Bull. Malays. Math. Sci. Soc. (2024) 47:149
https://doi.org/10.1007/s40840-024-01738-9

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Angles and Quasimöbius Mappings

Qingshan Zhou[1] · Tiantian Guan[1] · Zhiqiang Yang[2]

Received: 7 January 2024 / Revised: 25 May 2024 / Accepted: 25 June 2024 /
Published online: 29 July 2024

Abstract

In this paper we establish an angular characteristic for the class of quasimöbius mappings in metric spaces.

Keywords Angle · Quasimöbius mapping

Mathematics Subject Classification 30L10 · 30C65

1 Introduction and Main Result

It is known that there are many equivalent ways of defining quasiconformal mappings in Euclidean spaces and even in metric spaces, e.g. via the conformal moduli of quadrilaterals or rings, the extremal lengths, the distortion of infinitesimal spheres or balls or equilateral triangles, see [3, 10–12] and the references therein. These definitions play important roles in dealing with many research problems [1, 2, 12, 15] and are also useful in different situations, see [5, 10, 11, 13].

In 1965, Agard and Gehring [2] investigated the distortion of angles under planar quasiconformal mappings in order to generalize the fact that a conformal mapping is angle-preserving. For this reasoning, they introduced the notions of topological angles and inner angular measures between intersecting arcs, and they obtained some angular

Communicated by Rosihan M. Ali.

✉ Zhiqiang Yang
931490599@qq.com

Qingshan Zhou
qszhou1989@163.com ; q476308142@qq.com

Tiantian Guan
ttguan93@163.com

1 School of Mathematics and Big Data, Foshan University, Foshan 528000, Guangdong, People's Republic of China

2 MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, Hunan, People's Republic of China

characteristics for the class of quasiconformal mappings in the plane, see [2, Theorem 4]. Later, Agard demonstrated some equivalent conditions for quasiconformality in terms of topological angles and their measures in higher dimensional Euclidean spaces, see [1, Theorems 5.1 and 5.2].

In 2005, Aseev et al. in [4] generalized the idea of Agard and Gehring by introducing the following definition of an angle between sets in metric spaces.

Definition 1 Let A_1 and A_2 be two nonempty subsets of a metric space X with $A_1 \cap A_2 \neq \emptyset$ and $\mathrm{diam}(A_1 \cup A_2) > 0$. The angle between A_1 and A_2 is defined to be

$$\angle(A_1, A_2) = \inf_{x_1 \in A_1, x_2 \in A_2} \sup_{z \in A} \frac{|x_1 - x_2|}{|x_1 - z| + |x_2 - z|}, \tag{1}$$

where $A = A_1 \cap A_2$.

Remark 1 If $\mathrm{diam}(A_1 \cup A_2) = 0$, then $A_1 = A_2 = \{x\}$ for some $x \in X$. Thus the inequality (1) is true for all $c > 0$, which leads to $\angle(A_1, A_2) = \infty$. Hence, to exclude this possibility, we assume that $\mathrm{diam}(A_1 \cup A_2) > 0$, i.e., $A_1 \cup A_2$ contains at least two points. Note that $0 \le \angle(A_1, A_2) \le 1$. Next we give some examples.

(1) If $A_1 = A_2 = [0, 1] \subset \mathbb{R}$, then $\angle(A_1, A_2) = 0$.
(2) If $A_1 = A_2 = \{x, y\}$, then $\angle(A_1, A_2) = 1$.
(3) If $A_1 = A_2 = \{1, 2, 3\} \subset \mathbb{R}$, then $\angle(A_1, A_2) = 1/3$.
(4) If $A_1 = \mathbb{Q} \cap [0, 1]$ and $A_2 = ([0, 1]\backslash A_1) \cup \{0, 1\}$, then $\angle(A_1, A_2) = 0$.

In this paper, X and Y are assumed to be metric spaces. The polish notation $|x - y|$ is used for the distance between x and y in any metric space. The primes always stand for the images of points and sets under a mapping f. For example $x' = f(x)$ and $A' = f(A)$. Other notions and notations in this section will be explained in Sect. 2.

