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Some Sharp Bohr-Type Inequalities for Analytic Functions

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Abstract
This article focuses on the improvement of the classic Bohr's inequality for bounded analytic functions on the unit disk. We give some sharp versions of Bohr's inequality, generalizing the previous results.

Keywords Bohr radius · Bohr-type inequality · Bounded analytic functions · Schwarz lemma

Mathematics Subject Classification 30A10 · 30C45

1 Introduction

Bohr's theorem states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a bounded analytic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| = r < 1\}$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, then

$$B_0(f, r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \tag{1.1}$$

for $r \leq 1/3$, the constant $1/3$ is sharp and the value is called the classical Bohr radius. In 1914, the inequality was originally obtained by Bohr only for $r \leq 1/6$ [11]. Later, Riesz, Schur and Wiener proved the inequality (1.1) holds for $r \leq 1/3$ and showed that the constant $1/3$ cannot be improved. Other proofs about this inequality in [30, 31] and also see [4, 15].

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The inequality (1.1) is known and classical as Bohr inequality. Recently, several aspects of Bohr inequality and its improvements have created enormous interest in various settings. Such as even analytic function, alternating series and odd analytic function [10, 21], starlike logharmonic mappings [9], subordinating families and harmonic mappings [3], the classes of quasi-subordination and K-quasiregular harmonic mappings [24], Banach spaces and Banach algebras [12, 13], operator theory [25], several real or complex variables [5–8]. For more general results, see [27, 28].

Let

$$B_k(f,r) := \sum_{n=k}^{\infty} |a_n| r^n, \quad ||f_k||_r^2 := \sum_{n=k}^{\infty} |a_n|^2 r^{2n}, \quad and \quad f_1(z) := f(z) - a_0.$$

We denote by $S_r(f)$ the area of the image of the subdisk $|z| < r$ under the mapping f and take $S_r(f)$ as S_r for convenience.

Recently, Kayumov [20] and Liu [23] proved the following Bohr-type inequalities.

Theorem 1.1 [20] Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then

$$|f(z)|^2 + B_1(f,r) \leq 1 \quad for \quad |z| = r \leq 1/3,$$

the radius $1/3$ cannot be improved.

Theorem 1.2 [23] Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. Then

$$B_0(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)}||f_1||_r^2 + |f_1(z)| \leq 1 \quad for \quad |z| = r \leq 1/5,$$

the radius $1/5$ cannot be improved. Moreover,

$$|a_0|^2 + B_1(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)}||f_1||_r^2 + |f_1(z)| \leq 1 \quad for \quad |z| = r \leq 1/3,$$

the radius $1/3$ cannot be improved.

Theorem 1.3 [23] Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. Then

$$|f(z)| + B_1(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)}||f_1||_r^2 \leq 1 \quad for \quad |z| = r \leq r_{a_0} = \frac{2}{3+|a_0|+\sqrt{5}(1+|a_0|)},$$

the radius r_{a_0} is the best possible and $r_{a_0} \geq \sqrt{5} - 2$. Moreover,

$$|f(z)|^2 + B_1(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)}||f_1||_r^2 \leq 1 \quad |z| = r \leq r'_{a_0},$$

where r'_{a_0} is the unique positive root of the equation

$$(1-|a_0|^3)r^3-(1+2|a_0|)r^2-2r+1=0.$$

The radius r'_{a_0} is the best possible. Further, we have $1/3 < r'_{a_0} < \frac{1}{2+|a_0|}$.

There are various ways to generalize the classical Bohr inequality for bounded analytic functions. For instance, Huang and Hu extended the Bohr-type inequality by allowing Schwarz function in place of the initial coefficients in function's power series expansions in [18] and [16], respectively. In [32] and [17], the authors established some sharp Bohr-type inequalities with one parameter or involving convex combination. Kayumov-Ponnusamy [22] improved the Bohr inequality by adding the area $S_r(f)$ of the image of the subdisk $|z| < r$ under the mapping f. In addition, there is a harmonic analog of Bohr inequality and another improved version about $S_r(f)$ in [14, 22].

It is worth noting that initially, the Bohr radius was defined for analytic functions mapping the unit disk to the unit disk. However, subsequent research has extended its applicability to mappings from the unit disk to the punctured unit disk [2], the exterior of the closed unit disk [1], and various other domains [7].

This paper is motivated by Ismagilov's methods, which replace the constant term with the absolute value of the function and the square of the absolute value of the function, to obtain the sharp inequalities presented in [19]. Additionally, Ponnusamy's methods incorporate the classical lemma of Schwarz in the power series expansion of the function in [29]. Given these motivations, it is natural to generalize Theorems 1.1, 1.2 and 1.3 to Theorems 3.1, 3.2, 3.3 and 3.4, all of which yield sharp results.

The paper is organized as follows. In Sect. 2, we provide some key lemmas that play a crucial role in the proofs. In Sect. 3, we present the main results along with their proofs.

2 Some Lemmas

In order to establish our main results, we need the following some lemmas.

Lemma 2.1 (Schwarz-Pick lemma) *Let $\phi(z)$ be analytic in $\mathbb{D}$ and $|\phi(z)| < 1$ in $\mathbb{D}$. Then*

$$\frac{|\phi(z_1)-\phi(z_2)|}{|1-\overline{\phi(z_1)}\phi(z_2)|} \le \frac{|z_1-z_2|}{|1-\overline{z_1}z_2|} \quad for \quad z_1, z_2 \in \mathbb{D},$$

and equality holds for distinct $z_1, z_2 \in \mathbb{D}$ if and only if ϕ is a Möbius transformation. In particularly,

$$|\phi'(z)| \le \frac{1-|\phi(z)^2|}{1-|z^2|} \quad for \quad z \in \mathbb{D},$$

and equality holds for some $z \in \mathbb{D}$ if and only if ϕ is a Möbius transformation.

