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New Modular Equations of Composite Degrees and Partition Identities

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Abstract
In a recent study, Kim established a general identity which implies a generalization of the modular equations of degrees 3, 5, 11 and 23, and derived some identities for partitions. In this paper we provide proofs for some new modular equations of composite degrees and degree of 7 by methods of elementary algebra and Kim's generalization of theta-function identities. In addition, we derive many partition identities, which are proved depending upon these modular equations and reciprocation.

Keywords Partitions · Modular equations · Theta function identities

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1 Introduction

Ramanujan, a gifted mathematician, has done a lot of research in his life, one of which is the research on modular equation of degrees 3, 5 and 7. For each particular degree, Ramanujan derived a series of interesting identities concerned with theta-function of appropriate arguments. Professor Berndt sorted out a lot of work of Ramanujan, and found many modular equations of higher and composite degrees which are proved by employing the theory of theta-function identities, elementary algebra, geometrical construction and the theory of modular forms in [7] and [2]. By using the theory of vertex operator algebra, Milas [16] gave a new proof of the famous Ramanujan's modulus 5 modular equation from his "Lost Notebook".

Farkas and Kra [13] began an original study of partition identities arising from theta-function identities and established elegant theorem about colored partitions. These

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partition identities can also be proved by modular equations. Berndt realized that many of Ramanujan's modular equations yielded further interesting partition identities in [9]. Later, Baruah and Berndt continued to study the partition identity by using the modular equation in [4, 5]. Warnaar [20] generalized Farkas and Kra partition theorem, and Kim [14] gave a combinatorial proof with beautiful arguments. Sandon and Zanello conducted a series of studies on partitions, provided a unified combinatorial framework, and proposed 30 conjectures about colored partitions in [18] and [19]. These conjectures of Sandon and Zanello have been proved utilizing modular equations and theta-function identities by Berndt and the first author in the paper [10] and [11]. Baruah and Boruah [6] have also established many of the conjectures of Sandon and Zanello. Soon after, the first author [21] found some colored partition identities which did not belong to the general and unified combinatorial framework provided by Sandon and Zanello. In 2021, Kim [15] established a generalization of the modular equations of degrees 5, 11 and 23, which could be used to prove most of Sandon and Zanello's conjectures from a computational perspective. So far, no bijective proofs of Sandon and Zanello's conjectures has been found.

In this paper, we establish six modular equations of degrees 1, 3, 5, 15, and six modular equations of degrees 1, 3, 9, respectively. we also find two modular equations of degree 7 and formula for multiplier for degree 7. By employing elementary algebra and Kim's generalization in [15], we provide the proofs of those equations in Sect. 3. Next, proofs of twelve partition identities relying on new modular equations and reciprocation are also given in Sect. 4.

2 Preliminary Results

For any complex numbers a and $|q| < 1$, define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

and

$$[a_1,\ldots,a_n;q]_\infty = (a_1;q)_\infty(q/a_1;q)_\infty\ldots(a_n;q)_\infty(q/a_n;q)_\infty.$$

Recall that Ramanujan's theta-functions $\varphi(-q)$ and $f(-q)$, and his function $\chi(q)$ are defined by

$$\varphi(-q) := \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{2.1}$$

$$f(-q) := (q;q)_\infty, \tag{2.2}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{2.3}$$

The latter equality in (2.1) is a consequence of Jacobi's triple product identity. Recall the Euler's famous identity (see [1, 3, 12])

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty \tag{2.4}$$

i.e., the number of partitions of the positive integer n into odd parts is identical to the number of partitions of n into distinct parts. Naturally, we may get the following identity:

$$\chi(-q) = (q;q^2)_\infty = \frac{1}{(-q;q)_\infty}. \tag{2.5}$$

The complete elliptic integral of the first kind is defined for $|k| < 1$ by

$$K := K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

The number k is called the modulus. The complementary modulus k' is defined by $k' = \sqrt{1-k^2}$. Set $K' = K(k')$. Expanding the integrand in a binomial series and integrating termwise, we find that

$$K = \frac{\pi}{2}\, {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; k^2\Big),$$

where ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ denotes the ordinary hyper-geometric function.

In the context of the classical theory of elliptic functions, we recall some of the principal results [8, p. 123, Theorem 5.2.8]:

Set

$$x = k^2 = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{2.6}$$

If

$$q = e^{-y} := F(x) = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right) = \exp\left(-\pi \frac{K'}{K}\right), \tag{2.7}$$

then

$$\varphi^2(q) = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; x\Big) =: z, \tag{2.8}$$

where $\varphi(q)$ is defined by (2.1).

Let K, K', L, and L', denote the complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and $\ell' := \sqrt{1-\ell^2}$, respectively. Suppose that the equality

$$n\frac{K'}{K}=\frac{L'}{L} \tag{2.9}$$

holds for some positive integer n. A relation between k and ℓ induced by (2.9) is called a modular equation of degree n. Ramanujan recorded his modular equations in terms of α and β, where $\alpha=k^2$ and $\beta=\ell^2$. We often say that β has degree n over α.

If we further set $z_n:=\varphi^2(q^n)$, then the multiplier m of degree n is defined by

$$m:=\frac{z_1}{z_n}. \tag{2.10}$$

Theorem 2.1 *(Method of Reciprocation) If we replace α by $1-\beta$, β by $1-\alpha$, and m by n/m in a modular equation of degree n, then we obtain a new modular equation of the same degree.*

We need certain evaluations of Ramanujan for theta functions given in the following lemma [7, p. 124, Entry 12], [8, p. 127].

Lemma 2.2 *If α, q, and z are related by (2.6), (2.7) and (2.8), then*

$$f(-q)=2^{-1/6}\sqrt{z}(1-\alpha)^{1/6}(\alpha/q)^{1/24}, \tag{2.11}$$
$$f(-q^2)=2^{-1/3}\sqrt{z}\{\alpha(1-\alpha)/q\}^{1/12}, \tag{2.12}$$
$$f(-q^4)=2^{-2/3}\sqrt{z}(\alpha/q)^{1/6}(1-\alpha)^{1/24}, \tag{2.13}$$
$$\chi(q)=2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}, \tag{2.14}$$
$$\chi(-q)=2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}, \tag{2.15}$$
$$\chi(-q^2)=2^{1/3}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}. \tag{2.16}$$

Suppose that β has degree n over α. If we replace q by q^n above, then the same evaluations hold with α replaced by β and with $z=z_1$ replaced by z_n.

Recall the principle of duplication and the principle of dimidiation [7, p. 125], [8, p. 125], [17]:

Define, for $0<x'<1$,

$$x'=\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2 \tag{2.17}$$

from which it follows that

$$x:=\frac{4\sqrt{x'}}{(1+\sqrt{x'})^2}. \tag{2.18}$$

Furthermore, define

$$e^{-y'}:=F(x') \quad \text{and} \quad z':={}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x'\Big),$$

and there are the following formulas:

$$e^{-y} = e^{-y'/2}, \quad z = (1+\sqrt{x'})z' \quad \text{and} \quad z' = \frac{1}{2}(1+\sqrt{1-x})z. \tag{2.19}$$

Theorem 2.3 *Suppose that two sets of parameters,x, y, z and x', y', z', are related by the Eqs. (2.6)–(2.8) with x, y, z replaced x', y', z', respectively. Suppose they satisfy an equation of the form*

$$\Omega\big(x', y', z'\big) = 0,$$

and x is related to x' by (2.18). Then, by (2.17) and (2.19), we obtain an equation of the form

$$\Omega\left(\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, 2y, \frac{1}{2}(1+\sqrt{1-x})z\right) = 0, \tag{2.20}$$

Applying the principle of duplication to (2.14) and (2.16), we derive that

$$\chi(q^2) = 2^{1/12}(1+\sqrt{1-\alpha})^{1/4}(\alpha/q)^{-1/12}(1-\alpha)^{-1/48}, \tag{2.21}$$

$$\chi(-q^4) = 2^{5/12}(1+\sqrt{1-\alpha})^{1/4}(\alpha/q)^{-1/6}(1-\alpha)^{1/48}. \tag{2.22}$$

Theorem 2.4 *Suppose that two sets of parameters,x, y, z and x', y', z', are related by the Eqs. (2.6)–(2.8) with x, y, z replaced x', y', z', respectively. Suppose they satisfy an equation of the form*

$$\Omega\big(x', y', z'\big) = 0,$$

and we reverse the roles of x, y, z with those of x', y', z', respectively. Then, by (2.18). and (2.19), we obtain an equation of the form

$$\Omega\left(\frac{4\sqrt{x}}{(1+\sqrt{x})^2}, y/2, (1+\sqrt{x})z\right) = 0, \tag{2.23}$$

Applying the principle of dimidiation to (2.14) and (2.15), we derive that

$$\chi(q^{1/2}) = 2^{1/12}(1+\sqrt{\alpha})^{1/4}(\alpha/q)^{-1/48}(1-\alpha)^{-1/12}, \tag{2.24}$$

$$\chi(-q^{1/2}) = 2^{1/12}(1-\sqrt{\alpha})^{1/4}(\alpha/q)^{-1/48}(1-\alpha)^{-1/12}. \tag{2.25}$$

3 Some New Modular Equations

Theorem 3.1 *Let α, β, γ, and δ be of the first, third, fifth, and fifteenth degrees, respectively. Let m denote the multiplier connecting α and β, and m' be the multiplier relating γ and δ. Then*

$$(i)\quad \left(\frac{m}{m'}\right)^{3/2}=\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{3/8}-6\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}$$
$$-\left(\frac{\beta\gamma}{\alpha\delta}\right)^{3/8}-\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{3/8}. \tag{3.1}$$

$$(ii)\quad -\left(\frac{m}{m'}\right)^{1/2}=\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4}$$
$$+\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}\left(3-\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}-\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}\right). \tag{3.2}$$

$$(iii)\quad \left(\frac{m'}{m}\right)^{3/2}=\left(\frac{\alpha\delta}{\beta\gamma}\right)^{3/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{3/8}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{3/8}$$
$$+6\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4}. \tag{3.3}$$

$$(iv)\quad \left(\frac{m}{m'}\right)^{1/2}+\frac{m}{m'}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}=2\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4}$$
$$+\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{3/8}-\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{3/8}. \tag{3.4}$$

$$(v)\quad \left(\frac{m}{m'}\right)^{1/2}+2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}=-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{3/16}$$
$$\times\left(\left(\frac{(1-\sqrt{1-\beta})(1-\sqrt{1-\gamma})}{(1-\sqrt{1-\alpha})(1-\sqrt{1-\delta})}\right)^{1/4}-\left(\frac{(1+\sqrt{1-\beta})(1+\sqrt{1-\gamma})}{(1+\sqrt{1-\alpha})(1+\sqrt{1-\delta})}\right)^{1/4}\right). \tag{3.5}$$

$$(vi)\quad \frac{z_1}{z_5}=\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}\left(\frac{1+\sqrt{1-\gamma}}{1+\sqrt{1-\alpha}}\right)^{1/2}-4\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}$$
$$-\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}\left(\frac{1-\sqrt{1-\gamma}}{1-\sqrt{1-\alpha}}\right)^{1/2}+\left(\frac{\gamma}{\alpha}\right)^{1/2}. \tag{3.6}$$

Proof (i) Recall the partition identity given by [15]

$$[-q,-q^7,-q^{11},-q^{13};q^{30}]_\infty^3-[q,q^7,q^{11},q^{13};q^{30}]_\infty^3$$
$$=6q+q^3\left([-q^2,-q^4,-q^8,-q^{14};q^{30}]_\infty^3+[q^2,q^4,q^8,q^{14};q^{30}]_\infty^3\right). \tag{3.7}$$

