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# A Generalized Brezis–Lieb Lemma on Graphs and Its Application to Kirchhoff Type Equations

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## Abstract

In this paper, with the help of potential function, we extend the classical Brezis–Lieb lemma on Euclidean space to graphs, which can be applied to the following Kirchhoff equation

$$\begin{cases} -\left(1+b\int_{\mathbb{V}}|\nabla u|^2 d\mu\right)\Delta u+(\lambda V(x)+1)\,u=|u|^{p-2}u \ \text{ in } \mathbb{V}, \\ u\in W^{1,2}(\mathbb{V}), \end{cases}$$

on a connected locally finite graph $G=(\mathbb{V},\mathbb{E})$, where $b,\lambda>0$, $p>2$ and $V(x)$ is a potential function defined on $\mathbb{V}$. The purpose of this paper is four-fold. First of all, using the idea of the filtration Nehari manifold technique and a compactness result based on generalized Brezis–Lieb lemma on graphs, we prove that there admits a positive solution $u_{\lambda,b}\in E_\lambda$ with positive energy for $b\in(0,b^*)$ when $2<p<4$. In the sequel, when $p\geqslant 4$, a positive ground state solution $w_{\lambda,b}\in E_\lambda$ is also obtained by using standard variational methods. What's more, we explore various asymptotic behaviors of $u_{\lambda,b}, w_{\lambda,b}\in E_\lambda$ by separately controlling the parameters $\lambda\to\infty$ and $b\to 0^+$, as well as jointly controlling both parameters. Finally, we utilize iteration to obtain the $L^\infty$-norm estimates of the solution.

**Keywords** Locally finite graph · Kirchhoff equation · Brezis–Lieb lemma · Nehari manifold method

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## 1 Introduction

Consider the following Kirchhoff type equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u) \tag{1.1}$$

with a bounded domain $\Omega \subset \mathbb{R}^N$, which received much attention on mathematical studies, was originally introduced by Kirchhoff [18] to describe the transversal oscillations of a stretched string. Above equation is a general version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

with a strong physical background, where $L$ is the length of the string, $h$ is the area of cross section, $\rho$ is the mass density, $E$ is the Young's modulus of the material, and $P_0$ is the initial tension. Equation (1.2) can be also seen as a generalization of the classical D'Alembert's wave equation for free vibrations of elastic strings. For more details in the physical and mathematical background of Kirchhoff type equations, one can refer to [23]. In addition, some early classical studies of Kirchhoff equations can be seen in Bernstein [4]. Afterwards, Lions [20] proposed an abstract functional analysis framework for the Kirchhoff type equation (1.1) in 1978, which attracted the attention of several researchers. Subsequently, based on variational methods, the solvability of (1.1) has been thoroughly investigated when nonlinear $f$ satisfies various growth conditions, see [9, 24]. Moreover, there are numerous interesting results were built on the corresponding elliptic version, like

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x) u = f(x, u) \text{ in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.3}$$

where $N \geqslant 1$, $a, b > 0$, $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and $f \in C\left(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}\right)$. To deal with (1.3) in the case where $f(x, u) := f(u)$ with $N = 3$, via using the mountain pass theorem and Nehari manifold, He and Zou [17] established the existence of positive ground state solutions. Later, Li and Ye [19] obtained the existence of positive ground state solutions when $f(x, u) := |u|^{p-2}u$ with $3 < p < 6$ in (1.3). For more results about Kirchhoff type problems, we refer the readers to [7, 8, 21, 22, 27–30] and the references therein.

In recent years, analysis on graphs has begun to attract attention and has had several applications in different fields, data analysis, optimal transport, machine learning, etc [1, 10, 12]. Particularly, there are many interesting directions about graph in mathematics. It's remarkable that, Grigor'yan et al. [13, 15, 16] established the variational framework on graphs for some nonlinear equations, and the existence of solution was considered, where mountain pass theorem and a method of upper and lower solutions have been uesd. Furthermore, they pointed out that the Sobolev space $W^{m,p}(\mathbb{V})$ on a finite graph $G = (\mathbb{V}, \mathbb{E})$ is pre-compact. Soon after, Zhang and Zhao [33] introduced two Sobolev embedding theorems on graphs and investigated the equation

$$-\Delta u + (\lambda V(x) + 1)\, u = |u|^{p-2}u \ \text{ in } \mathbb{V}, \tag{1.4}$$

where $V(x) \geqslant 0$ satisfies following conditions:

$(V_1)$ $V(x) \geqslant 0$ on $\mathbb{V}$, and the potential well $\Omega = \{x \in \mathbb{V} : V(x) = 0\}$ is a non-empty, connected and bounded domain in $\mathbb{V}$.

$(V_2)$ There exists a vertex $x_0 \in \mathbb{V}$ such that $V(x) \to +\infty$ as $d(x, x_0) \to +\infty$. Specifically, they proved that there exists a ground state solution $u_\lambda$ of (1.4) and it converges in $W^{1,2}(\mathbb{V})$ to a ground state solution of the Dirichlet problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

as $\lambda \to \infty$ along a subsequence. For more interesting resuts on graphs, we refer the readers to [2, 3, 6, 26].

Now, We turn our attention to the Kirchhoff equation. Pan and Ji [25] used the constrained variational method to study

$$-\left(a + b\int_{\mathbb{V}} |\nabla u|^2 d\mu\right) \Delta u + c(x)u = f(u) \tag{1.5}$$

on a locally finite graph, where $f(u)$ satisfies super cubic growth condition and the other suitable assumptions. More precisely, the authors showed that (1.5) has a least energy sign-changing solution $u_b$, and its energy is strictly larger than twice that of least energy solutions firstly. Then, they proved that $u_b$ converges to a least energy sign-changing solution of the problem

$$-a\Delta u + c(x)u = f(u), \quad x \in \mathbb{V},$$

as $b \to 0^+$ along a subsequence.

Based on the above works, for the Kirchhoff equation on graphs, there is a lack of relevant results when the nonlinear terms are associated with subcubic growth. Thus, in the following article, we attempt to investigate the existence of positive solutions to the Kirchhoff type problem

$$\begin{cases} -\left(a + b\int_{\mathbb{V}} |\nabla u|^2 d\mu\right) \Delta u + (\lambda V(x) + 1)\, u = |u|^{p-2}u \ \text{ in } \mathbb{V}, \\ u \in W^{1,2}(\mathbb{V}), \end{cases} \tag{1.6}$$

on a connected locally finite graph $G = (\mathbb{V}, \mathbb{E})$, where $a > 0$ is a constant, $b$ and $\lambda$ are positive parameters, $p > 2$ and the potential $V(x)$ satisfies conditions $(V_1)$, $(V_2)$.

We shall consider (1.6) for $2 < p < 4$ and $p \geqslant 4$ separately. For the former, in order to overcome the barrier of lacking compactness on graphs, we introduce two embedding theorems based on steep potential. Moreover, to obtain compactness results for $2 < p < 4$, we extend Brezis–Lieb lemma on graphs. For the latter, we use standard Nehari manifold to seek the positive ground state solution for the equation (1.6). What's more, various asymptotic behaviors and the $L^\infty$-norm estimates of the solution are considered.

### 1.1 Notations

Let $G = (\mathbb{V}, \mathbb{E})$ be a graph which is locally finite and connected, where $\mathbb{V}$ and $\mathbb{E}$ denote the vertex set and the edge set, respectively. Here, a graph $G$ is said to be locally finite if for any $x \in \mathbb{V}$, there are only finite $y \in \mathbb{V}$ such that $xy \in \mathbb{E}$. And a graph is connected if any two vertices $x$ and $y$ can be connected via finite edges. For any $x, y \in \mathbb{V}$ with $xy \in \mathbb{E}$, we assume it has a positive symmetric weight on $G$, namely $w_{xy} = w_{yx} > 0$.

Then we define the measure $\mu : \mathbb{V} \to \mathbb{R}^+$ on the graph, which is a finite positive function on $\mathbb{V}$. And we call it a uniformly positive measure if there exists a constant $\mu_0 > 0$ such that $\mu(x) \geqslant \mu_0$ for all $x \in \mathbb{V}$. For any function $u : \mathbb{V} \to \mathbb{R}$, the $\mu$-Laplacian (or Laplacian for short) of $u$ is defined as

$$\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)), \tag{1.7}$$

where $y \sim x$ means $xy \in \mathbb{E}$ or $y$ is adjacent to $x$. We denote the gradient form of two functions $u$ and $v$ on the graph by

$$\Gamma(u, v)(x) := \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)).$$

Denote $\Gamma(u) = \Gamma(u, u)$ and $\nabla u \nabla v = \Gamma(u, v)$, then the length of gradient $u$ is represented by

$$|\nabla u|(x) := \sqrt{\Gamma(u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}.$$

The integral of a function $f$ over $\mathbb{V}$ is given as

$$\int_{\mathbb{V}} f d\mu = \sum_{x \in \mathbb{V}} \mu(x) f(x).$$

We denote the space of functions on $\mathbb{V}$ by $C(\mathbb{V})$. For $u \in C(\mathbb{V})$, its support set is defined as $\operatorname{supt}(u) = \{x \in \mathbb{V} : u(x) \neq 0\}$. Let $C_c(\mathbb{V})$ be the set of all functions with finite support and $W^{1,2}(\mathbb{V})$ be the completion of $C_c(\mathbb{V})$ under the norm

$$\|u\|_{W^{1,2}(\mathbb{V})} = \left(\int_{\mathbb{V}} \left(|\nabla u|^2 + u^2\right) d\mu\right)^{1/2}.$$

Obviously, $W^{1,2}(\mathbb{V})$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{V}} (\Gamma(u, v) + uv) d\mu, \quad \forall u, v \in W^{1,2}(\mathbb{V}).$$

What's more, the space $L^p(\mathbb{V})$ is defined as

$$L^p(\mathbb{V}) = \left\{u \in C(\mathbb{V}) : \|u\|_{L^p(\mathbb{V})} < \infty\right\},$$

where

$$\|u\|_{L^p(\mathbb{V})} = \begin{cases} \left(\sum\limits_{x\in\mathbb{V}} \mu(x)|u(x)|^p\right)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \sup\limits_{x\in\mathbb{V}} |u(x)| & \text{if } p = \infty. \end{cases}$$

Consider a domain $\Omega \subset \mathbb{V}$. The distance $d(x, y)$ of two vertices $x, y \in \Omega$ is defined by the minimal length of a path which connect $x$ and $y$. $\Omega$ is said a bounded domain in $\mathbb{V}$, if the distance $d(x, y)$ is bounded for any $x, y \in \Omega$. The boundary of $\Omega$ is defined as

$$\partial\Omega := \{y \notin \Omega : \exists x \in \Omega \text{ such that } xy \in \mathbb{E}\}$$

and the interior of $\Omega$ is denoted by $\Omega^\circ$. Obviously, we can see that $\Omega^\circ = \Omega$, which is different from the Euclidean case. Furthermore, the Hilbert space $W_0^{1,2}(\Omega)$ is the completion of $C_c(\Omega)$ under the norm

$$\|u\|_{W_0^{1,2}(\Omega)} = \left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_{\Omega} u^2 d\mu\right)^{\frac{1}{2}}.$$

Because of the formula for integral by parts which will be introduced in Sect. 2, there is an extral integral on $\partial\Omega$ for the gradient form of $u$.

And for more details about graphs, we refer the reader to [14].

Furthermore, if conditions $(V_1)$, $(V_2)$ are satisfied, we denote $m_\Omega$ the ground state energy of the equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $p > 2$ and $\Omega = \{x \in \mathbb{V} : V(x) = 0\}$.

### 1.2 Main Results

After explaining the notations, we introduce the main results achieved. In this paper, we always assume that $G = (\mathbb{V}, \mathbb{E})$ is a locally finite and connected graph with positive symmetric weight and uniformly positive measure. We consider Kirchhoff type problem (1.6) satisfying the conditions $(V_1)$ and $(V_2)$. To state our main results, without loss of generality, we may assume that $a = 1$. Consequently, we shall investigate the Kirchhoff type problem

$$\begin{cases} -\left(1 + b\int_{\mathbb{V}} |\nabla u|^2 d\mu\right) \Delta u + (\lambda V(x) + 1)\, u = |u|^{p-2}u \ \text{ in } \mathbb{V}, \\ u \in W^{1,2}(\mathbb{V}), \end{cases} \tag{$\mathcal{K}_{\lambda,b}$}$$

with the associated energy functional

$$I_{\lambda,b}(u) = \frac{1}{2}\int_{\mathbb{V}} \left[|\nabla u|^2 + (\lambda V(x) + 1)\, u^2\right] d\mu + \frac{b}{4}\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 - \frac{1}{p}\int_{\mathbb{V}} |u|^p \, d\mu.$$

Then it is natural to consider a function space

$$E_\lambda = \left\{u \in W^{1,2}(\mathbb{V}) : \int_{\mathbb{V}} (\lambda V(x) + 1)\, u^2 d\mu < \infty\right\}$$

with the norm

$$\|u\|_\lambda = \left\{\int_{\mathbb{V}} \left[|\nabla u|^2 + (\lambda V(x) + 1)\, u^2\right] d\mu\right\}^{1/2}.$$

The space $E_\lambda$ is also a Hilbert space and its inner product is

$$\langle u, v\rangle_\lambda = \int_{\mathbb{V}} [\Gamma(u, v) + (\lambda V(x) + 1)\, uv]\, d\mu, \quad \forall u, v \in E_\lambda.$$

We now summarize our main results as follows. The first main result is concerned with the existence of positive solutions.

