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Stability and Optimal Decay for the 3D Anisotropic MHD Equations

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Abstract
This paper focuses on the stability and decay rates of solutions to the three dimensional anisotropic magnetohydrodynamic equations with horizontal velocity dissipation and magnetic damping phenomenon. By fully exploiting the structure of the system, the energy methods and the method of bootstrapping argument, we prove the global stability of solutions to this system with initial data small in $H^3(\mathbb{R}^3)$. Furthermore, we make use of the integral representation approach to obtain the optimal decay rates of these global solutions and their derivatives. This result along with its proof offers an effective approach to the large-time behavior on partially dissipated systems and reveals the stabilizing phenomenon exhibited by electrically conducting fluids.

Keywords Partial dissipation · Stability · Decay rates · Anisotropic equations

Mathematics Subject Classification 35B35 · 35B40 · 35Q35 · 76D03 · 76E25

1 Introduction

The MHD equations govern the motion of electrically conducting fluids in the presence of a magnetic field such as plasmas, liquid metals and electrolytes (see e.g., [1–5]). In this paper, we consider the stability and large-time behavior of solutions to the

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following 3D incompressible MHD equations with anisotropic dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta_h u + B \cdot \nabla B, \ x \in \mathbb{R}^3, t > 0, \\ \partial_t B + u \cdot \nabla B + \eta B = B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x,0) = u_0(x), B(x,0) = B_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)^{\mathrm{T}}$, $B = (B_1, B_2, B_3)^{\mathrm{T}}$ and P represent the velocity field of the fluid, the magnetic field and the scalar pressure, respectively. $\nu > 0$ represents the kinematic viscosity and $\eta > 0$ the magnetic diffusivity. Here $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ is the Laplace operator involving the horizontal direction and we shall also write $\nabla_h = (\partial_1, \partial_2)$. The partial dissipative MHD equations arise in the modeling of reconnecting plasmas (see e.g., [6, 7]). In fact, in certain cases and under suitable scaling, certain components of the dissipation can become small and be ignored, such as the vertical dissipation is negligible as compared to the horizontal dissipation (see e.g., [8, 9]).

There are considerable number of scholars devoted their efforts to the global well-posedness problems of 3D MHD equations, and significant progress has been made (see e.g., [10–21]). The pioneering work of Lin and Zhang [15] devoted to the small data global well-posedness and stability problems on partially dissipated MHD systems. Chen et al. [14] established the global stability with only velocity dissipation or only magnetic diffusion in the periodic domain, when the initial magnetic field is close to a background magnetic field satisfiying the Diophantine condition. In addition, Wu and Zhu [11] solved the global stability for the 3D MHD equations with only horizontal velocity dissipation and vertical magnetic diffusion. Lin et al. [10] established the global well-posedness of a special 3D MHD system near a background magnetic field. Yang et al. [18] proved global regularity for the 3D MHD equations with fractional partial dissipation. Jiu et al. [21] focused on the unique weak solutions of the non-resisitive magnetohydrodynamic equations with fractional dissipation. The stability and large-time behavior problems on the MHD equations have recently attracted considerable interests at the same time. It should be mentioned that Schonbek et al. [22] proved the optimal decay rates of the full dissipative systems. Shang and Zhai [23] focused on the stability problem and large time behavior of solutions with horizontal dissipation. More recently, Zheng and Li [24] focused a global solution when the initial data is small in $H^3(\mathbb{R}^3)$, and obtained optimal decay rates. Lin et al. [25] proved the stability and large-time behavior with velocity dissipation in only one direction and horizontal magnetic diffusion. Lai et al. [20] established the stability and sharp decay estimates for 3D MHD equations with only vertical dissipation near a background magnetic field. This list is by no means exhaustive.

The goal here is to establish the stability and large-time behavior near a background magnetic field. We investigate small perturbations of the system (1.1) around the equilibrium state $(0, e_3)$. Thus, We can set $b = B - e_3$, yields

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta_h u + b \cdot \nabla b + \partial_3 b, \ x \in \mathbb{R}^3, t > 0, \\
\partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_3 u, \\
\nabla \cdot u = \nabla \cdot b = 0, \\
u(x,0) = u_0(x), b(x,0) = b_0(x).
\end{cases}
\tag{1.2}
$$

For notational convenience, we write ∂_1, ∂_2 and ∂_3 for the partial derivatives ∂_{x_1}, ∂_{x_2} and ∂_{x_3} respectively. Our motivation for this study comes from two sources. The first is to reveal and rigorously establish the stabilizing phenomenon exhibited by electrically conducting fluids. The second is to gain a better understanding of the stability and large-time behavior of anisotropic models, and to develop an efficient approach for obtaining optimal decay rates. More precisely, we have the following theorems.

Theorem 1.1 *Considering* (1.2) *with the initial data* $(u_0, b_0) \in H^3\left(\mathbb{R}^3\right)$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *There exists a constant* $\varepsilon = \varepsilon(\nu, \eta) > 0$, *if*

$$
\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon,
\tag{1.3}
$$

then (1.2) *has a unique global solution* (u, b) *satisfying, for any* $t > 0$,

$$
\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t \left(\|\nabla_h u\|_{H^3}^2 + \|b\|_{H^3}^2\right) d\tau \leq C\varepsilon^2.
\tag{1.4}
$$

To separate this linear parts in (1.2) from the nonlinear parts, we apply the Helmholtz-Leray projection $\mathbb{P} := I - \nabla \Delta^{-1} \nabla \cdot$ to the velocity equation in (1.2). Thus, (1.2) can be written as

$$
\begin{cases}
\partial_t u = \nu \Delta_h u + \partial_3 b + N_1, \\
\partial_t b = -\eta b + \partial_3 u + N_2,
\end{cases}
\tag{1.5}
$$

where

$$
N_1 = P(-u \cdot \nabla u + b \cdot \nabla b), \quad N_2 = -u \cdot \nabla b + b \cdot \nabla u.
$$

Differentiating (1.5) in time and making several substitutions, we find

$$
\begin{cases}
\partial_{tt} u - (\nu \Delta_h - \eta)\partial_t u - \partial_3^2 u - \nu\eta \Delta_h u = N_3, \\
\partial_{tt} b - (\nu \Delta_h - \eta)\partial_t b - \partial_3^2 b - \nu\eta \Delta_h b = N_4,
\end{cases}
\tag{1.6}
$$

where N_3 and N_4 are given by

$$
N_3 = (\partial_t + \eta) N_1 + \partial_3 N_2, \qquad N_4 = \partial_3 N_1 + (\partial_t - \nu \Delta_h) N_2.
$$

Clearly, u and b satisfy the same linear wave equation with different nonlinear parts, (1.6) exhibits much more regularization than its original counterpart in (1.2). In particular, the two terms $\partial_3^2 u$ and $\partial_3^2 b$ in (1.6), emerged from the interaction of the velocity and the magnetic field, generates the dissipation in the x_3-direction. The stabilizing and damping properties of (1.6) is a consequence of the background magnetic field and interactions within the MHD system. By exploiting these properties, we establish the following theorem assessing the large-time behavior of the solutions of (1.2).

Theorem 1.2 *Assume* $(u_0, b_0) \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$*, then there exists a sufficiently small constant* $\varepsilon > 0$ *such that, if*

$$\|(u_0, b_0)\|_{H^3 \cap L^1} \leq \varepsilon,$$

then the corresponding solution (u, b) *of* (1.2) *obtained in Theorem* 1.1 *obeys*

$$\begin{aligned}
&\|\big(u(t), b(t)\big)\|_{L^2(\mathbb{R}^3)} \leq c\varepsilon(1+t)^{-\frac{1}{2}}, \quad \|\big(\nabla u(t), \nabla b(t)\big)\|_{L^2(\mathbb{R}^3)} \leq c\varepsilon(1+t)^{-\frac{5}{4}},\\
&\|\big(\nabla\nabla_h u(t), \nabla\nabla_h b(t)\big)\|_{L^2(\mathbb{R}^3)} \leq c\varepsilon(1+t)^{-1}, \quad \|\big(\partial_3^2 u(t), \partial_3^2 b(t)\big)\|_{L^2(\mathbb{R}^3)} \leq c\varepsilon(1+t)^{-\frac{7}{8}}.
\end{aligned}$$

The proof of Theorem 1.2 is not trivial. The proof employs many other helpful strategies such as dividing the time integral involving the nonlinear terms into two parts such as

$$\begin{aligned}
\int_0^t \|\widehat{K_1}(t-\tau)\widehat{u \cdot \nabla u}\|_{L^2(\mathbb{R}^3)} d\tau = &\int_0^{\frac{t}{2}} \|\widehat{K_1}(t-\tau)\widehat{u \cdot \nabla u}\|_{L^2(\mathbb{R}^3)} d\tau\\
&+ \int_{\frac{t}{2}}^t \|\widehat{K_1}(t-\tau)\widehat{u \cdot \nabla u}\|_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned}$$

The decay of the first piece relies on the kernel function while the decay of the second piece comes from the nonlinear term.

The rest of this paper is divided into two sections. Section 2 details how we use the energy methods and bootstrapping arguments to prove the stability of (1.2) global solutions in proving Theorem 1.1. Section 3 shows how we adopt an effective approach to obtain the optimal decay rates in proving Theorem 1.2. Throughout the article, to simplify the notation, we will write $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^3)}$, $\|f\|_{L^p_{x_h}}$ for $\|f\|_{L^p_{x_1 x_2}}$ and $\|f\|_{H^s}$ for $\|f\|_{H^s(\mathbb{R}^3)}$.

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. We give several anisotropic upper bounds for products and triple products in the following lemma. It is a powerful tool for dealing with anisotropic equations. The proof of this lemma can be found in (see e.g., [11]).