Lemma 2.2 [26] Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. Then

$$B_1(f, r) + \frac{1 + |a_0|r}{(1 + |a_0|)(1 - r)} ||f_1||_r^2 \leq \left(1 - |a_0|^2\right) \frac{r}{1 - r} \quad for \quad r \in [0, 1).$$

The general version of this lemma is proofed in [23, Lemma 4].

Lemma 2.3 [19] Let $p \in \mathbb{N}$, $0 \leq m \leq p$ and $f(z) = \sum_{n=0}^{\infty} a_{pn+m} z^{pn+m}$ is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. Then

$$\sum_{n=1}^{\infty} |a_{pn+m}| r^{pn} \leq \begin{cases} r^p \frac{(1-|a_m|^2)}{1-r^p|a_m|}, & for \quad |a_m| \geq r^p, \\ r^p \frac{\sqrt{1-|a_m|^2}}{\sqrt{1-r^{2p}}}, & for \quad |a_m| < r^p. \end{cases}$$

Lemma 2.4 [22] Let $|b_0| < 1$ and $0 < r \leq 1/\sqrt{2}$. Suppose that $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic and satisfies the inequality $|g(z)| < 1$ in $\mathbb{D}$, $S_r(g)$ denotes the area of the image of the subdisk $|z| < r$ under the mapping g. Then

$$\frac{S_r(g)}{\pi} = \frac{1}{\pi} \int \int_{|z|<r} |g(z)'|^2 dxdy = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \leq r^2 \frac{(1 - |b_0|^2)^2}{(1 - |b_0|^2 r^2)^2}.$$

3 Main Results

In this section, we give four sharp Bohr-type inequalities for bounded analysis functions.

Theorem 3.1 *Assume that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic in* $\mathbb{D}$ *and* $|f(z)| \leq 1$ *in* $\mathbb{D}$. *Then*

$$|f(z)|^2 + B_1(f, r) + \frac{1 + |a_0|r}{(1 + |a_0|)(1 - r)} ||f_1||_r^2 + \frac{8}{9} \left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 \leq 1 \quad (3.1)$$

for $r \leq 1/3$, *where*

$$\lambda = \frac{6a^5 + 10a^4 - 260a^3 - 108a^2 + 1854a + 162}{324(2a - 1)(a + 1)^2} = 12.704586\ldots$$

and $a \approx 0.555991$ *is the unique positive root of the equation*

$$t^5 + 2t^4 + 8t^3 - 34t^2 - 761t + 432 = 0$$

in the interval $(0, 1)$. *The equality is attained for the function*

$$f(z) = \frac{a - z}{1 - az}.$$

Proof Since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$, $a := |a_0| \in [0, 1)$, by Schwarz-Pick lemma, Lemma 2.2 and Lemma 2.4, respectively, we have

$$\begin{aligned}
&|f(z)|^2 + B_1(f, r) + \frac{1+ar}{(1+a)(1-r)}||f_1||_r^2 + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda\left(\frac{S_r}{\pi}\right)^2 \\
&\leq \left(\frac{a+r}{1+ar}\right)^2 + (1-a^2)\frac{r}{1-r} + \frac{8}{9}\frac{(1-a^2)^2r^2}{(1-a^2r^2)^2} + \lambda\frac{(1-a^2)^4r^4}{(1-a^2r^2)^4} \\
&:= M(a, r).
\end{aligned}$$

Consider the function $M(a, r)$ which is an increasing function of r for $0 \leq r \leq 1/3$, then we have

$$\begin{aligned}
M(a, r) \leq & M(a, 1/3) \\
=& \left(\frac{1+3a}{a+3}\right)^2 + \frac{1-a^2}{2} + 8\frac{(1-a^2)^2}{(9-a^2)^2} + 81\lambda\frac{(1-a^2)^4}{(9-a^2)^4} \\
=& 1 - \frac{(1+a)(1-a)^3\Phi_1(a)}{2(9-a^2)^4},
\end{aligned}$$

where

$$\Phi_1(a) = 162(a-1)(1+a)^3\lambda + (-a^2 - 2a + 47)(3-a)^2(3+a)^2.$$

Now, we need to show that $\Phi_1(a) \geq 0$ holds for $a \in [0, 1)$. Let $\Phi_1'(a) = 0$, we obtain

$$\lambda = \frac{6a^5 + 10a^4 - 260a^3 - 108a^2 + 1854a + 162}{324(2a-1)(a+1)^2},$$

then

$$\Phi_1(a) = \frac{a^2-9}{2a-1}(a^5 + 2a^4 + 8a^3 - 34a^2 - 761a + 432).$$

One can verify that the function $\Phi_1(a)$ in the interval $[0, 1)$ has unique zero $a \approx 0.555991$ which is the unique positive root of the equation $t^5 + 2t^4 + 8t^3 - 34t^2 - 761t + 432 = 0$. Furthermore, by simple calculation, we obtain $\lambda = 12.704586\ldots$ and $\Phi_1'(a) = 0$. Namely, the function $\Phi_1(a)$ has exactly one stationary point $a = 0.555991\ldots$ in $[0, 1]$. Meanwhile, we have $\Phi_1(0) > 0$ and $\Phi_1(1) > 0$. Thus, $\Phi_1(a) \geq 0$ which proves that (3.1) holds for $r \leq 1/3$.