In order to find the characteristics of the identity, we organize it into the following form

$$\frac{(-q;q^2)_\infty^3(-q^{15};q^{30})_\infty^3}{(-q^3;q^6)_\infty^3(-q^5;q^{10})_\infty^3}-\frac{(q;q^2)_\infty^3(q^{15};q^{30})_\infty^3}{(q^3;q^6)_\infty^3(q^5;q^{10})_\infty^3}=6q$$
$$+q^3\left(\frac{(-q^2;q^2)_\infty^3(-q^{30};q^{30})_\infty^3}{(-q^6;q^6)_\infty^3(-q^{10};q^{10})_\infty^3}+\frac{(q^2;q^2)_\infty^3(q^{30};q^{30})_\infty^3}{(q^6;q^6)_\infty^3(q^{10};q^{10})_\infty^3}\right),$$

which can be transformed into

$$\frac{\chi^3(q)\chi^3(q^{15})}{\chi^3(q^3)\chi^3(q^5)}-\frac{\chi^3(-q)\chi^3(-q^{15})}{\chi^3(-q^3)\chi^3(-q^5)}=6q+q^3\left(\frac{\chi^3(-q^6)\chi^3(-q^{10})}{\chi^3(-q^2)\chi^3(-q^{30})}+\frac{f^3(-q^2)f^3(-q^{30})}{f^3(-q^6)f^3(-q^{10})}\right),$$

by (2.2), (2.3), and (2.5). Applying (2.12), (2.14)–(2.16) in Lemma 2.2, we obtain

$$\begin{aligned}&\frac{\{2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}\}^3\{2^{1/6}\{\delta(1-\delta)/q^{15}\}^{-1/24}\}^3}{\{2^{1/6}\{\beta(1-\beta)/q^3\}^{-1/24}\}^3\{2^{1/6}\{\gamma(1-\gamma)/q^5\}^{-1/24}\}^3}\\&-\frac{\{2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}\}^3\{2^{1/6}(1-\delta)^{1/12}(\delta/q^{15})^{-1/24}\}^3}{\{2^{1/6}(1-\beta)^{1/12}(\beta/q^3)^{-1/24}\}^3\{2^{1/6}(1-\gamma)^{1/12}(\gamma/q^5)^{-1/24}\}^3}=6q\\&+q^3\left\{\frac{\{2^{1/3}(1-\beta)^{1/24}(\beta/q^3)^{-1/12}\}^3\{2^{1/3}(1-\gamma)^{1/24}(\gamma/q^5)^{-1/12}\}^3}{\{2^{1/3}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}\}^3\{2^{1/3}(1-\delta)^{1/24}(\delta/q^{15})^{-1/12}\}^3}\right.\\&\left.+\frac{\{2^{-1/3}\sqrt{z_1}\{\alpha(1-\alpha)/q\}^{1/12}\}^3\{2^{-1/3}\sqrt{z_{15}}\{\delta(1-\delta)/q^{15}\}^{1/12}\}^3}{\{2^{-1/3}\sqrt{z_3}\{\beta(1-\beta)/q^3\}^{1/12}\}^3\{2^{-1/3}\sqrt{z_5}\{\gamma(1-\gamma)/q^5\}^{1/12}\}^3}\right\}.\end{aligned}$$

Multiplying both sides of the last identity by $q^{-1}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}$, we get

$$\begin{aligned}&\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{3/8}-\left(\frac{\beta\gamma}{\alpha\delta}\right)^{3/8}=6\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}\\&+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{3/8}+\left(\frac{z_1z_{15}}{z_3z_5}\right)^{3/2}\end{aligned}$$

from which (3.1) is apparent.

(ii) For brevity, let $A=(\frac{\beta\gamma}{\alpha\delta})^{1/8}$, $B=\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\}^{1/8}$ and $M=\sqrt{\frac{m}{m'}}$. we rewrite the identity (3.1) as follows:

$$M^3=(AB)^3-6(AB)^2-A^3-B^3. \tag{3.8}$$

Recall the modular equation given by [7, p. 384, Entry 11(ix)]

$$-M=A+B-AB. \tag{3.9}$$

Taking the third power of this identity (3.9), and adding the Eq. (3.8), after some algebraic manipulation and simplification, we deduce that

$$4AB=A+B-A^2-B^2+(A+B)AB. \tag{3.10}$$

Using the identity (3.9), we can finish the proof of (3.2).

(iii) Rewrite the modular equation given by [7, p. 384, Entry 11(viii)] as

$$\frac{1}{M}=\frac{1}{A}+\frac{1}{B}-\frac{1}{AB}. \tag{3.11}$$

Multiply both sides of the identity (3.11) by AB, and then take the cube of the resulting identity,

$$(AB)^3/M^3=A^3+B^3-1+3(A+B)\{AB-(A+B)+1\}. \tag{3.12}$$

Applying the Eq. (3.10), we obtain the following identity

$$(AB)^3/M^3 = A^3 + B^3 - 1 + 6AB.$$

Dividing both sides of the last identity by $(AB)^3$, we can complete the proof of (3.3).

(iv) Recall the partition identity given by [15]

$$\begin{aligned}
&[-q, -q^7, -q^{11}, -q^{13}; q^{30}]_\infty^2[-q^2, -q^4, -q^8, -q^{14}; q^{30}]_\infty \\
&\quad +[q, q^7, q^{11}, q^{13}; q^{30}]_\infty^2[q^2, q^4, q^8, q^{14}; q^{30}]_\infty \\
&\quad = 2 + q\Big([-q, -q^7, -q^{11}, -q^{13}; q^{30}]_\infty[-q^2, -q^4, -q^8, -q^{14}; q^{30}]_\infty^2 \\
&\qquad -[q, q^7, q^{11}, q^{13}; q^{30}]_\infty[q^2, q^4, q^8, q^{14}; q^{30}]_\infty^2\Big). \qquad (3.13)
\end{aligned}$$

The identity (3.13) can be expressed in the following form

$$\begin{aligned}
&\frac{(-q;q)_\infty(-q^{15};q^{15})_\infty(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^3)_\infty(-q^5;q^5)_\infty(-q^3;q^6)_\infty(-q^5;q^{10})_\infty} \\
&\quad +\frac{(q;q)_\infty(q^{15};q^{15})_\infty(q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^3;q^3)_\infty(q^5;q^5)_\infty(q^3;q^6)_\infty(q^5;q^{10})_\infty} = 2 \\
&\quad +q\left(\frac{(-q;q)_\infty(-q^{15};q^{15})_\infty(-q^2;q^2)_\infty(-q^{30};q^{30})_\infty}{(-q^3;q^3)_\infty(-q^5;q^5)_\infty(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty}\right. \\
&\quad \left.-\frac{(q;q)_\infty(q^{15};q^{15})_\infty(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty(q^5;q^5)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty}\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{\chi(-q^3)\chi(-q^5)\chi(q)\chi(q^{15})}{\chi(-q)\chi(-q^{15})\chi(q^3)\chi(q^5)} + \frac{f(-q)f(-q^{15})\chi(-q)\chi(-q^{15})}{f(-q^3)f(-q^5)\chi(-q^3)\chi(-q^5)} = 2 \\
&\quad +q\left(\frac{\chi(-q^3)\chi(-q^5)\chi(-q^6)\chi(-q^{10})}{\chi(-q)\chi(-q^{15})\chi(-q^2)\chi(-q^{30})} - \frac{f(-q)f(-q^{15})f(-q^2)f(-q^{30})}{f(-q^3)f(-q^5)f(-q^6)f(-q^{10})}\right),
\end{aligned}$$

by (2.2), (2.3), and (2.5). Utilizing (2.11), (2.12), (2.14)–(2.16) in Lemma 2.2, we deduce that

$$\begin{aligned}
&\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} + \sqrt{\frac{z_1 z_{15}}{z_3 z_5}}\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4} = 2 \\
&\quad +\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \frac{z_1 z_{15}}{z_3 z_5}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4}
\end{aligned}$$

which yields the desired result upon rearrangement.

(v) Consider the partition identity given by [15]

$$
\begin{aligned}
&[-q,-q^2,-q^7,-q^{11},-q^{13},-q^{14},-q^{17},-q^{19},-q^{22},-q^{23},-q^{26},-q^{29};q^{60}]_\infty\\
&\quad -[q,q^2,q^7,q^{11},q^{13},q^{14},q^{17},q^{19},q^{22},q^{23},q^{26},q^{29};q^{60}]_\infty\\
&\quad = 2q+q^2\Big([-q,-q^4,-q^7,-q^8,-q^{11},-q^{13},-q^{16},-q^{17},-q^{19},-q^{23},-q^{28},-q^{29};q^{60}]_\infty\\
&\quad +[q,q^4,q^7,q^8,q^{11},q^{13},q^{16},q^{17},q^{19},q^{23},q^{28},q^{29};q^{60}]_\infty\Big).
\end{aligned}
\tag{3.14}
$$

which can be transformed into

$$
\begin{aligned}
&\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty(-q^2;q^4)_\infty(-q^{30};q^{60})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty(-q^6;q^{12})_\infty(-q^{10};q^{20})_\infty}\\
&\quad -\frac{(q;q^2)_\infty(q^{15};q^{30})_\infty(q^2;q^4)_\infty(q^{30};q^{60})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty(q^6;q^{12})_\infty(q^{10};q^{20})_\infty}=2q\\
&\quad +q^2\left(\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty(-q^4;q^4)_\infty(-q^{60};q^{60})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty(-q^{12};q^{12})_\infty(-q^{20};q^{20})_\infty}\right.\\
&\quad \left.+\frac{(q;q^2)_\infty(q^{15};q^{30})_\infty(q^4;q^4)_\infty(q^{60};q^{60})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty(q^{12};q^{12})_\infty(q^{20};q^{20})_\infty}\right).
\end{aligned}
$$

Use (2.2), (2.3), and (2.5) to obtain the equation

$$
\begin{aligned}
&\frac{\chi(q)\chi(q^{15})\chi(q^2)\chi(q^{30})}{\chi(q^3)\chi(q^5)\chi(q^6)\chi(q^{10})}-\frac{\chi(-q)\chi(-q^{15})\chi(-q^2)\chi(-q^{30})}{\chi(-q^3)\chi(-q^5)\chi(-q^6)\chi(-q^{10})}=2q\\
&\quad +q^2\left(\frac{\chi(q)\chi(q^{15})\chi(-q^{12})\chi(-q^{20})}{\chi(q^3)\chi(q^5)\chi(-q^4)\chi(-q^{60})}+\frac{\chi(-q)\chi(-q^{15})f(-q^4)f(-q^{60})}{\chi(-q^3)\chi(-q^5)f(-q^{12})f(-q^{20})}\right).
\end{aligned}
$$

We utilize (2.13)–(2.16) in Lemma 2.2, and the identities (2.21) and (2.22), and eliminate q, rearrange terms, simplify to find that

$$
\begin{aligned}
&\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/16}\left(\frac{(1+\sqrt{1-\alpha})(1+\sqrt{1-\delta})}{(1+\sqrt{1-\beta})(1+\sqrt{1-\gamma})}\right)^{1/4}\\
&\quad -\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}=2\\
&\quad +\left(\frac{(1+\sqrt{1-\beta})(1+\sqrt{1-\gamma})}{(1+\sqrt{1-\alpha})(1+\sqrt{1-\delta})}\right)^{1/4}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/16}\\
&\quad +\sqrt{\frac{z_1z_{15}}{z_3z_5}}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}.
\end{aligned}
$$

Thus, (3.5) is evident.

