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Solutions to discrete nonlinear Kirchhoff–Choquard equations

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Abstract
In this paper, we study the discrete Kirchhoff–Choquard equation

$$-\left(a+b\int_{\mathbb{Z}^3}|\nabla u|^2d\mu\right)\Delta u+V(x)u=(R_\alpha * F(u))\,f(u),\quad x\in\mathbb{Z}^3,$$

where $a,\ b>0$, $\alpha\in(0,3)$ are constants and R_α is the Green's function of the discrete fractional Laplacian that behaves as the Riesz potential. Under some suitable assumptions on V and f, we prove the existence of nontrivial solutions and ground state solutions respectively by variational methods.

Keywords Nonlinear equations · Discrete Kirchhoff–Choquard problems · Existence · Ground state solutions · Variational methods

Mathematics Subject Classification 35J20 · 35J60 · 35R02

1 Introduction

The Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2d\mu\right)\Delta u+V(x)u=g(x,u),\quad u\in H^1(\mathbb{R}^3),\tag{1}$$

where $a,\ b>0$, has drawn lots of interest in recent years due to the appearance of $(\int_{\mathbb{R}^3}|\nabla u|^2\,d\mu)\Delta u$. For example, Wu [40] proved the existence of nontrivial solutions under general assumptions on g by the symmetric mountain pass theorem. Moreover, if $g(x,u)=g(u)$, He and Zou [14] showed the existence of ground state solutions

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under the Ambrosetti–Rabinowitz conditions on g by the Nehari manifold approach; Guo [11] also derived the existence of ground state solutions for g that does not satisfy the Ambrosetti–Rabinowitz conditions; Wu and Tang [41] verified the existence and concentration of ground state solutions under some assumptions on V and g by the sign-changing Nehari manifold method. In particular, for $g(u) = |u|^{p-1}u$, Sun and Zhang [34] obtained the uniqueness of ground state solutions for $p \in (3, 5)$. Li and Ye [21] established the existence of ground state solutions for $p \in (2, 5)$ based on a monotonicity trick and a new version of global compactness lemma. Later, Lü and Lu [29] extended the result of [21] to $p \in (1, 5)$ by different methods. For more related works, we refer the readers to [1, 2, 4–6, 13, 36].

In many physical applications, the Choquard-type nonlinearity $g(x, u) = (I_\alpha * F(u)) f(u)$ appears naturally, where I_α is the Riesz potential. Clearly, two nonlocal terms are involved in the Eq. (1), which means that the problem is not a pointwise identity any more. Thus, some mathematical difficulties have been provoked, which makes the research on these problems very meaningful. Recently, for $\alpha \in (1, 3)$, Zhou and Zhu [44] proved the existence of ground state solutions; Liang et al. [23] obtained the existence of multi-bump solutions. For $\alpha \in (0, 3)$, Chen et al. [3] proved the existence of ground state solutions under some hypotheses on V and f; Lü and Dai [28] established the existence and asymptotic behavior of ground state solutions by a Pohozaev-type constraint technique; Hu et al. [15] obtained two classes of ground state solutions under the general Berestycki-Lions conditions on f. Moreover, for $f(u) = |u|^{p-2}u$ with $p \in (2, 3 + \alpha)$, Lü [27] demonstrated the existence and asymptotic behavior of ground state solutions by the Nehari manifold and the concentration compactness principle. For more related works about the Choquard-type nonlinearity, we refer the readers to [8, 16, 24, 29, 42].

Nowadays, many researchers turn to study differential equations on graphs, especially for the nonlinear elliptic equations. See for examples [7, 12, 17, 18, 38, 43] for the discrete nonlinear Schrödiner equations. For the discrete nonlinear Choquard equations, we refer the readers to [22, 25, 26, 37]. Recently, Lü [30] proved the existence of ground state solutions for a class of Kirchhoff equations on lattice graphs $\mathbb{Z}^3$. To the best of our knowledge, there is no existence results for the Kirchhoff–Choquard equations on graphs. Motivated by the works mentioned above, in this paper, we would like to study a class of Kirchhoff-type equations with general convolution nonlinearity on lattice graphs $\mathbb{Z}^3$ and discuss the existence of solutions under different conditions on potential V.