Lemma 2.1 *Assume that* $f, \partial_1 f, \partial_3 f, \partial_1\partial_3 f, g, \partial_2 g, h, \partial_3 h \in L^2(\mathbb{R}^3)$. *Then*

$$\begin{aligned}
&\int |fgh| dx \leq C\|f\|_{L^2}^{\frac{1}{2}} \|\partial_1 f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}},\\
&\int |fgh| dx \leq C\|f\|_{L^2}^{\frac{1}{4}} \|\partial_1 f\|_{L^2}^{\frac{1}{4}} \|\partial_2 f\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_2 f\|_{L^2}^{\frac{1}{4}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_3 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}.
\end{aligned}$$

Proof of Theorem 1.1 Taking the L^2 inner product of Eqs. $(1.2)_1$ and $(1.2)_2$ with u and b, respectively, and integrate by parts to obtain

$$\|(u(t),b(t))\|_{L^2}^2+2\eta\int_0^t\|b\|_{L^2}^2\,d\tau+2\nu\int_0^t\|\nabla_h u\|_{L^2}^2 d\tau=\|(u_0,b_0)\|_{L^2}^2. \tag{2.1}$$

Applying the differential operator ∂_i^3 to the equations in (1.2), then dotting by $(\partial_i^3 u,\partial_i^3 b)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\left(\left\|\partial_i^3 u\right\|_{L^2}^2+\left\|\partial_i^3 b\right\|_{L^2}^2\right)+\nu\left\|\partial_i^3\nabla_h u\right\|_{L^2}^2+\eta\left\|\partial_i^3 b\right\|_{L^2}^2\\&:=I_1+I_2+I_3+I_4+I_5,\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}I_1&=\sum_{i=1}^{3}\int\partial_i^3\partial_3 b\cdot\partial_i^3 u+\partial_i^3\partial_3 u\cdot\partial_i^3 b dx, \qquad I_2=-\sum_{i=1}^{3}\int\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u dx,\\ I_3&=\sum_{i=1}^{3}\int[\partial_i^3(b\cdot\nabla b)-b\cdot\nabla\partial_i^3 b]\cdot\partial_i^3 u dx, \quad I_4=-\sum_{i=1}^{3}\int\partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b dx,\\ I_5&=\sum_{i=1}^{3}\int[\partial_i^3(b\cdot\nabla u)-b\cdot\nabla\partial_i^3 u]\cdot\partial_i^3 b dx.\end{aligned}$$

By integration by parts, $I_1=0$. Now we bound I_2, we decompose it into three parts,

$$\begin{aligned}I_2&=-\sum_{i=1}^{2}\int\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u dx-\int\partial_3^3(u_h\cdot\nabla_h u)\cdot\partial_3^3 u dx-\int\partial_3^3(u_3\partial_3 u)\cdot\partial_3^3 u dx\\&:=I_{21}+I_{22}+I_{23}.\end{aligned} \tag{2.3}$$

By Leibniz Formula and Hölder's inequality, we derive

$$\begin{aligned}I_{21}&=-\sum_{i=1}^{2}\sum_{k=1}^{3}C_3^k\int\partial_i^k u\cdot\nabla\partial_i^{3-k}u\cdot\partial_i^3 u dx\\&\le C\sum_{i=1}^{2}\sum_{k=1}^{3}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_1\partial_i^k u\|_{L^2}^{\frac12}\|\nabla\partial_i^{3-k}u\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_i^{3-k}u\|_{L^2}^{\frac12}\|\partial_i^3 u\|_{L^2}^{\frac12}\|\partial_3\partial_i^3 u\|_{L^2}^{\frac12}\\&\le C\|u\|_{H^3}\|\nabla_h u\|_{H^3}^2.\end{aligned} \tag{2.4}$$

Similarly, we have

$$I_{22}\le C\|u\|_{H^3}\|\nabla_h u\|_{H^3}^2. \tag{2.5}$$

By $\nabla \cdot u = 0$ and Lemma 2.1, we have

$$\begin{aligned}
I_{23} &= -\sum_{k=1}^{3} C_3^k \int \partial_3^k u_3 \cdot \partial_3^{4-k} u \cdot \partial_3^3 u dx \\
&\leq C \|\partial_3^{k-1} \nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3^k \nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3^{4-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3^{4-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_3^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3^3 u\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|u\|_{H^3} \|\nabla_h u\|_{H^3}^2. \qquad (2.6)
\end{aligned}$$

Combining (2.4), (2.5) and (2.6), we obtain

$$I_2 \leq C \|u\|_{H^3} \|\nabla_h u\|_{H^3}^2 .$$

By Hölder's inequality and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
I_3 &= \sum_{i=1}^{3} \sum_{k=1}^{2} C_3^k \int \partial_i^k b \cdot \nabla \partial_i^{3-k} b \cdot \partial_i^3 u dx + \sum_{i=1}^{3} \int \partial_i^3 b \cdot \nabla b \cdot \partial_i^3 u dx \\
&\leq C \sum_{i=1}^{3} \sum_{k=1}^{2} \|\partial_i^k b\|_{L^4} \|\nabla \partial_i^{3-k} b\|_{L^4} \|\partial_i^3 u\|_{L^2} + C \sum_{i=1}^{3} \|\partial_i^3 b\|_{L^2} \|\nabla b\|_{L^\infty} \|\partial_{i^3} u\|_{L^2} \\
&\leq C \|u\|_{H^3} \|b\|_{H^3}^2.
\end{aligned}$$

The estimation process for I_4 and I_5 are similarly to that for I_3, we obtain

$$I_4 + I_5 \leq C \|u\|_{H^3} \|b\|_{H^3}^2.$$

Inserting the above estimates in (2.2), integrating in time over $[0, t]$ and adding to (2.1),yields

$$\begin{aligned}
&\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + 2\nu \int_0^t \|\nabla_h u\|_{H^3}^2 d\tau + 2\eta \int_0^t \|b\|_{H^3}^2 d\tau \\
&\quad \leq \|u_0\|_{H^3}^2 + \|b_0\|_{H^3}^2 + C \int_0^t \|u\|_{H^3} (\|\nabla_h u\|_{H^3}^2 + \|b\|_{H^3}^2) d\tau. \qquad (2.7)
\end{aligned}$$

Let

$$E(t) = \sup_{0 \leq \tau \leq t} \{\|u(\tau)\|_{H^3}^2 + b(\tau))\|_{H^3}^2\} + C \int_0^t (\|\nabla_h u\|_{H^3}^2 + \|b\|_{H^3}^2) d\tau.$$

Then (2.7) implies

$$E(t) \leq E(0) + C_1 E^{\frac{3}{2}}(t). \qquad (2.8)$$

According to the bootstrapping argument, we assume that the initial data $\|(u_0, b_0)\|_{H^3} \leq \varepsilon \leq \frac{1}{4C_1^2}$, namely

$$E(0) \leq \frac{1}{16C_1^2}.$$

In fact, if we make the ansatz that,

$$E(t) \leq \frac{1}{4C_1^2}.$$

Then (2.8) implies

$$E(t) \leq C_1 E^{\frac{1}{2}}(t) \cdot E(t) + E(0) \leq \frac{1}{2} E(t) + E(0).$$

Consequently,

$$E(t) \leq \frac{1}{8C_1^2}.$$

The bootstrapping argument then implies that $T = \infty$ and asserts that for any time $t > 0$,

$$E(t) \leq \frac{1}{4C_1^2}.$$

which, in particular, implies the desire global bound on the solution (u, b). As a consequence, we obtain the global existence of solutions. The uniqueness is obvious due to the high regularity of the solution. □

3 Proof of Theorem 1.2

This section completes the proof of Theorem 1.2. It will be divided into four subsections after we present several lemmas and a proposition. The first one provides the exact $L^p - L^q$ decay rate for the heat operator associated with a fractional Laplacian (see e.g., [26]).

Lemma 3.1 *Assume $\alpha > 0$ and $\beta \geq 0$ are real numbers. Let $1 \leq p \leq q \leq \infty$. Then there exists a constant $C > 0$ such that, for any $t > 0$,*

$$\|\Lambda^\beta e^{-\Lambda^\alpha t} f\|_{L^q(\mathbb{R}^d)} \leq C t^{-\frac{\beta}{\alpha} - \frac{d}{\alpha}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

The second one provides an upper bound on a convolution integral, which can be proved similarly to Lemma 2.4 in (see e.g., [27]).

Lemma 3.2 *Let* $0 < s_1 \leq s_2$*. then, for a constant* $C > 0$*,*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}d\tau \leq \begin{cases} C(1+t)^{-s_1}, & if s_2 > 1, \\ C(1+t)^{-s_1}ln(1+t), & if s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & if s_2 < 1. \end{cases}$$

The last lemma offers upper bounds with optimal decay rates for a convolution type integral. Its proof can be found in many references (see e.g., [28, 29]).

Lemma 3.3 *For any* $c > 0$ *and* $s > 0$*,*

$$\int_0^t e^{-c(t-\tau)}(1+\tau)^{-s}d\tau \leq C(1+t)^{-s}.$$

We have separated the linear parts from the the nonlinear ones in (1.2) and obtained (1.5). Taking the Fourier transform of (1.5), we find

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} + \begin{pmatrix} \widehat{N_1} \\ \widehat{N_2} \end{pmatrix},$$

where A represents the multiplier matrix of the linear operators,

$$A = \begin{pmatrix} -\nu\xi_h^2 & i\xi_3 \\ i\xi_3 & -\eta \end{pmatrix}$$

with $|\xi_h|^2 = \xi_1^2 + \xi_2^2$. To diagonalize A, we compute the eigenvalues of A,

$$\lambda_1 = \frac{-(\eta+\nu\xi_h^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta+\nu\xi_h^2) + \sqrt{\Gamma}}{2}, \quad \Gamma = (\eta+\nu\xi_h^2)^2 - 4(\xi_3^2 + \nu\eta\xi_h^2).$$

By Duhamel's principle,

$$\begin{pmatrix} \widehat{u}(t) \\ \widehat{b}(t) \end{pmatrix} = e^{At} \begin{pmatrix} \widehat{u}_0 \\ \widehat{b}_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} \widehat{N_1}(\tau) \\ \widehat{N_2}(\tau) \end{pmatrix} d\tau. \tag{3.1}$$

Thus, we obtain the integral representation

$$\begin{cases} \widehat{u}(t) = \widehat{K_1}\widehat{u_0} + \widehat{K_2}\widehat{b_0} + \int_0^t \left(\widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right)d\tau, \\ \widehat{b}(t) = \widehat{K_2}\widehat{u_0} + \widehat{K_3}\widehat{b_0} + \int_0^t \left(\widehat{K_2}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{N_2}(\tau)\right)d\tau, \end{cases} \tag{3.2}$$

where

$$\widehat{K_1} = \eta G_1 + G_2, \quad \widehat{K_2} = i\xi_3 G_1, \quad \widehat{K_3} = -\eta G_1 + G_3 \tag{3.3}$$

with

$$G_1 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_2 = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_3 = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}.$$

We remark that when $\lambda_1 = \lambda_2$ or $\Gamma = 0$, the representation in (3.2) remains valid, the formulas of the kernel functions $\widehat{K_1}$ through $\widehat{K_3}$ are replaced by the corresponding limiting formulas

$$\widehat{K_1} = \eta \lim_{\lambda_2 \to \lambda_1} G_1 + \lim_{\lambda_2 \to \lambda_1} G_2 = \eta t e^{\lambda_1 t} + (1 + \lambda_1 t) e^{\lambda_1 t},$$
$$\widehat{K_2} = i\xi_3 t e^{\lambda_1 t}, \quad \widehat{K_3} = -\eta t e^{\lambda_1 t} + (1 - \lambda_1 t) e^{\lambda_1 t}.$$

The next proposition provides upper bounds for the kernel function $\widehat{K_1}$ through $\widehat{K_3}$, which plays a crucial role to prove Theorem 1.2. The kernel functions depend on the Fourier frequency and are anisotropic. Consequently we need to divide the frequency space into subsets and classify the behavior of the kernel function in each subsets.