(vi) Consider the partition identity given by [15]

$$[-q,-q^3,-q^4,-q^7,-q^8,-q^9;q^{20}]_\infty^2-[q,q^3,q^4,q^7,q^8,q^9;q^{20}]_\infty^2=4q$$
$$+q^2\Big([-q,-q^2,-q^3,-q^6,-q^7,-q^9;q^{20}]_\infty^2-[q,q^2,q^3,q^6,q^7,q^9;q^{20}]_\infty^2\Big). \tag{3.15}$$

The identity (3.15) can be expressed in the following form

$$\frac{(-q;q)_\infty^2}{(-q^5;q^{10})_\infty^2(-q^2;q^4)_\infty^2(-q^{20};q^{20})_\infty^2}-\frac{(q;q)_\infty^2}{(q^5;q^{10})_\infty^2(q^2;q^4)_\infty^2(q^{20};q^{20})_\infty^2}=4q$$
$$+q^2\left(\frac{(-q;q)_\infty^2}{(-q^5;q^{10})_\infty^2(-q^4;q^4)_\infty^2(-q^{10};q^{20})_\infty^2}-\frac{(q;q)_\infty^2}{(q^5;q^{10})_\infty^2(q^4;q^4)_\infty^2(q^{10};q^{20})_\infty^2}\right).$$

Substituting (2.2), (2.3), and (2.5) into foregoing equality, we arrive at

$$\frac{\chi^2(-q^{20})}{\chi^2(-q)\chi^2(q^5)\chi^2(q^2)}-\frac{f^2(-q)}{\chi^2(-q^5)\chi^2(-q^2)f^2(-q^{20})}=4q$$
$$+q^2\left(\frac{\chi^2(-q^4)}{\chi^2(q^5)\chi^2(-q)\chi^2(q^{10})}-\frac{f^2(-q)}{\chi^2(-q^5)f^2(-q^4)\chi^2(-q^{10})}\right).$$

Applying (2.11), (2.13)–(2.16) in Lemma 2.2, as well as the identities (2.21), and (2.22), and dividing both sides of the resulting identity by q, we deduce that

$$\left(\frac{1+\sqrt{1-\gamma}}{1+\sqrt{1-\alpha}}\right)^{1/2}\left(\frac{\alpha}{\gamma}\right)^{1/4}\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-\frac{z_1}{z_5}\left(\frac{\alpha}{\gamma}\right)^{1/4}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}=4$$
$$+\left(\frac{1+\sqrt{1-\alpha}}{1+\sqrt{1-\gamma}}\right)^{1/2}\left(\frac{\gamma}{\alpha}\right)^{1/4}\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-\left(\frac{\gamma}{\alpha}\right)^{1/4}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}.$$

By elementary algebra, we can complete the proof of (3.6). □

Theorem 3.2 *Let β and γ be of the third and ninth degrees, respectively, with respect to α. Let $m=z_1/z_3$ and $m'=z_3/z_9$, Then the following modular equations are valid:*

$$(i)\quad -3\frac{m}{m'}=\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/2}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/2}$$
$$+\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}\left(5-\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}-\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}\right). \tag{3.16}$$

$$(ii)\quad 27\left(\frac{m}{m'}\right)^3=\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{3/4}-12\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/2}$$
$$-\left(\frac{\beta^2}{\alpha\gamma}\right)^{3/4}-\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{3/4}. \tag{3.17}$$

$$(iii)\quad \left(\frac{m'}{m}\right)^3=\left(\frac{\alpha\gamma}{\beta^2}\right)^{3/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{3/4}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{3/4}$$

$$+12\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/2}. \tag{3.18}$$

$$(iv)\quad \frac{z_1}{z_9}-\sqrt{\frac{z_1}{z_9}}\left(\frac{\gamma}{\alpha}\right)^{1/8}=\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}-2\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}\left(\frac{\gamma}{\alpha}\right)^{1/8}-\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}\left(\frac{\gamma}{\alpha}\right)^{1/8}. \tag{3.19}$$

$$(v)\quad \frac{z_1}{z_3}=\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}\left(\left(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\right)^{1/2}-\left(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\right)^{1/2}\right)-\left(\frac{\beta}{\alpha}\right)^{1/2}. \tag{3.20}$$

$$(vi)\quad \sqrt{\frac{z_1}{z_9}}=\left(\frac{1-\gamma}{1-\alpha}\right)^{3/16}\left(\frac{1+\sqrt{1-\gamma}}{1+\sqrt{1-\alpha}}\right)^{1/4}-2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}$$
$$-\left(\frac{1-\gamma}{1-\alpha}\right)^{3/16}\left(\frac{1-\sqrt{1-\gamma}}{1-\sqrt{1-\alpha}}\right)^{1/4}+\left(\frac{\gamma}{\alpha}\right)^{1/4}. \tag{3.21}$$

Proof (i) For brevity, let $a=\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}$, $b=\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}$ and $M=m/m'$. Recall the identities given by [7, p. 232, (5.1)]

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8}=\frac{3+m}{2m},\quad \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}=\frac{3-m}{2m}, \tag{3.22}$$
$$\left(\frac{\beta^3}{\alpha}\right)^{1/8}=\frac{m-1}{2},\quad \left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}=\frac{m+1}{2}. \tag{3.23}$$

Employing the last identities, we find that

$$a=\left(\frac{\beta^3}{\alpha}\right)^{1/4}\left(\beta\gamma\right)^{-1/4}=\frac{(m-1)^2m'}{(3+m')(m'-1)},$$
$$b=\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4}\left((1-\beta)(1-\gamma)\right)^{-1/4}=\frac{(m+1)^2m'}{(3-m')(m'+1)},$$

In order to prove these formulas, we first need to express m and m' as rational functions of a parameter t given by [7, p. 354, (3.10) and (3.11)]

$$m^2=\frac{(1+2t)^4}{1+8t^3}\quad\text{and}\quad m'^2=1+8t^3. \tag{3.24}$$

It is easy to check that

$$a+b=\frac{1+2t}{t(1-t)}\quad\text{and}\quad ab=\frac{(1+2t)^2(t^2+t+1)}{t(1-t)(1+8t^3)}. \tag{3.25}$$

Now, pay attention to the following identity

$$a+b-(a+b)^2+(a+b)ab=(a+b)(1-a-b+ab)=4ab \tag{3.26}$$

where we have employed (3.25). Invoking the identity $a+b-ab=-3M$ by [7, p. 352, Entry 3(xii)], and substituting it into the identity (3.26), we can complete the proof of (3.16).

(ii) We consider the identity

$$a+b-ab=-3M \tag{3.27}$$

which when cubed yields

$$a^3+b^3-(ab)^3+3ab(a+b)(1-a-b+ab)=-27M^3.$$

Applying the identity (3.26), we achieve the desired result.

(iii) Rewrite the identity given by [7, p. 353, Entry 3(xiii)]

$$1/a+1/b-1/ab=1/M \tag{3.28}$$

Multiply both sides of the identity (3.28) by ab, take the third power of the resulting identity, and use the Eq. (3.26) to deduce that

$$a^3+b^3-1+12ab=\frac{(ab)^3}{M^3}. \tag{3.29}$$

Divide both sides of the foregoing equality by $(ab)^3$. Thus, the truth of (3.18) is manifest.

(iv) Since β and γ are of the third degree in α and β, respectively, it follows from (3.22) and (3.23) that

$$\left(\frac{\gamma}{\alpha}\right)^{1/8}=\left(\frac{\beta^3}{\alpha}\right)^{1/8}\left(\frac{\gamma}{\beta^3}\right)^{1/8}=\frac{m'(m-1)}{3+m'}$$
$$\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}=\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}\left(\frac{1-\gamma}{(1-\beta)^3}\right)^{1/8}=\frac{m'(m+1)}{3-m'}.$$

Applying (3.24), we can check that

$$\left(\frac{\gamma}{\alpha}\right)^{1/8}=\frac{1+2t}{2(1-t)}-\frac{\sqrt{1+8t^3}}{2(1-t)} \tag{3.30}$$
$$\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}=\frac{1+2t}{2(1-t)}+\frac{\sqrt{1+8t^3}}{2(1-t)}. \tag{3.31}$$

Employing (3.30) and (3.31), we find after a considerable amount of elementary algebra that

$$\begin{aligned}&\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}\left(\frac{\gamma}{\alpha}\right)^{1/8}-\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}+\frac{z_1}{z_9}\\&=\left(\frac{1+2t}{2(1-t)}+\frac{\sqrt{1+8t^3}}{2(1-t)}\right)^2\frac{t(1+2t)}{1-t}-\left(\frac{1+2t}{2(1-t)}+\frac{\sqrt{1+8t^3}}{2(1-t)}\right)^3+(1+2t)^2\\&=\frac{(1+2t)^2(1-3t)}{2(1-t)^2}-\frac{\sqrt{1+8t^3}(1+2t)(1+t)}{2(1-t)^2}.\end{aligned}$$

On the other hand, we can also check that

$$\begin{aligned}
&\sqrt{\frac{z_1}{z_9}}\Big(\frac{\gamma}{\alpha}\Big)^{1/8}-2\Big(\frac{1-\gamma}{1-\alpha}\Big)^{1/4}\Big(\frac{\gamma}{\alpha}\Big)^{1/8}\\
&=(1+2t)\bigg(\frac{1+2t}{2(1-t)}-\frac{\sqrt{1+8t^3}}{2(1-t)}\bigg)-\frac{2t(1+2t)}{1-t}\bigg(\frac{1+2t}{2(1-t)}+\frac{\sqrt{1+8t^3}}{2(1-t)}\bigg)\\
&=\frac{(1+2t)^2(1-3t)}{2(1-t)^2}-\frac{\sqrt{1+8t^3}(1+2t)(1+t)}{2(1-t)^2}.
\end{aligned}$$

Hence, the identity (3.19) can be established.

(v) By simple elementary algebra, (3.22) and (3.23), we can easily check that

$$1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}=2\big(\alpha\beta(1-\alpha)(1-\beta)\big)^{1/4}. \tag{3.32}$$

Multiplying both sides of (3.32) by 2 and extracting the square root on both sides of the resulting identity, we find that

$$\begin{aligned}
&\big((1+\sqrt{1-\beta})(1-\sqrt{1-\alpha})\big)^{1/2}-\big((1-\sqrt{1-\beta})(1+\sqrt{1-\alpha})\big)^{1/2}\\
&=2\big(\alpha\beta(1-\alpha)(1-\beta)\big)^{1/8}.
\end{aligned}$$

Thus, we obtain the following identity:

$$\bigg(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\bigg)^{1/2}-\bigg(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\bigg)^{1/2}=\frac{(2-2\sqrt{\alpha\beta}-2\sqrt{(1-\alpha)(1-\beta)})^{1/2}}{\alpha^{1/2}} \tag{3.33}$$

From (3.22) and (3.23)

$$m=2\Big(\frac{\beta}{\alpha^3}\Big)^{\frac18}\Big(\frac{(1-\beta)^3}{1-\alpha}\Big)^{\frac18}-\Big(\frac{\beta}{\alpha}\Big)^{\frac12}=\Big(\frac{1-\beta}{1-\alpha}\Big)^{\frac14}\bigg(\frac{2(\alpha\beta(1-\alpha)(1-\beta))^{1/8}}{\alpha^{\frac12}}\bigg)-\Big(\frac{\beta}{\alpha}\Big)^{\frac12}. \tag{3.34}$$

The truth of the equality (3.20) is now manifest from (3.32), (3.33) and (3.34).

(vi) Recall the partition identity given by [15]

$$\begin{aligned}
\frac{(-q;q)_\infty(-q^{18};q^{36})_\infty}{(-q^2;q^4)_\infty(-q^9;q^9)_\infty}-\frac{(q;q)_\infty(q^{18};q^{36})_\infty}{(q^2;q^4)_\infty(q^9;q^9)_\infty}&=2q\\
+q^2\bigg(\frac{(-q;q)_\infty(-q^{36};q^{36})_\infty}{(-q^4;q^4)_\infty(-q^9;q^9)_\infty}&-\frac{(q;q)_\infty(q^{36};q^{36})_\infty}{(q^4;q^4)_\infty(q^9;q^9)_\infty}\bigg).
\end{aligned} \tag{3.35}$$

which is equivalent to

$$\frac{\chi(-q^9)\chi(q^{18})}{\chi(-q)\chi(q^2)}-\frac{f(-q)\chi(-q^{18})}{f(-q^9)\chi(-q^2)}=2q+q^2\bigg(\frac{\chi(-q^9)\chi(-q^4)}{\chi(-q)\chi(-q^{36})}-\frac{f(-q)f(-q^{36})}{f(-q^4)f(-q^9)}\bigg).$$

by (2.2), (2.3), and (2.5). Applying (2.11), (2.13), (2.15), (2.16) in Lemma 2.2, and the identities (2.21) and (2.22), and eliminating q from the resulting identity, we check that

$$\left(\frac{1+\sqrt{1-\gamma}}{1+\sqrt{1-\alpha}}\right)^{1/4}\left(\frac{\alpha}{\gamma}\right)^{1/8}\left(\frac{1-\gamma}{1-\alpha}\right)^{1/16}-\sqrt{\frac{z_1}{z_9}}\left(\frac{\alpha}{\gamma}\right)^{1/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}=2$$
$$+\left(\frac{1+\sqrt{1-\alpha}}{1+\sqrt{1-\gamma}}\right)^{1/4}\left(\frac{\gamma}{\alpha}\right)^{1/8}\left(\frac{1-\gamma}{1-\alpha}\right)^{1/16}-\left(\frac{\gamma}{\alpha}\right)^{1/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}.$$