Let us first give some notations. Let $C(\mathbb{Z}^3)$ be the set of all functions on $\mathbb{Z}^3$ and $C_c(\mathbb{Z}^3)$ be the set of all functions on $\mathbb{Z}^3$ with finite support. We denote by the $\ell^p(\mathbb{Z}^3)$ the space of ℓ^p-summable functions on $\mathbb{Z}^3$. Moreover, for any $u \in C(\mathbb{Z}^3)$, we always write $\int_{\mathbb{Z}^3} f(x)\, d\mu = \sum\limits_{x \in \mathbb{Z}^3} f(x)$, where μ is the counting measure in $\mathbb{Z}^3$.

In this paper, we consider the following Kirchhoff–Choquard equation

$$-\left(a + b\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)\Delta u + V(x)u = (R_\alpha * F(u))\, f(u), \quad x \in \mathbb{Z}^3, \tag{2}$$

where $a, b > 0$ are constants, $\alpha \in (0, 3)$ and R_α represents the Green's function of the discrete fractional Laplacian, see [31, 37],

$$R_\alpha(x,y)=\frac{K_\alpha}{(2\pi)^3}\int_{\mathbb{T}^3} e^{i(x-y)\cdot k}\mu^{-\frac{\alpha}{2}}(k)\,dk,\quad x,y\in\mathbb{Z}^3,$$

which contains the fractional degree

$$K_\alpha=\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\mu^{\frac{\alpha}{2}}(k)\,dk,\ \mu(k)=6-2\sum_{j=1}^{3}\cos(k_j),$$

where $\mathbb{T}^3=[0,2\pi]^3$, $k=(k_1,k_2,k_3)\in\mathbb{T}^3$. We refer the readers to [9, 10, 20, 32, 33] for more results involved in the fractional calculus. Clearly, the Green's function R_α has no singularity at $x=y$. According to [31], the Green's function R_α behaves as $|x-y|^{\alpha-3}$ for $|x-y|\gg 1$. Here $\Delta u(x)=\sum\limits_{y\sim x}(u(y)-u(x))$ and $|\nabla u(x)|=\left(\frac{1}{2}\sum\limits_{y\sim x}(u(y)-u(x))^2\right)^{\frac{1}{2}}$.

Now we give assumptions on the potential V and the nonlinearity f:

(h_1) for any $x\in\mathbb{Z}^3$, there exists $V_0>0$ such that $V(x)\geq V_0$;
(h_2) there exists a point $x_0\in\mathbb{Z}^3$ such that $V(x)\to\infty$ as $|x-x_0|\to\infty$;
(h_3) $V(x)$ is τ-periodic in $x\in\mathbb{Z}^3$ with $\tau\in\mathbb{Z}$;
(f_1) $f(t)$ is continuous in $t\in\mathbb{R}$ and $f(t)=o(t)$ as $|t|\to 0$;
(f_2) there exist $c>0$ and $p>\frac{3+\alpha}{3}$ such that

$$|f(t)|\leq c(1+|t|^{p-1}),\quad t\in\mathbb{R};$$

(f_3) there exists $\theta>4$ such that

$$0\leq\theta F(t)=\theta\int_0^t f(s)\,ds\leq 2f(t)t,\quad t\in\mathbb{R};$$

(f_4) for any $u\in H\backslash\{0\}$,

$$\frac{\int_{\mathbb{Z}^3}(R_\alpha * F(tu))f(tu)u\,d\mu}{t^3}$$

is strictly increasing with respect $t\in(0,\infty)$.

