 0$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2} - \frac{1}{2}p(z), \qquad z \in \mathbb{D}. \tag{2.8}$$

By elementary calculations, we have

$$\begin{aligned} &a_2 = -\frac{1}{4}c_1, \qquad a_3 = \frac{1}{24}\big(-2c_2 + c_1^2\big), \qquad a_4 = \frac{1}{192}\big(-8c_3 + 6c_1c_2 - c_1^3\big), \\ &a_5 = \frac{1}{1920}\big(-48c_4 + 32c_1c_3 - 12c_1^2c_2 + 12c_2^2 + c_1^4\big). \end{aligned} \tag{2.9}$$

Thus, (1.7) and (2.9) give

$$17694720 \cdot H_{2,2}(F_f/2) = 1152\big(8c_2 - c_1^2\big)c_4 + 1536c_1c_2c_3 - 7680c_3^2 + 288c_1^3c_3 - 288c_1^2c_2^2 - 1024c_2^3 - c_1^6. \tag{2.10}$$

Since the class $\mathcal{G}$ and $\big|H_{2,2}(F_f/2)\big|$ are rotationally invariant, we may assume that $c_1 \in [0, 2]$. Thus, in view of (1.9) we assume that $\zeta_1 \in [0, 1]$. Using (2.10) and (1.9)-(1.12), we obtain

$$\begin{aligned}276480 \cdot H_{2,2}(F_f/2) = &\big[15\zeta_1^6 + 144(1-\zeta_1^2)\zeta_1^4\zeta_2 + 168(1-\zeta_1^2)(1-2\zeta_1^2)\zeta_1^2\zeta_2^2 \\ &+ 16(1-\zeta_1^2)(28 - 20\zeta_1^2 + 19\zeta_1^4)\zeta_2^3 + 96(1-\zeta_1^2)^2\zeta_1^2\zeta_2^4\big] \\ &+ 24(1-\zeta_1^2)(1-|\zeta_2|^2)\big[7\zeta_1^3 - 12(2+\zeta_1^2)\zeta_1\zeta_2 \\ &- 8(1-\zeta_1^2)\zeta_1\zeta_2^2\big]\zeta_3 - 48(1-\zeta_1^2)(1-|\zeta_2|^2) \\ &\big[(1-\zeta_1^2)(10+2|\zeta_2|^2) + 9\zeta_1^2\overline{\zeta_2}\big]\zeta_3^2 + 144(1-\zeta_1^2) \\ &(1-|\zeta_2|^2)(1-|\zeta_3|^2)\big[3\zeta_1^2 + 4(1-\zeta_1^2)\zeta_2\big]\zeta_4,\end{aligned}$$

for some $\zeta_1 \in [0, 1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$. Since $|\zeta_4| \le 1$, we have

$$\begin{aligned}&276480 \cdot \big|H_{2,2}(F_f/2)\big| \\ &\le \big|15\zeta_1^6 + 144(1-\zeta_1^2)\zeta_1^4\zeta_2 + 168(1-\zeta_1^2)(1-2\zeta_1^2)\zeta_1^2\zeta_2^2 \\ &+ 16(1-\zeta_1^2)(28-20\zeta_1^2+19\zeta_1^4)\zeta_2^3 + 96(1-\zeta_1^2)^2\zeta_1^2\zeta_2^4\big| \\ &+ 24(1-\zeta_1^2)(1-|\zeta_2|^2)\big|7\zeta_1^3 - 12(2+\zeta_1^2)\zeta_1\zeta_2 - 8(1-\zeta_1^2)\zeta_1\zeta_2^2\big| \cdot |\zeta_3| \\ &+ 48(1-\zeta_1^2)(1-|\zeta_2|^2)\big[\big|(1-\zeta_1^2)(10+2|\zeta_2|^2) + 9\zeta_1^2\overline{\zeta_2}\big| - 3\big|3\zeta_1^2 \\ &+ 4(1-\zeta_1^2)\zeta_2\big|\big] \cdot |\zeta_3|^2 + 144(1-\zeta_1^2)(1-|\zeta_2|^2)\big|3\zeta_1^2 + 4(1-\zeta_1^2)\zeta_2\big|.\end{aligned}$$

A. Suppose that

$$\big|(1-\zeta_1^2)(10+2|\zeta_2|^2) + 9\zeta_1^2\overline{\zeta_2}\big| - 3\big|3\zeta_1^2 + 4(1-\zeta_1^2)\zeta_2\big| \ge 0.$$