By elementary algebra, we easily deduce the result claimed in (3.21). □

Theorem 3.3 *If β is of the seventh degree in α, and m is the multiplier for degree 7, then*

$$(i)\quad \frac{(1+\sqrt{1-\alpha})^{1/4}(1+\sqrt{1-\beta})^{1/4}}{(\alpha\beta)^{1/8}((1-\alpha)(1-\beta))^{1/16}}-\frac{(1-\sqrt{1-\alpha})^{1/4}(1-\sqrt{1-\beta})^{1/4}}{(\alpha\beta)^{1/8}((1-\alpha)(1-\beta))^{1/16}}$$
$$=\sqrt{2}\left(1+\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/8}\right). \tag{3.36}$$
$$(ii)\quad m=\frac{z_1}{z_7}=\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}\left(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\right)^{1/2}+4\left(\frac{\beta}{\alpha^7}\right)^{1/24}\left(\frac{1-\beta}{1-\alpha}\right)^{1/6}$$
$$-\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}\left(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\right)^{1/2}-\left(\frac{\beta}{\alpha}\right)^{1/2}. \tag{3.37}$$

Proof (i) Recall the partition identity given by [15]

$$\frac{(-q;q)_\infty(-q^7;q^7)_\infty}{(-q^4;q^4)_\infty(-q^{28};q^{28})_\infty}-\frac{(q;q)_\infty(q^7;q^7)_\infty}{(q^4;q^4)_\infty(q^{28};q^{28})_\infty}=2q+2q^2\frac{(-q;q)_\infty(-q^7;q^7)_\infty}{(-q^2;q^4)_\infty(-q^{14};q^{28})_\infty}. \tag{3.38}$$

Employing (2.2), (2.3), and (2.5), we find that

$$\frac{\chi(-q^4)\chi(-q^{28})}{\chi(-q)\chi(-q^7)}-\frac{f(-q)f(-q^7)}{f(-q^4)f(-q^{28})}=2q+\frac{2q^2}{\chi(-q)\chi(-q^7)\chi(q^2)\chi(q^{14})}.$$

Applying (2.11), (2.13), (2.15) in Lemma 2.2, and the identities (2.21), and (2.22), and eliminating $\sqrt{2}q$, we check that

$$\frac{(1+\sqrt{1-\alpha})^{1/4}(1+\sqrt{1-\beta})^{1/4}}{(\alpha\beta)^{1/8}((1-\alpha)(1-\beta))^{1/16}}-\sqrt{2}\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/8}=\sqrt{2}$$
$$+\frac{(\alpha\beta)^{1/8}}{((1-\alpha)(1-\beta))^{1/16}(1+\sqrt{1-\alpha})^{1/4}(1+\sqrt{1-\beta})^{1/4}}$$

Using the square difference formula for the last equation, we arrive at the Eq. (3.36).

(ii) Consider Theorem 2.1 given by [15]

$$\begin{aligned}
&[-c^2, -ac^2, -bc^2, -abc^2, -d^2, -ad^2, -bd^2, -abd^2, -cd, -acd, -bcd, -abcd; q^2]_\infty \\
&\quad -[c^2, ac^2, bc^2, abc^2, d^2, ad^2, bd^2, abd^2, cd, acd, bcd, abcd; q^2]_\infty \\
&\quad = 2c^2 \frac{[a^2, b^2, (d/c)^3, (abc^2d^2)^2/q^2; q^2]_\infty}{[a, b, d/c, abc^2d^2; q^2]_\infty} \\
&\quad + \frac{(abc^2d^2)^3}{q^3}\left(\left[\frac{-q}{c^2}, \frac{-q}{ac^2}, \frac{-q}{bc^2}, \frac{-q}{abc^2}, \frac{-q}{d^2}, \frac{-q}{ad^2}, \frac{-q}{bd^2}, \frac{-q}{abd^2}, \frac{-q}{cd}, \frac{-q}{acd}, \frac{-q}{bcd}, \frac{-q}{abcd}; q^2\right]_\infty\right. \\
&\quad \left. -\left[\frac{q}{c^2}, \frac{q}{ac^2}, \frac{q}{bc^2}, \frac{q}{abc^2}, \frac{q}{d^2}, \frac{q}{ad^2}, \frac{q}{bd^2}, \frac{q}{abd^2}, \frac{q}{cd}, \frac{q}{acd}, \frac{q}{bcd}, \frac{q}{abcd}; q^2\right]_\infty\right)
\end{aligned} \tag{3.39}$$

Replace q by q^{14}, and let $a = q^{21}$, $b = q^7$, $c = q^{-1}$ and $d = q^{-5}$ in (3.39). Multiplying both sides of the resulting identity by q^{24}, we deduce that

$$\begin{aligned}
&q^3 \frac{(-q; q^2)_\infty(-q^2; q^4)_\infty^2}{(-q^7; q^{14})_\infty(-q^{14}; q^{28})_\infty^2} - q^3 \frac{(q; q^2)_\infty(q^2; q^4)_\infty^2}{(q^7; q^{14})_\infty(q^{14}; q^{28})_\infty^2} \\
&= -2\frac{(q^{14}; q^{28})_\infty^4}{(q^7; q^{14})_\infty^2} + \frac{(-q; q^2)_\infty(-q^4; q^4)_\infty^2}{(-q^7; q^{14})_\infty(-q^{28}; q^{28})_\infty^2} + \frac{(q; q^2)_\infty(q^4; q^4)_\infty^2}{(q^7; q^{14})_\infty(q^{28}; q^{28})_\infty^2}
\end{aligned}$$

which is equivalent to

$$q^3\left(\frac{\chi(q)\chi^2(q^2)}{\chi(q^7)\chi^2(q^{14})} - \frac{\chi(-q)\chi^2(-q^2)}{\chi(-q^7)\chi^2(-q^{14})}\right) = -2\frac{\chi^4(-q^{14})}{\chi^2(-q^7)} + \frac{\chi(q)\chi^2(-q^{28})}{\chi(q^7)\chi^2(-q^4)} + \frac{\chi(-q)f^2(-q^4)}{\chi(-q^7)f^2(-q^{28})}$$

by (2.2), (2.3), and (2.5). Employing (2.13)–(2.16), (2.21) and (2.22), and simplifying, we check that

$$\begin{aligned}
&\cdot \left(\frac{1-\alpha}{1-\beta}\right)^{-1/12}\left(\frac{\alpha/q}{\beta/q^7}\right)^{7/24}\left(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\right)^{1/2} - q^3\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha/q}{\beta/q^7}\right)^{-5/24} \\
&= -4(\beta/q^7)^{-1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{-1/12}\left(\frac{\alpha/q}{\beta/q^7}\right)^{7/24}\left(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\right)^{1/2} + \frac{z_1}{z_7}\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha/q}{\beta/q^7}\right)^{7/24}
\end{aligned}$$

Dividing both sides of the last equality by $\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha/q}{\beta/q^7}\right)^{7/24}$, and rearranging terms, we derive the equality (3.37). □

4 Proofs of Twelve Partition Identities Using New Modular Equations

Theorem 4.1 *Let $P_1(N)$ denote the number of partitions of N into distinct parts congruent to ± 2 modulo 6 having 3 copies, multiples of 6 having 6 copies, or odd parts not multiples of 3 having 6 copies, $P_2(N)$ denote the number of partitions of N into parts congruent to 3 modulo 6 having 12 copies or distinct even parts not multiples of 3 having 6 copies, $P_3(N)$ denote the number of partitions of N into parts congruent* *to* ± 1 *modulo* 6 *having* 3 *copies, congruent to* 3 *modulo* 6 *having* 6 *copies, or distinct parts congruent to* ± 2 *modulo* 6 *having* 3 *copies and multiples of* 6 *having* 6 *copies,* $P_4(N)$ *denote the number of partitions of* N *into parts congruent to* ± 1 *modulo* 6 *having* 3 *copies, congruent to* 3 *modulo* 6 *having* 6 *copies, or distinct parts that are multiples of* 6 *having* 12 *copies. Then, for all* $N \geq 3$,

$$P_1(N) - P_2(N) = 6P_3(N-1) + 8P_4(N-3).$$

Proof Utilizing (3.22) and (3.23), we first see that

$$\begin{aligned}&\left(\frac{\alpha}{\beta^3}\right)^{1/8}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} - \left(\frac{\beta^3}{\alpha}\right)^{1/4}\left(\frac{1-\alpha}{(1-\beta)^3}\right)^{1/8}\\ &= \frac{2}{m-1}\left(\frac{m+1}{2}\right)^2 - \left(\frac{m-1}{2}\right)^2\frac{2}{m+1} = \frac{3m^2+1}{m^2-1},\end{aligned}$$

where β has degree 3 over α. Indeed, by simple elementary algebra, we can find that

$$\left(\frac{\alpha}{\beta^3}\right)^{1/8}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} - \left(\frac{\beta^3}{\alpha}\right)^{1/4}\left(\frac{1-\alpha}{(1-\beta)^3}\right)^{1/8} = 3 + \left(\frac{\alpha(1-\alpha)}{\{\beta(1-\beta)\}^3}\right)^{1/8}. \tag{4.1}$$

Next, multiply both sides of (4.1) by $2q$ to obtain the equation

$$\begin{aligned}&\frac{\{2^{1/6}(1-\beta)^{1/12}(\beta/q^3)^{-1/24}\}^9}{\{2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}\}^3} - 8q^3\frac{\{2^{1/3}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}\}^3}{\{2^{1/3}(1-\beta)^{1/24}(\beta/q^3)^{-1/12}\}^9}\\ &= 6q + \frac{\{2^{1/6}\{\beta(1-\beta)/q^3\}^{-1/24}\}^9}{\{2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}\}^3}.\end{aligned}$$

From (2.14)–(2.16) in Lemma 2.2, we find that the foregoing identity can be written as

$$\frac{\chi^9(-q^3)}{\chi^3(-q)} - 8q^3\frac{\chi^3(-q^2)}{\chi^9(-q^6)} = 6q + \frac{\chi^9(q^3)}{\chi^3(q)}.$$

Hence, applying the definition of χ in (2.3), we obtain the q-product form of the last identity, namely,

$$\frac{(q^3;q^6)_\infty^9}{(q;q^2)_\infty^3} - \frac{(-q^3;q^6)_\infty^9}{(-q;q^2)_\infty^3} = 6q + 8q^3\frac{(-q^6;q^6)_\infty^9}{(-q^2;q^2)_\infty^3}.$$

Multiplying both sides of the foregoing equality by $\frac{(-q^2;q^2)_\infty^3(-q^6;q^6)_\infty^3}{(q;q^2)_\infty^3(q^3;q^6)_\infty^3}$, and applying Euler's identity (2.4), we have

$$\frac{(q^3;q^6)_\infty^6(-q^2;q^2)_\infty^3(-q^6;q^6)_\infty^3}{(q;q^2)_\infty^6} - \frac{(-q^2;q^2)_\infty^6}{(q^3;q^6)_\infty^{12}(-q^6;q^6)_\infty^6}$$

$$= 6q\frac{(-q^2;q^2)_\infty^3(-q^6;q^6)_\infty^3}{(q;q^2)_\infty^3(q^3;q^6)_\infty^3} + 8q^3\frac{(-q^6;q^6)_\infty^{12}}{(q;q^2)_\infty^3(q^3;q^6)_\infty^3}.$$

Equating the coefficients of q^N on both sides of this equation, we finish the proof. □

Remark: A modular equation can be obtained from Theorem 3.1 in Kim's paper [19], which is the reciprocal of the modular Eq. (4.1) in Theorem 4.1.