Next we show that the constant λ is sharp, we consider the function $f(z)$ given by

$$f(z) = \frac{a-z}{1-az} = a - \left(1-a^2\right)\sum_{n=1}^{\infty} a^{n-1}z^n, \quad z \in \mathbb{D},$$

where $a \in [0, 1)$. For this function, taking $z = -r$ and computing the value on the left side of inequality(3.1), we obtain

$$
\begin{aligned}
M_1(a,r) :=& |f(z)|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{1+|a_0|r}{(1+|a_0|)(1-r)} \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda_1 \left(\frac{S_r}{\pi}\right)^2 \\
=& \left(\frac{a+r}{1+ar}\right)^2 + \frac{(1-a^2)r}{1-ar} + \frac{(1-a^2)^2 r^2}{(1+a)(1-r)(1-ar)} + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda_1 \left(\frac{S_r}{\pi}\right)^2 \\
=& \left(\frac{a+r}{1+ar}\right)^2 + \frac{r(1-a^2)}{1-r} + \frac{8}{9}\frac{(1-a^2)^2 r^2}{(1-a^2r^2)^2} + \lambda_1 \frac{(1-a^2)^4 r^4}{(1-a^2r^2)^4}.
\end{aligned}
$$

For $r = 1/3$, the above expression becomes

$$
\begin{aligned}
M_1(a, 1/3) =& \left(\frac{1+3a}{a+3}\right)^2 + \frac{1-a^2}{2} + 8\frac{(1-a^2)^2}{(9-a^2)^2} + 81\lambda \frac{(1-a^2)^4}{(9-a^2)^4} \\
&+81(\lambda_1 - \lambda)\frac{(1-a^2)^4}{(9-a^2)^4}.
\end{aligned}
$$

Now choosing a as the positive root of the equation $t^5+2t^4+8t^3-34t^2-761t+432 = 0$. Then, we obtain

$$
M_1(a, 1/3) = 1 + 81(\lambda_1 - \lambda)\frac{(1-a^2)^4}{(9-a^2)^4}.
$$

Which is obviously greater than 1 in case $\lambda_1 > \lambda$. This proves the sharpness and the proof of Theorem 3.1 is complete. □

Remark 3.1 From the proof of Theorem 3.1, we have the following result: for any function $T(t) : [0, \infty) \to [0, \infty)$ such that $T(t) > 0$ for $t > 0$, there exist bounded analytic function $f : \mathbb{D} \to \mathbb{D}$ for which the following inequality

$$
|f(z)|^2 + B_1(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)} ||f_1||_r^2 + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 + T(S_r) \le 1 \quad for \quad r \le 1/3
$$

is wrong, the constant λ is in Theorem 3.1.

Theorem 3.2 *Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$
|a_0| + B_1(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)} ||f_1||_r^2 + |f_1(z)| + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 \le 1 \tag{3.2}
$$

for $r \leq 1/5$, *where*

$$\lambda = \frac{36a^7 - 189a^6 - 8310a^5 + 2125a^4 + 264800a^3 + 88125a^2 - 1988750a - 578125}{22500(3a-1)(a-1)(a+1)^3}$$
$$= 191.551761\ldots$$

and $a \approx 0.396199$, *is the unique positive root of the equation* $\psi(t) = 0$ *in the interval (0,1), where*

$$\psi(t) = 3t^7 - 6t^6 + 45t^5 - 1152t^4 - 39955t^3 - 14750t^2 + 258275t - 97500.$$

The equality is attained for the function

$$f(z) = \frac{a-z}{1-az}.$$

Proof Let $a := |a_0| \in [0, 1)$. By Lemma 2.3, for $m = 0$ and $p = 1$, we obtain the following inequalities:

$$\sum_{n=1}^{\infty} |a_n| r^n \leq \begin{cases} A(r) := \frac{r(1-a^2)}{1-ra} & for \quad a \geq r, \\ B(r) := \frac{r\sqrt{1-a^2}}{\sqrt{1-r^2}} & for \quad a < r. \end{cases} \tag{3.3}$$

At first we consider $a \geq 1/5$. In this case, using (3.3), Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} & a + B_1(f,r) + \frac{1+ar}{(1+a)(1-r)} ||f_1||_r^2 + |f_1(z)| + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda \left(\frac{S_r}{\pi}\right)^2 \\ & \leq a + (1-a^2)\frac{r}{1-r} + \sum_{n=1}^{\infty} |a_n| r^n + \frac{16}{9}\frac{(1-a^2)^2 r^2}{(1-a^2r^2)^2} + \lambda \frac{(1-a^2)^4 r^4}{(1-a^2r^2)^4} \\ & \leq a + (1-a^2)\frac{r}{1-r} + A(r) + \frac{16}{9}\frac{(1-a^2)^2 r^2}{(1-a^2r^2)^2} + \lambda \frac{(1-a^2)^4 r^4}{(1-a^2r^2)^4} \\ & := N(a,r). \end{aligned}$$

Obviously, the function $N(a,r)$ is an increasing function of r for $0 \le r \le 1/5$, then we have

$$\begin{aligned} N(a,r) \le& N(a,1/5)\\ =& a+\frac{1}{4}(1-a^2)+A(1/5)+\frac{400}{9}\frac{(1-a^2)^2}{(25-a^2)^2}+625\lambda\frac{(1-a^2)^4}{(25-a^2)^4}\\ =& \frac{1+4a-a^2}{4}+\frac{1-a^2}{5-a}+\frac{400}{9}\frac{(1-a^2)^2}{(25-a^2)^2}+625\lambda\frac{(1-a^2)^4}{(25-a^2)^4}\\ =& 1-\left(1-\frac{1+4a-a^2}{4}-\frac{1-a^2}{5-a}-\frac{400}{9}\frac{(1-a^2)^2}{(25-a^2)^2}-625\lambda\frac{(1-a^2)^4}{(25-a^2)^4}\right)\\ =& 1-\frac{(1-a)^2\Phi_2(a)}{36(25-a^2)^4}, \end{aligned}$$

where

$$\Phi_2(a)=-22500(1-a)^2(1+a)^4\lambda+(25-a^2)^2(9a^4-54a^3-2320a^2-1850a+10775).$$