Proposition 3.1 *We split $\mathbb{R}^3$ into two subsets, $\mathbb{R}^3 = A_1 \cup A_2$ with*

$$A_1 := \left\{ \xi \in R^3, \sqrt{\Gamma} \le \frac{\eta + \nu\xi_h^2}{2} \ i.e.,\ \nu\eta\xi_h^2 + \xi_3^2 \ge \frac{3}{16}(\eta + \nu\xi_h^2)^2 \right\},$$
$$A_2 := \left\{ \xi \in R^3, \sqrt{\Gamma} > \frac{\eta + \nu\xi_h^2}{2} \ i.e.,\ \nu\eta\xi_h^2 + \xi_3^2 < \frac{3}{16}(\eta + \nu\xi_h^2)^2 \right\}.$$

(1) For any $\xi \in A_1$, there is $c_0 > 0, C > 0$, we have

$$Re\lambda_1 \le -\frac{1}{2}(\eta + \nu\xi_h^2), \quad Re\lambda_2 \le -\frac{1}{4}(\eta + \nu\xi_h^2),$$
$$|G_1(t)| \le t e^{-\frac{1}{4}(\eta + \nu\xi_h^2)t}, \quad |K_i| \le C e^{-c_0(1+\xi_h^2)t}, \quad i = 1, 2, 3.$$

(2) For any $\xi \in A_2$, there is $c_0 > 0, C > 0$, we have

$$\lambda_1 < -\frac{3}{4}(\eta + \nu\xi_h^2), \quad \lambda_2 \le -\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2},$$
$$|G_1(t)| < \frac{2}{\eta + \nu\xi_h^2}\left(e^{-\frac{3}{4}(\eta + \nu\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2}t} \right),$$
$$|K_i| < C\left(e^{-\frac{3}{4}(\eta + \nu\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2}t} \right), \quad i = 1, 2, 3.$$

If we further divide A_2 into two subsets A_{21}, A_{22},

$$A_{21} = \{\xi \in A_2, \nu\xi_h^2 \le \eta\}, \quad A_{22} = \{\xi \in A_2, \nu\xi_h^2 > \eta\},$$

then, for $c_0 > 0$, $C > 0$, *and* $i = 1, 2, 3$.

$$|\widehat{K_i}(t)| \le C\left(e^{-c_0(1+\xi_h^2)t} + e^{-c_0|\xi|^2 t}\right), \quad \xi \in A_{21},$$
$$|\widehat{K_i}(t)| \le C\left(e^{-c_0(1+\xi_h^2)t} + e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})t}\right), \quad \xi \in A_{22}.$$

Proof of Proposition 3.1 (1) For $\xi \in A_1$,

$$\Gamma = (\eta + \nu\xi_h^2)^2 - 4(\xi_3^2 + \nu\eta\xi_h^2) \le (\eta + \nu\xi_h^2)^2 - \frac{3}{4}(\eta + \nu\xi_h^2)^2 = \frac{1}{4}(\eta + \nu\xi_h^2)^2.$$

Through the direct estimates and the mean-value theorem, we have

$$Re\lambda_1 \le -\frac{1}{2}(\eta + \nu\xi_h^2), \quad Re\lambda_2 \le -\frac{1}{4}(\eta + \nu\xi_h^2), \ \ |G_1| \le te^{-\frac{1}{4}(\eta+\nu\xi_h^2)t}.$$

To bound the kernel functions $\widehat{K_1}(t)$ and $\widehat{K_3}(t)$, there are two aspects to consider: When λ_1 is a real number, for some pure constants c_0 dependent of ν and η, then

$$|\widehat{K_1}(t)| \le \eta|G_1| + |\lambda_1 G_1| + |e^{\lambda_2 t}| \le Ce^{-c_0(1+\xi_h^2)t}.$$

where we have use the simple fact $xe^{-C_1 x} \le C_2$ for any $x \ge 0$ and $C_1 > 0$ and suitable $C_2 > 0$. When λ_1 is an imaginary number, then

$$|\lambda_1|^2 = \nu\eta\xi_h^2 + \xi_3^2, \quad -\Gamma = -(\eta + \nu\xi_h^2)^2 + 4|\lambda_1|^2.$$

Now we bound $|\lambda_1 G_1|$, we further divide it into two subsets: if $|\lambda_1| \le |\sqrt{\Gamma}|$, we have

$$|\lambda_1 G_1| \le \frac{|\lambda_1|}{|\sqrt{\Gamma}|}(|e^{\lambda_1 t}| + |e^{\lambda_2 t}|) \le Ce^{-\frac{1}{4}(\eta+\nu\xi_h^2)t}.$$

If $|\lambda_1| \ge |\sqrt{\Gamma}|$, we obtain

$$|\lambda_1 G_1| \le \frac{\eta + \nu\xi_h^2}{\sqrt{3}}|G_1| \le C(\eta + \nu\xi_h^2)te^{-\frac{1}{4}(\eta+\nu\xi_h^2)t} \le Ce^{-\frac{1}{8}(\eta+\nu\xi_h^2)t}.$$

Consequently, we derive

$$|\widehat{K_1}(t)| \le \eta|G_1| + |\lambda_1 G_1| + |e^{\lambda_2 t}| \le Ce^{-c_0(1+\xi_h^2)t}.$$

In summary, for $\xi \in A_1$, we have

$$|\widehat{K_1}(t)| \le Ce^{-c_0(1+\xi_h^2)t}.$$

Similarly, we obtain

$$|\widehat{K_3}(t)| \le Ce^{-c_0(1+\xi_h^2)t}.$$

Next we will estimate $\widehat{K_2}(t)$. we divide the consideration into two cases:

$$|\xi_3| \le |\sqrt{\Gamma}| \quad and \quad |\xi_3| \ge |\sqrt{\Gamma}|.$$

When $|\xi_3| \le |\sqrt{\Gamma}|$, we have

$$|\widehat{K_2}(t)| \le \frac{|\xi_3|}{|\sqrt{\Gamma}|}(|e^{\lambda_1 t}| + |e^{\lambda_2 t}|) \le Ce^{-c_0(1+\xi_h^2)t}.$$

When $|\xi_3| \ge |\sqrt{\Gamma}|$, yields

$$|\widehat{K_2}(t)| \le \frac{1}{\sqrt{3}}(\eta + \nu\xi_h^2)te^{-\frac{1}{8}(\eta+\nu\xi_h^2)t}e^{-\frac{1}{8}(\eta+\nu\xi_h^2)t} \le Ce^{-c_0(1+\xi_h^2)t}.$$

(2) For $\xi \in A_2$, we have $\frac{1}{2}(\eta + \nu\xi_h^2) < \sqrt{\Gamma} \le \eta + \nu\xi_h^2$. Then

$$-(\eta + \nu\xi_h^2) \le \lambda_1 < -\frac{3}{4}(\eta + \nu\xi_h^2),$$

$$\lambda_2 = \frac{-(\eta + \nu\xi_h^2) + \sqrt{\Gamma}}{2} = \frac{4(\nu\eta\xi_h^2 + \xi_3^2)}{-2(\eta + \nu\xi_h^2 + \sqrt{\Gamma})} \le -\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2}. \tag{3.4}$$

Therefore,

$$|G_1| \le \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} + e^{\lambda_2 t}) < \frac{2}{\eta + \nu\xi_h^2}\left(e^{-\frac{3}{4}(\eta+\nu\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}t}\right). \tag{3.5}$$

It follows that

$$|\widehat{K_1}(t)| \le \eta|G_1| + |\lambda_1 G_1| + |e^{\lambda_2 t}| < C\left(e^{-\frac{3}{4}(\eta+\nu\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}t}\right).$$

Similarly,

$$|\widehat{K_3}(t)| < C\left(e^{-\frac{3}{4}(\eta+\nu\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}t}\right).$$

Due to $\xi \in A_2$, we find

$$\xi_3^2 \le \frac{3}{16}(\eta+\nu\xi_h^2)^2 \quad or \quad \frac{|\xi_3|}{\eta+\nu\xi_h^2} \le \frac{\sqrt{3}}{4}.$$

Therefore,

$$|\widehat{K_2}| \le |\xi_3||G_1| < C\left(e^{-\frac{3}{4}(\eta+\nu\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}t}\right).$$

The further division of A_2 into A_{21} and A_{22} is to make the upper bound for $|\widehat{K_1}|$, $|\widehat{K_2}|$, and $|\widehat{K_3}|$ more definite. For $\xi \in A_{21}$, $\nu\xi_h^2 \le \eta$, we obtain

$$\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2} \ge \frac{\nu\eta\xi_h^2+\xi_3^2}{2\eta} = \frac{\nu\xi_h^2}{2} + \frac{\xi_3^2}{2\eta} \ge c_0|\xi|^2.$$

Therefore,

$$|\widehat{K_i}(t)| \le C(e^{-c_0(1+\xi_h^2)t} + e^{-c_0|\xi|^2t}), \quad i=1,2,3.$$