Theorem 4.2 *Let S denote the set of positive integers where multiples of 4 but not multiples of 3 occur in 2 copies, and odd numbers not multiples of 3 occur in 1 copy. Let T denote the set of positive integers that consists of 2 copies of integers congruent to 2 modulo 4, but not multiples of 3, and 1 copy of odd numbers not multiples of 3. Let $P_1(N)$ denote the number of partitions of N into an odd number of distinct elements of S, and let $P_2(N)$ denote the number of partitions of N into an even number of distinct elements of T. Then, for all $N \geq 2$,*

$$P_1(N) = P_2(N-1).$$

Proof First we recall the modular Eq. (3.20)

$$\frac{z_1}{z_3} = \left(\frac{1-\beta}{1-\alpha}\right)^{1/4}\left(\left(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\right)^{1/2} - \left(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\right)^{1/2}\right) - \left(\frac{\beta}{\alpha}\right)^{1/2}$$

where β has degree 3 over α. Rearranging terms and multiplying both sides of the resulting identity by $\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24}$, we obtain the equation

$$\left(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\right)^{1/2}\left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24} - \frac{z_1}{z_3}\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24}$$
$$= \left(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\right)^{1/2}\left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24} + q\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\beta/q^3}{\alpha/q}\right)^{5/24}. \tag{4.2}$$

First, from (2.14) and (2.22),

$$\frac{\chi(q)\chi^2(-q^{12})}{\chi(q^3)\chi^2(-q^4)} = \left(\frac{1+\sqrt{1-\beta}}{1+\sqrt{1-\alpha}}\right)^{1/2}\left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24}. \tag{4.3}$$

Second, from (2.13) and (2.15),

$$\frac{\chi(-q)f^2(-q^4)}{\chi(-q^3)f^2(-q^{12})} = \frac{z_1}{z_3}\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24}. \tag{4.4}$$

Third, from (2.14) and (2.21)

$$q\frac{\chi(q)\chi^2(q^2)}{\chi(q^3)\chi^2(q^6)} = q\left(\frac{1+\sqrt{1-\alpha}}{1+\sqrt{1-\beta}}\right)^{1/2}\left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\left(\frac{\beta/q^3}{\alpha/q}\right)^{5/24}$$

$$= \left(\frac{1-\sqrt{1-\beta}}{1-\sqrt{1-\alpha}}\right)^{1/2}\left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\left(\frac{\alpha/q}{\beta/q^3}\right)^{7/24}. \tag{4.5}$$

Fourth, from (2.15) and (2.16),

$$q\frac{\chi(-q)\chi^2(-q^2)}{\chi(-q^3)\chi^2(-q^6)} = q\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\beta/q^3}{\alpha/q}\right)^{5/24}. \tag{4.6}$$

Hence, from (4.3)–(4.6), we have

$$\frac{\chi(q)\chi^2(-q^{12})}{\chi(q^3)\chi^2(-q^4)} - \frac{\chi(-q)f^2(-q^4)}{\chi(-q^3)f^2(-q^{12})} = q\frac{\chi(q)\chi^2(q^2)}{\chi(q^3)\chi^2(q^6)} + q\frac{\chi(-q)\chi^2(-q^2)}{\chi(-q^3)\chi^2(-q^6)}. \tag{4.7}$$

Employing the definitions of f, and χ from (2.2), and (2.3), respectively, we can convert (4.7) into q-products, namely,

$$\frac{(-q;q^2)_\infty(-q^4;q^4)^2_\infty}{(-q^3;q^6)_\infty(-q^{12};q^{12})^2_\infty} - \frac{(q;q^2)_\infty(q^4;q^4)^2_\infty}{(q^3;q^6)_\infty(q^{12};q^{12})^2_\infty}$$
$$= q\left(\frac{(-q;q^2)_\infty(-q^2;q^4)^2_\infty}{(-q^3;q^6)_\infty(-q^6;q^{12})^2_\infty} + \frac{(q;q^2)_\infty(q^2;q^4)^2_\infty}{(q^3;q^6)_\infty(q^6;q^{12})^2_\infty}\right).$$

Equating the coefficients of q^N on both sides of the last equation, we finish the proof. □

Theorem 4.3 *Let S denote the set of partitions with even parts that are not multiples of 3 having 2 copies, with odd parts that are not multiples of 3 having 4 copies. Let T denote the set of partitions with distinct odd parts that are not multiples of 3 having 2 copies, or with distinct parts in multiples of 4 but not multiples of 3 having 2 copies, with parts congruent to 2 modulo 4 not multiples of 3 having 3 copies. If $P_1(N)$ denotes the number of partitions of N in S, and if $P_2(N)$ denotes the number of partitions of N in T. Then, for all $N \geq 1$,*

$$3P_1(N) = 2P_2(2N+1).$$

Proof Consider the modular Eq. (3.20). Applying Theorem 2.1, we can get

$$3\frac{z_3}{z_1} = \left(\frac{\alpha}{\beta}\right)^{1/4}\left(\left(\frac{1+\sqrt{\alpha}}{1+\sqrt{\beta}}\right)^{1/2} - \left(\frac{1-\sqrt{\alpha}}{1-\sqrt{\beta}}\right)^{1/2}\right) - \left(\frac{1-\alpha}{1-\beta}\right)^{1/2}. \tag{4.8}$$

Multiplying both sides of (4.8) by $\left\{\frac{1-\beta}{1-\alpha}\right\}^{1/2}$, rearranging terms and rewriting the resulting identity, we arrive at

$$3\left(\frac{\sqrt{z_3}(\beta(1-\beta)/q^3)^{1/12}}{\sqrt{z_1}(\alpha(1-\alpha)/q)^{1/12}}\right)^2\left(\frac{1-\beta}{1-\alpha}\right)^{1/3}\left(\frac{\beta/q^3}{\alpha/q}\right)^{-1/6} + 1$$

$$= q^{\frac{-1}{2}} \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1}{3}} \left(\frac{\beta/q^3}{\alpha/q}\right)^{\frac{-7}{24}} \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{-1}{6}} \left(\frac{\alpha/q}{\beta/q^3}\right)^{\frac{-1}{24}} \left(\left(\frac{1+\sqrt{\alpha}}{1+\sqrt{\beta}}\right)^{\frac{1}{2}} - \left(\frac{1-\sqrt{\alpha}}{1-\sqrt{\beta}}\right)^{\frac{1}{2}}\right).$$

Hence, from (2.12), (2.14)–(2.16), (2.24), and (2.25), we obtain

$$3\frac{f^2(-q^6)\chi^4(-q^3)}{f^2(-q^2)\chi^4(-q)} + 1 = q^{-1/2}\left(\frac{\chi^2(q^{1/2})}{\chi^2(q^{3/2})} - \frac{\chi^2(-q^{1/2})}{\chi^2(-q^{3/2})}\right)\frac{\chi^2(-q^6)\chi^3(-q^3)}{\chi^2(-q^2)\chi^3(-q)}.$$

Using the definitions of f and χ from (2.2) and (2.3), respectively, we can convert the last equation into q-products, namely,

$$3\frac{(q^6;q^6)_\infty^2(q^3;q^6)_\infty^4}{(q^2;q^2)_\infty^2(q;q^2)_\infty^4} + 1 = q^{-1/2}\left(\frac{(-q^{1/2};q)_\infty^2}{(-q^{3/2};q^3)_\infty^2} - \frac{(q^{1/2};q)_\infty^2}{(q^{3/2};q^3)_\infty^2}\right)\frac{(-q^2;q^2)_\infty^2(q^3;q^6)_\infty^3}{(-q^6;q^6)_\infty^2(q;q^2)_\infty^3}.$$

Multiplying both sides of the last identity by $q^{1/2}$, and replacing q by q^2, we derive that

$$3q\frac{(q^{12};q^{12})_\infty^2(q^6;q^{12})_\infty^4}{(q^4;q^4)_\infty^2(q^2;q^4)_\infty^4} + q = \left(\frac{(-q;q^2)_\infty^2}{(-q^3;q^6)_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^3;q^6)_\infty^2}\right)\frac{(-q^4;q^4)_\infty^2(q^6;q^{12})_\infty^3}{(-q^{12};q^{12})_\infty^2(q^2;q^4)_\infty^3}.$$

Equating the coefficients of q^{2N+1} on both sides of the last equation, we finish the proof. □

Theorem 4.4 *Let S denote the set of partitions with even parts that are not multiples of 5 having 2 copies, with distinct odd parts that are not multiples of 5 having 2 copies. Let T denote the set of partitions with distinct odd parts not multiples of 5 having 2 copies, or with distinct parts that are multiples of 4 not multiples of 5 having 2 copies. Let $P_1(N)$ denote the number of partitions of N in S. Let $P_2(N)$ denote the number of partitions of N into an odd number of parts in S. Let $P_3(N)$ denote the number of partitions of N in T. Then, for all $N \geq 1$,*

$$2P_1(N) + P_2(N) = P_3(2N+3).$$

Proof Taking the reciprocal of the equality (3.6), and rearranging terms, we deduce that

$$5\frac{z_5}{z_1} = \left(\frac{\alpha}{\gamma}\right)^{3/8}\left(\left(\frac{1+\sqrt{\alpha}}{1+\sqrt{\gamma}}\right)^{1/2} - \left(\frac{1-\sqrt{\alpha}}{1-\sqrt{\gamma}}\right)^{1/2}\right) - 4\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/4} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/2}, \tag{4.9}$$

where γ has degree 5 over α. Dividing both sides of the last equality by $\left(\frac{(1-\alpha)\alpha/q}{(1-\gamma)\gamma/q^5}\right)^{1/4}$, we get the equation

$$
\begin{aligned}
&5\left(\frac{\sqrt{z_5}(\gamma(1-\gamma)/q^5)^{1/12}}{\sqrt{z_1}(\alpha(1-\alpha)/q)^{1/12}}\right)^2\left(\frac{(\alpha(1-\alpha)/q)^{-1/24}}{(\gamma(1-\gamma)/q^5)^{-1/24}}\right)^2\\
&=q^{-3/2}\left(\left(\frac{(1+\sqrt{\alpha})^{1/4}(\alpha/q)^{-1/48}(1-\alpha)^{-1/12}}{(1+\sqrt{\gamma})^{1/4}(\gamma/q^5)^{-1/48}(1-\gamma)^{-1/12}}\right)^2-\left(\frac{(1-\sqrt{\alpha})^{1/4}(\alpha/q)^{-1/48}(1-\alpha)^{-1/12}}{(1-\sqrt{\gamma})^{1/4}(\gamma/q^5)^{-1/48}(1-\gamma)^{-1/12}}\right)^2\right)\\
&\times\left(\frac{(1-\gamma)^{1/24}(\gamma/q^5)^{-1/12}}{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}}\right)^2-4q^{-1}+\left(\frac{(1-\alpha)^{1/12}(\alpha/q)^{-1/24}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}}{(1-\gamma)^{1/12}(\gamma/q^5)^{-1/24}(1-\gamma)^{1/24}(\gamma/q^5)^{-1/12}}\right)^2.
\end{aligned}
$$

Utilizing (2.2), (2.14)–(2.16), and (2.24), (2.25), we arrive at

$$
5\frac{f^2(-q^{10})\chi^2(q)}{f^2(-q^2)\chi^2(q^5)}=q^{-3/2}\left(\frac{\chi^2(q^{1/2})}{\chi^2(q^{5/2})}-\frac{\chi^2(-q^{1/2})}{\chi^2(-q^{5/2})}\right)\frac{\chi^2(-q^{10})}{\chi^2(-q^2)}-4q^{-1}+\frac{\chi^2(-q)\chi^2(-q^2)}{\chi^2(-q^5)\chi^2(-q^{10})}.
$$

Referring to the definition of f and χ from (2.2) and (2.3), respectively, we can convert the foregoing identity into q-products, namely,

$$
\begin{aligned}
5\frac{(q^{10};q^{10})_\infty^2(-q;q^2)_\infty^2}{(q^2;q^2)_\infty^2(-q^5;q^{10})_\infty^2}&=q^{-3/2}\frac{(-q^2;q^2)_\infty^2}{(-q^{10};q^{10})_\infty^2}\left(\frac{(-q^{1/2};q)_\infty^2}{(-q^{5/2};q^5)_\infty^2}-\frac{(q^{1/2};q)_\infty^2}{(q^{5/2};q^5)_\infty^2}\right)\\
&-4q^{-1}+\frac{(q;q^2)_\infty^2(-q^{10};q^{10})_\infty^2}{(q^5;q^{10})_\infty^2(-q^2;q^2)_\infty^2}.
\end{aligned}
$$

Next, divide both sides of the resulting identity by $q^{-3/2}$, and substitute q by q^2, rearrange terms to obtain

$$
\begin{aligned}
&q^3\left(5\frac{(q^{20};q^{20})_\infty^2(-q^2;q^4)_\infty^2}{(q^4;q^4)_\infty^2(-q^{10};q^{20})_\infty^2}-\frac{(q^2;q^4)_\infty^2(-q^{20};q^{20})_\infty^2}{(q^{10};q^{20})_\infty^2(-q^4;q^4)_\infty^2}\right)\\
&=\frac{(-q^4;q^4)_\infty^2}{(-q^{20};q^{20})_\infty^2}\left(\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2}-\frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}\right)-4q.
\end{aligned}
$$

Equating the coefficients of q^{2N+3} on both sides of the last equation, we finish the proof. ◻