In [4], the authors observed that the notion of metric angles can be used to characterize the class of quasisymmetric mappings.

Theorem 1 ([4, Theorem 2.1]) *Suppose that $f : X \to Y$ is a homeomorphism. Then f is η-quasisymmetric if and only if there exist continuous strictly increasing functions φ and ψ on* $[0, 1]$ *such that $\varphi(0) = 0$ and $\psi(0) = 0$ and the inequalities $\angle(A_1', A_2') \ge \psi(\angle(A_1, A_2))$ and $\angle(A_1, A_2) \ge \varphi(\angle(A_1', A_2'))$ hold for all subsets $A_1, A_2 \subset X$.*

On the one hand, the positive angle condition is helpful to study the gluing of quasisymmetric mappings and quasimöbius mappings, see e.g. [4, 7, 8]. On the other hand, the procedures of sewing quasisymmetric mappings and quasimöbius mappings have recently found applications in the study of hyperbolic groups with planar boundaries and the planar Schönflies theorem for bilipschitz and quasisymmetric mappings, see [9, 14, 15].

The above background motivates us to consider the following question:

Question 2 *Is there an angular characteristic for the class of quasimöbius mappings in $\mathbb{R}^n$ or metric spaces?*

As a special class of quasimöbius mappings, the inversion transformation is obviously quasimöbius but not quasisymmetric. Therefore, one finds from Theorem 1 that a quasimöbius mapping does not preserve positive angles unless certain additional assumptions are imposed. In this paper, a concrete example will be presented to explain this phenomenon, see Example 1 in Sect. 4.

As our main result, we give an affirmative answer to Question 2 as follows.

Theorem 2 *Suppose that $f : X \to Y$ is a homeomorphism between metric spaces. Then f is θ-quasimöbius for some homeomorphism $\theta : [0, \infty) \to [0, \infty)$ with $\theta(0) = 0$ if and only if there is some continuous, strictly increasing function $\varphi : [0, 1] \to [0, 1]$ with $\varphi(0) = 0$ with the following property: for all subsets $A_1, A_2 \subset X$ satisfying*

$$\angle(A_1, A_2) \geq t \quad \textit{and} \quad \operatorname{diam}(A_1 \cap A_2) \geq s \min\{\operatorname{diam}(A_1), \operatorname{diam}(A_2)\},$$

for some $s, t > 0$, we have

$$\angle(A'_1, A'_2) \geq \varphi(ts).$$

φ and θ depend only on each other.

Remark 2 The necessity of Theorem 2 can be regarded as a generalization of [7, Lemma 3.2(i)] at two aspects. On the one hand, we do not need the intersection of two given sets to be uniformly perfect. On the other hand, the distortion of angles under metric spaces inversions, introduced by Buckley et al. [6], was investigated by Guan et al. in [7], and what we considered here are quasimöbius mappings. Indeed, our proof is quite different from theirs.

Moreover, one observes from Theorem 2 that the requirements (1) and (4) of [7, Theorem 1.1] is reasonable and in a sense necessary.

The rest of this paper is organized as follows. In Sect. 2, we recall some necessary terminology and several useful known results. Section 3 is devoted to the proof of Theorem 2. Finally, we show the properties of Example 1 in Sect. 4 regarding positive angles and quasimöbius mappings.

2 Preliminaries and Auxiliary Results

Let X be a metric space. The sets A, A_1 and A_2 stand for subsets of X. For a bounded set A, $\operatorname{diam}(A)$ means its diameter, and $x, y, \cdots$ mean the points in X. Given two real numbers s and t, we let

$$s \vee t = \max\{s, t\} \text{ and } s \wedge t = \min\{s, t\}.$$

Next, we recall the definition of quasimöbius mappings introduced by Väisälä in [16]. We refer to [5, 6, 9, 17] for more investigations and applications.

Definition 2 Suppose that $\theta : [0, \infty) \to [0, \infty)$ is a homeomorphism with $\theta(0) = 0$. A homeomorphism $f : X \to Y$ is called θ *-quasimöbius* if $r(a, b, c, d) \leq t$ implies

$$r(a', b', c', d') \leq \theta(t)$$

for any quadruple of distinct points (a, b, c, d) in X and any number $t \geq 0$, where

$$r(a, b, c, d) = \frac{|a - c||b - d|}{|a - b||c - d|}$$

denotes the *cross ratio* of (a, b, c, d).