For $\xi \in A_{22}$, $\nu\xi_h^2 > \eta$, we derive

$$\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2} \ge \frac{\nu\eta\xi_h^2+\xi_3^2}{2\nu\xi_h^2} = \frac{\eta}{2} + \frac{\xi_3^2}{2\nu\xi_h^2} \ge c_0\left(1+\frac{\xi_3^2}{\xi_h^2}\right).$$

Thus,

$$|\widehat{K_i}(t)| \le C(e^{-c_0(1+\xi_h^2)t} + e^{-c_0\left(1+\frac{\xi_3^2}{\xi_h^2}\right)t}), \quad i=1,2,3.$$

This completes the Proof of Proposition 3.1. We are ready to prove Theorem 1.2. □

Proof of Theorem 1.2 The proof of the desired decay estimate is obtained via the bootstrapping argument applied to the integral representation of u and b. We make the ansatz, for $1 \le t \le T$,

$$\begin{aligned}
&\|\big(u(t), b(t)\big)\|_{L^2} \le \tilde{c}\varepsilon(1+t)^{-\frac{1}{2}}, && \|\big(\nabla u(t), \nabla b(t)\big)\|_{L^2} \le \tilde{c}\varepsilon(1+t)^{-\frac{5}{4}},\\
&\|\big(\nabla\nabla_h u(t), \nabla\nabla_h b(t)\big)\|_{L^2} \le \tilde{c}\varepsilon(1+t)^{-1}, && \|\big(\partial_3^2 u(t), \partial_3^2 b(t)\big)\|_{L^2} \le \tilde{c}\varepsilon(1+t)^{-\frac{7}{8}},
\end{aligned} \tag{3.6}$$

where $\tilde{c}$ is a constant to be specified later in the following proof. We then show by using the ansatz in (3.6) and the integral representation of u and b that

$$\|\big(u(t), b(t)\big)\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{1}{2}}, \qquad \|\big(\nabla u(t), \nabla b(t)\big)\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{5}{4}},$$

$$\|\big(\nabla\nabla_h u(t), \nabla\nabla_h b(t)\big)\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-1}, \quad \|\big(\partial_3^2 u(t), \partial_3^2 b(t)\big)\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{7}{8}}. \tag{3.7}$$

The bootstrapping argument then implies that (3.7) indeed holds for all time $1 \le t < \infty$.

For the sake of clarity, the rest of this section is divided into four subsections with each subsection devoted to one of the inequalities in (3.7). □

3.1 Estimates of $\|(u(t), b(t))\|_{L^2}$

The goal of this subsection is to prove the first inequality in (3.7), namely

$$\|(u(t), b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{1}{2}}.$$

According to (3.2), we obtain

$$\begin{aligned}
\|u(t)\|_{L^2} =& \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2} + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2} d\tau \\
&+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2} d\tau := D_1 + D_2 + D_3 + D_4.
\end{aligned} \tag{3.8}$$

We first bound D_1. To do this, applying Lemma 3.1 and Proposition 3.1, we derive

$$\begin{aligned}
D_1 &\le C\|e^{-c_0(1+\xi_h^2)t}\widehat{u_0}\|_{L^2} + C\|e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2} + C\|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})t}\widehat{u_0}\|_{L^2} \\
&\le Ce^{-c_0 t}\|u_0\|_{L^2} + C\|e^{-c_0\Lambda^2 t}u_0\|_{L^2} && (3.9) \\
&\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2} + Ct^{-\frac{3}{4}}\|u_0\|_{L^1} \\
&\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2\cap L^1}, && (3.10)
\end{aligned}$$

where we have used the simple fact that $e^{-c_0 t}(1+t)^m \le C(c_0, m)$, for $t > 0, m = \frac{1}{2}$. Similarly,

$$D_2 \le C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2\cap L^1}. \tag{3.11}$$

Recalling that $N_1 = \mathbb{P}(-u\cdot\nabla u + b\cdot\nabla b)$ with $\mathbb{P}$ being the Leray projection onto divergence-free vector fields and using the fact that $\mathbb{P}$ is bounded on L^2, we have

$$\begin{aligned}
D_3 \le& C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2} d\tau + C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}|\widehat{b\cdot\nabla b}|\|_{L^2} d\tau \\
&+ C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2} d\tau + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{b\cdot\nabla b}|\|_{L^2} d\tau
\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t \|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2}d\tau+C\int_0^t \|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})(t-\tau)}|\widehat{b\cdot\nabla b}|\|_{L^2}d\tau\\
&:=D_{31}+D_{32}+D_{33}+D_{34}+D_{35}+D_{36}.
\end{aligned}
$$

D_{31} is further divided into two parts,

$$
\begin{aligned}
D_{31}&\le C\int_0^{\frac{t}{2}} \|e^{-c_0(1+\xi_h^2)(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2}d\tau+C\int_{\frac{t}{2}}^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2}d\tau\\
&:=D_{311}+D_{312}.
\end{aligned}
$$

By Gagliardo–Nirenberg inequality and Hölder's inequality, we have

$$
\begin{aligned}
D_{311}&\le C\int_0^{\frac{t}{2}} e^{-c_0(t-\tau)}\|u\cdot\nabla u\|_{L^2}d\tau\le Ce^{-\frac{c_0t}{2}}\int_0^{\frac{t}{2}}\|u\|_{H^3}^2d\tau\\
&\le Ce^{-\frac{c_0t}{2}}\frac{t}{2}\varepsilon^2\le C\varepsilon^2(1+t)^{-\frac{1}{2}},
\end{aligned}
$$

where we have used the fact that $\|u\|_{H^3}\le c\varepsilon$ and $e^{-\frac{c_0t}{2}}(1+t)^m\le C(m)$ for $t>0, m=\frac{3}{2}$. By Lemma 3.1 and Hölder's inequality, yield

$$
\begin{aligned}
D_{312}&\le C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}\|e^{-c_0\xi_h^2(t-\tau)}\widehat{u\cdot\nabla u}\|_{L^2}d\tau\\
&\le C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\left\|\|u\cdot\nabla u\|_{L^1_{x_h}}\right\|_{L^2_{x_3}}d\tau\\
&\le C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|u\|_{L^2_{x_h}L^\infty_{x_3}}\|\nabla u\|_{L^2}d\tau\\
&\le C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}d\tau\le C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{1}{2}},
\end{aligned}
$$

Where we have used $\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}d\tau=\int_0^{\frac{t}{2}}e^{-cs}s^{-\frac{1}{2}}ds\le C$ for $C>0$. D_{33} is naturally divided into two parts,

$$
\begin{aligned}
D_{33}&\le C\int_0^{\frac{t}{2}} \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2}d\tau+C\int_{\frac{t}{2}}^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|\|_{L^2}d\tau\\
&:=D_{331}+D_{332}.
\end{aligned}
$$

By Lemma 3.1 and Hölder's inequality, we have

$$
D_{331}\le C\int_0^{\frac{t}{2}}\||\xi|e^{-c_0|\xi|^2(t-\tau)}\widehat{u\otimes u}\|_{L^2}d\tau\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{5}{4}}\|u\otimes u\|_{L^1}d\tau
$$

$$\leq Ct^{-\frac{5}{4}} \int_0^{\frac{t}{2}} (\tilde{c}\varepsilon(1+\tau)^{-\frac{1}{2}})^2 d\tau \leq C\tilde{c}^2\varepsilon^2 t^{-\frac{5}{4}} ln(1+t/2) \leq C\tilde{c}^2\varepsilon^2 t^{-\frac{5}{4}+\sigma},$$

where we have used $t^{-\sigma} ln(1+\frac{t}{2}) \leq C(\sigma)$ for $\sigma > 0$ and for all $t \geq 1$. By Hölder's inequality,

$$\begin{aligned} D_{332} &\leq C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{3}{4}} \|u \cdot \nabla u\|_{L^1} d\tau \leq C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{3}{4}} \|u\|_{L^2} \|\nabla u\|_{L^2} d\tau \\ &\leq C\tilde{c}^2\varepsilon^2 (1+t)^{-\frac{7}{4}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{3}{4}} d\tau \leq C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{7}{4}} (t/2)^{\frac{1}{4}} \leq C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{3}{2}}. \end{aligned}$$

By Lemma 3.3 and Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} D_{35} &\leq C \int_0^t \|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})(t-\tau)} \widehat{u \cdot \nabla u}\|_{L^2} d\tau \leq C \int_0^t e^{-c_0(t-\tau)} \|u \cdot \nabla u\|_{L^2} d\tau \\ &\leq C \int_0^t e^{-c_0(t-\tau)} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2} d\tau \leq C\tilde{c}^2\varepsilon^2 \int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{7}{4}} d\tau \\ &\leq C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

By the same technique, D_{32}, D_{34}, D_{36} share similar estimates as D_{31}, D_{33}, D_{35}, respectively.
Combining all estimates above for D_{31} through D_{36}, we conclude

$$D_3 \leq C\varepsilon^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{3.12}$$

D_4 obeys the same upper bound as D_3, namely

$$D_4 \leq C\varepsilon^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{3.13}$$

Inserting the uppers (3.9), (3.11), (3.12) and (3.13) in (3.8) leads to

$$\|u(t)\|_{L^2} \leq C_1(1+t)^{-\frac{1}{2}} \|(u_0, b_0)\|_{L^1 \cap L^2} + C_2\varepsilon^2(1+t)^{-\frac{1}{2}} + C_3\tilde{c}^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

Therefore, if we choose $\tilde{c}$ and ε satisfying

$$C_1 \leq \frac{\tilde{c}}{8}, \quad C_2\varepsilon \leq \frac{\tilde{c}}{16}, \quad C_3\tilde{c}\varepsilon \leq \frac{1}{16},$$

then we obtain

$$\|u(t)\|_{L^2} \leq \frac{\tilde{c}}{4}\varepsilon(1+t)^{-\frac{1}{2}}.$$

The same upper bound hold for $\|b\|_{L^2}$. Therefore,

$$\|(u(t), b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{1}{2}}.$$

This completes the proof of the first inequality in (3.7).