Theorem 4.5 *Let S denote the set of partitions with distinct parts that congruent to 7 modulo 14 and congruent to 14 modulo 28 having 2 copies,, congruent to 1 modulo 2 not multiples of 7 and congruent to 2 modulo 4 not multiples of 7 having 1 copy. Let T denote the set of partitions with distinct parts that are multiples of 2 not multiples of 7 having 1 copy, and multiples of 14 having 2 copies, or with parts that congruent to 1 modulo 2 not multiples of 7 having 1 copy, congruent to 7 modulo 14 having 2 copies, If $P_1(N)$ denotes the number of partitions of N in S, and if $P_2(N)$ denotes the number of partitions of N in T, then, for all $N \geq 1$,*

$$
P_1(2N+3)=2P_2(N).
$$

Proof Consider the modular equation

$$\frac{(1+\sqrt{\beta})^{1/4}(1+\sqrt{\alpha})^{1/4}}{((1-\alpha)(1-\beta))^{1/8}(\alpha\beta)^{1/16}}-\frac{(1-\sqrt{\beta})^{1/4}(1-\sqrt{\alpha})^{1/4}}{((1-\alpha)(1-\beta))^{1/8}(\alpha\beta)^{1/16}}
=\sqrt{2}\left(1+\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)^{1/8}\right) \tag{4.10}$$

which is the reciprocal of the equality (3.36), and β has degree 7 over α. First, from (2.14) and (2.24),

$$\chi(q)\chi(q^7)\chi(q^{1/2})\chi(q^{7/2})=2^{1/2}q^{1/2}\frac{(1+\sqrt{\beta})^{1/4}(1+\sqrt{\alpha})^{1/4}}{((1-\alpha)(1-\beta))^{1/8}(\alpha\beta)^{1/16}}. \tag{4.11}$$

Second, from (2.14) and (2.25),

$$\chi(q)\chi(q^7)\chi(-q^{1/2})\chi(-q^{7/2})=2^{1/2}q^{1/2}\frac{(1-\sqrt{\beta})^{1/4}(1-\sqrt{\alpha})^{1/4}}{((1-\alpha)(1-\beta))^{1/8}(\alpha\beta)^{1/16}}. \tag{4.12}$$

Third, from (2.15) and (2.16),

$$\frac{1}{\chi(-q)\chi(-q^2)\chi(-q^7)\chi(-q^{14})}=2^{-1}q^{-1}\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)^{1/8} \tag{4.13}$$

Hence, from (4.11)–(4.13), we derive that

$$\chi(q)\chi(q^7)\{\chi(q^{1/2})\chi(q^{7/2})-\chi(-q^{1/2})\chi(-q^{7/2})\}=2q^{1/2}\left(1+\frac{2q}{\chi(-q)\chi(-q^2)\chi(-q^7)\chi(-q^{14})}\right).$$

Employing above the definition of χ from (2.3) and Euler's identity (2.4), we can find that

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty\{(-q^{1/2};q)_\infty(-q^{7/2};q^7)_\infty-(q^{1/2};q)_\infty(q^{7/2};q^7)_\infty\}
=2q^{1/2}+4q^{3/2}\frac{(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty}{(q;q^2)_\infty(q^7;q^{14})_\infty}.$$

By replacing q with q^2 in last equality, we deduce that

$$(-q^2;q^4)_\infty(-q^{14};q^{28})_\infty\{(-q;q^2)_\infty(-q^7;q^{14})_\infty-(q;q^2)_\infty(q^7;q^{14})_\infty\}$$
$$=2q+4q^3\frac{(-q^4;q^4)_\infty(-q^{28};q^{28})_\infty}{(q^2;q^4)_\infty(q^{14};q^{28})_\infty}.$$

Equate the coefficients of q^{2N+3} on both sides of the last equation to finish the proof. □

Theorem 4.6 *Let S denote the set of partitions with parts that are not multiples of 7 having 2 copies. Let T denote the set of partitions with distinct odd parts not multiples of 7 having 2 copies, or with distinct parts that are multiples of 4 not multiples of 7 having 3 copies, congruent to 2 modulo 4 not multiples of 7 having 1 copy. Let W denote the set of partitions with distinct parts congruent to 1 modulo 2 having 4 copies, multiples of 2 not multiples of 7 having 3 copies, multiples of 14 having 2 copies. Let $P_1(N)$ denote the number of partitions of N in S. Let $P_2(N)$ be the number of partitions of N into an even number of parts in S. Let $P_3(N)$ be the number of partitions of N in T. Let $P_4(N)$ denote the number of partitions of N in W. Then, for all $N\geq 1$,*

$$3P_1(N)+P_2(N)=-P_3(2N+3)+2P_4(N+1).$$

Proof Referring to the reciprocal of the formulas (3.37), we can check that

$$7\frac{z_7}{z_1}=\left(\frac{\alpha}{\beta}\right)^{\frac14}\left(\left(\frac{1-\sqrt{\alpha}}{1-\sqrt{\beta}}\right)^{\frac12}-\left(\frac{1+\sqrt{\alpha}}{1+\sqrt{\beta}}\right)^{\frac12}\right)+4\left(\frac{1-\alpha}{(1-\beta)^7}\right)^{\frac{1}{24}}\left(\frac{\alpha}{\beta}\right)^{\frac16}-\left(\frac{1-\alpha}{1-\beta}\right)^{\frac12} \tag{4.14}$$

where β has the degree 7 over α. Next, multiply both sides of (4.14) by $\left(\frac{1-\beta}{1-\alpha}\right)^{1/3}\left(\frac{\beta/q^7}{\alpha/q}\right)^{1/12}$ and rearrange terms to obtain the equation

$$7\left(\frac{\sqrt{z_7}(1-\beta)^{1/6}(\beta/q^7)^{1/24}}{\sqrt{z_1}(1-\alpha)^{1/6}(\alpha/q)^{1/24}}\right)^2+\left(\frac{(1-\alpha)^{1/12}(\alpha/q)^{-1/24}}{(1-\beta)^{1/12}(\beta/q^7)^{-1/24}}\right)^2$$
$$=q^{-3/2}\left(\left(\frac{(1-\sqrt{\alpha})^{1/4}(\alpha/q)^{-1/48}(1-\alpha)^{-1/12}}{(1-\sqrt{\beta})^{1/4}(\beta/q^7)^{-1/48}(1-\beta)^{-1/12}}\right)^2-\left(\frac{(1+\sqrt{\alpha})^{1/4}(\alpha/q)^{-1/48}(1-\alpha)^{-1/12}}{(1+\sqrt{\beta})^{1/4}(\beta/q^7)^{-1/48}(1-\beta)^{-1/12}}\right)^2\right)$$
$$\times\left(\frac{(1-\beta)^{1/24}(\beta/q^7)^{-1/12}}{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}}\right)^2\frac{(1-\beta)^{1/12}(\beta/q^7)^{-1/24}}{(1-\alpha)^{1/12}(\alpha/q)^{-1/24}}+4q^{-1}\frac{(1-\beta)^{\frac{1}{24}}(\beta/q^7)^{\frac{-1}{12}}(1-\alpha)^{\frac{-1}{4}}}{(1-\alpha)^{\frac{1}{24}}(\alpha/q)^{\frac{-1}{12}}}.$$

Utilizing (2.11), (2.14)–(2.16), (2.24) and (2.25), we arrive at

$$7\frac{f^2(-q^7)}{f^2(-q)}+\frac{\chi^2(-q)}{\chi^2(-q^7)}=q^{-3/2}\left(\frac{\chi^2(-q^{1/2})}{\chi^2(-q^{7/2})}-\frac{\chi^2(q^{1/2})}{\chi^2(q^{7/2})}\right)\frac{\chi^2(-q^{14})\chi(-q^7)}{\chi^2(-q^2)\chi(-q)}+4q^{-1}\frac{\chi(-q^{14})}{\chi(-q^2)}\frac{\chi^2(q)}{\chi^2(-q)},$$

Applying the definition of f and χ from (2.2) and (2.3), and Euler's identity (2.4), we can convert the last identity into q-products, namely,

$$7\frac{(q^7;q^7)_\infty^2}{(q;q)_\infty^2}+\frac{(-q^7;q^7)_\infty^2}{(-q;q)_\infty^2}=q^{-3/2}\left(\frac{(q^{1/2};q)_\infty^2}{(q^{7/2};q^7)_\infty^2}-\frac{(-q^{1/2};q)_\infty^2}{(-q^{7/2};q^7)_\infty^2}\right)\frac{(-q^2;q^2)_\infty^2(-q;q)_\infty}{(-q^{14};q^{14})_\infty^2(-q^7;q^7)_\infty}$$
$$+4q^{-1}\frac{(-q^2;q^2)_\infty(-q;q^2)_\infty^2(-q;q)_\infty^2}{(-q^{14};q^{14})_\infty}.$$

Replace q by q^2 and eliminate q^{-3} from the resulting identity to obtain that

$$7q^3\frac{(q^{14};q^{14})_\infty^2}{(q^2;q^2)_\infty^2}+q^3\frac{(-q^{14};q^{14})_\infty^2}{(-q^2;q^2)_\infty^2}=-\left(\frac{(-q;q^2)_\infty^2}{(-q^7;q^{14})_\infty^2}-\frac{(q;q^2)_\infty^2}{(q^7;q^{14})_\infty^2}\right)\frac{(-q^4;q^4)_\infty^2(-q^2;q^2)_\infty}{(-q^{28};q^{28})_\infty^2(-q^{14};q^{14})_\infty}$$
$$+4q\frac{(-q^4;q^4)_\infty^3(-q^2;q^4)_\infty^4}{(-q^{28};q^{28})_\infty}.$$

Equating the coefficients of q^{2N+3} on both sides of the last equation, we finish the proof. □

Theorem 4.7 *Let $P_1(N)$ denote the number of partitions of N into distinct odd parts that are not multiples of* 3 *having* 12 *copies or even parts congruent to* ±2, ±10,±14 *modulo* 36 *having* 12 *copies, and even parts congruent to* ±4, ±6,±8, ±16 *modulo* 36 *having* 6 *copies, $P_2(N)$ denote the number of partitions of N into odd parts that are multiples of* 3 *but not multiples of* 9 *having* 12 *copies or even parts not multiples of* 3 *having* 6 *copies, $P_3(N)$ denote the number of partitions of N into odd parts that are not multiples of* 9 *having* 6 *copies or parts multiples of* 6 *but not multiples of* 18 *having* 6 *copies, $P_4(N)$ denote the number of partitions of N into distinct odd parts not multiples of* 9 *having* 6 *copies or even parts congruent to* 2 *modulo* 4 *not multiples of* 6 *having* 18 *copies, and multiples of* 4 *but not multiples of* 3 *having* 6 *copies, $P_5(N)$ denote the number of partitions of N into odd parts not multiples of* 9 *having* 6 *copies or even parts not multiples of* 3 *having* 6 *copies. Then, for all $N\geq 3$,*

$$P_1(N)-P_2(N)=27P_3(N-3)+P_4(N-3)+12P_5(N-1).$$

Proof Recall the modular Eq. (3.17)

$$27\left(\frac{m}{m'}\right)^3=\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{3/4}-12\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/2}$$
$$-\left(\frac{\beta^2}{\alpha\gamma}\right)^{3/4}-\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{3/4}$$

where β is third degree over α, and γ is ninth degree over α. Next, multiply both sides of the last identity by $q\left(\frac{\alpha(1-\alpha)\gamma(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/2}$ to obtain the equation

$$27q^3\left(\frac{\sqrt{z_1}(\alpha(1-\alpha)/q)^{\frac{1}{12}}\sqrt{z_9}(\gamma(1-\gamma)/q^9)^{\frac{1}{12}}}{(\sqrt{z_3}(\beta(1-\beta)/q^3)^{\frac{1}{12}})^2}\right)^6=\left(\frac{(\alpha(1-\alpha)/q)^{\frac{-1}{24}}(\gamma(1-\gamma)/q^9)^{\frac{-1}{24}}}{((\beta(1-\beta)/q^3)^{\frac{-1}{24}})^2}\right)^6$$

$$-12q-\left(\frac{(1-\alpha)^{\frac{1}{12}}(\alpha/q)^{\frac{-1}{24}}(1-\gamma)^{\frac{1}{12}}(\gamma/q^9)^{\frac{-1}{24}}}{((1-\beta)^{\frac{1}{12}}(\beta/q^3)^{\frac{-1}{24}})^2}\right)^6-q^3\left(\frac{((1-\beta)^{\frac{1}{24}}(\beta/q^3)^{\frac{-1}{12}})^2}{(1-\alpha)^{\frac{1}{24}}(\alpha/q)^{\frac{-1}{12}}(1-\gamma)^{\frac{1}{24}}(\gamma/q^9)^{\frac{-1}{12}}}\right)^6.$$