Then

$$276480 \cdot \big|H_{2,2}(F_f/2)\big| \le \varphi(\zeta_1, |\zeta_2|),$$

where $\varphi:\ \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned}\varphi(x, y) := &15x^6 + 24(1-x^2)(20 - 20x^2 + 7x^3) \\ &+ 144(1-x^2)(4x + 3x^2 + 2x^3 + x^4)y \\ &+ 24(1-x^2)(-16 + 8x + 16x^2 - 15x^3 + 7x^2 \cdot |1-2x^2|)y^2 \\ &+ 16(1-x^2)(28 - 36x - 47x^2 - 18x^3 + 19x^4)y^3 \\ &+ 96(1-x^2)^2(-1 - 2x + x^2)y^4.\end{aligned}$$

We show that $\varphi(x, y) \le 480$ for $(x, y) \in [0, 1] \times [0, 1]$.

I. On the vertices of $[0, 1] \times [0, 1]$, we have

$$\varphi(0,0) = 480, \quad \varphi(0,1) = 448, \quad \varphi(1,0) = \varphi(1,1) = 15.$$

II. On the sides of $[0, 1] \times [0, 1]$, we get

$$\begin{aligned}
&\varphi(0, y) = 32(15 - 12y^2 + 14y^3 - 3y^4) \le \varphi(0,0) = 480, \quad y \in (0,1),\\
&\varphi(x, 0) = 15x^6 + 24(1 - x^2)(20 - 20x^2 + 7x^3) \le \varphi(0,0) = 480, \quad x \in (0,1),\\
&\varphi(1, y) = 15, \quad y \in (0,1),\\
&\varphi(x, 1) = 15x^6 + 8(1 - x^2)(56 - 28x^2 + 44x^4 + 21|x^2 - 2x^4|)\\
&\qquad \le \varphi(0,1) = 448, \ x \in (0,1).
\end{aligned}$$

III. It remains to consider the set $(0, 1) \times (0, 1)$.

If $1 - 2x^2 \ge 0$. Then all the real solutions ($x \ne 0, \pm 1$) of the system of equations

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} =& (-1920x + 504x^2 + 1920x^3 - 840x^4 + 90x^5)\\
&+ 144(4 + 6x - 6x^2 - 8x^3 - 10x^4 - 6x^5)y\\
&+ 24(8 + 78x - 69x^2 - 148x^3 + 75x^4 + 84x^5)y^2\\
&+ 16(-36 - 150x + 54x^2 + 264x^3 + 90x^4 - 114x^5)y^3\\
&+ 192(-1 + 3x + 6x^2 - 6x^3 - 5x^4 + 3x^5)y^4 = 0,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \varphi}{\partial y} =& 48(1 - x^2)\big[3(4x + 3x^2 + 2x^3 + x^4)\\
&+ (-16 + 8x + 23x^2 - 15x^3 - 14x^4)y\\
&+ (28 - 36x - 47x^2 - 18x^3 + 19x^4)y^2\\
&+ 8(-1 - 2x + 2x^2 + 2x^3 - x^4)y^3\big] = 0,
\end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -55.9185, \\ y_1 \approx 0.774424, \end{cases} \quad \begin{cases} x_2 \approx -31.3008, \\ y_2 \approx 0.399471, \end{cases} \quad \begin{cases} x_3 \approx -0.761847, \\ y_3 \approx -11.6168, \end{cases}$$

$$\begin{cases} x_4 \approx 1.45524, \\ y_4 \approx 6.40123, \end{cases} \quad \begin{cases} x_5 \approx -1.62457, \\ y_5 \approx 2.67252, \end{cases} \quad \begin{cases} x_6 \approx 0.116028, \\ y_6 \approx 0.909375, \end{cases}$$

$$\begin{cases} x_7 \approx -0.94848, \\ y_7 \approx -0.331374, \end{cases} \quad \begin{cases} x_8 \approx -0.894531, \\ y_8 \approx 0.380566. \end{cases}$$