In [5], Bonk and Kleiner introduced the following useful notation of a quadruple of distinct points (a, b, c, d):

$$< a, b, c, d >= \frac{|a - c| \wedge |b - d|}{|a - b| \wedge |c - d|}.$$

Obviously, if we change the positions of b and c, then we obtain

$$r(a, b, c, d) = \frac{1}{r(a, c, b, d)} \quad \text{and} \quad < a, b, c, d >= \frac{1}{< a, c, b, d >}.$$

The relation between $r(a, b, c, d)$ and $< a, b, c, d >$ states as follows.

Lemma 3 ([5, Lemma 2.3]) *Let* $\eta_0(t) = 3(t \vee \sqrt{t})$ *for* $t \geq 0$. *Then for every quadruple of distinct points* (a, b, c, d) *in a metric space* X *we have*

$$< a, b, c, d >\leq \eta_0(r(a, b, c, d)).$$

If we change the positions of b and c, then we get

$$< a, b, c, d >\geq \eta_1(r(a, b, c, d))$$

with $\eta_1(t) = \frac{1}{\eta_0(1/t)}$, see [5, Page 133] or [18, Lemma 2.1]. By Lemma 3, a direct computation gives the following equivalent condition for quasimöbius mappings.

Lemma 4 *A homeomorphism* $f : X \to Y$ *is* θ*-quasimöbius for some homeomorphism* $\theta : [0, \infty) \to [0, \infty)$ *with* $\theta(0) = 0$ *if and only if there exists a homeomorphism* $\theta_1 : [0, \infty) \to [0, \infty)$ *with* $\theta_1(0) = 0$ *such that* $< a, b, c, b >\leq t$ *implies*

$$< a', b', c', d' >\leq \theta_1(t)$$

for any quadruple of distinct points (a, b, c, d) *in* X *and any number* $t \geq 0$, *where* θ *and* θ_1 *depend on each other.*

3 Proof of Theorem 2

In this section, we assume that $f : X \to Y$ is a homeomorphism between metric spaces.

The proof of Theorem 2 is divided into two parts. We first show the necessity which is stated in the following lemma.

Lemma 5 *Let $\theta : [0, \infty) \to [0, \infty)$ be a homeomorphism with $\theta(0) = 0$. If f is θ-quasimöbius, then there is a continuous, strictly increasing function $\varphi : [0, 1] \to [0, 1]$ with $\varphi(0) = 0$ such that for all subsets $A_1, A_2 \subset X$ satisfying*

$$\angle(A_1, A_2) \geq t > 0 \quad \textit{and} \quad \operatorname{diam}(A_1 \cap A_2) \geq s(\operatorname{diam}(A_1) \wedge \operatorname{diam}(A_2)),$$

we have

$$\angle(A_1', A_2') \geq \varphi(ts), \tag{6}$$

where φ depends only on θ.

Proof We only need to show that

$$\angle(A_1, A_2) \geq t > 0 \quad \text{and} \quad \operatorname{diam}(A_1 \cap A_2) \geq s(\operatorname{diam}(A_1) \wedge \operatorname{diam}(A_2))$$

imply that

$$\frac{|x_1' - x_2'|}{\inf_{z' \in A}\{|x_1' - z'| + |z' - x_2'|\}} \geq \varphi(ts), \tag{7}$$

for all $x_1 \in A_1$ and $x_2 \in A_2$ with $x_1 \neq x_2$, where $A = A_1 \cap A_2$ and

$$\varphi(ts) = \frac{1}{2\theta_1(\frac{3}{st}) + 1}.$$

Note that the function θ_1 is from Lemma 4 which depends only on θ.

If either $x_1 \in A$ or $x_2 \in A$, then it is easy to get

$$\frac{|x_1' - x_2'|}{\inf_{z' \in A'}\{|x_1' - z'| + |z' - x_2'|\}} \geq \frac{|x_1' - x_2'|}{|x_1' - x_2'|} = 1,$$

which implies (7).