**Theorem 1.1** *(i) Suppose that $2 < p < 4$, $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ and conditions $(V_1)$, $(V_2)$ hold. Then there exists $b^* > 0$ such that for every $b \in (0, b^*)$ and $\lambda > 0$, Eq. $(\mathcal{K}_{\lambda,b})$ has one positive solution $u_{\lambda,b} \in E_\lambda$ satisfying*

$$0 < \mu_0^{\frac{1}{2}} \leqslant \left\|u_{\lambda,b}\right\|_\lambda < \sqrt{\frac{2pm_\Omega}{p-2}}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}$$

*and*

$$0 < \frac{p-2}{4p}\mu_0 \leqslant I_{\lambda,b}\left(u_{\lambda,b}\right) < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}.$$

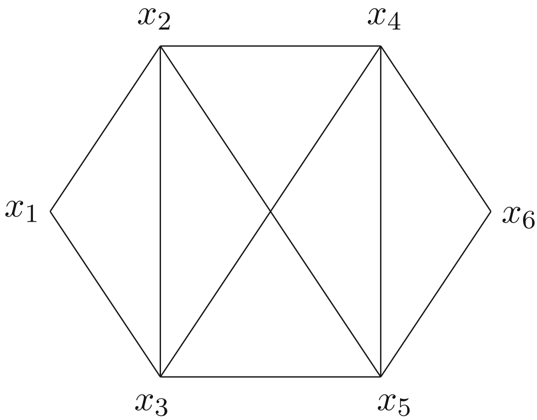


**Fig. 1** The graph $G_6$

*(ii) Suppose that* $p = 4$ *and conditions* $(V_1), (V_2)$ *hold. Then there exists* $\hat{b} > 0$ *such that for every* $b \in (0, \hat{b})$, $\lambda > 0$, *Eq.* $(\mathcal{K}_{\lambda,b})$ *has a positive ground state solution* $w_{\lambda,b} \in E_\lambda$ *satisfying* $I_{\lambda,b}\left(w_{\lambda,b}\right) = l_{\lambda,b} > 0$.

*(iii) Suppose that* $p > 4$ *and conditions* $(V_1), (V_2)$ *hold. Then for every* $\lambda > 0$, $b > 0$, *Eq.* $(\mathcal{K}_{\lambda,b})$ *has a positive ground state solution* $w_{\lambda,b} \in E_\lambda$ *satisfying* $I_{\lambda,b}\left(w_{\lambda,b}\right) = l_{\lambda,b} > 0$.

***Remark 1.1*** To illustrate that the condition $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ can be achieved, we consider a finite connected graph $G_6 = (\mathbb{V}, \mathbb{E})$ as shown in the Fig. 1. The vertex set $\mathbb{V}$ is $\{x_1, x_2, \cdots, x_6\}$ and the edge set $\mathbb{E}$ is $\{x_{12}, x_{13}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}, x_{46}, x_{56}\}$, where $x_{ij}$ represents the edge connecting vertices $x_i$ and $x_j$. For simplicity, we take the measure $\mu$ satisfying $\mu(x_i) = 1$ for $i = 1, 2, \cdots, 6$ and take the weight $w_{x_i x_j} = w_{x_j x_i} = 1$ for all $x_{ij} \in \mathbb{E}$. Hence, $G_6$ is a finite and connected graph with positive symmetric weight and uniformly positive measure.

Next, let $V(x)$ be

$$V(x_i) = \begin{cases} 0 & \text{if } i = 1, \\ 1 & \text{if } i = 2, 3, 4, 5, 6. \end{cases}$$

We consider the equation (1.8) in a Hilbert space $W_0^{1,2}(\Omega)$, with corresponding functional

$$I_\Omega(u) = \frac{1}{2}\left(\int_{\Omega \cup \partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu\right) - \frac{1}{p}\int_\Omega |u|^p d\mu.$$

Clearly, the potential well is $\Omega = \{x_1\}$ with boundary $\partial\Omega = \{x_2, x_3\}$. Note that

$$u_1(x) = \begin{cases} 1 & \text{if } x = x_1, \\ 0 & \text{if } x = x_2, x_3, \end{cases}$$

satisfying $u_1(x) \in \mathcal{N}_\Omega$, and so

$$0 < m_\Omega := \inf_{u \in \mathcal{N}_\Omega} I_\Omega(u) \leqslant \frac{1}{2} - \frac{1}{p},$$

where

$$\mathcal{N}_\Omega := \left\{ u \in W_0^{1,2}(\Omega) \backslash \{0\} : \|u\|^2_{W_0^{1,2}(\Omega)} = \|u\|^p_{L^p(\Omega)} \right\}.$$

Thus, we can take $\mu_0 = 1$ satisfying $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$.

Note that Zhang and Du [32] applied the truncation technique to handle the similar Kirchhoff equation in $\mathbb{R}^3$ space successfully, however, this technique is not applicable to graphs. Therefore, we need to identify an alternative approach. Inspired by Sun and Wu [29], we apply a novel constraint method, which will be introduced in Sect. 3, to obtain the boundedness of the Palais-Smale sequences. Furthermore, to overcome the compactness, we extend the classical Brezis–Lieb lemma [5] on Euclidean space to a connected locally finite graph in Sect. 2.2.

To study the asymptotic behavior of $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$ as $\lambda \to \infty$, it is advisable to consider the Dirichlet problem

$$\begin{cases} -\left(1 + b\int_\Omega |\nabla u|^2 d\mu\right) \Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathcal{K}_{\infty,b})$$

Similar to Eq. $(\mathcal{K}_{\lambda,b})$, the Dirichlet problem $\left(\mathcal{K}_{\infty,b}\right)$ also has a positive solution under some assumptions.

**Theorem 1.2** *(i) Suppose that $2 < p < 4$, $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ and conditions $(V_1)$, $(V_2)$ hold. Then there exists $b^* > 0$ such that for every $b \in (0, b^*)$, Eq. $\left(\mathcal{K}_{\infty,b}\right)$ has one positive solution $u_{\infty,b} \in W_0^{1,2}(\Omega)$.*

*(ii) Suppose that $p = 4$ and conditions $(V_1)$, $(V_2)$ hold. Then there exists $\hat{b} > 0$ such that for every $b \in (0, \hat{b})$, Eq. $(\mathcal{K}_{\infty,b})$ has a positive ground state solution $w_{\infty,b} \in W_0^{1,2}(\Omega)$.*

*(iii) Suppose that $p > 4$ and conditions $(V_1)$, $(V_2)$ hold. Then for every $b > 0$, Eq. $(\mathcal{K}_{\infty,b})$ has a positive ground state solution $w_{\infty,b} \in W_0^{1,2}(\Omega)$.*

***Remark 1.2*** For convenience, we use the same notations $b^*$, $\hat{b}$ here as in Theorem 1.1.

Next, we have the following result:

**Theorem 1.3** *(i) Suppose that $2 < p < 4$, $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ and conditions $(V_1)$, $(V_2)$ hold. Then there exists $b_* \in (0, b^*)$, for any sequence $\lambda_n \to \infty$, pass to a subsequence, $u_{\lambda_n,b} \in E_{\lambda_n}$ converges in $W^{1,2}(\mathbb{V})$ to a positive solution of Eq. $\left(\mathcal{K}_{\infty,b}\right)$ with $b \in (0, b_*)$ fixed, where $u_{\lambda_n,b} \in E_{\lambda_n}$ is the positive solution of Eq. $(\mathcal{K}_{\lambda_n,b})$ obtained by Theorem 1.1(i).*

*(ii)* *Suppose that* $p = 4$*, and conditions* $(V_1), (V_2)$ *hold. Then, for any sequence* $\lambda_n \to \infty$*, pass to a subsequence,* $w_{\lambda_n,b} \in E_{\lambda_n}$ *converges in* $W^{1,2}(\mathbb{V})$ *to a positive solution of Eq.* $\left(\mathcal{K}_{\infty,b}\right)$ *with* $b \in (0, \hat{b})$ *fixed, where* $w_{\lambda_n,b} \in E_{\lambda_n}$ *is the positive ground state solution of Eq.* $(\mathcal{K}_{\lambda_n,b})$ *obtained by Theorem 1.1(ii).*

*(iii)* *Suppose that* $p > 4$*, and conditions* $(V_1), (V_2)$ *hold. Then, for any sequence* $\lambda_n \to \infty$*, pass to a subsequence,* $w_{\lambda_n,b} \in E_{\lambda_n}$ *converges in* $W^{1,2}(\mathbb{V})$ *to a positive solution of Eq.* $\left(\mathcal{K}_{\infty,b}\right)$ *with* $b \in (0, +\infty)$ *fixed, where* $w_{\lambda_n,b} \in E_{\lambda_n}$ *is the positive ground state solution of Eq.* $(\mathcal{K}_{\lambda_n,b})$ *obtained by Theorem 1.1(iii).*

After exploring the asymptotic behavior of $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$ as $\lambda \to \infty$, we turn our attention to study asymptotic behavior of $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$ as $b \to 0^+$, thus

$$\begin{cases} -\Delta u + (\lambda V(x) + 1)\, u = |u|^{p-2}u \ \text{ in } \mathbb{V}, \\ u \in W^{1,2}(\mathbb{V}), \end{cases} \tag{$\mathcal{K}_{\lambda,0}$}$$

is considered naturally. According to [33], if conditions $(V_1)$ , $(V_2)$ are satisfied, Eq. $(\mathcal{K}_{\lambda,0})$ has a positive solution for $p > 2$.

**Theorem 1.4** *(i) Suppose that* $2 < p < 4$*,* $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ *and conditions* $(V_1), (V_2)$ *hold. Let* $u_{\lambda,b} \in E_\lambda$ *be the positive solution of Eq.* $\left(\mathcal{K}_{\lambda,b}\right)$ *obtained by theorem 1.1(i). Then for each* $\lambda \in (0, \infty)$ *fixed, up to a subsequence,* $u_{\lambda,b} \to u_{\lambda,0}$ *in* $E_\lambda$ *as* $b \to 0^+$*, where* $u_{\lambda,0} \in E_\lambda$ *is a positive solution of Eq.* $(\mathcal{K}_{\lambda,0})$*.*

*(ii)* *Suppose that* $p = 4$ *and conditions* $(V_1), (V_2)$ *hold. Let* $w_{\lambda,b} \in E_\lambda$ *be the positive ground state solution of Eq.* $\left(\mathcal{K}_{\lambda,b}\right)$ *obtained by Theorem 1.1(ii). Then for each* $\lambda \in (0, \infty)$ *fixed, up to a subsequence,* $w_{\lambda,b} \to w_{\lambda,0}$ *in* $E_\lambda$ *as* $b \to 0^+$*, where* $w_{\lambda,0} \in E_\lambda$ *is a positive solution of Eq.* $(\mathcal{K}_{\lambda,0})$*.*

*(iii)* *Suppose that* $p > 4$ *and conditions* $(V_1), (V_2)$ *hold. Let* $w_{\lambda,b} \in E_\lambda$ *be the positive ground state solution of Eq.* $\left(\mathcal{K}_{\lambda,b}\right)$ *obtained by Theorem 1.1(iii). Then for each* $\lambda \in (0, \infty)$ *fixed, up to a subsequence,* $w_{\lambda,b} \to w_{\lambda,0}$ *in* $E_\lambda$ *as* $b \to 0^+$*, where* $w_{\lambda,0} \in E_\lambda$ *is a positive solution of Eq.* $(\mathcal{K}_{\lambda,0})$*.*

Next, we present a theorem that describes the asymptotic behavior of $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$ as $\lambda \to \infty$ and $b \to 0^+$. According to [33], if conditions $(V_1)$ , $(V_2)$ are satisfied, then

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{K}_{\infty,0}$}$$

has a positive solution for $p > 2$.

**Theorem 1.5** *(i) Suppose that* $2 < p < 4$*,* $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ *and conditions* $(V_1), (V_2)$ *hold. Let* $u_{\lambda,b} \in E_\lambda$ *be the positive solution of Eq.* $\left(\mathcal{K}_{\lambda,b}\right)$ *obtained by Theorem 1.1(i). Then up to a subsequence,* $u_{\lambda,b} \to u_{\infty,0}$ *in* $W^{1,2}(\mathbb{V})$ *as* $b \to 0^+$ *and* $\lambda \to \infty$*, where* $u_{\infty,0} \in W_0^{1,2}(\Omega)$ *is a positive solution of Eq.* $\left(\mathcal{K}_{\infty,0}\right)$*.*

*(ii)* *Suppose that* $p = 4$*, and conditions* $(V_1), (V_2)$ *hold. Let* $w_{\lambda,b} \in E_\lambda$ *be the positive ground state solution of Eq.* $\left(\mathcal{K}_{\lambda,b}\right)$ *obtained by Theorem 1.1(ii). Then up to a*

*subsequence,* $w_{\lambda,b} \to w_{\infty,0}$ *in* $W^{1,2}(\mathbb{V})$ *as* $b \to 0^+$ *and* $\lambda \to \infty$*, where* $w_{\infty,0} \in W_0^{1,2}(\Omega)$ *is a positive solution of Eq.* $\left(\mathcal{K}_{\infty,0}\right)$.