Now, we show that $\Phi_2(a)\ge 0$ holds for $a\in[1/5,1)$. Let $\Phi_2'(a)=0$, we obtain

$$\lambda=\frac{36a^7-189a^6-8310a^5+2125a^4+264800a^3+88125a^2-1988750a-578125}{22500(3a-1)(a-1)(a+1)^3}.$$

Then

$$\Phi_2(a)=\frac{3(a^2-25)}{1-3a}\psi(a),$$

where

$$\begin{aligned} \psi(t)=& 3t^7-6t^6+45t^5-1152t^4-39955t^3-14750t^2\\ &+258275t-97500. \end{aligned}$$

Clearly, the function $\Phi_2(a)$ has unique zero $a\approx 0.396199$ in the interval $[1/5,1)$ which is the positive root of the equation $\psi(t)=0$. Thus, we obtain $\lambda=191.551761\ldots$ and $\Phi_2'(a)=0$. On the other hand, we have $\Phi_2(1/5)>0$ and $\Phi_2(1)>0$. Hence, $\Phi_2(a)\ge 0$ which proves that (3.2) holds for $a\ge 1/5$ and $r\le 1/5$.

Next, we consider $a < 1/5$. Combining (3.3), Lemma 2.2 and Lemma 2.4, we deduce that

$$\begin{aligned} &a + B_1(f,r) + \frac{1+ar}{(1+a)(1-r)}||f_1||_r^2 + |f_1(z)| + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda\left(\frac{S_r}{\pi}\right)^2 \\ &\le a + (1-a^2)\frac{r}{1-r} + B(r) + \frac{16}{9}\frac{(1-a^2)^2r^2}{(1-a^2r^2)^2} + \lambda\frac{(1-a^2)^4r^4}{(1-a^2r^2)^4} \\ &:= N^*(a,r). \end{aligned}$$

Observe that $N^*(a,r)$ is an increasing function of r for $0 \le r \le 1/5$, then we obtain

$$\begin{aligned} N^*(a,r) \le& N^*(a,1/5) \\ \le& \frac{1+4a-a^2}{4} + \frac{\sqrt{1-a^2}}{\sqrt{24}} + \frac{400}{9}\frac{(1-a^2)^2}{(25-a^2)^2} + 625\lambda\frac{(1-a^2)^4}{(25-a^2)^4} \\ :=& \Psi_1(a). \end{aligned}$$

Obviously,

$$\begin{aligned} \Psi_1'(a) =& 1 - \frac{a}{2} - \frac{1}{\sqrt{24}}\frac{a}{\sqrt{1-a^2}} + \frac{400}{9}\frac{96a(a^2-1)}{(25-a^2)^3} + 625\lambda\frac{192a(a^2-1)^3}{(25-a^2)^5} \\ =& 1 - X(a,\lambda), \end{aligned}$$

where

$$X(a,\lambda) = \frac{a}{2} + \frac{1}{\sqrt{24}}\frac{a}{\sqrt{1-a^2}} + \frac{400}{9}\frac{96a(1-a^2)}{(25-a^2)^3} + 625\lambda\frac{192a(1-a^2)^3}{(25-a^2)^5}.$$

Observe that the function $X(a,\lambda)$ is an increasing function of a for $a \in [0,1/5)$. Then, we have $X(a,\lambda) \le X(1/5,\lambda) < 1$. Thus $\Psi_1'(a) \ge 0$. It follows that $\Psi_1(a)$ is an increasing function of a for $a \in [0,1/5)$ and so $\Psi_1(a) \le \Psi_1(1/5) \approx 0.9677 < 1$. Therefore, inequality (3.2) holds for $a < 1/5$ and for $r \le 1/5$.

To show the sharpness of the constant λ, we consider the function $f(z)$ given by

$$f(z) = \frac{a-z}{1-az} = a - (1-a^2)\sum_{n=1}^{\infty} a^{n-1}z^n, \quad z \in \mathbb{D}, \tag{3.4}$$

where $a \in [0,1)$. Taking $z = r$ and computing the value on the left side of inequality (3.2), we have

$$N_1(a,r) := a + \frac{r}{1-r}(1-a^2) + A(r) + \frac{16}{9}\left(\frac{S_r}{\pi}\right) + \lambda_1\left(\frac{S_r}{\pi}\right)^2.$$

For $r = 1/5$, the above expression becomes

$$N_1(a, 1/5) = \frac{1+4a-a^2}{4} + \frac{1-a^2}{5-a} + \frac{400}{9}\frac{(1-a^2)^2}{(25-a^2)^2} + 625\lambda\frac{(1-a^2)^4}{(25-a^2)^4} +625(\lambda_1-\lambda)\frac{(1-a^2)^4}{(25-a^2)^4}.$$

Now choosing a as the positive root of the equation $\psi(t) = 0$, we obtain that

$$N_1(a, 1/5) = 1 + 625(\lambda_1-\lambda)\frac{(1-a^2)^4}{(25-a^2)^4}.$$

The above equality is greater than 1 for $\lambda_1 > \lambda$. This proves the sharpness and the proof of Theorem 3.2 is complete. □

One can replace $|a_0|$ by $|a_0|^2$ in Theorem 3.2, but this will increase the Bohr radius. Namely, the following theorem is valid.