Thus, it is left to consider the case that $x_1 \in A_1 \setminus A$ and $x_2 \in A_2 \setminus A$. Without loss of generality, we also assume that $\operatorname{diam}(A_1) \leq \operatorname{diam}(A_2)$. Since $\angle(A_1, A_2) \geq t$, there exists a point $x \in A$ for which

$$|x_1 - x_2| \geq \frac{t}{2}(|x_1 - x| \vee |x_2 - x|). \tag{8}$$

By the assumption $\operatorname{diam}(A) \geq s \operatorname{diam}(A_1)$, there is a point $y \in A$ such that

$$|x-y| \geq \frac{1}{3}\operatorname{diam}(A) \geq \frac{s}{3}|x_1-x|.$$

This, together with (8) and the fact that $t, s \in [0,1]$, shows that

$$< x_1, x, x_2, y >= \frac{|x_1-x_2| \wedge |x-y|}{|x_1-x| \wedge |x_2-y|} \geq \frac{t}{2} \wedge \frac{s}{3} \geq \frac{st}{3}.$$

Then we have

$$< x_1, x_2, x, y >= \frac{1}{< x_1, x, x_2, y >} \leq \frac{3}{st}.$$

Since f is θ-quasimöbius, it follows from Lemma 4 that

$$< x_1', x_2', x', y' > \leq \theta_1\left(\frac{3}{st}\right),$$

where the function θ_1 depends only on θ.

Thus we get

$$< x_1', x', x_2', y' >= \frac{1}{< x_1', x_2', x', y' >} \geq \frac{1}{\theta_1(\frac{3}{st})},$$

which guarantees that

$$\frac{|x_1'-x_2'|}{|x_1'-x'|} \vee \frac{|x_1'-x_2'|}{|x_2'-y'|} \geq < x_1', x', x_2', y' > \geq \frac{1}{\theta_1(\frac{3}{st})}.$$

Therefore, we have

$$\begin{aligned}
&\frac{|x_1'-x_2'|}{\inf_{z' \in A'}\{|x_1'-z'|+|z'-x_2'|\}}\\
&\geq \left(\frac{|x_1'-x_2'|}{|x_1'-x'|+|x'-x_2'|}\right) \vee \left(\frac{|x_1'-x_2'|}{|x_1'-y'|+|y'-x_2'|}\right)\\
&\geq \left(\frac{|x_1'-x_2'|}{2|x_1'-x'|+|x_1'-x_2'|}\right) \vee \left(\frac{|x_1'-x_2'|}{2|y'-x_2'|+|x_1'-x_2'|}\right)\\
&= \frac{|x_1'-x_2'|}{2(|x_1'-x'| \wedge |x_2'-y'|)+|x_1'-x_2'|}\\
&\geq \frac{1}{2\theta_1(\frac{3}{ts})+1}.
\end{aligned}$$

Let

$$\varphi(ts) = \begin{cases} 0 & \text{for } ts = 0, \\ \left(2\theta_1(\frac{3}{ts}) + 1\right)^{-1} & \text{for } ts \in (0, +\infty). \end{cases}$$

Hence we obtain (7) and then $\angle(A_1', A_2') \geq \varphi(ts)$. The lemma follows. □

Next, we prove the sufficiency.

Lemma 9 *Suppose* $\varphi : [0, 1] \to [0, 1]$ *is a continuous, strictly increasing function with* $\varphi(0) = 0$. *If for all subsets* $A_1, A_2 \subset X$ *satisfying*

$$\angle(A_1, A_2) \geq t > 0 \quad \textit{and} \quad \operatorname{diam}(A_1 \cap A_2) \geq s(\operatorname{diam}(A_1) \wedge \operatorname{diam}(A_2)),$$

we have

$$\angle(A_1', A_2') \geq \varphi(ts),$$

then f *is* θ*-quasimöbius for a homeomorphism* $\theta : [0, \infty) \to [0, \infty)$ *with* $\theta(0) = 0$ *depending only on* φ.