Hence, from (2.12), (2.14)–(2.16), we find that

$$27q^3\frac{f^6(-q^2)f^6(-q^{18})}{f^{12}(-q^6)}=\frac{\chi^6(q)\chi^6(q^9)}{\chi^{12}(q^3)}-12q-\frac{\chi^6(-q)\chi^6(-q^9)}{\chi^{12}(-q^3)}-q^3\frac{\chi^{12}(-q^6)}{\chi^6(-q^2)\chi^6(-q^{18})}.$$

Employing the identity (2.2), (2.3), and (2.5), and rearranging terms, we arrive at

$$\begin{aligned}&\frac{(-q;q^2)_\infty^6(-q^9;q^{18})_\infty^6}{(-q^3;q^6)_\infty^{12}}-\frac{(q;q^2)_\infty^6(q^9;q^{18})_\infty^6}{(q^3;q^6)_\infty^{12}}\\&=q^3\left(27\frac{(q^2;q^2)_\infty^6(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^{12}}+\frac{(-q^2;q^2)_\infty^6(-q^{18};q^{18})_\infty^6}{(-q^6;q^6)_\infty^{12}}\right)+12q.\end{aligned}$$

Multiplying both sides of foregoing identity by $\frac{(q^9;q^{18})_\infty^6(q^6;q^6)_\infty^6}{(q;q^2)_\infty^6(q^2;q^2)_\infty^6}$, and using Euler's identity (2.4), we deduce that

$$\begin{aligned}&\frac{(-q;q^2)_\infty^{12}}{(-q^3;q^6)_\infty^{12}}\frac{(q^{18};q^{36})_\infty^6}{(q^2;q^4)_\infty^6}\frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^6}-\frac{(q^9;q^{18})_\infty^{12}}{(q^3;q^6)_\infty^{12}}\frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^6}\\&=27q^3\frac{(q^9;q^{18})_\infty^6}{(q;q^2)_\infty^6}\frac{(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^6}+q^3\frac{(-q;q^2)_\infty^6}{(-q^9;q^{18})_\infty^6}\frac{(q^6;q^{12})_\infty^{12}}{(q^2;q^4)_\infty^{12}}\frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^6}+12q\frac{(q^9;q^{18})_\infty^6(q^6;q^6)_\infty^6}{(q;q^2)_\infty^6(q^2;q^2)_\infty^6}.\end{aligned}$$

Equating the coefficients of q^N on both sides of the last equation, we finish the proof. □

Theorem 4.8 *Let $P_1(N)$ denote the number of partitions of N into odd parts that are not multiples of 9 having 6 copies or even parts that are not multiples of 3 having 6 copies, $P_2(N)$ denote the number of partitions of N into distinct odd parts that are not multiples of 3 having 6 copies or even parts congruent to 6 modulo 12 not multiples of 18 having 18 copies, and multiples of 12 but not multiples of 36 having 6 copies, $P_3(N)$ denote the number of partitions of N into odd parts not multiples of 3 having 12 copies or multiples of 6 that are not multiples of 18 having 6 copies, $P_4(N)$ denote the number of partitions of N into distinct odd parts multiples of 3 but not multiples of 9 having 12 copies or even parts congruent to 2 modulo 4 not multiples of 18 having 6 copies, and multiples of 6 but not multiples of 18 having 6 copies, $P_5(N)$ denote the number of partitions of N into odd parts not multiples of 9 having 6 copies or multiples of 6 that are not multiples of 18 having 6 copies. Then, for all $N \geq 3$,*

$$P_1(N)-P_2(N)=P_3(N-3)-P_4(N-3)+12P_5(N-2).$$

Proof Consider the modular Eq. (3.18)

$$\left(\frac{m'}{m}\right)^3=\left(\frac{\alpha\gamma}{\beta^2}\right)^{3/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{3/4}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{3/4}$$

$$+12\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/2}$$

where β and γ are of third and ninth degrees. Multiplying both sides of the last equality by $q^2\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/2}$, we can check that

$$\left(\frac{(\sqrt{z_3}(\beta(1-\beta)/q^3)^{\frac{1}{12}})^2}{\sqrt{z_1}(\alpha(1-\alpha)/q)^{\frac{1}{12}}\sqrt{z_9}(\gamma(1-\gamma)/q^9)^{\frac{1}{12}}}\right)^6=q^3\left(\frac{((1-\beta)^{\frac{1}{12}}(\beta/q^3)^{\frac{-1}{24}})^2}{(1-\alpha)^{\frac{1}{12}}(\alpha/q)^{\frac{-1}{24}}(1-\gamma)^{\frac{1}{12}}(\gamma/q^9)^{\frac{-1}{24}}}\right)^6$$
$$+\left(\frac{(1-\alpha)^{\frac{1}{24}}(\alpha/q)^{\frac{-1}{12}}(1-\gamma)^{\frac{1}{24}}(\gamma/q^9)^{\frac{-1}{12}}}{((1-\beta)^{\frac{1}{24}}(\beta/q^3)^{\frac{-1}{12}})^2}\right)^6-q^3\left(\frac{((\beta(1-\beta)/q^3)^{\frac{-1}{24}})^2}{(\alpha(1-\alpha)/q)^{\frac{-1}{24}}(\gamma(1-\gamma)/q^9)^{\frac{-1}{24}}}\right)^6+12q^2$$

which can be transformed into

$$\frac{f^{12}(-q^6)}{f^6(-q^2)f^6(-q^{18})}=q^3\frac{\chi^{12}(-q^3)}{\chi^6(-q)\chi^6(-q^9)}+\frac{\chi^6(-q^2)\chi^6(-q^{18})}{\chi^{12}(-q^6)}-q^3\frac{\chi^{12}(q^3)}{\chi^6(q)\chi^6(q^9)}+12q^2$$

by (2.12), (2.14)–(2.16). Employing the definition of f and χ from (2.2) and (2.3), and (2.5), respectively, and rearranging terms, we can derive a reformulation of the last equation into q-products, namely,

$$\frac{(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^6(q^{18};q^{18})_\infty^6}-\frac{(-q^6;q^6)_\infty^{12}}{(-q^2;q^2)_\infty^6(-q^{18};q^{18})_\infty^6}$$
$$=q^3\left(\frac{(q^3;q^6)_\infty^{12}}{(q;q^2)_\infty^6(q^9;q^{18})_\infty^6}-\frac{(-q^3;q^6)_\infty^{12}}{(-q;q^2)_\infty^6(-q^9;q^{18})_\infty^6}\right)+12q^2$$

In order to find a beautiful partition interpretation, we convert the above equality by multiplying it by $\frac{(q^{18};q^{18})_\infty^6(q^9;q^{18})_\infty^6}{(q^6;q^6)_\infty^6(q;q^2)_\infty^6}$.

$$\frac{(q^9;q^{18})_\infty^6}{(q;q^2)_\infty^6}\frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^6}-\frac{(-q;q^2)_\infty^6}{(-q^9;q^{18})_\infty^6}\frac{(q^{18};q^{36})_\infty^{12}}{(q^6;q^{12})_\infty^{12}}\frac{(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^6}$$
$$=q^3\frac{(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^6}\frac{(q^3;q^6)_\infty^{12}}{(q;q^2)_\infty^{12}}-q^3\frac{(q^{18};q^{36})_\infty^6}{(q^2;q^4)_\infty^6}\frac{(-q^3;q^6)_\infty^{12}}{(-q^9;q^{18})_\infty^{12}}\frac{(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^6}+12q^2\frac{(q^{18};q^{18})_\infty^6(q^9;q^{18})_\infty^6}{(q^6;q^6)_\infty^6(q;q^2)_\infty^6}.$$

Equating the coefficients of q^N on both sides of the last equation, we finish the proof. □

Theorem 4.9 *Let S denote the set of positive integers that consists of 2 copies of odd numbers but not multiples of 9 and 1 copy of even numbers not multiples of 9. Let T denote the set of positive integers that consists of 2 copies of even numbers but not multiples of 9 and 1 copy of odd numbers not multiples of 9. Let $D_S(N)$ and $D_T(N)$ denote the number of partitions of N into an odd number of distinct elements of S and T, respectively. Then, for all $N \geq 2$,*

$$D_S(N-1)=D_T(N).$$

Proof We consider the equality (3.19),

$$\frac{z_1}{z_9}-\sqrt{\frac{z_1}{z_9}}\left(\frac{\gamma}{\alpha}\right)^{1/8}=\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}-2\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}\left(\frac{\gamma}{\alpha}\right)^{1/8}-\left(\frac{1-\gamma}{1-\alpha}\right)^{3/8}\left(\frac{\gamma}{\alpha}\right)^{1/8}$$

where γ has degree 9 over α. Next, multiply both sides of the foregoing identity by $q\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}\left(\frac{\alpha}{\gamma}\right)^{1/8}$ to obtain

$$\begin{aligned}
&\frac{\{\sqrt{z_1}\{\alpha(1-\alpha)/q\}^{1/12}\}^2(1-\alpha)^{1/12}(\alpha/q)^{-1/24}}{\{\sqrt{z_9}\{\gamma(1-\gamma)/q^9\}^{1/12}\}^2(1-\gamma)^{1/12}(\gamma/q^9)^{-1/24}}\\
&\quad-q\,\frac{\{(1-\alpha)^{1/12}(\alpha/q)^{-1/24}\}^2\sqrt{z_1}\{\alpha(1-\alpha)/q\}^{1/12}}{\{(1-\gamma)^{1/12}(\gamma/q^9)^{-1/24}\}^2\sqrt{z_9}\{\gamma(1-\gamma)/q^9\}^{1/12}}\\
&\quad=\frac{\{\alpha(1-\alpha)/q\}^{-1/24}\{(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}\}^2}{\{\gamma(1-\gamma)/q^9\}^{-1/24}\{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}\}^2}-2q\\
&\quad-q\,\frac{\{\{\alpha(1-\alpha)/q\}^{-1/24}\}^2(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}}{\{\{\gamma(1-\gamma)/q^9\}^{-1/24}\}^2(1-\alpha)^{1/24}(\alpha/q)^{-1/12}},
\end{aligned}$$

Using (2.12) and (2.14)–(2.16), we find that

$$\frac{f^2(-q^2)\chi(-q)}{f^2(-q^{18})\chi(-q^9)}-q\,\frac{\chi^2(-q)f(-q^2)}{\chi^2(-q^9)f(-q^{18})}=\frac{\chi(q)\chi^2(-q^{18})}{\chi(q^9)\chi^2(-q^2)}-2q-q\,\frac{\chi^2(q)\chi(-q^{18})}{\chi^2(q^9)\chi(-q^2)},$$

which can be transformed into

$$\begin{aligned}
&q\,\frac{(-q;q^2)_\infty^2(-q^2;q^2)_\infty}{(-q^9;q^{18})_\infty^2(-q^{18};q^{18})_\infty}-q\,\frac{(q;q^2)_\infty^2(q^2;q^2)_\infty}{(q^9;q^{18})_\infty^2(q^{18};q^{18})_\infty}\\
&\quad=\frac{(-q^2;q^2)_\infty^2(-q;q^2)_\infty}{(-q^{18};q^{18})_\infty^2(-q^9;q^{18})_\infty}-\frac{(q^2;q^2)_\infty^2(q;q^2)_\infty}{(q^{18};q^{18})_\infty^2(q^9;q^{18})_\infty}-2q.
\end{aligned}$$

by the definitions of f and χ from (2.2) and (2.3), respectively, and rearranging terms. Equating the coefficients of q^N on both sides of the last equation, we finish the proof. □