*(iii) Suppose that* $p > 4$*, and conditions* $(V_1)$, $(V_2)$ *hold. Let* $w_{\lambda,b} \in E_\lambda$ *be the positive ground state solution of Eq.* $\left(\mathcal{K}_{\lambda,b}\right)$ *obtained by Theorem 1.1(iii). Then up to a subsequence,* $w_{\lambda,b} \to w_{\infty,0}$ *in* $W^{1,2}(\mathbb{V})$ *as* $b \to 0^+$ *and* $\lambda \to \infty$*, where* $w_{\infty,0} \in W_0^{1,2}(\Omega)$ *is a positive solution of Eq.* $\left(\mathcal{K}_{\infty,0}\right)$.

Finally in this subsection, we give the estimate of solutions.

**Theorem 1.6** *Let* $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$ *be obtained by Theorem 1.1. Then there exist* $C_1, C_2 > 0$ *(independent of* $\lambda$*) satisfying*

$$\left\|u_{\lambda,b}\right\|_{L^\infty(\mathbb{V})} \leqslant C_1 \text{ and } \left\|w_{\lambda,b}\right\|_{L^\infty(\mathbb{V})} \leqslant C_2$$

*for all* $\lambda > 0$.

The paper is organized as follows: in Sect. 1, we describe the development of Kirchhoff type equation and some research results about it on Euclidean Space. Following that, the research results of discrete equation in recent years are presented. What's more, we provide an explanation of main notations and present the main results achieved, including the existence of positive solution and its asymptotic behavior for Eq. $(\mathcal{K}_{\lambda,b})$. In Sect. 2, in order to derive the main results, we undertake essential preparations. This includes introducing the formula for integral by parts and the Sobolev embedding theorem on graphs. Subsequently, we introduce a generalized Brezis–Lieb lemma on graphs. In Sect. 3, we filtrate the Nehari manifold $\mathbf{N}_{\lambda,b}$ associated with Eq. $(\mathcal{K}_{\lambda,b})$ laying the foundation of the proof in next section. In Sect. 4, we show that the existence of the positive solution $u_{\lambda,b} \in E_\lambda$ for Eq. $(\mathcal{K}_{\lambda,b})$ when $2 < p < 4$, where we have employed calculus of variations. In Sect. 5, we prove the multiple results regarding the asymptotic behavior of $u_{\lambda,b} \in E_\lambda$ by controlling parameters $\lambda$ and $b$. In Sect. 6, by applying standard variational methods, we show that the existence of the positive ground state solution $w_{\lambda,b} \in E_\lambda$ for Eq. $(\mathcal{K}_{\lambda,b})$ when $p \geqslant 4$. In Sect. 7, we analyze the asymptotic behavior of the solution $w_{\lambda,b} \in E_\lambda$ by controlling parameters $\lambda$ and $b$. In Sect. 8, we give the estimate involving the $L^\infty$-norm of solutions by using iteration.

## 2 Preliminaries

We will introduce some useful results on graphs.

### 2.1 Formula for Integral by Parts and Embedding Theorems

In this subsection, we shall present two lemmas about integral by parts on graphs first. According to the work of Zhang and Zhao [33], we have the following lemmas.

**Lemma 2.1** *Suppose that* $u \in W^{1,2}(\mathbb{V})$ *and its Laplacian* $\Delta u$ *is defined by* (1.7)*. Then for any* $v \in C_c(\mathbb{V})$*, we have*

$$\int_{\mathbb{V}} \nabla u \cdot \nabla v d\mu = \int_{\mathbb{V}} \Gamma(u, v) d\mu = -\int_{\mathbb{V}} \Delta u v d\mu.$$

**Lemma 2.2** *Suppose that* $u \in W^{1,2}(\mathbb{V})$ *and its Laplacian* $\Delta u$ *is well-defined. Let* $v$ *be a function which belongs to* $C_c(\Omega)$*, where* $\Omega \subset \mathbb{V}$ *is a bounded domain. Then we have*

$$\int_{\Omega \cup \partial\Omega} \nabla u \cdot \nabla v d\mu = \int_{\Omega \cup \partial\Omega} \Gamma(u, v) d\mu = -\int_{\Omega} \Delta u v d\mu.$$

After that, since the graph has no concept of dimension, the Sobolev embedding theorem becomes unusual. For this, we introduce two compactness results related to $E_\lambda$.

**Lemma 2.3** *Assume that* $\lambda > 0$ *and* $V(x)$ *satisfies conditions* $(V_1)$, $(V_2)$*. Then* $E_\lambda$ *is weakly pre-compact and* $E_\lambda$ *compactly embedded into* $L^q(\mathbb{V})$ *for any* $q \in [2, \infty]$ *and the embedding is independent of* $\lambda$*. Namely, there exists a constant* $C$ *depending on* $\mu_0$ *and* $q$ *such that for any* $u \in E_\lambda$, $\|u\|_{L^q(\mathbb{V})} \leqslant C\|u\|_\lambda$*. Particularly, there holds*

$$\|u\|_\lambda \geqslant \mu_0^{\frac{q-2}{2q}} \|u\|_{L^q(\mathbb{V})} \quad \text{for } 2 \leqslant q < +\infty. \tag{2.1}$$

*Moreover, for any bounded sequence* $\{u_n\} \subset E_\lambda$*, there exists* $u \in E_\lambda$ *such that, going if necessary to a subsequence,*

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_\lambda, \\ u_n(x) \to u(x) & \forall x \in \mathbb{V}, \\ u_n \to u & \text{in } L^q(\mathbb{V}). \end{cases}$$

***Proof*** We assume that $u \in E_\lambda$, take any vertex $x_1 \in \mathbb{V}$ and fix it, there holds

$$\begin{aligned} \|u\|_\lambda^2 &= \int_{\mathbb{V}} \left[ |\nabla u|^2 + (\lambda V(x) + 1) u^2 \right] d\mu \\ &\geqslant \int_{\mathbb{V}} u^2 d\mu = \sum_{x \in \mathbb{V}} \mu(x) u^2(x) \geqslant \mu_0 u^2(x_1), \end{aligned}$$

which implies

$$u(x_1) \leqslant \sqrt{\frac{1}{\mu_0}} \|u\|_\lambda.$$

Thus $E_\lambda \hookrightarrow L^\infty(\mathbb{V})$ continuously and the embedding is independent of $\lambda$. When $2 \leqslant q < +\infty$, we have

$$\begin{aligned}\|u\|_\lambda^q &= \left\{\int_{\mathbb{V}}\left[|\nabla u|^2 + (\lambda V(x)+1)\,u^2\right]d\mu\right\}^{\frac{q}{2}} \geqslant \left(\int_{\mathbb{V}} u^2 d\mu\right)^{\frac{q}{2}}\\ &= \left[\sum_{x\in\mathbb{V}}\mu(x)u^2(x)\right]^{\frac{q}{2}} \geqslant \sum_{x\in\mathbb{V}}\mu^{\frac{q}{2}}(x)|u(x)|^q\\ &\geqslant \mu_0^{\frac{q}{2}-1}\sum_{x\in\mathbb{V}}\mu(x)|u(x)|^q = \mu_0^{\frac{q}{2}-1}\int_{\mathbb{V}}|u|^q d\mu,\end{aligned}$$

so we get $\|u\|_\lambda \geqslant \mu_0^{\frac{q-2}{2q}}\|u\|_{L^q(\mathbb{V})}$. Then we obtain the continuous embedding $E_\lambda \hookrightarrow L^q(\mathbb{V})$ for any $2 \leqslant q \leqslant \infty$.

The rest of the proof is similar to the lemma 2.6 in [33], so we omit it here. □

For the space $W_0^{1,2}(\Omega)$, we have another embedding theorem as following.

**Lemma 2.4** (See [33], Lemma 2.7) *Assume that $\Omega$ is a bounded domain in $\mathbb{V}$. Then $W_0^{1,2}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for any $q \in [1,\infty]$. Namely, there exists a constant $C$ depending only on $q$ and $\Omega$ such that for any $u \in W_0^{1,2}(\Omega)$, $\|u\|_{L^q(\Omega)} \leqslant C\|u\|_{W_0^{1,2}(\Omega)}$. Moreover, for any bounded sequence $\{u_n\} \subset W_0^{1,2}(\Omega)$, there exists $u \in W_0^{1,2}(\Omega)$ such that, going if necessary to a subsequence,*

$$\begin{cases} u_n \rightharpoonup u & in\ W_0^{1,2}(\Omega),\\ u_n(x) \to u(x) & \forall x \in \Omega,\\ u_n \to u & in\ L^q(\Omega).\end{cases}$$

## 2.2 A Generalization of Brezis–Lieb Lemma

After introducing two embedding theorems, we establish a generalization of Brezis–Lieb lemma on graphs as following.

**Lemma 2.5** *Suppose that conditions $(V_1)$ and $(V_2)$ hold. If $\{u_n\} \subset E_\lambda$ is bounded and there exists $u \in E_\lambda$ such that $u_n \rightharpoonup u$, then there holds*

$$\lim_{n\to\infty}\left(\|u_n\|_{L^q(\mathbb{V})}^q - \|u_n - u\|_{L^q(\mathbb{V})}^q\right) = \|u\|_{L^q(\mathbb{V})}^q$$

*for $2 \leqslant q < \infty$.*

***Proof*** We assume that $\{u_n\}$ has a upper bound $M$ in $E_\lambda$, then

$$\|u\|_\lambda \leqslant \varliminf_{n\to\infty}\|u_n\|_\lambda \leqslant M.$$

Fix $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $\alpha, \beta \in \mathbb{R}$,

$$\left||\alpha+\beta|^q - |\alpha|^q\right| \leqslant \varepsilon|\alpha|^q + C_\varepsilon|\beta|^q.$$

If we let

$$f_n^\varepsilon := \left(\left||u_n|^q - |u_n-u|^q - |u|^q\right| - \varepsilon|u_n-u|^q\right)^+$$

and take $\alpha = u_n - u,\ \beta = u$, then

$$\begin{aligned} f_n^\varepsilon &= \left(\left||\alpha+\beta|^q - |\alpha|^q - |\beta|^q\right| - \varepsilon|\alpha|^q\right)^+ \\ &\leqslant (\left||\alpha+\beta|^q - |\alpha|^q\right| + |\beta|^q - \varepsilon|\alpha|^q)^+ \\ &\leqslant \left(\varepsilon|\alpha|^q + C_\varepsilon|\beta|^q + |\beta|^q - \varepsilon|\alpha|^q\right)^+ \\ &= \left[(1+C_\varepsilon)|\beta|^q\right]^+ = (1+C_\varepsilon)|\beta|^q = (1+C_\varepsilon)|u|^q, \end{aligned}$$

so there exists a constant $C$ such that

$$\begin{aligned} \int_{\mathbb{V}} V(x)\left|f_n^\varepsilon\right| d\mu &\leqslant (1+C_\varepsilon)\int_{\mathbb{V}} V(x)|u|^q d\mu \\ &\leqslant \frac{(1+C_\varepsilon)\,C}{\lambda^{\frac{q}{2}}}\|u\|_\lambda^q \leqslant \frac{(1+C_\varepsilon)\,C}{\lambda^{\frac{q}{2}}} M^q. \end{aligned}$$

In view of $(V_2)$, let $x_0 \in \mathbb{V}$ be fixed, there exists some $R > 0$ such that

$$V(x) \geqslant \frac{(1+C_\varepsilon)\,CM^q}{\lambda^{\frac{q}{2}}\varepsilon} \text{ when } \operatorname{dist}(x,x_0) > R.$$

Hence, we obatin

$$\int_{\operatorname{dist}(x,x_0)>R} \left|f_n^\varepsilon\right| d\mu \leqslant \frac{\lambda^{\frac{q}{2}}\varepsilon}{(1+C_\varepsilon)\,CM^q}\int_{\operatorname{dist}(x,x_0)>R} V(x)\left|f_n^\varepsilon\right| d\mu \leqslant \varepsilon. \quad (2.2)$$

Since $\{u_n\} \subset E_\lambda$ is also bounded in $L^2(\mathbb{V})$ and we have the weak convergence in $L^2(\mathbb{V})$, then

$$\lim_{n\to\infty}\int_{\mathbb{V}} (u_n - u)\,\varphi d\mu = \lim_{n\to\infty}\sum_{x\in\mathbb{V}} \mu(x)\,[u_n(x) - u(x)]\,\varphi(x) = 0 \qquad (2.3)$$

for any $\varphi \in L^2(\mathbb{V})$. Take any $x_1 \in \mathbb{V}$ and let

$$\varphi_1(x) = \begin{cases} 1 & \text{if } x = x_1, \\ 0 & \text{if } x \neq x_1, \end{cases}$$

which belongs to $L^2(\mathbb{V})$. By substituting $\varphi_1$ into (2.3), we get $\lim_{n\to\infty} \mu(x_1)[u_n(x_1) - u(x_1)] = 0$. Thus, $\lim_{n\to\infty} u_n(x) = u(x)$ for any $x \in \mathbb{V}$, and so $f_n^\varepsilon(x) \to 0$ for any $x \in \mathbb{V}$. Moreover, because $\{x \in \mathbb{V} : \text{dist}(x, x_0) \leqslant R\}$ is a finite set, there holds

$$\lim_{n\to+\infty} \int_{\text{dist}(x,x_0)\leqslant R} \left| f_n^\varepsilon \right| d\mu = 0. \tag{2.4}$$