3.2 Estimates of $\|(\nabla u(t), \nabla b(t))\|_{L^2}$

The goal of this subsection is to prove the second inequality in (3.7), namely

$$\|(\nabla u(t), \nabla b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{5}{4}}.$$

Applying ∇ to (3.2), yields

$$\begin{cases} \widehat{\nabla u}(t) = \widehat{K_1}\widehat{\nabla u_0} + \widehat{K_2}\widehat{\nabla b_0} + \int_0^t \big(\widehat{K_1}(t-\tau)\widehat{\nabla N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{\nabla N_2}(\tau)\big)d\tau, \\ \widehat{\nabla b}(t) = \widehat{K_2}\widehat{\nabla u_0} + \widehat{K_3}\widehat{\nabla b_0} + \int_0^t \big(\widehat{K_2}(t-\tau)\widehat{\nabla N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{\nabla N_2}(\tau)\big)d\tau. \end{cases} \tag{3.14}$$

According to (3.14), we obtain

$$\begin{aligned} \|\nabla u(t)\|_{L^2} &\leq \|\widehat{K_1}(t)\widehat{\nabla u_0}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\nabla b_0}\|_{L^2} + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\nabla N_1}(\tau)\|_{L^2}d\tau \\ &\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\nabla N_2}(\tau)\|_{L^2}d\tau := E_1 + E_2 + E_3 + E_4. \end{aligned} \tag{3.15}$$

By Lemma 3.1 and Proposition 3.1, we derive

$$\begin{aligned} E_1 &\leq C\|e^{-c_0(1+\xi_h^2)t}\widehat{\nabla u_0}\|_{L^2} + C\|e^{-c_0|\xi|^2t}\widehat{\nabla u_0}\|_{L^2} + C\|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})t}\widehat{\nabla u_0}\|_{L^2} \\ &\leq Ce^{-c_0t}\|\nabla u_0\|_{L^2} + C\||\xi|e^{-c_0|\xi|^2t}\widehat{u_0}\|_{L^2} \\ &\leq Ce^{-c_0t}\|\nabla u_0\|_{L^2} + Ct^{-\frac{5}{4}}\|u_0\|_{L^1} \\ &\leq C(1+t)^{-\frac{5}{4}}\|u_0\|_{H^1\cap L^1}, \end{aligned}$$

where we have used $e^{-c_0t}(1+t)^m \leq C(c_0, m)$, for $t > 0, m = \frac{5}{4}$. Similarly, we obtain

$$E_2 \leq C(1+t)^{-\frac{5}{4}}\|b_0\|_{H^1\cap L^1}.$$

For E_3, we still reformulate it into several parts,

$$
\begin{aligned}
E_3 \leq\ & C \int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau + C \int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)} \nabla \widehat{(b\cdot\nabla b)}\|_{L^2} d\tau \\
& + C \int_0^t \|e^{-c_0|\xi|^2(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau + C \int_0^t \|e^{-c_0|\xi|^2(t-\tau)} \nabla \widehat{(b\cdot\nabla b)}\|_{L^2} d\tau \\
& + C \int_0^t \|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau + C \int_0^t \|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})(t-\tau)} \nabla \widehat{(b\cdot\nabla b)}\|_{L^2} d\tau \\
:=\ & E_{31} + E_{32} + E_{33} + E_{34} + E_{35} + E_{36}.
\end{aligned}
$$

E_{31} is further divided into two parts:

$$
\begin{aligned}
E_{31} =\ & C \int_0^{\frac{t}{2}} \|e^{-c_0(1+\xi_h^2)(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau + C \int_{\frac{t}{2}}^t \|e^{-c_0(1+\xi_h^2)(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau \\
:=\ & E_{311} + E_{312}.
\end{aligned}
$$

By Gagliardo–Nirenberg inequality and $e^{-\frac{c_0 t}{2}}(1+t)^m \leq C(c_0, m)$ for $t > 0$, $m = \frac{9}{4}$, we have

$$
E_{311} \leq C \int_0^{\frac{t}{2}} e^{-c_0(t-\tau)} \|\nabla(u\cdot\nabla u)\|_{L^2} d\tau \leq C e^{-\frac{c_0 t}{2}} \frac{t}{2} \varepsilon^2 \leq C\varepsilon^2 (1+t)^{-\frac{5}{4}}.
$$

By Lemma 3.1 and Hölder's inequality, we obtain

$$
\begin{aligned}
E_{312} \leq\ & C \int_{\frac{t}{2}}^t e^{-c_0(t-\tau)} \|e^{-c_0\xi_h^2(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau \\
\leq\ & C \int_{\frac{t}{2}}^t e^{-c_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} \left\| (\|\nabla u\cdot\nabla u\|_{L^1_{x_h}} + \|u\cdot\nabla^2 u\|_{L^1_{x_h}}) \right\|_{L^2_{x_3}} d\tau \\
\leq\ & C \int_{\frac{t}{2}}^t e^{-c_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} (\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2} + \|u\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}) d\tau \\
\leq\ & C\tilde{c}^2\varepsilon^2 \int_{\frac{t}{2}}^t e^{-c_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{5}{4}} d\tau \leq C\tilde{c}^2\varepsilon^2 (1+t)^{-\frac{5}{4}},
\end{aligned}
$$

where we have used $\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}} d\tau \leq C$ for $C > 0$. Next we proceed to estimate E_{33}. In order to get better large-time behavior, it is further divided into two parts,

$$
\begin{aligned}
E_{33} \leq\ & C \int_0^{\frac{t}{2}} \|e^{-c_0|\xi|^2(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau + C \int_{\frac{t}{2}}^t \|e^{-c_0|\xi|^2(t-\tau)} \nabla \widehat{(u\cdot\nabla u)}\|_{L^2} d\tau \\
:=\ & E_{331} + E_{332}.
\end{aligned}
$$

By Lemma 3.2, Hölder's inequality, we have

$$
\begin{aligned}
E_{331} &\le C\int_0^{\frac{t}{2}} \||\xi|^2 e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2}d\tau \le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{7}{4}}\|u\otimes u\|_{L^1}d\tau\\
&\le C\tilde{c}^2\varepsilon^2 t^{-\frac{7}{4}} ln(1+t/2) \le C\tilde{c}^2\varepsilon^2 t^{-\frac{7}{4}+\sigma} \le C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{5}{4}},
\end{aligned}
$$

where we have used $t^{-\sigma} ln(1+t/2) \le C(\sigma)$ for $\sigma > 0$ and all $t \ge 1$. By Lemma 3.2, and Hölder's inequality, we obtain

$$
\begin{aligned}
E_{332} &\le C\int_{\frac{t}{2}}^{t} \||\xi|e^{-c_0|\xi|^2(t-\tau)}|\widehat{\nabla(u\otimes u)}\|_{L^2}d\tau \le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{7}{8}}\|\nabla(u\otimes u)\|_{L^{\frac{4}{3}}}d\tau\\
&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{7}{8}}\|\nabla u\|_{L^2}\|u\|_{L^4}d\tau \le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{7}{8}}\|\nabla u\|_{L^2}\|u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{3}{4}}d\tau\\
&\le C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Now we turn to estimate E_{35}. By Lemma 3.3 and Gagliardo–Nirenberg inequality,

$$
\begin{aligned}
E_{35} &\le C\int_0^t e^{-c_0(t-\tau)}\|\nabla(u\cdot\nabla u)\|_{L^2}d\tau\\
&\le C\int_0^t e^{-c_0(t-\tau)}(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}})d\tau\\
&\le C(\tilde{c}^2+\tilde{c}^{\frac{3}{2}})\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{5}{4}}d\tau \le C(\tilde{c}^2+\tilde{c}^{\frac{3}{2}})\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Similarly, E_{32}, E_{34}, E_{36} obey the same bounds as E_{31}, E_{33}, E_{35}, respectively. Then,

$$
E_3 \le C\varepsilon^2(1+t)^{-\frac{5}{4}} + C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{5}{4}} + C\tilde{c}^{\frac{3}{2}}\varepsilon^2(1+t)^{-\frac{5}{4}}.
$$

Similarly, E_4 obeys the same upper bound.

$$
E_4 \le C\varepsilon^2(1+t)^{-\frac{5}{4}} + C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{5}{4}} + C\tilde{c}^{\frac{3}{2}}\varepsilon^2(1+t)^{-\frac{5}{4}}.
$$

Combining all estimate above for E_1 through E_4, we conclude

$$
\begin{aligned}
\|\nabla u(t)\|_{L^2} &\le C_1(1+t)^{-\frac{5}{4}}\|(u_0,b_0)\|_{H^1\cap L^1} + C_2\varepsilon^2(1+t)^{-\frac{5}{4}} + C_3\tilde{c}^{\frac{3}{2}}\varepsilon^2(1+t)^{-\frac{5}{4}}\\
&\quad + C_4\tilde{c}^2\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Therefore, if we choose $\tilde{c}$ and δ satisfying

$$
C_1 \le \frac{\tilde{c}}{8},\quad C_2\varepsilon \le \frac{\tilde{c}}{16},\quad C_3\tilde{c}^{\frac{1}{2}}\varepsilon \le \frac{1}{32},\quad C_4\tilde{c}\varepsilon \le \frac{1}{32},
$$

then we obtain

$$\|\nabla u(t)\|_{L^2} \le \frac{\tilde{c}}{4}\varepsilon(1+t)^{-\frac{5}{4}}.$$

Similarly, $\|\nabla b\|_{L^2}$ obeys the same bound. Therefore,

$$\|(\nabla u(t), \nabla b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{5}{4}}.$$

This completes the proof of the second inequality in (3.7).