Thus (x_6, y_6) is the unique critical point of φ in $(0, \sqrt{2}/2] \times (0, 1)$ with

$$\varphi(x_6, y_6) \approx 383.3639 < 480.$$

If $1 - 2x^2 < 0$. Then all the real solutions ($x \neq 0, \pm 1$) of the system of equations

$$\begin{aligned}\frac{\partial \varphi}{\partial x} =& (-1920x + 504x^2 + 1920x^3 - 840x^4 + 90x^5)\\ &+ 144(4 + 6x - 6x^2 - 8x^3 - 10x^4 - 6x^5)y\\ &+ 24(8 + 50x - 69x^2 + 20x^3 + 75x^4 - 84x^5)y^2\\ &+ 16(-36 - 150x + 54x^2 + 264x^3 + 90x^4 - 114x^5)y^3\\ &+ 192(-1 + 3x + 6x^2 - 6x^3 - 5x^4 + 3x^5)y^4 = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \varphi}{\partial y} =& 48(1 - x^2)\big[3(4x + 3x^2 + 2x^3 + x^4)\\ &+ (-16 + 8x + 9x^2 - 15x^3 + 14x^4)y\\ &+ (28 - 36x - 47x^2 - 18x^3 + 19x^4)y^2\\ &+ 8(-1 - 2x + 2x^2 + 2x^3 - x^4)y^3\big] = 0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -0.766314, \\ y_1 \approx -11.6419, \end{cases} \quad \begin{cases} x_2 \approx 2.70716, \\ y_2 \approx 7.14005, \end{cases} \quad \begin{cases} x_3 \approx -1.36535, \\ y_3 \approx 4.72423, \end{cases}$$
$$\begin{cases} x_4 \approx -0.957006, \\ y_4 \approx -0.450247, \end{cases} \quad \begin{cases} x_5 \approx 0.10236, \\ y_5 \approx 0.87329, \end{cases} \quad \begin{cases} x_6 \approx -0.899079, \\ y_6 \approx 0.308825. \end{cases}$$

Thus the function φ has no critical point in $(\sqrt{2}/2, 1) \times (0, 1)$.

B. Suppose that

$$\left|(1 - \zeta_1^2)(10 + 2|\zeta_2|^2) + 9\zeta_1^2\overline{\zeta_2}\right| - 3\left|3\zeta_1^2 + 4(1 - \zeta_1^2)\zeta_2\right| < 0.$$

Then

$$276480 \cdot \left|H_{2,2}(F_f/2)\right| \leq \psi(\zeta_1, |\zeta_2|),$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned}\psi(x, y) := {}& 15x^6 + 24(1-x^2)(18x^2+7x^3)\\ &+ 144(1-x^2)(4+4x-4x^2+2x^3+x^4)y\\ &+ 24(1-x^2)(8x-18x^2-15x^3+7x^2\cdot|1-2x^2|)y^2\\ &+ 16(1-x^2)(-8-36x+16x^2-18x^3+19x^4)y^3\\ &+ 96(1-x^2)^2(-2x+x^2)y^4.\end{aligned}$$

We show that $\psi(x, y) \le 448$ for $(x, y) \in [0, 1] \times [0, 1]$.
I. On the vertices of $[0, 1] \times [0, 1]$, we have

$$\psi(0,0) = 0, \quad \psi(0,1) = 448, \quad \psi(1,0) = \psi(1,1) = 15.$$

II. On the sides of $[0, 1] \times [0, 1]$, we get

$$\begin{aligned}\psi(0, y) &= 64(9y-2y^3) \le \psi(0,1) = 448, \quad y \in (0,1),\\ \psi(x, 0) &= 15x^6 + 24(1-x^2)(18x^2+7x^3) \le \psi(x_0, 0) \approx 140.341,\\ &\quad x_0 \approx 0.733049,\ x \in (0,1),\\ \psi(1, y) &= 15, \quad y \in (0,1),\\ \psi(x, 1) &= 15x^6+8(1-x^2)(56-28x^2+44x^4+21|x^2-2x^4|) \le 448, \quad x \in (0,1).\end{aligned}$$

III. It remains to consider the set $(0, 1) \times (0, 1)$.
If $1 - 2x^2 \ge 0$. Then all the real solutions ($x \ne 0, \pm 1$) of the system of equations