We conclude from (2.2) and (2.4) that $\lim_{n\to+\infty} \int_{\mathbb{V}} \left| f_n^\varepsilon \right| d\mu = 0$, then

$$\lim_{n\to+\infty} \int_{\mathbb{V}} f_n^\varepsilon d\mu = 0.$$

Note that

$$\left| |u_n|^q - |u_n - u|^q - |u|^q \right| \leqslant f_n^\varepsilon + \varepsilon |u_n - u|^q,$$

we obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{\mathbb{V}} \left| |u_n|^q - |u_n - u|^q - |u|^q \right| d\mu &\leqslant \overline{\lim_{n\to\infty}} \int_{\mathbb{V}} f_n^\varepsilon d\mu + \varepsilon \overline{\lim_{n\to\infty}} \int_{\mathbb{V}} |u_n - u|^q \, d\mu \\ &\leqslant 0 + \varepsilon \cdot 2^q \overline{\lim_{n\to\infty}} \int_{\mathbb{V}} \left( |u_n|^q + |u|^q \right) d\mu.\end{aligned}$$

Now let $\varepsilon \to 0$, there exists

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{V}} \left| |u_n|^q - |u_n - u|^q - |u|^q \right| d\mu = 0,$$

which implies that

$$\lim_{n\to\infty} \int_{\mathbb{V}} \left( |u_n|^q - |u_n - u|^q - |u|^q \right) d\mu = 0.$$

We finish the proof now. □

***Remark 2.1*** If $\|u_n\|_{W_0^{1,2}(\Omega)}$ is bounded and there exists $u \in W_0^{1,2}(\Omega)$ satisfying $u_n \rightharpoonup u$, then $\lim_{n\to\infty} u_n(x) = u(x)$ for all $x \in \Omega$ is obviously, which implies that

$$\lim_{n\to\infty} \left( \|u_n\|_{L^q(\Omega)}^q - \|u_n - u\|_{L^q(\Omega)}^q \right) = \|u\|_{L^q(\Omega)}^q$$

for $q \geqslant 1$.

## 3 The Construction of a Noval Constraint Manifold

To find the solutions of Eq. $(\mathcal{K}_{\lambda,b})$, it is naturally to consider the critical points of the functional $I_{\lambda,b}(u)$. A direct calculation shows that

$$\begin{aligned}\left\langle I'_{\lambda,b}(u),\varphi\right\rangle &= \left[1+b\left(\int_{\mathbb{V}}|\nabla u|^2 d\mu\right)\right]\int_{\mathbb{V}}\nabla u\nabla\varphi d\mu \\ &\quad +\int_{\mathbb{V}}(\lambda V(x)+1)\,u\varphi d\mu-\int_{\mathbb{V}}|u|^{p-2}u\varphi d\mu\end{aligned}$$

for all $\varphi\in E_\lambda$, where $I'_{\lambda,b}$ denotes the Fréchet derivative of $I_{\lambda,b}$. Next, we can define the Nehari manifold

$$\mathbf{N}_{\lambda,b}:=\left\{u\in E_\lambda\backslash\{0\}:\left\langle I'_{\lambda,b}(u),u\right\rangle=0\right\},$$

then $u\in\mathbf{N}_{\lambda,b}$ if and only if $\|u\|_\lambda^2+b\left(\int_{\mathbb{V}}|\nabla u|^2 d\mu\right)^2=\int_{\mathbb{V}}|u|^p d\mu$.

Motived by Sun and Wu [29], we filtrate the Nehari manifold $\mathbf{N}_{\lambda,b}$ with $2<p<4$. For each $r\in[2,+\infty)$, suppose that conditions $(V_1)$ and $(V_2)$ hold, in view of (2.1), we find

$$\int_{\mathbb{V}}|u|^r d\mu\leqslant\mu_0^{-\frac{r-2}{2}}\|u\|_\lambda^r\ \text{ for }\lambda>0.$$

Hence, for $u\in\mathbf{N}_{\lambda,b}$, there holds

$$\|u\|_\lambda^2\leqslant\|u\|_\lambda^2+b\left(\int_{\mathbb{V}}|\nabla u|^2 d\mu\right)^2=\int_{\mathbb{V}}|u|^p d\mu\leqslant\ \mu_0^{-\frac{p-2}{2}}\|u\|_\lambda^p,$$

which implies that

$$\int_{\mathbb{V}}|u|^p d\mu\geqslant\|u\|_\lambda^2\geqslant\mu_0\ \text{ for all }u\in\mathbf{N}_{\lambda,b}. \tag{3.1}$$

It's noteworthy that the Nehari manifold $\mathbf{N}_{\lambda,b}$ is closely linked to the behavior of fibrering map of the form $h_{b,u}:t\to I_{\lambda,b}(tu)$ as

$$h_{b,u}(t)=\frac{t^2}{2}\|u\|_\lambda^2+\frac{bt^4}{4}\left(\int_{\mathbb{V}}|\nabla u|^2 d\mu\right)^2-\frac{t^p}{p}\int_{\mathbb{V}}|u|^p d\mu\ \text{ for }t>0.$$

For $u\in E_\lambda$, it is easy to see

$$h'_{b,u}(t)=t\|u\|_\lambda^2+bt^3\left(\int_{\mathbb{V}}|\nabla u|^2 d\mu\right)^2-t^{p-1}\int_{\mathbb{V}}|u|^p d\mu$$

and

$$h''_{b,u}(t) = \|u\|_\lambda^2 + 3bt^2 \left( \int_{\mathbb{V}} |\nabla u|^2 d\mu \right)^2 - (p-1)t^{p-2} \int_{\mathbb{V}} |u|^p d\mu.$$

Thus, for $u \in E_\lambda \backslash \{0\}$ and $t > 0$, $h'_{b,u}(t) = 0$ holds if and only if $tu \in \mathbf{N}_{\lambda,b}$. Especially, we can see that $h'_{b,u}(1) = 0$ holds if and only if $u \in \mathbf{N}_{\lambda,b}$. According to [31], we may decompose $\mathbf{N}_{\lambda,b}$ into three disjoint parts

$$\mathbf{N}^+_{\lambda,b} := \left\{ u \in \mathbf{N}_{\lambda,b} : h''_{b,u}(1) > 0 \right\},$$
$$\mathbf{N}^0_{\lambda,b} := \left\{ u \in \mathbf{N}_{\lambda,b} : h''_{b,u}(1) = 0 \right\},$$
$$\mathbf{N}^-_{\lambda,b} := \left\{ u \in \mathbf{N}_{\lambda,b} : h''_{b,u}(1) < 0 \right\}.$$

Then we have the following result.

**Lemma 3.1** *Suppose that $u_0$ is a local minimizer for $I_{\lambda,b}$ on $\mathbf{N}_{\lambda,b}$ and $u_0 \notin \mathbf{N}^0_{\lambda,b}$. Then $I'_{\lambda,b}(u_0) = 0$ in $E_\lambda{}^{-1}$.*

***Proof*** If $u_0$ is a local minimizer for $I_{\lambda,b}$ on $\mathbf{N}_{\lambda,b}$, then $u_0$ is a solution of the optimization problem

$$\text{minimize } I_{\lambda,b}(u) \text{ subject to } \gamma(u) = 0,$$

where $\gamma(u) = \|u\|_\lambda^2 + b\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 - \int_{\mathbb{V}} |u|^p d\mu$. It follows from Lagrange multiplier rule that $I'_{\lambda,b}(u_0) = \theta\gamma'(u_0)$ for some $\theta \in \mathbb{R}$. Hence, we have

$$\left\langle I'_{\lambda,b}(u_0), u_0 \right\rangle = \theta \left\langle \gamma'(u_0), u_0 \right\rangle.$$

For $u_0 \in \mathbf{N}_{\lambda,b}$, there holds

$$\begin{aligned} 0 &= \left\langle I'_{\lambda,b}(u_0), u_0 \right\rangle \\ &= \|u_0\|_\lambda^2 + b\left( \int_{\mathbb{V}} |\nabla u_0|^2 d\mu \right)^2 - \int_{\mathbb{V}} |u_0|^p d\mu. \end{aligned} \tag{3.2}$$

The condition $u_0 \notin \mathbf{N}^0_{\lambda,b}$ means that

$$\|u_0\|_\lambda^2 + 3b\left( \int_{\mathbb{V}} |\nabla u_0|^2 d\mu \right)^2 - (p-1) \int_{\mathbb{V}} |u_0|^p d\mu \neq 0. \tag{3.3}$$

So combining (3.2) and (3.3) gives

$$(p-2)\|u_0\|_\lambda^2 - b(4-p)\left( \int_{\mathbb{V}} |\nabla u_0|^2 d\mu \right)^2 \neq 0. \tag{3.4}$$

Thus, by (3.2) and (3.4), we have

$$\begin{aligned}\left\langle \gamma'\left(u_0\right), u_0\right\rangle &= 2\left\|u_0\right\|_\lambda^2 + 4b\left(\int_{\mathbb{V}} |\nabla u_0|^2\, d\mu\right)^2 - p\int_{\mathbb{V}} |u_0|^p\, d\mu \\ &= -(p-2)\left\|u_0\right\|_\lambda^2 + b(4-p)\left(\int_{\mathbb{V}} |\nabla u_0|^2\, d\mu\right)^2 \\ &\neq 0,\end{aligned}$$

which implies that $\theta = 0$, and so $I'_{\lambda,b}\left(u_0\right) = 0$. The proof is completed. □

**Lemma 3.2** *Suppose that* $2 < p < 4$ *and conditions* $(V_1)$, $(V_2)$ *hold. Then* $I_{\lambda,b}$ *is coercive and bounded below on* $\mathbf{N}^-_{\lambda,b}$. *Furthermore, for all* $u \in \mathbf{N}^-_{\lambda,b}$, *there holds*

$$I_{\lambda,b}(u) \geqslant \frac{p-2}{4p}\mu_0.$$

***Proof*** Note that

$$\begin{aligned}h''_{b,u}(1) &= \|u\|_\lambda^2 + 3b\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 - (p-1)\int_{\mathbb{V}} |u|^p d\mu \\ &= -2\|u\|_\lambda^2 + (4-p)\int_{\mathbb{V}} |u|^p d\mu\end{aligned} \tag{3.5}$$

for all $u \in \mathbf{N}_{\lambda,b}$. So combining (3.1) and (3.5) implies that

$$\begin{aligned}I_{\lambda,b}(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 - \frac{1}{p}\int_{\mathbb{V}} |u|^p d\mu \\ &= \frac{1}{4}\|u\|_\lambda^2 - \frac{4-p}{4p}\int_{\mathbb{V}} |u|^p d\mu \\ &\geqslant \frac{p-2}{4p}\|u\|_\lambda^2 \geqslant \frac{p-2}{4p}\mu_0\end{aligned} \tag{3.6}$$

for all $u \in \mathbf{N}^-_{\lambda,b}$. This completes the proof. □

Next, we attempt to do more analysis with $\mathbf{N}^-_{\lambda,b}$. Suppose that conditions $(V_1)$, $(V_2)$ hold. For any $u \in \mathbf{N}_{\lambda,b}$ with $I_{\lambda,b}(u) < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}$, it is easy to obtain

$$\begin{aligned}\frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} > I_{\lambda,b}(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 - \frac{1}{p}\int_{\mathbb{V}} |u|^p\, d\mu \\ &= \frac{p-2}{2p}\|u\|_\lambda^2 - \frac{b(4-p)}{4p}\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 \\ &\geqslant \frac{p-2}{2p}\|u\|_\lambda^2 - \frac{b(4-p)}{4p}\|u\|_\lambda^4.\end{aligned}$$

Hence, if $0 < b < \frac{(p-2)^2}{2pm_\Omega(4-p)}\left(\frac{4-p}{2}\right)^{\frac{2}{p-2}}$, one has two positive numbers $D_1$ and $D_2$ satisfying

$$\sqrt{\frac{pm_\Omega}{p-2}}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} < D_1 < \sqrt{\frac{2pm_\Omega}{p-2}}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} < D_2$$

such that

$$\|u\|_\lambda < D_1 \text{ or } \|u\|_\lambda > D_2.$$

Clearly, one can see that $D_1 \to \infty$ as $p \to 2^+$ or $p \to 4^-$. Therefore, one has

$$\begin{aligned}\mathbf{N}_{\lambda,b}\left(\frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}\right) &:= \left\{u \in \mathbf{N}_{\lambda,b} : I_{\lambda,b}(u) < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}\right\}\\ &= \mathbf{N}^{(1)}_{\lambda,b} \cup \mathbf{N}^{(2)}_{\lambda,b},\end{aligned}$$

where

$$\mathbf{N}^{(1)}_{\lambda,b} := \left\{u \in \mathbf{N}_{\lambda,b}\left(\frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}\right) : \|u\|_\lambda < D_1\right\}$$

and

$$\mathbf{N}^{(2)}_{\lambda,b} := \left\{u \in \mathbf{N}_{\lambda,b}\left(\frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}\right) : \|u\|_\lambda > D_2\right\}.$$

Moreover, for $0 < b < \frac{(p-2)^2}{2pm_\Omega(4-p)}\left(\frac{4-p}{2}\right)^{\frac{2}{p-2}}$, there holds

$$\|u\|_\lambda < D_1 < \sqrt{\frac{2pm_\Omega}{p-2}}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} \quad \text{for all } u \in \mathbf{N}^{(1)}_{\lambda,b}. \tag{3.7}$$

If $\mu_0 \geqslant \frac{2p}{p-2}m_\Omega$, combining (3.5) and (3.7) gives

$$\begin{aligned}h''_{b,u}(1) &= -2\|u\|^2_\lambda + (4-p)\int_{\mathbb{V}}|u|^p d\mu\\ &\leqslant -2\|u\|^2_\lambda + (4-p)\mu_0^{-\frac{p-2}{2}}\|u\|^p_\lambda\\ &< 0\end{aligned}$$

for all $u \in \mathbf{N}^{(1)}_{\lambda,b}$, and this shows that $\mathbf{N}^{(1)}_{\lambda,b} \subset \mathbf{N}^-_{\lambda,b}$. Therefore, the following statement is true.