Theorem 3.3 *Assume that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic in* $\mathbb{D}$ *and* $|f(z)| \le 1$ *in* $\mathbb{D}$. *Then*

$$|a_0|^2 + B_1(f,r) + \frac{1+|a_0|r}{(1+|a_0|)(1-r)}||f_1||_r^2 + |f_1(z)| + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda\left(\frac{S_r}{\pi}\right)^2 \le 1 \tag{3.5}$$

for $r \le 1/3$, *where*

$$\lambda = \frac{3a^5 + 10a^4 - 14a^3 - 108a^2 - 117a + 162}{162(1-2a)(1+a)^2} = 3.702103\ldots$$

and $a \approx 0.468007$, *is the unique positive root of the equation* $\phi(t) = 0$ *in the interval (0,1), where*

$$\phi(t) = t^5 + 3t^4 + 10t^3 + 46t^2 + 149t - 81.$$

The equality is attained for the function

$$f(z) = \frac{a-z}{1-az}.$$

Proof Let $a := |a_0| \in [0, 1)$. Firstly, we consider the first part. Namely, $a \geq 1/3$, then by (3.3), Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} & a^2 + B_1(f, r) + \frac{1 + ar}{(1+a)(1-r)} ||f_1||_r^2 + |f_1(z)| + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda\left(\frac{S_r}{\pi}\right)^2 \\ & \leq a^2 + (1-a^2)\frac{r}{1-r} + A(r) + \frac{8}{9}\frac{(1-a^2)^2 r^2}{(1-a^2r^2)^2} + \lambda\frac{(1-a^2)^4 r^4}{(1-a^2r^2)^4} \\ & := R(a, r). \end{aligned}$$

According to the monotony of r for $0 \leq r \leq 1/3$, we have

$$\begin{aligned} R(a, r) \leq & R(a, 1/3) \\ = & \frac{1 + a^2}{2} + \frac{1 - a^2}{3 - a} + 8\frac{(1-a^2)^2}{(9-a^2)^2} + 81\lambda\frac{(1-a^2)^4}{(9-a^2)^4} \\ = & 1 - \frac{(1+a)(1-a)^3\Phi_3(a)}{2(9-a^2)^4}, \end{aligned}$$

where

$$\Phi_3(a) = 162(a-1)(1+a)^3\lambda + (9-a^2)^2(a^2 + 4a + 11).$$

Next, we show that $\Phi_3(a) \geq 0$ holds for $a \in [1/3, 1)$. Let $\Phi_3'(a) = 0$, we obtain

$$\lambda = \frac{3a^5 + 10a^4 - 14a^3 - 108a^2 - 117a + 162}{162(1-2a)(1+a)^2}.$$

Then

$$\Phi_3(a) = \frac{(a^2 - 9)}{1 - 2a}\phi(a),$$

where

$$\phi(t) = t^5 + 3t^4 + 10t^3 + 46t^2 + 149t - 81.$$

The function $\Phi_3(a)$ in the interval $[1/3, 1)$ has unique zero $a \approx 0.468007$. This value corresponds to the positive root of the equation $\phi(t) = 0$. In the same way, we obtain $\lambda = 3.702103\ldots$ and $\Phi'(a) = 0$. Meanwhile, $\Phi_3(1/3) > 0$ and $\Phi_3(1) > 0$ is obvious. Thus, $\Phi_3(a) \geq 0$ which proves that (3.5) holds for $a \geq 1/3$ and $r \leq 1/3$.

Secondly, we consider the case $a < 1/3$, then combining (3.3), Lemma 2.2 and Lemma 2.4, we obtain that

$$\begin{aligned}
&a^2 + B_1(f,r) + \frac{1+ar}{(1+a)(1-r)}||f_1||_r^2 + |f_1(z)| + \frac{8}{9}\left(\frac{S_r}{\pi}\right) + \lambda\left(\frac{S_r}{\pi}\right)^2 \\
&\le a^2 + (1-a^2)\frac{r}{1-r} + B(r) + \frac{8}{9}\frac{(1-a^2)^2 r^2}{(1-a^2r^2)^2} + \lambda\frac{(1-a^2)^4 r^4}{(1-a^2r^2)^4} \\
&:= R^*(a,r).
\end{aligned}$$

Observe that $R^*(a,r)$ is an increasing function of r for $0 \le r \le 1/3$, then we obtain

$$\begin{aligned}
R^*(a,r) \le & R^*(a,1/3) \\
\le & \frac{1+a^2}{2} + \frac{\sqrt{1-a^2}}{\sqrt{8}} + 8\frac{(1-a^2)^2}{(9-a^2)^2} + 81\lambda\frac{(1-a^2)^4}{(9-a^2)^4} \\
:= & \Psi_2(a).
\end{aligned}$$

Routine and straightforward calculations show that the last expression $\Psi_2(a)$ maximizes at $a = 1/3$ for $a \in [0, 1/3)$. Then $\Psi_2(a) \le \Psi_2(1/3) \approx 0.9989 < 1$. This proves that (3.5) holds for $a < 1/3$ and $r \le 1/3$.