$$\begin{aligned}\frac{\partial \psi}{\partial x} = {}& (864x + 504x^2 - 1728x^3 - 840x^4 + 90x^5)\\ &+ 144(4-16x-6x^2+20x^3-10x^4-6x^5)y\\ &+ 24(8-22x-69x^2-12x^3+75x^4+84x^5)y^2\\ &+ 16(-36+48x+54x^2+12x^3+90x^4-114x^5)y^3\\ &+ 192(-1+x+6x^2-4x^3-5x^4+3x^5)y^4 = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \psi}{\partial y} = {}& 48(1-x^2)\big[3(4+4x-4x^2+2x^3+x^4)\\ &+ (8x-11x^2-15x^3-14x^4)y\\ &+ (-8-36x+16x^2-18x^3+19x^4)y^2\\ &+ 8(-2x+x^2+2x^3-x^4)y^3\big] = 0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -55.4838, \\ y_1 \approx 0.776748, \end{cases} \quad \begin{cases} x_2 \approx -29.6139, \\ y_2 \approx 0.387681, \end{cases} \quad \begin{cases} x_3 \approx 0.321635, \\ y_3 \approx -4.5939, \end{cases}$$
$$\begin{cases} x_4 \approx 1.55353, \\ y_4 \approx 3.22173, \end{cases} \quad \begin{cases} x_5 \approx -1.73153, \\ y_5 \approx 3.21781, \end{cases} \quad \begin{cases} x_6 \approx -1.90529, \\ y_6 \approx -0.222475, \end{cases}$$
$$\begin{cases} x_7 \approx -0.761626, \\ y_7 \approx -0.266647, \end{cases} \quad \begin{cases} x_8 \approx -0.570421, \\ y_8 \approx 0.104924, \end{cases} \quad \begin{cases} x_9 \approx -0.925824, \\ y_9 \approx 0.541028. \end{cases}$$

Thus the function ψ has no critical point in $(0, \sqrt{2}/2] \times (0, 1)$.

If $1 - 2x^2 < 0$. Then all the real solutions ($x \neq 0, \pm 1$) of the system of equations

$$\begin{aligned} \frac{\partial \psi}{\partial x} =& (864x + 504x^2 - 1728x^3 - 840x^4 + 90x^5) \\ &+ 144(4 - 16x - 6x^2 + 20x^3 - 10x^4 - 6x^5)y \\ &+ 24(8 - 50x - 69x^2 + 156x^3 + 75x^4 - 84x^5)y^2 \\ &+ 16(-36 + 48x + 54x^2 + 12x^3 + 90x^4 - 114x^5)y^3 \\ &+ 192(-1 + x + 6x^2 - 4x^3 - 5x^4 + 3x^5)y^4 = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \psi}{\partial y} =& 48(1 - x^2)\big[3(4 + 4x - 4x^2 + 2x^3 + x^4) \\ &+ (8x - 25x^2 - 15x^3 + 14x^4)y \\ &+ (-8 - 36x + 16x^2 - 18x^3 + 19x^4)y^2 \\ &+ 8(-2x + x^2 + 2x^3 - x^4)y^3\big] = 0, \end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx 2.62485, \\ y_1 \approx 7.99341, \end{cases} \quad \begin{cases} x_2 \approx -1.41281, \\ y_2 \approx 5.38615, \end{cases} \quad \begin{cases} x_3 \approx 0.337397, \\ y_3 \approx -4.50435, \end{cases}$$
$$\begin{cases} x_4 \approx -3.00552, \\ y_4 \approx -0.50500, \end{cases} \quad \begin{cases} x_5 \approx -0.772503, \\ y_5 \approx -0.286793, \end{cases} \quad \begin{cases} x_6 \approx -0.56427, \\ y_6 \approx 0.109389, \end{cases}$$
$$\begin{cases} x_7 \approx -0.980877, \\ y_7 \approx 0.459454. \end{cases}$$

Thus the function ψ has no critical point in $(\sqrt{2}/2, 1) \times (0, 1)$.

Summarizing, we see that the bounds obtained in Parts A and B give

$$\left|H_{2,2}(F_f/2)\right| \le \frac{1}{276480} \cdot 480 = \frac{1}{576}.$$

We finally note that equality in (2.6) holds for the function $f \in \mathcal{G}$ defined by (1.1), and satisfying (2.8) with

$$p(z) := \frac{1+z^3}{1-z^3}, \qquad z \in \mathbb{D},$$

for which $a_2 = a_3 = a_5 = 0$ and $a_4 = -1/12$. This completes the proof of the theorem. □

We finally prove the sharp bounds of $|H_{3,1}(f)|$ for functions $f \in \mathcal{G}$.

Theorem 2.3 *If $f \in \mathcal{G}$ be of the form (1.1), then*

$$\left|H_{3,1}(f)\right| \le \frac{19}{2160}. \tag{2.11}$$

The result is sharp for

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1-2z^2}{1-z^2}, \qquad z \in \mathbb{D}, \tag{2.12}$$

that is, $f(z) = z - z^3/6 - z^5/40 + \cdots$.