**Lemma 3.3** *Suppose that* $2 < p < 4$, $0 < b < \frac{(p-2)^2}{2pm_\Omega(4-p)}\left(\frac{4-p}{2}\right)^{\frac{2}{p-2}}$, $\mu_0 \geqslant \frac{2p}{p-2}m_\Omega$ *and conditions* $(V_1)$, $(V_2)$ *hold, then* $\mathbf{N}_{\lambda,b}^{(1)} \subset \mathbf{N}_{\lambda,b}^{-}$ *is sub-manifold. Moreover, each local minimizer of the functional* $I_{\lambda,b}$ *in the sub-manifold* $\mathbf{N}_{\lambda,b}^{(1)}$ *is a critical point of* $I_{\lambda,b}$ *in* $E_\lambda$.

For $u \in E_\lambda \backslash \{0\}$, we define

$$T_\lambda(u) = \left(\frac{\|u\|_\lambda^2}{\int_{\mathbb{V}} |u|^p d\mu}\right)^{\frac{1}{p-2}}$$

for convenience.

**Lemma 3.4** *Suppose that* $2 < p < 4$, $\mu_0 \geqslant \frac{2p}{p-2}m_\Omega$ *and conditions* $(V_1)$, $(V_2)$ *hold. For every* $b > 0$ *and* $u \in E_\lambda \backslash \{0\}$ *satisfying*

$$\int_{\mathbb{V}} |u|^p d\mu > \frac{p}{4-p}\left[\frac{b(4-p)}{p-2}\right]^{\frac{p-2}{2}} \|u\|_\lambda^p,$$

*there exist two constants* $t_b^+$ *and* $t_b^-$ *which satisfy*

$$T_\lambda(u) < t_b^- < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(u) < t_b^+$$

*such that* $t_b^\pm u \in \mathbf{N}_{\lambda,b}^\pm$.

***Proof*** For each $u \in E_\lambda \backslash \{0\}$ and $t > 0$, we define

$$m(t) = t^{-2}\|u\|_\lambda^2 - t^{p-4}\int_{\mathbb{V}} |u|^p d\mu \ \text{ for } t > 0,$$

then

$$\begin{aligned} h'_{b,u}(t) &= t\|u\|_\lambda^2 + bt^3\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 - t^{p-1}\int_{\mathbb{V}} |u|^p d\mu \\ &= t^3\left[m(t) + b\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2\right]. \end{aligned}$$

This implies that $tu \in \mathbf{N}_{\lambda,b}$ if and only if $m(t) + b\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 = 0$. Based on a calculation, there hold

$$m\left(T_\lambda(u)\right) = 0, \ \lim_{t\to 0^+} m(t) = +\infty, \ \lim_{t\to+\infty} m(t) = 0$$

and

$$m'(t) = t^{-3}\left[-2\|u\|_\lambda^2 + (4-p)t^{p-2}\int_{\mathbb{V}} |u|^p d\mu\right].$$

Note that $m(t)$ is decreasing when $0 < t < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(u)$ and is increasing when $t > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(u)$. Hence, when

$$\int_{\mathbb{V}} |u|^p d\mu > \frac{p}{4-p}\left[\frac{b(4-p)}{p-2}\right]^{\frac{p-2}{2}} \|u\|_\lambda^p,$$

we have

$$\begin{aligned}
\inf_{t>0} m(t) &= m\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(u)\right) \\
&= -\left(\frac{p-2}{4-p}\right)\left(\frac{2\|u\|_\lambda^2}{(4-p)\int_{\mathbb{V}} |u|^p d\mu}\right)^{\frac{-2}{p-2}} \|u\|_\lambda^2 \\
&< -b\left(\frac{p}{2}\right)^{\frac{2}{p-2}} \|u\|_\lambda^4 < -b\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2,
\end{aligned}$$

here we have applied $\left(\frac{p}{2}\right)^{\frac{2}{p-2}} > 1$. Then there exist two constants $t_b^+, t_b^- > 0$ satisfying

$$T_\lambda(u) < t_b^- < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(u) < t_b^+$$

such that

$$m\left(t_b^\pm\right) + b\left(\int_{\mathbb{V}} |\nabla u|^2 d\mu\right)^2 = 0,$$

and so $t_b^\pm u \in \mathbf{N}_{\lambda,b}$. Note that

$$\begin{aligned}
m'(t) &= -2t^{-3}\|u\|_\lambda^2 - (p-4)t^{p-5}\int_{\mathbb{V}} |u|^p d\mu \\
&= t^{-5}\left[-2t^2\|u\|_\lambda^2 + (4-p)t^p\int_{\mathbb{V}} |u|^p d\mu\right] \\
&= t^{-5}\left[-2\|tu\|_\lambda^2 + (4-p)\int_{\mathbb{V}} |tu|^p d\mu\right],
\end{aligned}$$

then through calculation, one has

$$h''_{b,t_b^- u}(1) = -2 \left\| t_b^- u \right\|_\lambda^2 + (4-p) \int_{\mathbb{V}} \left| t_b^- u \right|^p d\mu$$
$$= \left( t_b^- \right)^5 m' \left( t_b^- \right) < 0.$$

Similarly, there holds $h''_{b,t_b^+ u}(1) > 0$ and so $t_b^\pm u \in \mathbf{N}_{\lambda,b}^\pm$. Then the proof is completed. □

Next we focus on whether $\mathbf{N}_{\lambda,b}^{(1)}$ is non-empty. To see this, let's consider the Dirichlet problem (1.8) in a Hilbert space $W_0^{1,2}(\Omega)$, with corresponding functional

$$I_\Omega(u) = \frac{1}{2} \left( \int_{\Omega \cup \partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu \right) - \frac{1}{p} \int_\Omega |u|^p d\mu.$$

According to [33], suppose that $p > 2$ and conditions $(V_1)$, $(V_2)$ hold, then there exists a ground solution $w_\Omega$ satisfying

$$0 < m_\Omega := \inf_{u \in \mathcal{N}_\Omega} I_\Omega(u) = \frac{1}{2} \|w_\Omega\|_{W_0^{1,2}(\Omega)}^2 - \frac{1}{p} \|w_\Omega\|_{L^p(\Omega)}^p,$$

where

$$\mathcal{N}_\Omega := \left\{ u \in W_0^{1,2}(\Omega) \backslash \{0\} : \|u\|_{W_0^{1,2}(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p \right\}.$$

Furthermore, we obtain

$$\|w_\Omega\|_\lambda = \|w_\Omega\|_{W_0^{1,2}(\Omega)} = \sqrt{\frac{2pm_\Omega}{p-2}}$$

and

$$\|w_\Omega\|_{L^p(\mathbb{V})} = \|w_\Omega\|_{L^p(\Omega)} = \left( \frac{2pm_\Omega}{p-2} \right)^{\frac{1}{p}}.$$

Then for

$$0 < b < b^* := \frac{(p-2)^2}{2pm_\Omega(4-p)} \left( \frac{4-p}{p} \right)^{\frac{2}{p-2}} < \frac{(p-2)^2}{2pm_\Omega(4-p)} \left( \frac{4-p}{2} \right)^{\frac{2}{p-2}},$$

it has

$$\int_{\mathbb{V}} |w_\Omega|^p d\mu > \frac{p}{4-p} \left[ \frac{b(4-p)}{p-2} \right]^{\frac{p-2}{2}} \|w_\Omega\|_\lambda^p.$$

Consider lemma 3.4, it is easy to obtain positive constants $t_b^-$ and $t_b^+$ satisfying

$$T_\lambda(w_\Omega) < t_b^- < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(w_\Omega) < t_b^+$$

such that $t_b^\pm w_\Omega \in \mathbf{N}_{\lambda,b}^\pm$, where

$$T_\lambda(w_\Omega) = \left(\frac{\|w_\Omega\|_\lambda^2}{\int_{\mathbb{V}} |w_\Omega|^p \, d\mu}\right)^{\frac{1}{p-2}} = 1.$$

Moreover, when $1 < t_b^- < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}$, there hold

$$\begin{aligned} I_{\lambda,b}\left(t_b^- w_\Omega\right) &= \frac{\left(t_b^-\right)^2}{4} \|w_\Omega\|_\lambda^2 - \frac{4-p}{4p} \left(t_b^-\right)^p \int_{\mathbb{V}} |w_\Omega|^p \, d\mu \\ &= \frac{1}{4}\left[\left(t_b^-\right)^2 - \frac{4-p}{p}\left(t_b^-\right)^p\right] \frac{2pm_\Omega}{p-2} \\ &< \frac{1}{4}\left[\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} - \frac{4-p}{p}\left(\frac{2}{4-p}\right)^{\frac{p}{p-2}}\right] \frac{2pm_\Omega}{p-2} \\ &= \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} \end{aligned} \tag{3.8}$$

and

$$\left\|t_b^- w_\Omega\right\|_\lambda < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} T_\lambda(w_\Omega) \|w_\Omega\|_\lambda = \sqrt{\frac{2pm_\Omega}{p-2}} \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}},$$

so $t_b^- w_\Omega \in \mathbf{N}_{\lambda,b}^{(1)}$. Thus, we obtain the following statement.

**Lemma 3.5** *Suppose that $2 < p < 4$, $\mu_0 > \frac{2p}{p-2} m_\Omega$ and conditions $(V_1)$, $(V_2)$ hold. When*

$$0 < b < b^* := \frac{(p-2)^2}{2pm_\Omega(4-p)} \left(\frac{4-p}{p}\right)^{\frac{2}{p-2}},$$

*the sub-manifold $\mathbf{N}_{\lambda,b}^{(1)}$ is non-empty.*

## 4 The Proof of Theorem 1.1(i)

Now, we are ready to investigate the compactness for the functional $I_{\lambda,b}(u)$. In the following key proposition, we shall show that for $c$ sufficiently small, the sequence

$\{u_n\} \subset \mathbf{N}_{\lambda,b}^{(1)}$ satisfying $I_{\lambda,b}(u_n) \to c$ and $\left\| I'_{\lambda,b}(u_n) \right\|_{E_\lambda^{-1}} \to 0$ has a convergent subsequence.

**Proposition 4.1** *Suppose that* $2 < p < 4$, $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ *and conditions* $(V_1)$, $(V_2)$ *hold. Then* $I_{\lambda,b}$ *satisfies* $(PS)_c$*-condition in* $\mathbf{N}_{\lambda,b}^{(1)}$ *with* $c < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}$ *for all* $\lambda > 0$ *and* $0 < b < b^*$.

***Proof*** Let $\{u_n\} \subset \mathbf{N}_{\lambda,b}^{(1)}$ be a $(PS)_c$-sequence for $I_{\lambda,b}$ with $c < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}$. Then we have $\|u_n\|_\lambda < D_1$ by (3.7). Passing to a subsequence if necessary, there exists $u_0 \in E_\lambda$ such that

$$u_n \rightharpoonup u_0 \ \text{ in } E_\lambda, \tag{4.1}$$

$$u_n(x) \to u_0(x) \ \ \forall x \in \mathbb{V}, \tag{4.2}$$

$$u_n \to u_0 \ \text{ in } L^p(\mathbb{V}), \tag{4.3}$$

where we have used Lemma 2.3. Then, we aim to prove that $v_n := u_n - u_0 \to 0$ strongly in $E_\lambda$. We conclude from Lemma 2.5 that

$$\int_{\mathbb{V}} |v_n|^p \, d\mu = \int_{\mathbb{V}} |u_n|^p \, d\mu - \int_{\mathbb{V}} |u_0|^p \, d\mu + o_n(1). \tag{4.4}$$

Since $\|u_n\|_\lambda$ is bounded, we infer that, up to a subsequence,

$$\lim_{n\to\infty} \int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu = A,$$

where $A$ is a positive constant.