By (f_1) and (f_2), we have that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|f(t)|\leq\varepsilon|t|+C_\varepsilon|t|^{p-1},\quad t\in\mathbb{R}. \tag{3}$$

Hence

$$|F(t)|\leq\varepsilon|t|^2+C_\varepsilon|t|^p,\quad t\in\mathbb{R}. \tag{4}$$

Let $H^1(\mathbb{Z}^3)$ be the completion of $C_c(\mathbb{Z}^3)$ with respect to the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{Z}^3} \left(|\nabla u|^2 + u^2\right) d\mu\right)^{\frac{1}{2}}.$$

Let $V(x) \geqslant V_0 > 0$, we introduce a new subspace

$$H = \left\{u \in H^1(\mathbb{Z}^3) : \int_{\mathbb{Z}^3} V(x)u^2 \, d\mu < \infty\right\}$$

with the norm

$$\|u\| = \left(\int_{\mathbb{Z}^3} \left(a|\nabla u|^2 + V(x)u^2\right) d\mu\right)^{\frac{1}{2}},$$

where a is a positive constant. The space H is a Hilbert space with the inner product

$$(u, v) = \int_{\mathbb{Z}^3} (a\nabla u \nabla v + V uv) \, d\mu.$$

Since $V(x) \geq V_0 > 0$, we have

$$\|u\|_2^2 \leq \frac{1}{V_0} \int_{\mathbb{Z}^3} V(x)u^2(x) \, d\mu \leq \frac{1}{V_0}\|u\|^2.$$

Moreover, we have

$$\|u\|_q \leq \|u\|_p, \quad q \geq p,$$

which can be seen in [19, Lemma 2.1]. Therefore, for any $u \in H$ and $q \geq 2$, the above two inequalities imply

$$\|u\|_q \leq \|u\|_2 \leq C\|u\|. \tag{5}$$

The energy functional $J(u) : H \to \mathbb{R}$ associated to the Eq. (2) is given by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{Z}^3} \left(a|\nabla u|^2 + V(x)u^2\right) d\mu + \frac{b}{4}\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 \\ &\quad -\frac{1}{2}\int_{\mathbb{Z}^3} (R_\alpha * F(u))F(u) \, d\mu. \end{aligned}$$

Moreover, for any $\phi \in H$, one gets easily that

$$\begin{aligned} \left\langle J'(u), \phi\right\rangle &= \int_{\mathbb{Z}^3} (a\nabla u \nabla \phi + V(x)u\phi) \, d\mu + b\int_{\mathbb{Z}^3} |\nabla u|^2 \, d\mu \int_{\mathbb{Z}^3} \nabla u \nabla \phi \, d\mu \\ &\quad - \int_{\mathbb{Z}^3} (R_\alpha * F(u)) f(u)\phi \, d\mu. \end{aligned}$$

We say that $u \in H$ is a nontrivial solution to the Eq. (2), if u is a nonzero critical point of J, i.e. $J'(u) = 0$ with $u \neq 0$. A ground state solution to the Eq. (2) means that u is a nonzero critical point of J with the least energy, that is,

$$J(u) = \inf_{\mathcal{N}} J > 0,$$

where

$$\mathcal{N} = \left\{u \in H\backslash\{0\} : \left\langle J'(u), u\right\rangle = 0\right\}$$

is the Nehari manifold.

Now we state our main results.

Theorem 1.1 *Let* (h_1)*,* (h_2) *and* (f_1)*-*(f_3) *hold. Then the Eq.* (2) *has a nontrivial solution.*

Theorem 1.2 *Let* (h_1)*,* (h_2) *and* (f_1)*-*(f_4) *hold. Then the Eq.* (2) *has a ground state solution.*

Theorem 1.3 *Let* (h_1)*,* (h_3) *and* (f_1)*-*(f_4) *hold. Then the Eq.* (2) *has a ground state solution.*

The rest of this paper is organized as follows. In Sect. 2, we present some preliminary results on graphs. In Sect. 3, we prove Theorem 1.1 by the mountain pass theorem. In Sect. 4, we prove Theorem 1.2 based on the mountain pass theorem and Nehari manifold approach. In Sect. 5, we prove Theorem 1.3 by the method of generalized Nehari manifold.

2 Preliminaries

In this section, we introduce the basic settings on graphs and give some basic results.