Lastly, to show the sharpness of the constant λ, in the same way, we consider the function $f(z)$ is same as (3.4) and compute the value on the left side of inequality (3.5), then we get

$$R_1(a,1/3) := \frac{1+a^2}{2} + \frac{1-a^2}{3-a} + 8\frac{(1-a^2)^2}{(9-a^2)^2} + 81\lambda\frac{(1-a^2)^4}{(9-a^2)^4} + 81(\lambda_1-\lambda)\frac{(1-a^2)^4}{(9-a^2)^4}.$$

Choose a as the positive root of the equation $\phi(t) = 0$. Thus, we obtain that

$$R_1(a,1/3) = 1 + 81(\lambda_1-\lambda)\frac{(1-a^2)^4}{(9-a^2)^4},$$

which is obviously greater than 1 for $\lambda_1 > \lambda$. This proves λ is sharp and the proof is complete. □

Remark 3.2 From the proof of Theorem 3.2 and Theorem 3.3, one cannot replace $8/9$ by $16/9$, otherwise the inequality (3.5) is false.

Theorem 3.4 *For $k \ge 2$, assume that $f(z) = \sum_{n=k}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|f(z)| + B_k(f,r) + (\frac{r^{-k}}{1+|a_k|} + \frac{r^{1-k}}{1-r})||f_k||_r^2 \le 1 \tag{3.6}$$

for $|z| = r \le r_k$, where r_k is the unique positive root in $(0,1)$ of the equation

$$3r^{k+1} - 5r^k - 2r + 2 = 0.$$

Table 1 r_k is the unique root of the equation $3r^{k+1} - 5r^k - 2r + 2 = 0$ in $(0, 1)$

k	r_k	k	r_k	k	r_k	k	r_k	k	r_k
2	0.526255	3	0.622372	4	0.681398	5	0.721997	6	0.751956
7	0.775145	8	0.793727	9	0.809013	10	0.821851	15	0.864511
20	0.889031	25	0.905228	30	0.916843	50	0.942822	100	0.966213

Table 2 r'_k is the unique root of the equation $3r^{2k+1} - 5r^{2k} - 2r + 2 = 0$ in $(0, 1)$

k	r'_k	k	r'_k	k	r'_k	k	r'_k	k	r'_k
2	0.681398	3	0.751956	4	0.793727	5	0.821851	6	0.842302
7	0.857959	8	0.870396	9	0.880553	10	0.889031	15	0.916843
20	0.932560	25	0.942822	30	0.950117	50	0.966213	100	0.980391

The radius r_k is the best possible. Moreover,

$$|f(z)|^2 + r^k B_k(f, r) + \left(\frac{1}{1+|a_k|} + \frac{r}{1-r}\right) ||f_k||_r^2 \leq 1 \tag{3.7}$$

for $|z| = r \leq r'_k$, where r'_k is the unique positive root in $(0, 1)$ of the equation

$$3r^{2k+1} - 5r^{2k} - 2r + 2 = 0.$$

The radius r'_k is the best possible.

Before proving the Theorem 3.4, we present the value of r_k and r'_k for certain values of $k \geq 2$ in Tables 1 and 2.

Proof of Theorem 3.4 For the first part of the theorem, we have $f(z) = \sum_{n=k}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. According to the classical lemma of Schwarz we may write $f(z) = z^k g(z)$, where $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $\mathbb{D}$ and $|g(z)| \leq 1$ in $\mathbb{D}$. Then, for $n \geq 0$, we have $a_{n+k} = b_n$. Now apply Schwarz-Pick lemma and Lemma 2.2 to the function $g(z)$, we obtain

$$|g(z)| + \sum_{n=1}^{\infty} |b_n| r^n + \left(\frac{1}{1+|b_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} \leq \frac{|b_0| + r}{1 + |b_0| r} + \frac{(1 - |b_0|^2) r}{1 - r}.$$

By $a_{n+k} = b_n$, we get

$$|g(z)| + \sum_{n=1}^{\infty} |a_{n+k}| r^n + \left(\frac{1}{1+|a_k|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_{n+k}|^2 r^{2n} \leq \frac{|a_k| + r}{1 + |a_k| r} + \frac{(1 - |a_k|^2) r}{1 - r}.$$

Thus, we have

$$
\begin{aligned}
& r^k|g(z)| + \sum_{n=k+1}^{\infty}|a_n|r^n + \left(\frac{r^{-k}}{1+|a_k|} + \frac{r^{1-k}}{1-r}\right)\sum_{n=k+1}^{\infty}|a_n|^2r^{2n} \\
& \leq \left[\frac{|a_k|+r}{1+|a_k|r} + \frac{(1-|a_k|^2)r}{1-r}\right]r^k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|f(z)| + \sum_{n=k}^{\infty}|a_n|r^n + \left(\frac{r^{-k}}{1+|a_k|} + \frac{r^{1-k}}{1-r}\right)\sum_{n=k}^{\infty}|a_n|^2r^{2n} \leq \left[\frac{|a_k|+r}{1+|a_k|r} + \frac{(1-|a_k|^2)r}{1-r}\right]r^k \\
+|a_k|r^k + \left(\frac{r^{-k}}{1+|a_k|} + \frac{r^{1-k}}{1-r}\right)|a_k|^2r^{2k}.
\end{aligned}
$$

Now, we only need to show that

$$
\left[\frac{|a_k|+r}{1+|a_k|r} + \frac{(1-|a_k|^2)r}{1-r}\right]r^k + |a_k|r^k + \left(\frac{r^{-k}}{1+|a_k|} + \frac{r^{1-k}}{1-r}\right)|a_k|^2r^{2k} \leq 1
$$

holds for $r \leq r_k$. Namely,

$$
r^k\left[\frac{|a_k|+r}{1+|a_k|r} + \frac{(1-|a_k|^2)r}{1-r} + |a_k| + \frac{|a_k|^2}{1+|a_k|} + \frac{r|a_k|^2}{1-r}\right] \leq 1.
$$