Proof Choose $x, x_1, y, x_2 \in X$ arbitrarily. Let

$$t =< x, x_1, y, x_2 >= \frac{|x - y| \wedge |x_1 - x_2|}{|x - x_1| \wedge |x_2 - y|}. \tag{10}$$

Without loss of generality, we may assume that

$$|x' - y'| \geq |x_1' - x_2'|. \tag{11}$$

Let $A_1 = \{x_1, x, y\}$ and $A_2 = \{x_2, x, y\}$. Then we get $A = A_1 \cap A_2 = \{x, y\}$. It follows from (10) that

$$|x - y| \geq t(|x - x_1| \wedge |x_2 - y|).$$

An elementary computation shows that

$$(t + 1)|x - y| \geq t\big((|x_1 - x| + |x - y|) \wedge (|x_2 - y| + |x - y|)\big),$$

which leads to

$$(t + 1) \operatorname{diam}(A) \geq t(\operatorname{diam}(A_1) \wedge \operatorname{diam}(A_2)).$$

Thus we have

$$\operatorname{diam}(A) \geq \frac{t}{t + 1}(\operatorname{diam}(A_1) \wedge \operatorname{diam}(A_2)). \tag{12}$$

Moreover, by (10), we obtain

$$\begin{aligned}&\big(|x_1-x|+|x-x_2|\big)\wedge\big(|x_1-y|+|y-x_2|\big)\\&\quad\le\big(2|x_1-x|+|x_1-x_2|\big)\wedge\big(2|x_2-y|+|x_1-x_2|\big)\\&\quad=2(|x_1-x|\wedge|x_2-y|)+|x_1-x_2|\\&\quad=\frac{2}{t}(|x-y|\wedge|x_1-x_2|)+|x_1-x_2|\\&\quad\le\frac{2+t}{t}|x_1-x_2|\end{aligned}$$

which ensures that

$$\angle(A_1,A_2)=\frac{|x_1-x_2|}{|x_1-x|+|x-x_2|}\vee\frac{|x_1-x_2|}{|x_1-y|+|y-x_2|}\ge\frac{t}{t+2}$$

and therefore, by the assumption and (12), we have

$$\angle(A_1',A_2')\ge\varphi\left(\frac{t^2}{(t+1)(t+2)}\right). \tag{13}$$

Since

$$\angle(A_1',A_2')=\frac{|x_1'-x_2'|}{|x_1'-x'|+|x'-x_2'|}\vee\frac{|x_1'-x_2'|}{|x_1'-y'|+|y'-x_2'|},$$

it follows from (11) that

$$\angle(A_1',A_2')\le\frac{|x_1'-x_2'|}{|x_1'-x'|\wedge|x_2'-y'|}=\frac{|x_1'-x_2'|\wedge|x'-y'|}{|x_1'-x'|\wedge|x_2'-y'|}=<x',x_1',y',x_2'>.$$

This, together with (13), ensures that

$$T=<x',x_1',y',x_2'>\ge\varphi\left(\frac{t^2}{(t+1)(t+2)}\right)\ge\varphi\left(\frac{t^2}{(t+2)^2}\right),$$

which guarantees that

$$t\le\psi^{-1}\circ\varphi^{-1}(T),$$

where

$$\psi^{-1}(u)=\frac{2\sqrt{u}}{1-\sqrt{u}}.$$

Note that $T\to 0$ implies that $t\to 0$.

Now, Lemma 4 asserts that f^{-1} is θ_1-quasimöbius with θ_1 depending only on φ. Since the inverse of a quasimöbius mapping is also quasimöbius (cf. [16]), we know that f is quasimöbius as well. □

Proof of Theorem 2 Theorem 2 follows from Lemmas 5 and 9. □

4 An Example

The following example shows that a quasimöbius mapping does not preserve positive angles. Let $\mathbb{C}$ be the complex plane and let $z = t_1 + it_2$ be a point in $\mathbb{C}$.

Example 1 Consider the inversion transformation $u : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ with

$$u(z) = \frac{z}{|z|^2}, \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

Let $A_1 = \{n\}_{n=1}^{\infty}$ and $A_2 = \{1\} \cup \{-n\}_{n=1}^{\infty}$. Then we have the following:

(1) $u(z)$ is θ-quasimöbius with $\theta(t) = t$;
(2) $\angle(A_1, A_2) = 1$;
(3) $\angle(u(A_1), u(A_2)) = 0$.