Theorem 4.10 *Let S denote the set of partitions with parts not multiples of 9 having 2 copies. Let T denote the set of partitions with even parts not multiples of 9 having 1 copy. Let U denote the set of partitions with distinct even parts not multiples of 9 having 2 copies, or with odd parts not multiples of 9 having 1 copy. Let V denote the set of partitions with distinct parts not multiples of 9 having 1 copy. Let $P_1(N)$ denote the number of partitions of N in S. Let $P_2(N)$ be the number of partitions of N in T. Let $P_3(N)$ be the number of partitions of N in U. Let $P_4(N)$ denote the number of partitions of N in V. Then, for all $N \geq 1$,*

$$9P_1(N)-3P_2(N)=2P_3(2N+3)-2P_4(N+1).$$

Proof By the reciprocal of the equality (3.19), we get

$$9\frac{z_9}{z_1}-3\sqrt{\frac{z_9}{z_1}}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}=\left(\frac{\alpha}{\gamma}\right)^{3/8}-2\left(\frac{\alpha}{\gamma}\right)^{1/4}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}-\left(\frac{\alpha}{\gamma}\right)^{3/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} \tag{4.15}$$

where γ has degree 9 over α. Next, multiply both sides of the identity (4.15) by $q\left(\frac{\gamma}{\alpha}\right)^{1/4}\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}$ to obtain

$$\begin{aligned}&9q^3\frac{\{\sqrt{z_9}\{\gamma(1-\gamma)/q^9\}^{1/12}\}^2(1-\alpha)^{1/24}(\alpha/q)^{-1/12}}{\{\sqrt{z_1}\{\alpha(1-\alpha)/q\}^{1/12}\}^2(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}}\\&\quad-3q^3\frac{\{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}\}^2\sqrt{z_9}\{\gamma(1-\gamma)/q^9\}^{1/12}}{\{(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}\}^2\sqrt{z_1}\{\alpha(1-\alpha)/q\}^{1/12}}\\&=\frac{(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}(1-\gamma)^{1/12}(\gamma/q^9)^{-1/24}}{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}}-2q\\&\quad-\frac{\{\gamma(1-\gamma)/q^9\}^{-1/24}(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}}{\{\alpha(1-\alpha)/q\}^{-1/24}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}},\end{aligned}$$

Using (2.12) and (2.14)–(2.16), we find that

$$9q^3\frac{f^2(-q^{18})\chi(-q^2)}{f^2(-q^2)\chi(-q^{18})}-3q^3\frac{\chi^2(-q^2)f(-q^{18})}{\chi^2(-q^{18})f(-q^2)}=\frac{\chi(-q^{18})\chi(-q^9)}{\chi(-q^2)\chi(-q)}-2q-\frac{\chi(q^9)\chi(-q^{18})}{\chi(q)\chi(-q^2)},$$

which can be transformed into

$$\begin{aligned}&9q^3\frac{(q^{18};q^{18})_\infty^2(-q^{18};q^{18})_\infty}{(q^2;q^2)_\infty^2(-q^2;q^2)_\infty}-3q^3\frac{(q^{18};q^{18})_\infty(-q^{18};q^{18})_\infty^2}{(q^2;q^2)_\infty(-q^2;q^2)_\infty^2}\\&=\frac{(-q^2;q^2)_\infty(q^9;q^{18})_\infty}{(-q^{18};q^{18})_\infty(q;q^2)_\infty}-2q-\frac{(-q^9;q^{18})_\infty(-q^2;q^2)_\infty}{(-q;q^2)_\infty(-q^{18};q^{18})_\infty}.\end{aligned}$$

by the definitions of f and χ from (2.2) and (2.3), respectively. Multiplying both sides of the forgoing equality by $\frac{(-q^2;q^2)_\infty}{(-q^{18};q^{18})_\infty}$, we have

$$\begin{aligned}&9q^3\frac{(q^{18};q^{18})_\infty^2}{(q^2;q^2)_\infty^2}-3q^3\frac{(q^{36};q^{36})_\infty}{(q^4;q^4)_\infty}\\&=\frac{(-q^2;q^2)_\infty^2}{(-q^{18};q^{18})_\infty^2}\left(\frac{(q^9;q^{18})_\infty}{(q;q^2)_\infty}-\frac{(-q^9;q^{18})_\infty}{(-q;q^2)_\infty}\right)-2q\frac{(-q^2;q^2)_\infty}{(-q^{18};q^{18})_\infty}.\end{aligned}$$

Equating the coefficients of q^{2N+3} on both sides of the last equation, we finish the proof. □

Theorem 4.11 *Let S denote the set of partitions with distinct odd parts not multiples of 9, or with even parts not multiples of 9. Let T denote the set of partitions with* *distinct odd parts not multiples of* 9, *or with distinct parts that are multiples of* 4 *not multiples of* 9. *Let* $P_1(N)$ *denote the number of partitions of* N *in* S. *Let* $P_2(N)$ *be the number of partitions of* N *into an odd number of parts in* S. *let* $P_3(N)$ *be the number of partitions of* N *in* T. *Then, for all* $N \geq 1$,

$$P_1(N) + P_2(N) = P_3(2N + 3).$$

Proof By the reciprocal of the equality (3.21), we have

$$3\sqrt{\frac{z_9}{z_1}}=\left(\frac{\alpha}{\gamma}\right)^{3/16}\left(\left(\frac{1+\sqrt{\alpha}}{1+\sqrt{\gamma}}\right)^{1/4}-\left(\frac{1-\sqrt{\alpha}}{1-\sqrt{\gamma}}\right)^{1/4}\right)-2\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4} \tag{4.16}$$

where γ has degree 9 over α. Multiplying both sides of (4.16) by $\left(\frac{\gamma(1-\gamma)/q^9}{\alpha(1-\alpha)/q}\right)^{1/8}$ and rearranging the resulting identity, we obtain

$$\begin{aligned}
&3\frac{\sqrt{z_9}(\gamma(1-\gamma)/q^9)^{1/12}}{\sqrt{z_1}(\alpha(1-\alpha)/q)^{1/12}}\frac{(\alpha(1-\alpha)/q)^{-1/24}}{(\gamma(1-\gamma)/q^9)^{-1/24}}-\frac{(1-\alpha)^{1/8}(\alpha/q)^{-1/8}}{(1-\gamma)^{1/8}(\gamma/q^9)^{-1/8}}\\
&=q^{-3/2}\frac{(1+\sqrt{\alpha})^{1/4}(1-\alpha)^{-1/12}(\alpha/q)^{-1/48}}{(1+\sqrt{\gamma})^{1/4}(1-\gamma)^{-1/12}(\gamma/q^9)^{-1/48}}\frac{(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}}{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}}\\
&\quad-q^{-3/2}\frac{(1-\sqrt{\alpha})^{1/4}(1-\alpha)^{-1/12}(\alpha/q)^{-1/48}}{(1-\sqrt{\gamma})^{1/4}(1-\gamma)^{-1/12}(\gamma/q^9)^{-1/48}}\frac{(1-\gamma)^{1/24}(\gamma/q^9)^{-1/12}}{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}}-2q^{-1}.
\end{aligned}$$

Hence, from (2.12), (2.14)–(2.16), (2.24) and (2.25), we check that

$$3\frac{f(-q^{18})\chi(q)}{f(-q^2)\chi(q^9)}-\frac{\chi(-q)\chi(-q^2)}{\chi(-q^9)\chi(-q^{18})}=q^{-3/2}\frac{\chi(-q^{18})}{\chi(-q^2)}\left(\frac{\chi(q^{1/2})}{\chi(q^{9/2})}-\frac{\chi(-q^{1/2})}{\chi(-q^{9/2})}\right)-2q^{-1}$$

Canceling $q^{-3/2}$, and applying the definitions of f and χ from (2.2), (2.3) and (2.5), we can rewrite the last equation in the form

$$\begin{aligned}
&3q^{3/2}\frac{(q^{18};q^{18})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty(-q^9;q^{18})_\infty}-q^{3/2}\frac{(-q^{18};q^{18})_\infty(q;q^2)_\infty}{(-q^2;q^2)_\infty(q^9;q^{18})_\infty}\\
&=\frac{(-q^2;q^2)_\infty}{(-q^{18};q^{18})_\infty}\left(\frac{(-q^{1/2};q)_\infty}{(-q^{9/2};q^9)_\infty}-\frac{(q^{1/2};q)_\infty}{(q^{9/2};q^9)_\infty}\right)-2q^{1/2}.
\end{aligned}$$

Substituting q by q^2, we get

$$\begin{aligned}
&3q^3\frac{(q^{36};q^{36})_\infty(-q^2;q^4)_\infty}{(q^4;q^4)_\infty(-q^{18};q^{36})_\infty}-q^3\frac{(-q^{36};q^{36})_\infty(q^2;q^4)_\infty}{(-q^4;q^4)_\infty(q^{18};q^{36})_\infty}\\
&=\frac{(-q^4;q^4)_\infty}{(-q^{36};q^{36})_\infty}\left(\frac{(-q;q^2)_\infty}{(-q^9;q^{18})_\infty}-\frac{(q;q^2)_\infty}{(q^9;q^{18})_\infty}\right)-2q.
\end{aligned}$$

Equate the coefficients of q^{2N+3} on both sides of the last equation to complete the proof. □

Theorem 4.12 *Let S denote the set of 3 copies of positive integers where no multiples of 3 and 5 occur. Let $P_1(N)$ denote the number of partitions of N into an odd number of even elements of S, and let $P_2(N)$ denote the number of partitions of N into an odd number of odd elements of S. Then, for all $N \geq 3$,*

$$P_1(N) = P_2(N-3).$$

Proof Recall the modular equation (3.3)

$$\left(\frac{z_3 z_5}{z_1 z_{15}}\right)^{3/2} = \left(\frac{\alpha\delta}{\beta\gamma}\right)^{3/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{3/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{3/8}$$
$$+6\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4},$$

where α, β, γ, and δ are of the first, third, fifth, and fifteenth degrees, respectively. Multiplying both sides of the last equality by $q^2\left(\frac{\beta(1-\beta)\gamma(1-\gamma)}{\alpha(1-\alpha)\delta(1-\delta)}\right)^{1/4}$ and rearranging the resulting identity, we obtain

$$\left(\frac{\sqrt{z_3}(\beta(1-\beta)/q^3)^{1/12}\sqrt{z_5}(\gamma(1-\gamma)/q^5)^{1/12}}{\sqrt{z_1}(\alpha(1-\alpha)/q)^{1/12}\sqrt{z_{15}}(\delta(1-\delta)/q^{15})^{1/12}}\right)^3$$
$$-\left(\frac{(1-\alpha)^{1/24}(\alpha/q)^{-1/12}(1-\delta)^{1/24}(\delta/q^{15})^{-1/12}}{(1-\beta)^{1/24}(\beta/q^3)^{-1/12}(1-\gamma)^{1/24}(\gamma/q^5)^{-1/12}}\right)^3$$
$$= q^3\left(\frac{(1-\beta)^{1/12}(\beta/q^3)^{-1/24}(1-\gamma)^{1/12}(\gamma/q^5)^{-1/24}}{(1-\alpha)^{1/12}(\alpha/q)^{-1/24}(1-\delta)^{1/12}(\delta/q^{15})^{-1/24}}\right)^3$$
$$-q^3\left(\frac{(\beta(1-\beta)/q^3)^{-1/24}(\gamma(1-\gamma)/q^5)^{-1/24}}{(\alpha(1-\alpha)/q)^{-1/24}(\delta(1-\delta)/q^{15})^{-1/24}}\right)^3 + 6q^2.$$

Hence, from (2.12), (2.14)–(2.16), we arrive at

$$\frac{f^3(-q^6)f^3(-q^{10})}{f^3(-q^2)f^3(-q^{30})} - \frac{\chi^3(-q^2)\chi^3(-q^{30})}{\chi^3(-q^6)\chi^3(-q^{10})} = q^3\frac{\chi^3(-q^3)\chi^3(-q^5)}{\chi^3(-q)\chi^3(-q^{15})} - q^3\frac{\chi^3(q^3)\chi^3(q^5)}{\chi^3(q)\chi^3(q^{15})} + 6q^2$$

Using the definitions of f and χ from (2.2) and (2.3), we can rewrite the last equation in terms of q-products, namely,

$$\frac{(q^6;q^6)_\infty^3(q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty^3(q^{30};q^{30})_\infty^3} - \frac{(-q^6;q^6)_\infty^3(-q^{10};q^{10})_\infty^3}{(-q^2;q^2)_\infty^3(-q^{30};q^{30})_\infty^3}$$
$$= q^3\frac{(q^3;q^6)_\infty^3(q^5;q^{10})_\infty^3}{(q;q^2)_\infty^3(q^{15};q^{30})_\infty^3} - q^3\frac{(-q^3;q^6)_\infty^3(-q^5;q^{10})_\infty^3}{(-q;q^2)_\infty^3(-q^{15};q^{30})_\infty^3} + 6q^2.$$

Equate the coefficients of q^N on both sides of the last equation to finish the proof. □

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