Proof Let the function $f \in \mathcal{G}$ given by (1.1). Thus, (1.4) and (2.9) give

$$\begin{aligned} 552960 \cdot H_{3,1}(f) = & 288\left(4c_2 + c_1^2\right)c_4 - 288c_1c_2c_3 - 960c_3^2 + 48c_1^3c_3 \\ & - 84c_1^2c_2^2 + 32c_2^3 - 12c_1^4c_2 - c_1^6. \end{aligned} \tag{2.13}$$

Since the class $\mathcal{G}$ and $\left|H_{3,1}(f)\right|$ are rotationally invariant, we may assume that $c_1 \in [0, 2]$. Thus, in view of (1.9) we assume that $\zeta_1 \in [0, 1]$. Using (2.13) and (1.9)-(1.12), we obtain

$$\begin{aligned} 8640 \cdot H_{3,1}(f) = & (1-\zeta_1^2)\big\{\big[36\zeta_1^4\zeta_2 + 3(1-25\zeta_1^2)\zeta_1^2\zeta_2^2 \\ & + 4(19 - 5\zeta_1^2 + 13\zeta_1^4)\zeta_2^3 + 12(1-\zeta_1^2)\zeta_1^2\zeta_2^4\big] \\ & + 12(1-|\zeta_2|^2)\big[6\zeta_1^3 - (11 + 7\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2\big]\zeta_3 \\ & - 12(1-|\zeta_2|^2)\big[(1-\zeta_1^2)(5 + |\zeta_2|^2) + 9\zeta_1^2\overline{\zeta_2}\big]\zeta_3^2 \\ & + 36(1-|\zeta_2|^2)(1-|\zeta_3|^2)\big[3\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2\big]\zeta_4\big\}, \end{aligned}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$. Since $|\zeta_4| \leq 1$, we have

$$\begin{aligned}8640 \cdot \left|H_{3,1}(f)\right| \leq (1-\zeta_1^2)\big\{\big|36\zeta_1^4\zeta_2 + 3(1-25\zeta_1^2)\zeta_1^2\zeta_2^2 + 4(19-5\zeta_1^2+13\zeta_1^4)\zeta_2^3 \\ + 12(1-\zeta_1^2)\zeta_1^2\zeta_2^4\big| + 12(1-|\zeta_2|^2)\big|6\zeta_1^3 - (11+7\zeta_1^2)\zeta_1\zeta_2 \\ - 2(1-\zeta_1^2)\zeta_1\zeta_2^2\big| \cdot |\zeta_3| + 12(1-|\zeta_2|^2)\big[\big|(1-\zeta_1^2)(5+|\zeta_2|^2) \\ + 9\zeta_1^2\overline{\zeta_2}\big| - 3\big|3\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2\big|\big] \cdot |\zeta_3|^2 \\ + 36(1-|\zeta_2|^2)\big|3\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2\big|\big\}.\end{aligned}$$

A. Suppose that

$$\left|(1-\zeta_1^2)(5+|\zeta_2|^2) + 9\zeta_1^2\overline{\zeta_2}\right| - 3\left|3\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2\right| \geq 0.$$

Then

$$8640 \cdot \left|H_{3,1}(f)\right| \leq h(\zeta_1, |\zeta_2|),$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned}h(x,y) := &\, 12(1-x^2)(5-5x^2+6x^3) \\ &+ 12(1-x^2)(11x+9x^2+7x^3+3x^4)y \\ &+ 3(1-x^2)(-16+8x+16x^2-32x^3+x^2 \cdot |1-25x^2|)y^2 \\ &+ 4(1-x^2)(19-33x-32x^2-21x^3+13x^4)y^3 \\ &+ 12(1-x^2)^2(-1-2x+x^2)y^4.\end{aligned}$$

We show that $h(x,y) \leq 76$ for $(x,y) \in [0,1] \times [0,1]$.