Proof Obviously, u is a Möbius mapping and so (1) holds true. Part (2) follows easily from that $|a_1 - a_2| = |a_1 - 1| + |a_2 - 1|$ for all $a_1 \in A_1$ and $a_2 \in A_2$ as $a_2 \leq 1 \leq a_1$.

For (3), note that $u(A_1) = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and $u(A_2) = \{1\} \cup \{-\frac{1}{n}\}_{n=1}^{\infty}$. Thus, we have

$$\angle(u(A_1), u(A_2)) = \lim_{n\to\infty} \frac{\frac{2}{n}}{|1 - \frac{1}{n}| + |1 + \frac{1}{n}|} = 0,$$

which implies the statement (3). □

Acknowledgements Qingshan Zhou was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515012441). Tiantian Guan was partly supported by NNSF of China (No. 12201115), by the Guangdong Basic and Applied Basic Research Foundation (No. 2021A1515110484), and Research Fund of Guangdong-Hong Kong-Macao Joint Laboratory for Intelligent Micro-Nano Optoelectronic Technology (No. 2020B1212030010)

Declarations

Conflict of interest The author declares to have no Conflict of interest.

References

1. Agard, S.: Angles and quasiconformal mappings in space. J. Anal. Math. **22**, 177–200 (1969)
2. Agard, S.B., Gehring, F.W.: Angles and quasiconformal mappings. Proc. Lond. Math. Soc. **3**, 1–21 (1965)
3. Ackermann, C., Haïssinsky, P., Hinkkanen, A.: Equilateral triangle skew condition for quasiconformality. Indiana Univ. Math. J. **68**, 1591–1608 (2019)

4. Aseev, V.V., Kuzin, D.G., Tetenov, A.V.: Angles between sets and the gluing of quasisymmetric mappings in metric spaces. Izv. Vyssh. Uchebn. Zaved. Mat. **10**, 3–13 (2005)
5. Bonk, M., Kleiner, B.: Quasisymmetric parametrizations of two-dimensional metric spheres. Invent. Math. **150**, 127–183 (2002)
6. Buckley, M.S., Herron, A.D., Xie, X.: Metric space inversions, quasihyperbolic distance, and uniform spaces. Indiana Univ. Math. J. **57**, 837–890 (2008)
7. Guan, T., Huang, M., Wang, X.: An extension property of quasimöbius mappings in metric spaces. Ann. Acad. Sci. Fenn. Math. **45**, 199–213 (2020)
8. Haïssinsky, P.: A sewing problem in metric spaces. Ann. Acad. Sci. Fenn. Math. **34**, 319–345 (2009)
9. Haïssinsky, P.: Hyperbolic groups with planar boundaries. Invent. Math. **201**, 239–307 (2015)
10. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–61 (1998)
11. Heinonen, J., Koskela, P.: Definitions of quasiconformality. Invent. Math. **120**, 61–79 (1995)
12. Lehto, O., Virtanen, K.I.: Quasiconformal Mappings in the Plane, p. 258. Springer-Verlag, New York-Heidelberg (1973)
13. Rajala, K.: Uniformization of two-dimensional metric surfaces. Invent. Math. **207**, 1301–1375 (2017)
14. Vellis,V.: Extension properties of planar uniform domains, arXiv:1609.08763v4
15. Väisälä, J.: Lectures on n-Dimensional Quasiconformal Mappings, Lecture Notes in Mathematics, 229. Springer-Verlag, pp. 144 (1971)
16. Väisälä, J.: Quasi-Möbius maps. J. Anal. Math. **44**, 218–234 (1984)
17. Wang, X., Zhou, Q.: Quasimöbius maps, weakly quasimöbius maps and uniform perfectness in quasi-metric spaces. Ann. Acad. Sci. Fenn. Math. **42**, 257–284 (2017)
18. Zhou, Q., Li, Y., Xiao, A.: Uniform perfectness for quasi-metric spaces. Filomat **34**, 1975–1987 (2020)