For any $\varphi \in E_\lambda$, by (4.1) and (4.2),

$$\begin{aligned} o_n(1) &= \left\langle I'_{\lambda,b}(u_n), \varphi \right\rangle \\ &= \langle u_n, \varphi \rangle_\lambda + b \int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu \int_{\mathbb{V}} \nabla u_n \nabla \varphi d\mu - \int_{\mathbb{V}} |u_n|^{p-2} u_n \varphi d\mu \\ &= \langle u_0, \varphi \rangle_\lambda + bA \int_{\mathbb{V}} \nabla u_0 \nabla \varphi d\mu - \int_{\mathbb{V}} |u_0|^{p-2} u_0 \varphi d\mu + o_n(1). \end{aligned}$$

Then take $\varphi = u_0$ in above equality, there holds

$$\|u_0\|_\lambda^2 + bA \int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu - \int_{\mathbb{V}} |u_0|^p \, d\mu = o_n(1). \tag{4.5}$$

By $u_n \in \mathbf{N}_{\lambda,b}^{(1)} \subset \mathbf{N}_{\lambda,b}$, we have

$$\|u_n\|_\lambda^2 + b\left(\int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu\right)^2 - \int_{\mathbb{V}} |u_n|^p \, d\mu = 0. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we yield that

$$\begin{aligned}
o_n(1) &= \|u_n\|_\lambda^2 + b\left(\int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu\right)^2 - \int_{\mathbb{V}} |u_n|^p \, d\mu - \|u_0\|_\lambda^2 \\
&\quad - bA \int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu + \int_{\mathbb{V}} |u_0|^p \, d\mu \\
&= \|v_n\|_\lambda^2 + 2\langle v_n, u_0\rangle_\lambda + b\left(\int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu\right)^2 - bA \int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu \\
&\quad - \int_{\mathbb{V}} |v_n|^p \, d\mu + o_n(1) \\
&= \|v_n\|_\lambda^2 + b \int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu \left(\int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu - \int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu\right) \\
&\quad - \int_{\mathbb{V}} |v_n|^p \, d\mu + o_n(1) \\
&= \|v_n\|_\lambda^2 + b \int_{\mathbb{V}} |\nabla u_n|^2 \, d\mu \int_{\mathbb{V}} |\nabla v_n|^2 \, d\mu - \int_{\mathbb{V}} |v_n|^p \, d\mu + o_n(1) \\
&\geqslant \|v_n\|_\lambda^2 - \int_{\mathbb{V}} |v_n|^p \, d\mu + o_n(1).
\end{aligned}$$

It's easy to see $\|v_n\|_\lambda \to 0$, where we have applied (4.3). Then the proof is completed. □

According to lemma 3.2, 3.3 and 3.5, we can define

$$m_{\lambda,b} = \inf_{u \in \mathbf{N}_{\lambda,b}^{(1)}} I_{\lambda,b}(u)$$

when $0 < b < b^*$. By using (3.6) and (3.8), we deduce that

$$0 < \frac{p-2}{4p}\mu_0 \leqslant m_{\lambda,b} < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}. \tag{4.7}$$

Obviously, there exists a sequence $\{u_n\} \subset \mathbf{N}_{\lambda,b}^{(1)}$ such that

$$I_{\lambda,b}(u_n) = m_{\lambda,b} + o_n(1) \tag{4.8}$$

and

$$I'_{\lambda,b}(u_n) = o_n(1) \text{ in } E_\lambda{}^{-1} \tag{4.9}$$

from the Ekeland variational principle [11].

We have made sufficient preparations to prove the theorem 1.1(i) and we give the proof now.

It from (4.7), (4.8), (4.9) and proposition 4.1 that $I_{\lambda,b}$ satisfies the $(PS)_{m_{\lambda,b}}$ condition in $E_\lambda$ for all $0 < b < b^*$ and $\lambda > 0$. So we can find a subsequence $\{u_n\}$ and $u_{\lambda,b} \in E_\lambda$ such that $u_n \to u_{\lambda,b}$ strongly in $E_\lambda$ for all $0 < b < b^*$ and $\lambda > 0$. Thus, $u_{\lambda,b} \in E_\lambda$ is a minimizer for $I_{\lambda,b}$ in $\mathbf{N}^{(1)}_{\lambda,b}$. Since $\left|u_{\lambda,b}\right| \in \mathbf{N}^{(1)}_{\lambda,b}$ and

$$I_{\lambda,b}\left(\left|u_{\lambda,b}\right|\right) = I_{\lambda,b}\left(u_{\lambda,b}\right) = m_{\lambda,b},$$

we can derive that $u_{\lambda,b} \in E_\lambda$ is a non-negative solution for Eq. $(\mathcal{K}_{\lambda,b})$ easily, by using lemma 3.3. If $u_{\lambda,b}(x_1) = 0$ for some $x_1 \in \mathbb{V}$, then $\Delta u_{\lambda,b}(x_1) = 0$, i.e.

$$\begin{aligned} 0 = \Delta u_{\lambda,b}(x_1) &= \frac{1}{\mu(x_1)} \sum_{x \sim x_1} w_{xx_1}\left[u_{\lambda,b}(x) - u_{\lambda,b}(x_1)\right] \\ &= \frac{1}{\mu(x_1)} \sum_{x \sim x_1} w_{xx_1} u_{\lambda,b}(x), \end{aligned}$$

which means $\sum\limits_{x \sim x_1} w_{xx_1} u_{\lambda,b}(x) = 0$ with $w_{xx_1} > 0$, $u_{\lambda,b}(x) \geqslant 0$ for $x \sim x_1$. Hence there holds $u_{\lambda,b}(x) = 0$ for all $x \sim x_1$, we deduce from the arbitrariness of $x_1$ that $u_{\lambda,b}(x) \equiv 0$ in $\mathbb{V}$, which implies that $u_{\lambda,b}$ is a positive solution. Moreover, we deduce from (3.1) that any critical point $u_{\lambda,b} \in \mathbf{N}^{(1)}_{\lambda,b} \subset \mathbf{N}_{\lambda,b}$ satisfying $\|u_{\lambda,b}\|_\lambda \geqslant \mu_0^{\frac{1}{2}} > 0$. The proof is now finished. □

## 5 Asymptotic Behavior of Positive Solution $u_{\lambda,b}$

After obtaining the existence of the positive solution $u_{\lambda,b} \in E_\lambda$ of Eq. $\left(\mathcal{K}_{\lambda,b}\right)$ for $2 < p < 4$, we turn to study the asymptotic behavior of $u_{\lambda,b} \in E_\lambda$ obtained by the theorem 1.1(i).

### 5.1 Asymptotic Behavior as $\lambda \to \infty$

In this subsection, we investigate the asymptotic behavior of $u_{\lambda,b} \in E_\lambda$ of Eq. $\left(\mathcal{K}_{\lambda,b}\right)$ as $\lambda \to \infty$ and give the proof of theorem 1.2(i) and 1.3(i).

To deal with Eq. $\left(\mathcal{K}_{\infty,b}\right)$, it is naturally to consider

$$\begin{aligned} I_{\infty,b}(u) = &\frac{1}{2}\left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega |u|^2\, d\mu\right) + \frac{b}{4}\left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu\right)^2 \\ &-\frac{1}{p}\int_\Omega |u|^p\, d\mu \end{aligned}$$

with corresponding Nehari manifold

$$\mathbf{N}_{\infty,b} := \left\{u \in W_0^{1,2}(\Omega)\backslash\{0\} : \|u\|^2_{W_0^{1,2}(\Omega)} + b\left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu\right)^2 = \int_\Omega |u|^p\, d\mu\right\}.$$

Similar to the filtration of Nehari manifold $\mathbf{N}_{\lambda,b}$, we can define

$$\mathbf{N}_{\infty,b}^{(1)} := \left\{ u \in \mathbf{N}_{\infty,b} : I_{\infty,b}(u) < \frac{m_\Omega}{2}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}, \|u\|_{W_0^{1,2}(\Omega)} < D_1 \right\}$$

and

$$m_{\infty,b} = \inf_{u \in \mathbf{N}_{\infty,b}^{(1)}} I_{\infty,b}(u).$$

Just as what we have done for Eq. $(\mathcal{K}_{\lambda,b})$, we can get a positive solution $u_{\infty,b} \in W_0^{1,2}(\Omega)$ of Eq. $(\mathcal{K}_{\infty,b})$ for $2 < p < 4$, which achieves $m_{\infty,b}$ of the functional $I_{\infty,b}(u)$ in $\mathbf{N}_{\infty,b}^{(1)}$. Thus, we prove the theorem 1.2(i). □

Before proving the theorem 1.3(i), we establish the following lemma.

**Lemma 5.1** *Suppose that* $2 < p < 4$, $\mu_0 \geqslant \frac{2p}{p-2} m_\Omega$ *and conditions* $(V_1)$, $(V_2)$ *hold. Then there exists* $b_* \in (0, b^*)$ *such that* $m_{\lambda,b} \to m_{\infty,b}$ *as* $\lambda \to \infty$ *for all* $b \in (0, b_*)$ *fixed.*

***Proof*** It is easy to see $m_{\lambda,b} < m_{\infty,b}$ for any $\lambda > 0$. If not, we can find a nontrivial solution $u_{\lambda,b} \in E_\lambda$ of Eq. $(\mathcal{K}_{\lambda,b})$ which vanishes outside $\Omega$. So we have $u_{\lambda,b}(x_0) = 0$ for some $x_0 \in \mathbb{V}$, obviously there holds $\Delta u_{\lambda,b}(x_0) = 0$. In view of the maximum principle, we may deduce that $u_{\lambda,b}(x) \equiv 0$ in $\mathbb{V}$, which leads to a contradiction.

Taking a sequence $\lambda_n \to \infty$ such that

$$\lim_{n\to\infty} m_{\lambda_n,b} = M \leqslant m_{\infty,b},$$

where $m_{\lambda_n,b}$ is associated with the positive solution $u_{\lambda_n,b} \in \mathbf{N}_{\lambda_n,b}^{(1)}$ of Eq. $(\mathcal{K}_{\lambda_n,b})$ obtained by theorem 1.1(i). We may deduce from (4.7) that $M > 0$. In addition, $\{u_{\lambda_n,b}\}$ is uniformly bounded in $E_\lambda$. Going if necessary to a subsequence, we have $u_0 \in E_\lambda$ such that

$$u_{\lambda_n,b} \rightharpoonup u_0 \quad \text{in } E_\lambda, \tag{5.1}$$

$$u_{\lambda_n,b}(x) \to u_0(x) \ \ \forall x \in \mathbb{V}, \tag{5.2}$$

and for any $q \geqslant 2$,

$$u_{\lambda_n,b} \to u_0 \quad \text{in } L^q(\mathbb{V}), \tag{5.3}$$

by applying lemma 2.3. Then we shall prove that $u_0|_{\Omega^c} = 0$, otherwise we can assume $u_0(x_1) \neq 0$ for some $x_1 \notin \Omega$. In view of $u_{\lambda_n,b} \in \mathbf{N}_{\lambda_n,b}^{(1)}$, there holds

$$I_{\lambda_n,b}(u_{\lambda_n,b}) \geqslant \frac{p-2}{4p}\|u_{\lambda_n,b}\|_{\lambda_n}^2 \geqslant \frac{p-2}{4p}\int_{\mathbb{V}} \lambda_n V(x) u_{\lambda_n,b}^2 d\mu$$

$$\geqslant \frac{p-2}{4p}\lambda_n \mu\left(x_1\right) V\left(x_1\right) u_{\lambda_n,b}^2\left(x_1\right),$$

and then $\lim_{n\to\infty} I_{\lambda_n,b}\left(u_{\lambda_n,b}\right) = +\infty$ by using $V\left(x_1\right) > 0$, $u_{\lambda_n,b}\left(x_1\right) \to u_0(x_1) \neq 0$ and $\mu\left(x_1\right) > 0$. Hence, we get a contradiction to the fact that $m_{\lambda_n,b} < m_{\infty,b} < +\infty$. Furthermore, in view of (4.7), $u_0 \not\equiv 0$ is clearly.