3.3 Estimates of $\|(\nabla\nabla_h u(t), \nabla\nabla_h b(t))\|_{L^2}$

The goal of this subsection is to prove the third inequality in (3.7), namely

$$\|(\nabla\nabla_h u(t), \nabla\nabla_h b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-1}.$$

Applying $\nabla\nabla_h$ to (3.2), yields

$$\begin{cases} \widehat{\nabla\nabla_h u}(t) = \widehat{K_1}\widehat{\nabla\nabla_h u_0} + \widehat{K_2}\widehat{\nabla\nabla_h b_0} + \int_0^t \big(\widehat{K_1}(t-\tau)\widehat{\nabla\nabla_h N_1}(\tau) \\ +\widehat{K_2}(t-\tau)\widehat{\nabla\nabla_h N_2}(\tau)\big)d\tau, \\ \widehat{\nabla\nabla_h b}(t) = \widehat{K_2}\widehat{\nabla\nabla_h u_0} + \widehat{K_3}\widehat{\nabla\nabla_h b_0} + \int_0^t \big(\widehat{K_2}(t-\tau)\widehat{\nabla\nabla_h N_1}(\tau) \\ +\widehat{K_3}(t-\tau)\widehat{\nabla\nabla_h N_2}(\tau)\big)d\tau. \end{cases} \tag{3.16}$$

According to (3.16), we obtain

$$\begin{aligned} \|\nabla\nabla_h u(t)\|_{L^2} \le\ & \|\widehat{K_1}(t)\widehat{\nabla\nabla_h u_0}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\nabla\nabla_h b_0}\|_{L^2} \\ & + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\nabla\nabla_h N_1}(\tau)\|_{L^2}d\tau \\ & + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\nabla\nabla_h N_2}(\tau)\|_{L^2}d\tau := H_1 + H_2 + H_3 + H_4. \end{aligned} \tag{3.17}$$

By Lemma 3.1 and Proposition 3.1, we have

$$\begin{aligned} H_1 &\le C\|e^{-c_0(1+\xi_h^2)t}\widehat{\nabla\nabla_h u_0}\|_{L^2} + C\|e^{-c_0|\xi|^2 t}\widehat{\nabla\nabla_h u_0}\|_{L^2} + C\|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})t}\widehat{\nabla\nabla_h u_0}\|_{L^2} \\ &\le Ce^{-c_0 t}\|\nabla\nabla_h u_0\|_{L^2} + C\||\xi|^2 e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2} \\ &\le Ce^{-c_0 t}\|\nabla\nabla_h u_0\|_{L^2} + Ct^{-\frac{7}{4}}\|u_0\|_{L^1} \\ &\le C(1+t)^{-1}\|u_0\|_{H^2\cap L^1}, \end{aligned}$$

where we have used $e^{-c_0 t}(1+t)^m \le C(c_0, m)$ for $t > 0, m = 1$. By similar estimate as H_1,

$$H_2 \le C(1+t)^{-1}\|b_0\|_{H^2\cap L^1}.$$

The bound for H_3 is more complicated, we first decompose it as follows

$$\begin{aligned}
H_3 &\le C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2}d\tau + C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(b\cdot\nabla b)}\|_{L^2}d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2}d\tau + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\nabla\widehat{\nabla_h(b\cdot\nabla b)}\|_{L^2}d\tau \\
&\quad + C\int_0^t \|\widehat{K_1}(t-\tau)\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2(A_{22})}d\tau + C\int_0^t \|\widehat{K_1}(t-\tau)\nabla\widehat{\nabla_h(b\cdot\nabla b)}\|_{L^2(A_{22})}d\tau \\
&:= H_{31} + H_{32} + H_{33} + H_{34} + H_{35} + H_{36}.
\end{aligned} \tag{3.18}$$

We divide H_{31} into two parts:

$$\begin{aligned}
H_{31} &= C\int_0^{\frac{t}{2}} \|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2}d\tau \\
&:= H_{311} + H_{312},
\end{aligned}$$

By Hölder's inequality and Proposition 3.1, we derive

$$H_{311} \le C\int_0^{\frac{t}{2}} e^{-c_0(t-\tau)}\|\nabla\nabla_h(u\cdot\nabla u)\|_{L^2}d\tau \le Ce^{-\frac{c_0 t}{2}}\frac{t}{2}\varepsilon^2 \le C\varepsilon^2(1+t)^{-1},$$

where we have used $e^{-c_0 t}(1+t)^m \le C(c_0, m)$ for $t > 0, m = 2$. By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned}
H_{312} &\le C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}\Big\|\||\xi_h|e^{-c_0\xi_h^2(t-\tau)}\nabla\widehat{(u\cdot\nabla u)}\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}d\tau \\
&\le C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\big(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla_h\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla_h\nabla u\|_{L^2}^{\frac{1}{4}} \\
&\quad + \|u\|_{L^2}^{\frac{1}{4}}\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\nabla_h u\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla^2 u\|_{L^2}^{\frac{1}{2}}\big)d\tau \\
&\le C(\tilde{c}^{\frac{7}{4}} + \tilde{c}^{\frac{3}{2}})\varepsilon^2\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}d\tau \le C(\tilde{c}^{\frac{7}{4}} + \tilde{c}^{\frac{3}{2}})\varepsilon^2(1+t)^{-1}.
\end{aligned}$$

Next we proceed to estimate H_{33}. By Lemma 3.2 and Hölder's inequality, we have

$$\begin{aligned}
H_{33} &\le C\int_0^t \||\xi|^3 e^{-c_0|\xi|^2(t-\tau)}\widehat{u\otimes u}\|_{L^2}d\tau \le C\int_0^t (1+t-\tau)^{-\frac{9}{4}}\|u\otimes u\|_{L^1}d\tau \\
&\le C\tilde{c}^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{9}{4}}(1+\tau)^{-1}d\tau \le C\tilde{c}^2\varepsilon^2(1+t)^{-1}.
\end{aligned}$$

The bounds in Proposition 3.1 are not sufficient for estimating H_{35}, so we derive some alternative upper bounds. Recall that

$$A_{22} = \{\xi \in \mathbb{R}^3, \nu\eta\xi_h^2 + \xi_3^2 < \frac{3}{16}(\eta + \nu\xi_h^2)^2, \nu\xi_h^2 > \eta\}.$$

G_2 and G_3 can be written as

$$G_2 = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = \lambda_2 G_1 + e^{\lambda_1 t}, \quad G_3 = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1.$$

By (3.4) and (3.5), we obtain the new upper bounds for K_1 and K_2,

$$\begin{aligned} |\widehat{K_1}| &\le |e^{\lambda_1 t} + (\lambda_2 + \eta) G_1| \le e^{-c_0(1+\xi_h^2)t} + C\left(\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2} + \eta\right)|G_1| \\ &\le e^{-c_0(1+\xi_h^2)t} + \frac{2C}{\eta + \nu\xi_h^2}\left(\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2} + \eta\right)\left(e^{-c_0(1+\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2}t}\right), \end{aligned} \tag{3.19}$$

$$|\widehat{K_2}| \le |\xi_3||G_1| \le \frac{2|\xi_3|}{\eta + \nu\xi_h^2}\left(e^{-c_0(1+\xi_h^2)t} + e^{-\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2}t}\right). \tag{3.20}$$

To bound H_{35}, we use new bounds in (3.19),

$$\begin{aligned} H_{35} &\le C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2} d\tau \\ &\quad + C\int_0^t \Big\|\big(\frac{\nu\eta\xi_h^2 + \xi_3^2}{(\eta + \nu\xi_h^2)^2} + 1\big)e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\Big\|_{L^2} d\tau \\ &\quad + C\int_0^t \Big\|\frac{1}{\eta + \nu\xi_h^2}\big(\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2} + \eta\big)e^{-\frac{\nu\eta\xi_h^2 + \xi_3^2}{\eta + \nu\xi_h^2}(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\Big\|_{L^2} d\tau \\ &:= H_{351} + H_{352} + H_{353}. \end{aligned}$$

H_{351} obeys the same bounds as H_{31}. Due to $\xi \in A_{22}$, $\nu\eta\xi_h^2 + \xi_3^2 < \frac{3}{16}(\eta + \nu\xi_h^2)^2$, H_{352} can be bounded similarly as H_{31},

$$H_{352} \le C\int_0^t \Big\|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\Big\|_{L^2} d\tau \le C(1 + \tilde{c}^{\frac{3}{2}} + \tilde{c}^{\frac{7}{4}})\varepsilon^2(1+t)^{-1}.$$

We rewrite H_{353} into two parts,

$$
\begin{aligned}
H_{353} \le C \int_0^t \Big\| \frac{|\xi|^2}{(1+\xi_h^2)^2} e^{-c_0 \frac{|\xi|^2}{1+\xi_h^2}(t-\tau)} \nabla \widehat{\nabla_h (u\cdot\nabla u)} \Big\|_{L^2} d\tau \\
+ C \int_0^t \Big\| \frac{|\xi|}{1+\xi_h^2} e^{-c_0 \frac{|\xi|^2}{1+\xi_h^2}(t-\tau)} \widehat{\nabla_h (u\cdot\nabla u)} \Big\|_{L^2} d\tau := H_{3531} + H_{3532}.
\end{aligned}
$$

Due to $e^{-c_0 t}(1+t)^m \le C(c_0, m)$ for any $m > 0, t > 0$, we have

$$
\begin{aligned}
H_{3531} &\le C \int_0^t \Big\| \frac{|\xi|^4}{(1+\xi_h^2)^2} (t-\tau)^2 (t-\tau)^{-2} e^{-c_0 \frac{|\xi|^2}{1+\xi_h^2}(t-\tau)} \widehat{u\cdot\nabla u} \Big\|_{L^2} d\tau \\
&\le C \int_0^t (1+t-\tau)^{-2} \|u\cdot\nabla u\|_{L^2} d\tau \le C\tilde{c}^2 \varepsilon^2 (1+t)^{-2}.
\end{aligned}
$$

Now we turn to estimate H_{3532}. By Lemma 3.3 and Gagliardo–Nirenberg inequality,

$$
H_{3532} \le C \int_0^t \Big\| \frac{|\xi|^2}{1+\xi_h^2} (t-\tau)(t-\tau)^{-1} e^{-c_0 \frac{|\xi|^2}{1+\xi_h^2}(t-\tau)} \widehat{u\cdot\nabla u} \Big\|_{L^2} d\tau \le C\tilde{c}^2 \varepsilon^2 (1+t)^{-1}.
$$

Consequently,

$$
H_{35} \le C\varepsilon^2 (1+t)^{-1} + C\tilde{c}^{\frac{3}{2}} \varepsilon^2 (1+t)^{-1} + C\tilde{c}^{\frac{7}{4}} \varepsilon^2 (1+t)^{-1} + C\tilde{c}^2 \varepsilon^2 (1+t)^{-1}.
$$