I. On the vertices of $[0,1] \times [0,1]$, we have

$$h(0,0) = 60, \quad h(0,1) = 76, \quad h(1,0) = h(1,1) = 0.$$

II. On the sides of $[0,1] \times [0,1]$, we get

$$\begin{aligned}&h(0,y) = 76y^3 + 12(1-y^2)(5+y^2) \leq h(0,1) = 76, \quad y \in (0,1), \\ &h(x,0) = 12(1-x^2)(5-5x^2+6x^3) \leq h(0,0) = 60, \quad x \in (0,1), \\ &h(1,y) = 0, \quad y \in (0,1), \\ &h(x,1) = (1-x^2)(76-8x^2+76x^4+3|x^2-25x^4|) \leq h(0,1) = 76, \quad x \in (0,1).\end{aligned}$$

III. It remains to consider the set $(0,1) \times (0,1)$.

If $1-25x^2 \geq 0$. Then all the real solutions ($x \neq 0, \pm 1$) of the system of equations

$$\begin{aligned}\frac{\partial h}{\partial x} =& 12(-20x + 18x^2 + 20x^3 - 30x^4)\\ &+ 12(11 + 18x - 12x^2 - 24x^3 - 35x^4 - 18x^5)y\\ &+ 3(8 + 66x - 120x^2 - 168x^3 + 160x^4 + 150x^5)y^2\\ &+ 4(-33 - 102x + 36x^2 + 180x^3 + 105x^4 - 78x^5)y^3\\ &+ 24(-1 + 3x + 6x^2 - 6x^3 - 5x^4 + 3x^5)y^4 = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial h}{\partial y} =& 6(1 - x^2)\big[2(11x + 9x^2 + 7x^3 + 3x^4)\\ &+ (-16 + 8x + 17x^2 - 32x^3 - 25x^4)y\\ &+ 2(19 - 33x - 32x^2 - 21x^3 + 13x^4)y^2\\ &+ 8(-1 - 2x + 2x^2 + 2x^3 - x^4)y^3\big] = 0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -9502.13,\\ y_1 \approx 0.845903,\end{cases} \quad \begin{cases} x_2 \approx -0.731928,\\ y_2 \approx -20.5476,\end{cases} \quad \begin{cases} x_3 \approx 1.35957,\\ y_3 \approx 15.3823,\end{cases}$$

$$\begin{cases} x_4 \approx -14.6979,\\ y_4 \approx 0.668684,\end{cases} \quad \begin{cases} x_5 \approx -13.3345,\\ y_5 \approx 0.419256,\end{cases} \quad \begin{cases} x_6 \approx -1.63900,\\ y_6 \approx 5.10765,\end{cases}$$

$$\begin{cases} x_7 \approx 0.276393,\\ y_7 \approx 0.848735,\end{cases} \quad \begin{cases} x_8 \approx 0.429674,\\ y_8 \approx 0.674311,\end{cases} \quad \begin{cases} x_9 \approx -0.905519,\\ y_9 \approx -0.339299,\end{cases}$$

$$\begin{cases} x_{10} \approx -0.840874,\\ y_{10} \approx 0.380896,\end{cases} \quad \begin{cases} x_{11} \approx 0.0345448,\\ y_{11} \approx 0.0570942.\end{cases}$$

Thus (x_{11}, y_{11}) is the unique critical point of h in $(0, 1/5] \times (0, 1)$ with

$$h(x_{11}, y_{11}) \approx 60.0155 < 76.$$

If $1-25x^2 < 0$. Then all the real solutions ($x \neq 0, \pm 1$) of the system of equations

$$\begin{aligned}\frac{\partial h}{\partial x} =& 12(-20x + 18x^2 + 20x^3 - 30x^4)\\ &+ 12(11 + 18x - 12x^2 - 24x^3 - 35x^4 - 18x^5)y\\ &+ 3(8 + 62x - 120x^2 + 40x^3 + 160x^4 - 150x^5)y^2\\ &+ 4(-33 - 102x + 36x^2 + 180x^3 + 105x^4 - 78x^5)y^3\\ &+ 24(-1 + 3x + 6x^2 - 6x^3 - 5x^4 + 3x^5)y^4 = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial h}{\partial y} =& 6(1-x^2)\big[2(11x+9x^2+7x^3+3x^4)\\&+(-16+8x+15x^2-32x^3+25x^4)y\\&+2(19-33x-32x^2-21x^3+13x^4)y^2\\&+8(-1-2x+2x^2+2x^3-x^4)y^3\big]=0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases}x_1\approx -0.735039,\\ y_1\approx -20.4190,\end{cases}\quad \begin{cases}x_2\approx 1.54150,\\ y_2\approx 11.5368,\end{cases}\quad \begin{cases}x_3\approx -1.44859,\\ y_3\approx 7.65791,\end{cases}$$

$$\begin{cases}x_4\approx -0.926545,\\ y_4\approx -0.564185,\end{cases}\quad \begin{cases}x_5\approx 0.26956,\\ y_5\approx 0.879153,\end{cases}\quad \begin{cases}x_6\approx 0.524719,\\ y_6\approx 0.683744,\end{cases}$$

$$\begin{cases}x_7\approx -0.85181,\\ y_7\approx 0.256192,\end{cases}\quad \begin{cases}x_8\approx 0.0345845,\\ y_8\approx 0.0571628.\end{cases}$$