Consider that $\Omega$ is a finite set and (5.2), we get

$$\begin{aligned}
\lim_{n\to\infty}\int_{\Omega\cup\partial\Omega}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu &= \lim_{n\to\infty}\frac{1}{2}\sum_{x\in\Omega\cup\partial\Omega}\sum_{y\sim x} w_{xy}\left[u_{\lambda_n,b}(y)-u_{\lambda_n,b}(x)\right]^2\\
&= \lim_{n\to\infty}\frac{1}{2}\sum_{x\in\Omega\cup\partial\Omega}\sum_{y\sim x} w_{xy}\left[u_0(y)-u_0(x)\right]^2\\
&= \int_{\Omega\cup\partial\Omega}|\nabla u_0|^2 d\mu,
\end{aligned}$$

then

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu\\
&\quad\geqslant \underline{\lim_{n\to\infty}}\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu \geqslant \int_{\Omega\cup\partial\Omega}|\nabla u_0|^2 d\mu = \int_{\mathbb{V}}|\nabla u_0|^2 d\mu,
\end{aligned}\tag{5.4}$$

where we have used $u_0|_{\Omega^c} = 0$. Notice that

$$\begin{aligned}
&\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2 d\mu + \int_{\Omega} u_0^2\, d\mu + b\left(\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2 d\mu\right)^2\\
&\quad\leqslant \int_{\mathbb{V}}\left(|\nabla u_0|^2 + u_0^2\right) d\mu + b\left(\int_{\mathbb{V}}|\nabla u_0|^2 d\mu\right)^2\\
&\quad\leqslant \underline{\lim_{n\to\infty}}\left\{\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x)+1)\, u_{\lambda_n,b}^2\right] d\mu + b\left(\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\}\\
&\quad= \underline{\lim_{n\to\infty}}\int_{\mathbb{V}}\left|u_{\lambda_n,b}\right|^p d\mu = \int_{\mathbb{V}}|u_0|^p d\mu = \int_{\Omega}|u_0|^p d\mu
\end{aligned}$$

by (5.4) and $u_0|_{\Omega^c} = 0$. We obtain $t \in (0, 1]$ such that $tu_0 \in \mathbf{N}_{\infty,b}^{(1)}$ for $b$ sufficiently small since $\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2 d\mu$ has a bound independent of parameter $b$, i.e.

$$\int_{\Omega\cup\partial\Omega}|t\nabla u_0|^2 d\mu + \int_{\Omega}|tu_0|^2 d\mu + b\left(\int_{\Omega\cup\partial\Omega}|t\nabla u_0|^2 d\mu\right)^2 = \int_{\Omega}|tu_0|^p d\mu.$$

Hence, the following two cases are considered: $0 < t < 1$ and $t = 1$.

For the former, by (5.4), we have

$$m_{\infty,b} \leqslant I_{\infty,b}\left(tu_0\right)$$

$$
\begin{aligned}
&= \frac{p-2}{2p}\left(\int_{\Omega\cup\partial\Omega} |t\nabla u_0|^2\, d\mu + \int_{\Omega} |tu_0|^2\, d\mu\right) - b\frac{4-p}{4p}\left(\int_{\Omega\cup\partial\Omega} |t\nabla u_0|^2\, d\mu\right)^2 \\
&< \frac{p-2}{2p}\left[\int_{\mathbb{V}} \left(|\nabla u_0|^2 + u_0^2\right) d\mu\right] - \overline{\lim_{n\to\infty}}\, b\frac{4-p}{4p}\left(\int_{\mathbb{V}} |\nabla u_{\lambda_n,b}|^2\, d\mu\right)^2 \\
&\leqslant \underline{\lim_{n\to\infty}} \left\{\frac{p-2}{2p}\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x)+1)\, u_{\lambda_n,b}^2\right] d\mu \right. \\
&\quad \left. - b\frac{4-p}{4p}\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\} \\
&= \underline{\lim_{n\to\infty}}\, I_{\lambda_n,b}\left(u_{\lambda_n,b}\right) = M
\end{aligned}
$$

for $b$ small enough, which contradicts $\lim_{n\to\infty} m_{\lambda_n,b} = M \leqslant m_{\infty,b}$.

For the latter, we deduce from (5.4) that

$$
\begin{aligned}
&\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2\, d\mu + \int_{\Omega} u_0^2 d\mu + b\left(\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2\, d\mu\right)^2 \\
&\quad = \underline{\lim_{n\to\infty}} \left\{\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x)+1)\, u_{\lambda_n,b}^2\right] d\mu + b\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\}
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
&\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2\, d\mu + \int_{\Omega} u_0^2\, d\mu + b\left(\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2\, d\mu\right)^2 \\
&\quad \leqslant \underline{\lim_{n\to\infty}} \left\{\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x)+1)\, u_{\lambda_n,b}^2\right] d\mu + b\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\} \\
&\quad \leqslant \overline{\lim_{n\to\infty}} \left\{\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x)+1)\, u_{\lambda_n,b}^2\right] d\mu + b\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\} \\
&\quad = \overline{\lim_{n\to\infty}} \int_{\mathbb{V}} \left|u_{\lambda_n,b}\right|^p d\mu = \int_{\mathbb{V}} |u_0|^p\, d\mu = \int_{\Omega} |u_0|^p\, d\mu,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2\, d\mu + \int_{\Omega} u_0^2\, d\mu + b\left(\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2\, d\mu\right)^2 \\
&\quad = \overline{\lim_{n\to\infty}} \left\{\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x)+1)\, u_{\lambda_n,b}^2\right] d\mu + b\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\}.
\end{aligned}
\tag{5.6}
$$

So the combination of (5.5) and (5.6) gives

$$\lim_{n\to\infty}\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2+(\lambda_n V(x)+1)\,u_{\lambda_n,b}^2\right]d\mu=\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2\,d\mu+\int_{\Omega}u_0^2\,d\mu \tag{5.7}$$

and

$$\lim_{n\to\infty}\left(\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2=\left(\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2\,d\mu\right)^2. \tag{5.8}$$

According to (5.7) and (5.8), there holds

$$\begin{aligned}
m_{\infty,b}&\leqslant I_{\infty,b}(u_0)\\
&=\frac{p-2}{2p}\left(\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2\,d\mu+\int_{\Omega}|u_0|^2\,d\mu\right)-b\frac{4-p}{4p}\left(\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2\,d\mu\right)^2\\
&=\lim_{n\to\infty}\left\{\frac{p-2}{2p}\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2+(\lambda_n V(x)+1)\,u_{\lambda_n,b}^2\right]d\mu-b\frac{4-p}{4p}\left(\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\}\\
&=\lim_{n\to\infty}I_{\lambda_n,b}\left(u_{\lambda_n,b}\right)=M.
\end{aligned}$$

Consequently, we conclude from the above two cases that $m_{\infty,b}=M$, and so $\lim_{n\to\infty}m_{\lambda_n,b}=m_{\infty,b}$. This completes the proof. □

Based on the above discussion, the theorem 1.3(i) can be proved. Next, We need to prove that for any sequence $\lambda_n\to\infty$, the positive solution $u_{\lambda_n,b}\in\mathbf{N}^{(1)}_{\lambda_n,b}$ of Eq. $\left(\mathcal{K}_{\lambda_n,b}\right)$ satisfying $I_{\lambda_n,b}\left(u_{\lambda_n,b}\right)=m_{\lambda_n,b}$ converges in $W^{1,2}(\mathbb{V})$ to a positive solution $u_{\infty,b}\in W_0^{1,2}(\Omega)$ of Eq. $\left(\mathcal{K}_{\infty,b}\right)$ obtained by theorem 1.2(i) along a subsequence.

Similar to the discussion in the above lemma, there hold (5.1), (5.2), (5.3), $u_0|_{\Omega^c}=0$ and $u_0\not\equiv 0$ obviously. Then we aim to show that as $n\to\infty$, up to a subsequence, there hold

$$\lambda_n\int_{\mathbb{V}}V(x)u_{\lambda_n,b}^2 d\mu\to 0 \tag{5.9}$$

and

$$\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu\to\int_{\mathbb{V}}|\nabla u_0|^2\,d\mu. \tag{5.10}$$

If not, let's analyze it in two cases that (5.9) and (5.10) aren't valid respectively.

For the former, we assume that (5.9) doesn't hold, then $\lim\limits_{n\to\infty} \lambda_n \int_{\mathbb{V}} V(x)u^2_{\lambda_n,b}d\mu = \delta_1 > 0$, and so

$$\begin{aligned}
&\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 \, d\mu + \int_{\Omega} u_0^2 \, d\mu + b\left(\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 \, d\mu\right)^2 \\
&\quad < \int_{\mathbb{V}} \left(|\nabla u_0|^2 + u_0^2\right) d\mu + \delta_1 + b\left(\int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu\right)^2 \\
&\quad \leqslant \varliminf_{n\to\infty} \left\{\int_{\mathbb{V}} \left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x) + 1)\, u^2_{\lambda_n,b}\right] d\mu + b\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\} \\
&\quad = \lim_{n\to\infty} \int_{\mathbb{V}} \left|u_{\lambda_n,b}\right|^p d\mu = \int_{\mathbb{V}} |u_0|^p \, d\mu = \int_{\Omega} |u_0|^p \, d\mu, \qquad (5.11)
\end{aligned}$$

where we have used (5.4) and $u_0|_{\Omega^c} = 0$.

For the latter, we assume that (5.10) doesn't hold, then

$$\lim_{n\to\infty} \left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2 = \left(\int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu\right)^2 + \delta_2$$

with $\delta_2 > 0$, thus

$$\begin{aligned}
&\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 \, d\mu + \int_{\Omega} u_0^2 \, d\mu + b\left(\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 \, d\mu\right)^2 \\
&\quad < \int_{\mathbb{V}} \left(|\nabla u_0|^2 + u_0^2\right) d\mu + b\left(\int_{\mathbb{V}} |\nabla u_0|^2 \, d\mu\right)^2 + b\delta_2 \\
&\quad \leqslant \varliminf_{n\to\infty} \left\{\int_{\mathbb{V}} \left[\left|\nabla u_{\lambda_n,b}\right|^2 + (\lambda_n V(x) + 1)\, u^2_{\lambda_n,b}\right] d\mu + b\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\} \\
&\quad = \lim_{n\to\infty} \int_{\mathbb{V}} \left|u_{\lambda_n,b}\right|^p d\mu = \int_{\mathbb{V}} |u_0|^p \, d\mu = \int_{\Omega} |u_0|^p \, d\mu, \qquad (5.12)
\end{aligned}$$

by applying (5.4) and $u_0|_{\Omega^c} = 0$.

We deduce from (5.11) and (5.12) that there exists $t \in (0, 1)$ such that $tu_0 \in \mathbf{N}^{(1)}_{\infty,b}$ with a sufficiently small parameter $b$, i.e.

$$\int_{\Omega\cup\partial\Omega} |t\nabla u_0|^2 \, d\mu + \int_{\Omega} |tu_0|^2 \, d\mu + b\left(\int_{\Omega\cup\partial\Omega} |t\nabla u_0|^2 \, d\mu\right)^2 = \int_{\Omega} |tu_0|^p \, d\mu.$$

Thus, in view of (5.9) and (5.10), we get

$$\begin{aligned}
m_{\infty,b} &\leqslant I_{\infty,b}(tu_0) \\
&= \frac{p-2}{2p}\left(\int_{\Omega\cup\partial\Omega} |t\nabla u_0|^2 \, d\mu + \int_{\Omega} |tu_0|^2 \, d\mu\right) - b\frac{4-p}{4p}\left(\int_{\Omega\cup\partial\Omega} |t\nabla u_0|^2 \, d\mu\right)^2 \\
&< \frac{p-2}{2p} \int_{\mathbb{V}} \left(|\nabla u_0|^2 + u_0^2\right) d\mu - \varlimsup_{n\to\infty} b\frac{4-p}{4p}\left(\int_{\mathbb{V}} \left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2
\end{aligned}$$

$$\leqslant \lim_{n\to\infty}\left\{\frac{p-2}{2p}\int_{\mathbb{V}}\left[\left|\nabla u_{\lambda_n,b}\right|^2+(\lambda_n V(x)+1)\,u_{\lambda_n,b}^2\right]d\mu-b\frac{4-p}{4p}\left(\int_{\mathbb{V}}\left|\nabla u_{\lambda_n,b}\right|^2 d\mu\right)^2\right\}$$
$$=\lim_{n\to\infty} I_{\lambda_n,b}\left(u_{\lambda_n,b}\right)=m_{\infty,b}$$

for $b$ sufficiently small, which leads to a contradiction. So we find that $u_0$ is a solution of Eq. $\left(\mathcal{K}_{\infty,b}\right)$. Furthermore, lemma 5.1 gives that $u_0$ is a positive solution of Eq. $\left(\mathcal{K}_{\infty,b}\right)$, which achieves $m_{\infty,b}$ of the functional $I_{\infty,b}(u)$ in $\mathbf{N}_{\infty,b}^{(1)}$. This completes the proof. □

***Remark 5.1*** According to the proof of lemma 5.1 and the proof of this theorem, we can take the same $b_* \in (0, b^*)$ in lemma 5.1 and this theorem.

### 5.2 Asymptotic Behavior as $b \to 0^+$

In this subsection, we focus on reasearching asymptotic behavior of $u_{\lambda,b} \in E_\lambda$ as $b \to 0^+$. Based on the theorem 1.1(i), we give the proof of theorem 1.4(i).