By the same technique, H_{32}, H_{34}, H_{36} share similar estimates as H_{31}, H_{33}, H_{35}, respectively.
Inserting the bounds of $H_{31} - H_{36}$ in (3.18), yields

$$
H_3 \le C(1 + \tilde{c}^{\frac{7}{4}} + \tilde{c}^{\frac{3}{2}} + \tilde{c}^2) \varepsilon^2 (1+t)^{-1}.
$$

The estimation of many terms of H_4 is similar to that of H_3, which we first expand as follows

$$
\begin{aligned}
H_4 \le{}& C \int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)} \nabla \widehat{\nabla_h (u\cdot\nabla b)}\|_{L^2} d\tau + C \int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)} \nabla \widehat{\nabla_h (b\cdot\nabla u)}\|_{L^2} d\tau \\
&+ C \int_0^t \|e^{-c_0|\xi|^2(t-\tau)} \nabla \widehat{\nabla_h (u\cdot\nabla b)}\|_{L^2} d\tau + C \int_0^t \|e^{-c_0|\xi|^2(t-\tau)} \nabla \widehat{\nabla_h (b\cdot\nabla u)}\|_{L^2} d\tau \\
&+ C \int_0^t \|\widehat{K_2}(t-\tau) \nabla \widehat{\nabla_h (u\cdot\nabla b)}\|_{L^2(A_{22})} d\tau + C \int_0^t \|\widehat{K_2}(t-\tau) \nabla \widehat{\nabla_h (b\cdot\nabla u)}\|_{L^2(A_{22})} d\tau \\
:={}& H_{41} + H_{42} + H_{43} + H_{44} + H_{45} + H_{46}.
\end{aligned}
$$

H_{41}, H_{42}, H_{43} and H_{44} can be estimated with a nearly same argument as H_{31}, H_{32}, H_{33} and H_{34}, respectively. We use new bounds to estimate H_{45},

$$H_{45} \leq C\int_0^t \Big\|\frac{|\xi_3|}{1+\xi_h^2}e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla b)}\Big\|_{L^2}d\tau$$
$$+C\int_0^t \Big\|\frac{|\xi_3|}{1+\xi_h^2}e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla b)}\Big\|_{L^2}d\tau := H_{451}+H_{452}.$$

Since $\xi\in A_{22}$, $|\xi|^2\leq C(1+\xi_h^2)^2$. By the same process as H_{31},

$$H_{451} \leq C\int_0^t \Big\|e^{-c_0(1+\xi_h^2)(t-\tau)}\nabla\widehat{\nabla_h(u\cdot\nabla b)}\Big\|_{L^2}d\tau \leq C(1+\tilde c^{\frac{3}{2}}+\tilde c^{\frac{7}{4}})\varepsilon^2(1+t)^{-1}.$$

By Lemma 3.2 and Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}H_{452} &\leq C\int_0^t \Big\|\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)(t-\tau)^{-1}e^{-c_0\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)}\widehat{\nabla_h(u\cdot\nabla b)}\Big\|_{L^2}d\tau\\
&\leq C\int_0^t (1+t-\tau)^{-1}\|\nabla_h(u\cdot\nabla b)\|_{L^2}d\tau\\
&\leq C\int_0^t (1+t-\tau)^{-1}(\|\nabla_h u\|_{L^2}^{\frac14}\|\nabla\nabla_h u\|_{L^2}^{\frac34}\|\nabla b\|_{L^2}^{\frac14}\|\nabla^2 b\|_{L^2}^{\frac34}+\|\nabla u\|_{L^2}\\
&\quad\times\|\nabla_h\nabla b\|_{L^2}^{\frac12}\|\nabla_h\nabla^2 b\|_{L^2}^{\frac12})d\tau\\
&\leq C\tilde c^2\varepsilon^2(1+t)^{-1}+C\tilde c^{\frac32}\varepsilon^2(1+t)^{-1}.\end{aligned}$$

Consequently,

$$H_4 \leq C(1+\tilde c^{\frac32}+\tilde c^{\frac74}+\tilde c^2)\varepsilon^2(1+t)^{-1}.$$

Substituting the bounds of H_1-H_4 into (3.17), yields

$$\|\nabla\nabla_h u(t)\|_{L^2}\leq C_1(1+t)^{-1}\|(u_0,b_0)\|_{L^1\cap H^2}+C_2\varepsilon^2(1+t)^{-1}+C_3(\tilde c^{\frac74}$$
$$+\tilde c^{\frac32}+\tilde c^2)\varepsilon^2(1+t)^{-1}.$$

Therefore, if we choose $\tilde c$ and δ satisfying

$$C_1\leq\frac{\tilde c}{8},\quad C_2\varepsilon\leq\frac{\tilde c}{16},\quad C_3(\tilde c^{\frac74}+\tilde c^{\frac32}+\tilde c^2)\varepsilon\leq\frac{\tilde c}{16},$$

then we deduce

$$\|\nabla\nabla_h u(t)\|_{L^2}\leq\frac{\tilde c}{4}\varepsilon(1+t)^{-1}.$$

A similar bound holds for $\|\nabla\nabla_h b\|_{L^2}$. Therefore,

$$\|(\nabla\nabla_h u(t), \nabla\nabla_h b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\varepsilon(1+t)^{-1}.$$

This completes the proof of the third inequality in (3.7).

3.4 Estimates of $\|(\partial_3^2 u(t), \partial_3^2 b(t))\|_{L^2}$

The goal of this subsection is to prove the last inequality in (3.7), namely

$$\|(\partial_3^2 u(t), \partial_3^2 b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{7}{8}}.$$

Applying ∂_3^2 to (3.2), yields

$$\begin{cases} \widehat{\partial_3^2 u(t)} = \widehat{K_1}\widehat{\partial_3^2 u_0} + \widehat{K_2}\widehat{\partial_3^2 b_0} + \int_0^t \big(\widehat{K_1}(t-\tau)\widehat{\partial_3^2 N_1(\tau)} + \widehat{K_2}(t-\tau)\widehat{\partial_3^2 N_2(\tau)}\big)d\tau, \\ \widehat{\partial_3^2 b(t)} = \widehat{K_2}\widehat{\partial_3^2 u_0} + \widehat{K_3}\widehat{\partial_3^2 b_0} + \int_0^t \big(\widehat{K_2}(t-\tau)\widehat{\partial_3^2 N_1(\tau)} + \widehat{K_3}(t-\tau)\widehat{\partial_3^2 N_2(\tau)}\big)d\tau. \end{cases} \tag{3.21}$$

According to (3.21), we obtain

$$\begin{aligned} \|\partial_3^2 u(t)\|_{L^2} &\leq \|\widehat{K_1}(t)\widehat{\partial_3^2 u_0}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\partial_3^2 b_0}\|_{L^2} + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_3^2 N_1(\tau)}\|_{L^2}d\tau \\ &\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_3^2 N_2(\tau)}\|_{L^2}d\tau := S_1 + S_2 + S_3 + S_4. \end{aligned}$$

By Lemma 3.1 and Proposition 3.1, we have

$$\begin{aligned} S_1 &\leq C\|e^{-c_0(1+\xi_h^2)t}\widehat{\partial_3^2 u_0}\|_{L^2} + C\|e^{-c_0|\xi|^2 t}\widehat{\partial_3^2 u_0}\|_{L^2} + C\|e^{-c_0(1+\frac{\xi_3^2}{\xi_h^2})t}\widehat{\partial_3^2 u_0}\|_{L^2} \\ &\leq Ce^{-c_0 t}\|\partial_3^2 u_0\|_{L^2} + C\||\xi|^2 e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2} \\ &\leq Ce^{-c_0 t}\|\partial_3^2 u_0\|_{L^2} + Ct^{-\frac{7}{4}}\|u_0\|_{L^1} \\ &\leq C(1+t)^{-\frac{7}{8}}\|u_0\|_{H^2\cap L^1}, \end{aligned} \tag{3.22}$$

where we have used $e^{-c_0 t}(1+t)^m \leq C(c_0, m)$ for $t > 0, m = \frac{7}{8}$. S_2 can be bounded similarly,

$$S_2 \leq C(1+t)^{-\frac{7}{8}}\|b_0\|_{H^2\cap L^1}. \tag{3.23}$$

The bound for S_3 is more complicated, we first decompose it as follows

$$S_3 \leq C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\|_{L^2}d\tau + C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(b\cdot\nabla b)}\|_{L^2}d\tau$$

$$
\begin{aligned}
&+C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\|_{L^2}d\tau + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\partial_3^2\widehat{(b\cdot\nabla b)}\|_{L^2}d\tau\\
&+C\int_0^t \|\widehat{K_1}(t-\tau)\partial_3^2\widehat{(u\cdot\nabla u)}\|_{L^2(A_{22})}d\tau + C\int_0^t \|\widehat{K_1}(t-\tau)\partial_3^2\widehat{(b\cdot\nabla b)}\|_{L^2(A_{22})}d\tau\\
:=&S_{31}+S_{32}+S_{33}+S_{34}+S_{35}+S_{36}.
\end{aligned} \tag{3.24}
$$

We further write

$$
\begin{aligned}
S_{31} &= C\int_0^{\frac{t}{2}} \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\|_{L^2}d\tau + C\int_{\frac{t}{2}}^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\|_{L^2}d\tau.\\
&= S_{311}+S_{312}.
\end{aligned}
$$

By Hölder's inequality and $e^{-c_0t}(1+t)^m \le C(c_0,m)$, for $t>0$, $m=\frac{15}{8}$, we deduce

$$
S_{311} \le C\int_0^{\frac{t}{2}} e^{-c_0(t-\tau)}\|\partial_3^2(u\cdot\nabla u)\|_{L^2}d\tau \le Ce^{-\frac{c_0t}{2}}\frac{t}{2}\varepsilon^2 \le C\varepsilon^2(1+t)^{-\frac{7}{8}}.
$$