Let $a := |a_k| \in [0, 1]$,

$$
\begin{aligned}
C(a) &:= \frac{a+r}{1+ar} + \frac{(1-a^2)r}{1-r} + a + \frac{a^2}{1+a} + \frac{ra^2}{1-r} \\
&= \frac{a+r}{1+ar} + a + \frac{a^2}{1+a} + \frac{r}{1-r}.
\end{aligned}
$$

Obviously, $C(a)$ is an increasing function of a. Then, we have

$$
r^kC(a) \leq r^kC(1) = r^k\frac{5-3r}{2(1-r)}.
$$

It is sufficient for us to show the above inequality is less than or equals to 1 for $r \leq r_k$. It is equivalent to show $D(r) \geq 0$, where $D(r) = 3r^{k+1} - 5r^k - 2r + 2$. Because $D'(r) = -2 - r^{k-1}[k(5-3r) - 3r] \leq 0$ holds for $k \geq 2$ and $0 \leq r \leq 1$, then $D(r)$ is a decreasing function of r. It is also easy to verify that $D(0)D(1) < 0$. Hence, r_k is unique root of $D(r)$ and $D(r) \geq 0$ for $r \leq r_k$.

Next we show the sharpness of the radius r_k. Let $a \in [0, 1)$,

$$f(z) = z^k \left(\frac{a+z}{1+az}\right) = az^k + (1-a^2)\sum_{n=1}^{\infty}(-a)^{n-1}z^{n+k}, \quad z \in \mathbb{D}.$$

For this function, taking $z = r$, we obtain

$$\begin{aligned}
&|f(z)| + B_k(f,r) + (\frac{r^{-k}}{1+|a_k|} + \frac{r^{1-k}}{1-r})||f_{k+1}||_r^2 \\
&= r^k\frac{a+r}{1+ar} + ar^k + \sum_{n=k}^{\infty}(1-a^2)a^{n-k}r^{n+1} + \left(\frac{r^{-k}}{1+a} + \frac{r^{1-k}}{1-r}\right)\left[a^2r^{2k} + \sum_{n=k}^{\infty}(1-a^2)^2a^{2(n-k)}r^{2(n+1)}\right] \\
&= r^k\frac{a+r}{1+ar} + ar^k + \frac{(1-a^2)r^{k+1}}{1-ar} + \left(\frac{r^{-k}}{1+a} + \frac{r^{1-k}}{1-r}\right)\left[a^2r^{2k} + \frac{(1-a^2)^2r^{2k+2}}{1-a^2r^2}\right] \\
&= r^k\left[\frac{a+r}{1+ar} + a + \frac{a^2}{1+a} + \frac{r}{1-r}\right].
\end{aligned}$$

Comparison of the above expression with $C(a)$, allowing $a \to 1^-$ delivers the radius r_k is the best possible.

For the second part of the theorem, as in the previous case, let $f(z) = z^k g(z)$, where $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $\mathbb{D}$ and $|g(z)| \le 1$ in $\mathbb{D}$. Then, we have $a_{n+k} = b_n$ for $n \ge 0$. Similarly, applying Schwarz-Pick lemma and Lemma 2.2 to the function $g(z)$, we have

$$|g(z)|^2 + \sum_{n=1}^{\infty}|b_n|r^n + \left(\frac{1}{1+|b_0|} + \frac{r}{1-r}\right)\sum_{n=1}^{\infty}|b_n|^2r^{2n} \le \left(\frac{|b_0|+r}{1+|b_0|r}\right)^2 + \frac{(1-|b_0|^2)r}{1-r}.$$

Namely,

$$\begin{aligned}
&|f(z)|^2 + \sum_{n=k}^{\infty}|a_n|r^{n+k} + \left(\frac{1}{1+|a_k|} + \frac{r}{1-r}\right)\sum_{n=k}^{\infty}|a_n|^2r^{2n} \\
&\le \left[\left(\frac{|a_k|+r}{1+|a_k|r}\right)^2 + \frac{(1-|a_k|^2)r}{1-r}\right]r^{2k} \\
&\quad + |a_k|r^{2k} + \left(\frac{1}{1+|a_k|} + \frac{r}{1-r}\right)|a_k|^2r^{2k}.
\end{aligned}$$

Now we need to proof the side of last inequality is less than or equals to 1 for $r \le r_k'$, let $a := a_k$, it follows that

$$r^{2k}\left[\left(\frac{a+r}{1+ar}\right)^2 + a + \frac{a^2}{1+a} + \frac{r}{1-r}\right] \le r^{2k}\left[\frac{5-3r}{2(1-r)}\right] \le 1.$$

The equation

$$E(r) := 3r^{2k+1} - 5r^{2k} - 2r + 2 = 0$$

has a unique root in (0, 1) for $k \geq 2$, since $E(0) > 0$, $E(1) < 0$ and $E'(r) \leq 0$ for fix $r \in [0, 1]$. Thus, inequality (3.7) holds for $r \leq r_k'$.

The proof of sharpness is similar with inequality (3.6), we omit it. Therefore, the proof of Theorem 3.4 is complete. □

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Declarations

Conflict of interest No potential Conflict of interest was reported by the authors.