Fix $\lambda \in (0,\infty)$, for any sequence $b_n \to 0$, let $u_{\lambda,b_n} \in E_\lambda$ be the positive solution of Eq. $\left(\mathcal{K}_{\lambda,b_n}\right)$ obtained by theorem 1.1(i). Note that

$$0<\mu_0^{\frac{1}{2}}\leqslant\left\|u_{\lambda,b_n}\right\|_\lambda<\sqrt{\frac{2pm_\Omega}{p-2}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}}, \tag{5.13}$$

going to a subsequence if necessary, there exists $u_0 \in E_\lambda$ such that $u_{\lambda,b_n} \rightharpoonup u_0$ in $E_\lambda$. As we have discussed in Sect. 4, we can deduce $u_{\lambda,b_n} \to u_0$ in $E_\lambda$. Now we just have to prove that $u_0$ is a positive solution of Eq. $\left(\mathcal{K}_{\lambda,0}\right)$. Since $\left\langle I'_{\lambda,b_n}\left(u_{\lambda,b_n}\right),\varphi\right\rangle=0$ for all $\varphi \in E_\lambda$, it is easy to check that

$$\int_{\mathbb{V}}\nabla u_0\nabla\varphi+(\lambda V(x)+1)\,u_0\varphi d\mu=\int_{\mathbb{V}}|u_0|^{p-2}u_0\varphi d\mu,$$

which implies that $u_0$ is a nonnegative solution of Eq. $\left(\mathcal{K}_{\lambda,0}\right)$. And we have $u_0 \neq 0$ by (5.13). Furthermore, consider the maximum principle, we may obtain $u_0 > 0$ in $\mathbb{V}$. The proof is now finished. □

### 5.3 Asymptotic Behavior as $\lambda \to \infty$ and $b \to 0^+$

Finally, we explore the asymptotic behavior of the positive solution $u_{\lambda,b}$ as $\lambda \to \infty$ and $b \to 0^+$ in this subsection. And we are ready to give the proof of theorem 1.5(i).

Fix a sufficiently small $b$, for any sequence $\lambda_n \to \infty$, we may obtain the positive solution $u_{\lambda_n,b}$ of Eq. $\left(\mathcal{K}_{\lambda_n,b}\right)$ by using theorem 1.1(i). Passing to a subsequence, $u_{\lambda_n,b} \in E_{\lambda_n}$ converges to the positive solution $u_{\infty,b} \in W_0^{1,2}(\Omega)$ of Eq. $\left(\mathcal{K}_{\infty,b}\right)$ as $\lambda_n \to \infty$ by applying theorem 1.3(i). Then, in view of the proof of theorem 1.4(i), we derive that $u_{\infty,b} \in W_0^{1,2}(\Omega)$ converges to a positive solution of Eq. $\left(\mathcal{K}_{\infty,0}\right)$ as $b \to 0^+$.

Alternatively, up to a subsequence, we can consider fixing $\lambda \in (0, \infty)$ and letting $b \to 0^+$ subsequently, after then letting $\lambda \to \infty$. We may get similar result and the proof of theorem 1.5(i) is complete. □

## 6 The Proof of Theorem 1.1(ii) and (iii)

For $p \geqslant 4$, it is easy to prove that the existence of a ground state solution for Eq. $(\mathcal{K}_{\lambda,b})$ by standard variational methods. First, we give several useful preliminary results about energy functional $I_{\lambda,b}$ and Nehari manifold $\mathbf{N}_{\lambda,b}$.

**Lemma 6.1** *(i) Suppose that $p = 4$ and conditions $(V_1)$, $(V_2)$ hold. Then there exists $\hat{b} > 0$ such that for any $b \in (0, \hat{b})$ and any $\lambda > 0$, $\mathbf{N}_{\lambda,b}$ is non-empty.*
*(ii) Suppose that $p > 4$ and conditions $(V_1)$, $(V_2)$ hold, Then $\mathbf{N}_{\lambda,b}$ is non-empty for any $b > 0, \lambda > 0$.*

***Proof*** (i) For any $u \in E_\lambda \backslash \{0\}$, there holds

$$\left\langle I'_{\lambda,b}(tu), tu \right\rangle = t^2 \|u\|_\lambda^2 + bt^4 \left( \int_{\mathbb{V}} |\nabla u|^2 d\mu \right)^2 - t^p \int_{\mathbb{V}} |u|^p d\mu.$$

Hence, for sufficiently small $b$, it can be find $t_0 \in (0, +\infty)$ such that $\left\langle I'_{\lambda,b}(t_0 u), t_0 u \right\rangle = 0$, which means that $\mathbf{N}_{\lambda,b}$ is non-empty.
(ii) Similarly, we can prove (ii). □

**Lemma 6.2** *(i) Suppose that $p = 4$ and conditions $(V_1)$, $(V_2)$ hold, then $l_{\lambda,b} := \inf\limits_{u \in \mathbf{N}_{\lambda,b}} I_{\lambda,b}(u) > 0$ for any $b \in (0, \hat{b})$ and $\lambda > 0$.*
*(ii) Suppose that $p > 4$ and conditions $(V_1)$, $(V_2)$ hold, then $l_{\lambda,b} > 0$ for any $b > 0$ and $\lambda > 0$.*

***Proof*** (i) Consider (3.1), we deduce that

$$\begin{aligned} I_{\lambda,b}(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4} \left( \int_{\mathbb{V}} |\nabla u|^2 d\mu \right)^2 - \frac{1}{p} \int_{\mathbb{V}} |u|^p d\mu \\ &= \left( \frac{1}{2} - \frac{1}{4} \right) \|u\|_\lambda^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{V}} |u|^p d\mu \\ &\geqslant \frac{1}{4}\|u\|_\lambda^2 \geqslant \frac{1}{4}\mu_0 \end{aligned}$$

for any $u \in \mathbf{N}_{\lambda,b}$, so $l_{\lambda,b} \geqslant \frac{1}{4}\mu_0 > 0$.
(ii) We may prove (ii) similarly. □

**Lemma 6.3** *Suppose that $p \geqslant 4$ and conditions $(V_1)$, $(V_2)$ hold. Then for each $\lambda > 0$, $b > 0$, $I_{\lambda,b}$ satisfies the $(PS)_c$ condition in $\mathbf{N}_{\lambda,b}$ for any $c \in \mathbb{R}$.*

***Proof*** To prove that $I_{\lambda,b}$ satisfies $(PS)_c$ condition, we can assume $\{u_n\} \subset E_\lambda$ such that

$$I_{\lambda,b}(u_n) = \frac{1}{2}\|u_n\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{V}} |\nabla u_n|^2 d\mu\right)^2 - \frac{1}{p}\int_{\mathbb{V}} |u_n|^p d\mu = c + o_n(1) \tag{6.1}$$

and

$$\begin{aligned}\left\langle I'_{\lambda,b}(u_n), \varphi\right\rangle &= \langle u_n, \varphi\rangle_\lambda + b\int_{\mathbb{V}} |\nabla u_n|^2 d\mu \int_{\mathbb{V}} \nabla u_n \nabla\varphi d\mu - \int_{\mathbb{V}} |u_n|^{p-2} u_n\varphi d\mu \\ &= o_n(1)\|\varphi\|_\lambda \end{aligned}\tag{6.2}$$

for any $\varphi \in E_\lambda$. Replacing $\varphi$ by $u_n$ in (6.2), there holds

$$\|u_n\|_\lambda^2 + b\left(\int_{\mathbb{V}} |\nabla u_n|^2 d\mu\right)^2 - \int_{\mathbb{V}} |u_n|^p d\mu = o_n(1)\|u_n\|_\lambda . \tag{6.3}$$

Combining (6.1) and (6.3) gives

$$\|u_n\|_\lambda^2 = -\left(1 - \frac{4}{p}\right)\int_{\mathbb{V}} |u_n|^p d\mu + 4c + o_n(1)\|u_n\|_\lambda + o_n(1) \leqslant 4c + o_n(1),$$

which implies that $\{u_n\}$ is bounded in $E_\lambda$. Then we omit the rest of the proof due to its similarity with proposition 4.1. Now, the proof is completed. □

Based on the above discussion, We may provide the proof of theorem 1.1(ii) and (iii). In view of the Ekeland variational principle [11], there exists a sequence $\{u_n\} \subset \mathbf{N}_{\lambda,b}$ such that

$$I_{\lambda,b}(u_n) = l_{\lambda,b} + o_n(1) \tag{6.4}$$

and

$$I'_{\lambda,b}(u_n) = o_n(1) \text{ in } {E_\lambda}^{-1} \tag{6.5}$$

obviously. We yield from lemmas 6.1, 6.2, 6.3 and (6.4), (6.5) that there exists $w_{\lambda,b} \in E_\lambda$ such that $u_n \to w_{\lambda,b}$ strongly in $E_\lambda$. Hence, $w_{\lambda,b} \in E_\lambda$ is a minimizer for $I_{\lambda,b}$ in $\mathbf{N}_{\lambda,b}$. Note that $|w_{\lambda,b}| \in \mathbf{N}_{\lambda,b}$ and

$$I_{\lambda,b}(|w_{\lambda,b}|) = I_{\lambda,b}(w_{\lambda,b}) = l_{\lambda,b},$$

we obtain a positive ground solution $w_{\lambda,b} \in E_\lambda$ for Eq. $(\mathcal{K}_{\lambda,b})$ satisfying $I_{\lambda,b}(w_{\lambda,b}) = l_{\lambda,b} > 0$ by the maximum principle. This ends the proof of theorem 1.1(ii) and (iii). □

Similar to the above discussion, we can obtain the positive ground solution $w_{\infty,b} \in W_0^{1,2}(\Omega)$ of Eq. $(\mathcal{K}_{\infty,b})$ for $p \geqslant 4$. And the remaining proof of the theorem 1.2 is omitted. □

## 7 Asymptotic Behavior of Positive Ground Solution $w_{\lambda,b}$

After studying the existence of positive ground state solution $w_{\lambda,b} \in E_\lambda$ for Eq. $(\mathcal{K}_{\lambda,b})$, just as what we have discussed in Sect. 5, we may investigate the asymptotic behavior of $w_{\lambda,b} \in E_\lambda$ analogously, so we omit the proof of remaining part of theorems 1.3, 1.4, 1.5 here.

## 8 The Estimate of the Solution $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$

After exploring the existence and the asymptotic behavior of the solution $u_{\lambda,b}, w_{\lambda,b} \in E_\lambda$, we are interested in the estimate involving the $L^\infty$-norm of solutions. For this purpose, we have the following research. And the proof of theorem 1.6 is as following.

Let $\|\cdot\|_q := \|\cdot\|_{L^q(\mathbb{V})}$ for convenience. We assume $m > 0$ satisfying $m > p$, then

$$\sum_{x\in\mathbb{V}} \left|u_{\lambda,b}(x)\right|^{m\tau} \mu(x) \leqslant \mu_0^{\frac{p-m}{p}} \left[\sum_{x\in\mathbb{V}} \left|u_{\lambda,b}(x)\right|^{p\tau} \mu(x)\right]^{\frac{m\tau}{p\tau}}$$

for any $\tau > 1$, it follows

$$\left\|u_{\lambda,b}\right\|_{m\tau} \leqslant \mu_0^{\frac{p-m}{mp\tau}} \left\|u_{\lambda,b}\right\|_{p\tau}. \tag{8.1}$$

Set $\sigma = \frac{m}{p} > 1$. When $\tau = \sigma$ in (8.1), there holds

$$\left\|u_{\lambda,b}\right\|_{m\sigma} \leqslant \mu_0^{\frac{p-m}{mp\sigma}} \left\|u_{\lambda,b}\right\|_{m}.$$

Arguing by iteration, let $\tau = \sigma^j$ in (8.1), we can deduce that

$$\begin{aligned} \left\|u_{\lambda,b}\right\|_{m\sigma^j} &\leqslant \mu_0^{\frac{p-m}{mp}\left(\frac{1}{\sigma}+\frac{1}{\sigma^2}+\cdots+\frac{1}{\sigma^j}\right)} \left\|u_{\lambda,b}\right\|_{m} \\ &\leqslant \mu_0^{\frac{p-m}{mp}\frac{1}{1-\sigma}} \left\|u_{\lambda,b}\right\|_{m}. \end{aligned} \tag{8.2}$$

Note that $\left\|u_{\lambda,b}\right\|_{p+1} \leqslant \mu_0^{-\frac{1}{p(p+1)}} \left\|u_{\lambda,b}\right\|_{p}$, let $j \to \infty$ and take $m = p+1$ in (8.2), we find

$$\left\|u_{\lambda,b}\right\|_{\infty} \leqslant \mu_0^{\frac{1}{p(p+1)\left(\frac{p+1}{p}-1\right)}} \left\|u_{\lambda,b}\right\|_{p+1} \leqslant \mu_0^{\frac{p-1}{p(p+1)}} \left\|u_{\lambda,b}\right\|_{p}.$$

Considering (2.1), (3.6) and $m_{\lambda,b} < m_{\infty,b}$, we obtain

$$\|u_{\lambda,b}\|_\infty \leqslant \mu_0^{\frac{p-1}{p(p+1)}} \cdot \mu_0^{-\frac{p-2}{2p}} \|u_{\lambda,b}\|_\lambda \leqslant \mu_0^{\frac{3p-p^2}{2p(p+1)}} \left(\frac{4p}{p-2} m_{\lambda,b}\right)^{\frac{1}{2}}$$
$$\leqslant \mu_0^{\frac{3p-p^2}{2p(p+1)}} \left(\frac{4p}{p-2} m_{\infty,b}\right)^{\frac{1}{2}}.$$

Similarly, we may analyze the case of $w_{\lambda,b} \in E_\lambda$. Hence, we finish the proof. □

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## Declarations

**Conflict of interest** The authors declare no Conflict of interest.

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