Thus (x_5, y_5) and (x_6, y_6) are the critical points of h in $(1/5, 1)\times(0, 1)$ with

$$h(x_5, y_5)\approx 71.02494<76,\qquad h(x_6, y_6)\approx 73.94978<76.$$

B. Suppose that

$$\left|(1-\zeta_1^2)(5+|\zeta_2|^2)+9\zeta_1^2\overline{\zeta_2}\right|-3\left|3\zeta_1^2+2(1-\zeta_1^2)\zeta_2\right|<0.$$

Then

$$8640\cdot\left|H_{3,1}(f)\right|\le g(\zeta_1, |\zeta_2|),$$

where $g:\ \mathbb{R}^2\to\mathbb{R}$ is defined by

$$\begin{aligned}g(x, y) :=& 36(1-x^2)(3x^2+2x^3)\\&+12(1-x^2)(6+11x-6x^2+7x^3+3x^4)y\\&+3(1-x^2)(8x-36x^2-32x^3+x^2\cdot|1-25x^2|)y^2\\&+4(1-x^2)(1-33x+13x^2-21x^3+13x^4)y^3\\&+12(1-x^2)^2(-2x+x^2)y^4.\end{aligned}$$

We show that $g(x, y)\le 76$ for $(x, y)\in[0, 1]\times[0, 1]$.

I. On the vertices of $[0, 1]\times[0, 1]$, we have

$$g(0, 0)=0,\quad g(0, 1)=76,\quad g(1, 0)=g(1, 1)=0.$$

II. On the sides of $[0, 1] \times [0, 1]$, we get

$$
\begin{aligned}
&g(0, y) = 72y + 4y^3 \le g(0, 1) = 76, \quad y \in (0, 1),\\
&g(x, 0) = 36x^2(1 - x^2)(3 + 2x) \le g(x_0, 0) \approx 39.9705, \quad x_0 \approx 0.733044,\ x \in (0, 1),\\
&g(1, y) = 0, \quad y \in (0, 1),\\
&g(x, 1) = (1 - x^2)(76 - 8x^2 + 76x^4 + 3|x^2 - 25x^4|) \le g(0, 1) = 76, \quad x \in (0, 1).
\end{aligned}
$$

III. It remains to consider the set $(0, 1) \times (0, 1)$.

If $1 - 25x^2 \ge 0$. Then all the real solutions ($x \neq 0, \pm 1$) of the system of equations

$$
\begin{aligned}
\frac{\partial g}{\partial x} = &\,36(6x + 6x^2 - 12x^3 - 10x^4)\\
&+ 12(11 - 24x - 12x^2 + 36x^3 - 35x^4 - 18x^5)y\\
&+ 3(8 - 70x - 120x^2 + 40x^3 + 160x^4 + 150x^5)y^2\\
&+ 4(-33 + 24x + 36x^2 + 105x^4 - 78x^5)y^3\\
&+ 24(-1 + x + 6x^2 - 4x^3 - 5x^4 + 3x^5)y^4 = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial g}{\partial y} = &\,6(1 - x^2)\big[2(6 + 11x - 6x^2 + 7x^3 + 3x^4)\\
&+ (8x - 35x^2 - 32x^3 - 25x^4)y\\
&+ 2(1 - 33x + 13x^2 - 21x^3 + 13x^4)y^2\\
&+ 8(-2x + x^2 + 2x^3 - x^4)y^3\big] = 0,
\end{aligned}
$$

by a numerical computation are the following

$$
\begin{cases} x_1 \approx -9589.92,\\ y_1 \approx 1.38296, \end{cases}
\quad
\begin{cases} x_2 \approx 1.41932,\\ y_2 \approx 11.2938, \end{cases}
\quad
\begin{cases} x_3 \approx -14.5113,\\ y_3 \approx 0.695679, \end{cases}
$$