References

1. Abu-Muhanna, Y., Ali, R.M.: Bohr's phenomenon for analytic functions into the exterior of a compact convex body. J. Math. Anal. Appl. **379**(2), 512–517 (2011)
2. Abu-Muhanna, Y., Ali, R.M., Ng, Z.C.: Bohr radius for the punctured disk. Math. Nachr. **290**(16), 2434–2443 (2017)
3. Abu-Muhanna, Y., Ali, R.M., Ng, Z.C., Hasni, S.M.: Bohr radius for subordinating families of analytic functions and bounded harmonic mappings. J. Math. Anal. Appl. **420**(1), 124–136 (2014)
4. Abu-Muhanna, Y., Ali, R. M., Ponnusamy, S.: On the Bohr inequality. In *Progress in approximation theory and applicable complex analysis*, volume 117 of *Springer Optim. Appl.*, pp. 269–300. Springer, Cham, (2017)
5. Aizenberg, L.: Multidimensional analogues of Bohr's theorem on power series. Proc. Amer. Math. Soc. **128**(4), 1147–1155 (2000)
6. Aizenberg, L.: Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series. In *Selected topics in complex analysis*, volume 158 of *Oper. Theory Adv. Appl.*, pp. 87–94. Birkhäuser, Basel, (2005)
7. Aizenberg, L.: Generalization of results about the Bohr radius for power series. Studia Math. **180**(2), 161–168 (2007)
8. Aizenberg, L., Aytuna, A., Djakov, P.: Generalization of a theorem of Bohr for bases in spaces of holomorphic functions of several complex variables. J. Math. Anal. Appl. **258**(2), 429–447 (2001)
9. Ali, R.M., Abdulhadi, Z., Ng, Z.C.: The Bohr radius for starlike logharmonic mappings. Complex Var. Elliptic Equ. **61**(1), 1–14 (2016)
10. Ali, R.M., Barnard, R.W., Solynin, AYu.: A note on Bohr's phenomenon for power series. J. Math. Anal. Appl. **449**(1), 154–167 (2017)
11. Bohr, H.: A theorem concerning power series. Proc. London Math. Soc. **2**(13), 1–5 (1914)
12. Defant, A., García, D., Maestre, M.: Bohr's power series theorem and local Banach space theory. J. Reine Angew. Math. **557**, 173–197 (2003)
13. Dixon, P.G.: Banach algebras satisfying the non-unital von Neumann inequality. Bull. London Math. Soc. **27**(4), 359–362 (1995)
14. Evdoridis, S., Ponnusamy, S., Rasila, A.: Improved Bohr's inequality for locally univalent harmonic mappings. Indag. Math. **30**(1), 201–213 (2019)
15. Garcia, S.R., Mashreghi, J., Ross, W.T.: Finite Blaschke products and their connections. Springer, Cham (2018)
16. Hu, X.J., Wang, Q.H., Long, B.Y.: Bohr-type inequalities for bounded analytic functions of Schwarz functions. AIMS Math. **6**(12), 13608–13621 (2021)
17. Hu, X.J., Wang, Q.H., Long, B.Y.: Bohr-type inequalities with one parameter for bounded analytic functions of Schwarz functions. Bull. Malays. Math. Sci. Soc. **45**(1), 575–591 (2022)

18. Huang, Y., Liu, M.S., Ponnusamy, S.: Refined Bohr-type inequalities with area measure for bounded analytic functions. Anal. Math. Phys. **10**(4), 1–21 (2020)
19. Ismagilov, A., Kayumov, I.R., Ponnusamy, S.: Sharp Bohr type inequality. J. Math. Anal. Appl. **489**(1), 124147 (2020)
20. Kayumov, I. R., Ponnusamy, S.: Bohr–Rogosinski radius for analytic functions. (2017). arXiv:1708.05585
21. Kayumov, I.R., Ponnusamy, S.: Bohr inequality for odd analytic functions. Comput. Methods Funct. Theory **17**(4), 679–688 (2017)
22. Kayumov, I.R., Ponnusamy, S.: Improved version of Bohr's inequality. C. R. Math. Acad. Sci. Paris **356**(3), 272–277 (2018)
23. Liu, G., Liu, Z.H., Ponnusamy, S.: Refined Bohr inequality for bounded analytic functions. Bull. Sci. Math. **173**, 103054 (2021)
24. Liu, M.S., Ponnusamy, S., Wang, J.: Bohr's phenomenon for the classes of quasi-subordination and K-quasiregular harmonic mappings. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**(3), 1–15 (2020)
25. Paulsen, V.I., Popescu, G., Singh, D.: On Bohr's inequality. Proc. London Math. Soc. (3) **85**(2), 493–512 (2002)
26. Ponnusamy, S., Vijayakumar, R., Wirths, K.J.: New inequalities for the coefficients of unimodular bounded functions. Results Math. **75**(3), 107 (2020)
27. Ponnusamy, S., Vijayakumar, R., Wirths, K.J.: Modifications of Bohr's inequality in various settings. Houston J. Math. **47**(4), 807–831 (2021)
28. Ponnusamy, S., Vijayakumar, R., Wirths, K.J.: Improved Bohr's phenomenon in quasi-subordination classes. J. Math. Anal. Appl. **506**(1), 125645 (2022)
29. Ponnusamy, S., Wirths, K.J.: Bohr type inequalities for functions with a multiple zero at the origin. Comput. Methods Funct. Theory **20**(3–4), 559–570 (2020)
30. Sidon, S.: Über einen Satz von Herrn Bohr. Math. Z. **26**(1), 731–732 (1927)
31. Tomić, M.: Sur un théorème de H. Bohr. Math. Scand. **11**, 103–106 (1962)
32. Wu, L., Wang, Q.H., Long, B.Y.: Some Bohr-type inequalities with one parameter for bounded analytic functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116**(2), 1–13 (2022)