By Hölder's inequality and Lemma 2.1,

$$
\begin{aligned}
S_{312} \le\,& C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}\left\|\|e^{-c_0\xi_h^2(t-\tau)}\partial_3^2(u\cdot\nabla u)\|_{L^2_{x_h}}\right\|_{L^2_{x_3}}d\tau\\
\le\,& C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\left\|\left(\|\partial_3^2u\cdot\nabla u\|_{L^1_{x_h}} + \|\partial_3u\cdot\nabla\partial_3u\|_{L^1_{x_h}} + \|u\cdot\nabla\partial_3^2u\|_{L^1_{x_h}}\right)\right\|_{L^2_{x_3}}d\tau\\
\le\,& C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\big\|\big(\|\partial_3^2u\|_{L^2_{x_h}}\|\nabla u\|_{L^2_{x_h}} + \|\partial_3u\|_{L^2_{x_h}}\|\nabla\partial_3u\|_{L^2_{x_h}} + \|u\|_{L^2_{x_h}}\\
&\times\|\nabla\partial_3^2u\|_{L^2_{x_h}}\big)\big\|_{L^2_{x_3}}d\tau\\
\le\,& C\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\big(\|\partial_3^2u\|_{L^2}^{\frac{1}{2}}\|\partial_3^3u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2} + \|\partial_3u\|_{L^2}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3^2u\|_{L^2}^{\frac{1}{2}}\\
&+\|u\|_{L^2}^{\frac{1}{2}}\|\partial_3u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3^2u\|_{L^2}\big)d\tau\\
\le\,& C(\tilde{c}^{\frac{3}{2}}+\tilde{c})\varepsilon^2\int_{\frac{t}{2}}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{7}{8}}d\tau \le C(\tilde{c}^{\frac{3}{2}}+\tilde{c})\varepsilon^2(1+t)^{-\frac{7}{8}}.
\end{aligned}
$$

Next we proceed to estimate S_{33}. By Lemma 3.2 and Hölder's inequality, we have

$$
\begin{aligned}
S_{33} &\le C\int_0^t \||\xi|^3e^{-c_0|\xi|^2(t-\tau)}\widehat{u\otimes u}\|_{L^2}d\tau \le C\int_0^t (1+t-\tau)^{-\frac{9}{4}}\|u\otimes u\|_{L^1}d\tau\\
&\le C\tilde{c}^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{9}{4}}(1+\tau)^{-1}d\tau \le C\tilde{c}^2\varepsilon^2(1+t)^{-1}.
\end{aligned}
$$

To bound S_{35}, we use new bounds in (3.19),

$$\begin{aligned} S_{35} \le\, & C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\|_{L^2}d\tau \\ & + C\int_0^t \Big\|\Big(\frac{\nu\eta\xi_h^2+\xi_3^2}{(\eta+\nu\xi_h^2)^2}+1\Big)e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\Big\|_{L^2}d\tau \\ & + C\int_0^t \Big\|\frac{1}{\eta+\nu\xi_h^2}\Big(\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}+\eta\Big)e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\Big\|_{L^2}d\tau \\ := & S_{351}+S_{352}+S_{353}. \end{aligned}$$

S_{351} obeys the same bounds as S_{31}. Since $\xi\in A_{22}$, we have $\nu\eta\xi_h^2+\xi_3^2<\frac{3}{16}(\eta+\nu\xi_h^2)^2$. By the same process as S_{31}, we deduce

$$S_{352} \le C\int_0^t \Big\|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\Big\|_{L^2}d\tau \le C(1+\tilde{c}+\tilde{c}^{\frac{3}{2}})\varepsilon^2(1+t)^{-\frac{7}{8}}.$$

We rewrite S_{353} into two parts,

$$\begin{aligned} S_{353} \le\, & C\int_0^t \Big\|\frac{|\xi|^2}{(1+\xi_h^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla u)}\Big\|_{L^2}d\tau \\ & + C\int_0^t \Big\|\frac{|\xi|}{1+\xi_h^2}e^{-c_0\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)}\partial_3\widehat{(u\cdot\nabla u)}\Big\|_{L^2}d\tau := S_{3531}+S_{3532}. \end{aligned}$$

Due to $e^{-c_0t}(1+t)^m\le C(c_0,m)$ for any $m>0$, $t>0$, we have

$$S_{3531} \le C\int_0^t \Big\|\frac{|\xi|^4}{(1+\xi_h^2)^2}(t-\tau)^2(t-\tau)^{-2}e^{-c_0\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)}\widehat{u\cdot\nabla u}\Big\|_{L^2}d\tau \le C\tilde{c}^2\varepsilon^2(1+t)^{-2}.$$

Now we turn to estimate S_{3532}. By Lemma 3.2 and Gagliardo–Nirenberg inequality, we obtain

$$S_{3532} \le C\int_0^t \Big\|\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)(t-\tau)^{-1}e^{-c_0\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)}\widehat{u\cdot\nabla u}\Big\|_{L^2}d\tau \le C\tilde{c}^2\varepsilon^2(1+t)^{-1}.$$

Consequently,

$$S_{35} \le C\varepsilon^2(1+t)^{-\frac{7}{8}}+C\tilde{c}\varepsilon^2(1+t)^{-\frac{7}{8}}+C\tilde{c}^{\frac{3}{2}}\varepsilon^2(1+t)^{-\frac{7}{8}}+C\tilde{c}^2\varepsilon^2(1+t)^{-\frac{7}{8}}.$$

By the same technique, S_{32}, S_{34}, S_{36} share similar estimates as S_{31}, S_{33}, S_{35}, respectively.
Substituting the bounds of $S_{31}-S_{36}$ into (3.24), yields

$$S_3 \le C(1+\tilde{c}+\tilde{c}^{\frac{3}{2}}+\tilde{c}^2)\varepsilon^2(1+t)^{-\frac{7}{8}}. \tag{3.25}$$

The estimation of many terms of S_4 is similar to that of S_3, which we first expand as follows

$$\begin{aligned}
S_4 \leq\ & C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla b)}\|_{L^2}d\tau + C\int_0^t \|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(b\cdot\nabla u)}\|_{L^2}d\tau\\
&+C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla b)}\|_{L^2}d\tau + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\partial_3^2\widehat{(b\cdot\nabla u)}\|_{L^2}d\tau\\
&+C\int_0^t \|\widehat{K_2}(t-\tau)\partial_3^2\widehat{(u\cdot\nabla b)}\|_{L^2(A_{22})}d\tau + C\int_0^t \|\widehat{K_2}(t-\tau)\partial_3^2\widehat{(b\cdot\nabla u)}\|_{L^2(A_{22})}d\tau\\
:=\ & S_{41}+S_{42}+S_{43}+S_{44}+S_{45}+S_{46}.
\end{aligned}$$

S_{41}, S_{42}, S_{43} and S_{44} can be estimated with a nearly same argument as S_{31}, S_{32}, S_{33} and S_{34}, respectively. We use new bounds to estimate H_{45},

$$\begin{aligned}
S_{45} \leq\ & C\int_0^t \Big\|\frac{|\xi_3|}{1+\xi_h^2}e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla b)}\Big\|_{L^2}d\tau\\
&+C\int_0^t \Big\|\frac{|\xi_3|}{1+\xi_h^2}e^{-\frac{\nu\eta\xi_h^2+\xi_3^2}{\eta+\nu\xi_h^2}(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla b)}\Big\|_{L^2}d\tau := S_{451}+S_{452}.
\end{aligned}$$

Since $\xi\in A_{22}$, $|\xi|^2\leq C(1+\xi_h^2)^2$. By the same process as S_{31},

$$S_{451}\leq C\int_0^t \Big\|e^{-c_0(1+\xi_h^2)(t-\tau)}\partial_3^2\widehat{(u\cdot\nabla b)}\Big\|_{L^2}d\tau \leq C(1+\tilde{c}+\tilde{c}^{\frac{3}{2}})\varepsilon^2(1+t)^{-\frac{7}{8}}.$$

By Lemma 3.2 and using the simple fact that $e^{-c_0t}(1+t)^m \leq C(c_0,m)$ for any $m>0, t>0$,

$$\begin{aligned}
S_{452} &\leq C\int_0^t \Big\|\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)(t-\tau)^{-1}e^{-c_0\frac{|\xi|^2}{1+\xi_h^2}(t-\tau)}\partial_3\widehat{(u\cdot\nabla b)}\Big\|_{L^2}d\tau\\
&\leq C\int_0^t (1+t-\tau)^{-1}\|\partial_3(u\cdot\nabla b)\|_{L^2}d\tau\\
&\leq C\int_0^t (1+t-\tau)^{-1}\big(\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\nabla\partial_3 u\|_{L^2}^{\frac{3}{4}}\|\nabla b\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla b\|_{L^2}^{\frac{3}{4}}\\
&\quad+\|\nabla u\|_{L^2}\|\partial_3\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla^2 b\|_{L^2}^{\frac{1}{2}}\big)d\tau\\
&\leq C\tilde{c}^2\varepsilon^2(1+t)^{-1}+C\tilde{c}^{\frac{3}{2}}\varepsilon^2(1+t)^{-1}.
\end{aligned}$$

Consequently,

$$S_4\leq C(1+\tilde{c}+\tilde{c}^{\frac{3}{2}}+\tilde{c}^2)\varepsilon^2(1+t)^{-\frac{7}{8}}. \tag{3.26}$$

Combining (3.22), (3.23), (3.25) and (3.26), we obtain

$$
\begin{aligned}
&\|\partial_3^2 u(t)\|_{L^2} \le C_1(1+t)^{-\frac{7}{8}}\|(u_0, b_0)\|_{L^1\cap H^2}\\
&+ C_2\varepsilon^2(1+t)^{-\frac{7}{8}} + C_3(\tilde{c}+\tilde{c}^{\frac{3}{2}}+\tilde{c}^2)\varepsilon^2(1+t)^{-\frac{7}{8}}.
\end{aligned}
$$

Therefore, if we choose $\tilde{c}$ and δ satisfying

$$
C_1 \le \frac{\tilde{c}}{8}, \quad C_2\varepsilon \le \frac{\tilde{c}}{16}, \quad C_3(\tilde{c}+\tilde{c}^{\frac{3}{2}}+\tilde{c}^2)\varepsilon \le \frac{\tilde{c}}{16},
$$

then we deduce

$$
\|\partial_3^2 u(t)\|_{L^2} \le \frac{\tilde{c}}{4}\varepsilon(1+t)^{-\frac{7}{8}}.
$$

The same upper bound holds for $\|\partial_3^2 b\|_{L^2}$. Thus,

$$
\|(\partial_3^2 u(t), \partial_3^2 b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\varepsilon(1+t)^{-\frac{7}{8}}.
$$

This completes the proof of the last inequality in (3.7).

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest regarding the publication of this paper.

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