$$
\begin{cases} x_4 \approx -12.4904,\\ y_4 \approx 0.366752, \end{cases}
\quad
\begin{cases} x_5 \approx -0.0961383,\\ y_5 \approx -5.70471, \end{cases}
\quad
\begin{cases} x_6 \approx -1.68318,\\ y_6 \approx 6.24783, \end{cases}
$$

$$
\begin{cases} x_7 \approx -1.51857,\\ y_7 \approx -0.274771, \end{cases}
\quad
\begin{cases} x_8 \approx 0.380253,\\ y_8 \approx 0.809067, \end{cases}
\quad
\begin{cases} x_9 \approx 0.539943,\\ y_9 \approx 0.658839, \end{cases}
$$

$$
\begin{cases} x_{10} \approx 0.776016,\\ y_{10} \approx -1.50891, \end{cases}
\quad
\begin{cases} x_{11} \approx 0.572503,\\ y_{11} \approx -1.28199, \end{cases}
\quad
\begin{cases} x_{12} \approx -0.699973,\\ y_{12} \approx -0.30267, \end{cases}
$$

$$
\begin{cases} x_{13} \approx -0.409285,\\ y_{13} \approx 0.170843, \end{cases}
\quad
\begin{cases} x_{14} \approx -0.861525,\\ y_{14} \approx 0.540418. \end{cases}
$$

Thus the function g has no critical point in $(0, 1/5] \times (0, 1)$.

If $1-25x^2<0$. Then all the real solutions ($x\neq 0, \pm 1$) of the system of equations

$$\begin{aligned}\frac{\partial g}{\partial x} =& 36(6x+6x^2-12x^3-10x^4)\\ &+12(11-24x-12x^2+36x^3-35x^4-18x^5)y\\ &+3(8-74x-120x^2+248x^3+160x^4-150x^5)y^2\\ &+4(-33+24x+36x^2+105x^4-78x^5)y^3\\ &+24(-1+x+6x^2-4x^3-5x^4+3x^5)y^4=0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial y} =& 6(1-x^2)\big[2(6+11x-6x^2+7x^3+3x^4)\\ &+(8x-37x^2-32x^3+25x^4)y\\ &+2(1-33x+13x^2-21x^3+13x^4)y^2\\ &+8(-2x+x^2+2x^3-x^4)y^3\big]=0,\end{aligned}$$

by a numerical computation are the following

$$\begin{cases}x_1\approx 2.27466,\\ y_1\approx 9.76748,\end{cases}\quad \begin{cases}x_2\approx -1.49035,\\ y_2\approx 8.66161,\end{cases}\quad \begin{cases}x_3\approx -0.0959782,\\ y_3\approx -5.70831,\end{cases}$$

$$\begin{cases}x_4\approx -2.76974,\\ y_4\approx -0.612726,\end{cases}\quad \begin{cases}x_5\approx 0.328097,\\ y_5\approx 0.878105,\end{cases}\quad \begin{cases}x_6\approx 0.605801,\\ y_6\approx 0.667028,\end{cases}$$

$$\begin{cases}x_7\approx -0.723959,\\ y_7\approx -0.372024,\end{cases}\quad \begin{cases}x_8\approx -0.91188,\\ y_8\approx 0.413182,\end{cases}\quad \begin{cases}x_9\approx -0.414826,\\ y_9\approx 0.173701.\end{cases}$$

Thus (x_5, y_5) is the unique critical point of g in $(1/5, 1)\times(0, 1)$ with

$$g(x_5, y_5)\approx 70.12399<76, \qquad g(x_6, y_6)\approx 73.93581<76.$$

Summarizing, we see that the bounds obtained in Parts A and B give

$$\left|H_{3,1}(f)\right|\leq \frac{1}{8640}\cdot 76=\frac{19}{2160}.$$

We finally note that equality in (2.6) holds for the function $f\in\mathcal{G}$ defined by (1.1), and satisfying (2.8) with

$$p(z):=\frac{1+z^2}{1-z^2}, \qquad z\in\mathbb{D},$$

for which $a_2=a_4=0$, $a_3=-1/6$ and $a_5=-1/40$. This completes the proof of the theorem. □

Acknowledgements The present investigation was supported by the *Foundation of Educational Committee of Hunan Province* under Grant no. 18B388 of the P. R. China. The authors would like to thank the referees for their valuable comments and suggestions, which was essential to improve the quality of this paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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