Let $G = (\mathbb{V}, \mathbb{E})$ be a connected, locally finite graph, where $\mathbb{V}$ denotes the vertex set and $\mathbb{E}$ denotes the edge set. We call vertices x and y neighbors, denoted by $x \sim y$, if there exists an edge connecting them, i.e. $(x, y) \in \mathbb{E}$. For any $x, y \in \mathbb{V}$, the distance $d(x, y)$ is defined as the minimum number of edges connecting x and y, namely

$$d(x, y) = \inf\{k : x = x_0 \sim \cdots \sim x_k = y\}.$$

Let $B_r(a) = \{x \in \mathbb{V} : d(x, a) \leq r\}$ be the closed ball of radius r centered at $a \in \mathbb{V}$. For brevity, we write $B_r := B_r(0)$.

In this paper, we consider, the natural discrete model of the Euclidean space, the integer lattice graph. The 3-dimensional integer lattice graph, denoted by $\mathbb{Z}^3$, consists of the set of vertices $\mathbb{V} = \mathbb{Z}^3$ and the set of edges $\mathbb{E} = \{(x, y) : x,\ y \in \mathbb{Z}^3,\ \sum_{i=1}^{3} |x_i - y_i| = 1\}$. In the sequel, we denote $|x - y| := d(x, y)$ on the lattice graph $\mathbb{Z}^3$.

For $u, v \in C(\mathbb{Z}^3)$, we define the Laplacian of u as

$$\Delta u(x) = \sum_{y \sim x} (u(y) - u(x)),$$

and the gradient form Γ as

$$\Gamma(u, v)(x) = \frac{1}{2} \sum_{y \sim x} (u(y) - u(x))(v(y) - v(x)).$$

We write $\Gamma(u) = \Gamma(u, u)$ and denote the length of the gradient as

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2} \sum_{y \sim x} (u(y) - u(x))^2 \right)^{\frac{1}{2}}.$$

The space $\ell^p(\mathbb{Z}^3)$ is defined as

$$\ell^p(\mathbb{Z}^3) = \left\{ u \in C(\mathbb{Z}^3) : \|u\|_p < \infty \right\},$$

where

$$\|u\|_p = \begin{cases} \left(\sum\limits_{x \in \mathbb{Z}^3} |u(x)|^p \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup\limits_{x \in \mathbb{Z}^3} |u(x)|, & p = \infty. \end{cases}$$

The following discrete Hardy-Littlewood-Sobolev (HLS for abbreviation) inequality plays a key role in this paper, see [22, 37].

Lemma 2.1 *Let* $0 < \alpha < 3$, $1 < r, s < \infty$ *and* $\frac{1}{r} + \frac{1}{s} + \frac{3-\alpha}{3} = 2$. *We have the discrete HLS inequality*

$$\int_{\mathbb{Z}^3} (R_\alpha * u)(x) v(x) \, d\mu \le C_{r,s,\alpha} \|u\|_r \|v\|_s, \quad u \in \ell^r(\mathbb{Z}^3), \ v \in \ell^s(\mathbb{Z}^3). \tag{6}$$

And an equivalent form is

$$\|R_\alpha * u\|_{\frac{3r}{3-\alpha r}} \le C_{r,\alpha} \|u\|_r, \quad u \in \ell^r(\mathbb{Z}^3), \tag{7}$$

where $1 < r < \frac{3}{\alpha}$.

Denote

$$I(u) := \frac{1}{2}\int_{\mathbb{Z}^3}(R_\alpha * F(u))F(u)\,d\mu, \quad u \in H.$$

Then for any $\phi \in H$, we have

$$\langle I'(u), \phi\rangle = \int_{\mathbb{Z}^3}(R_\alpha * F(u))f(u)\phi\,d\mu.$$

Lemma 2.2 *Let* (f_1)-(f_3) *hold. Then*

(i) I *is weakly lower semicontinuous;*
(ii) I' *is weakly continuous.*

Proof Let $u_n \rightharpoonup u$ in H. Then $\{u_n\}$ is bounded in H, and hence bounded in $\ell^\infty(\mathbb{Z}^3)$. Therefore, by diagonal principle, there exists a subsequence of $\{u_n\}$ (still denoted by itself) such that

$$u_n \to u, \quad \text{pointwise in } \mathbb{Z}^3. \tag{8}$$

(i) By Fatou's lemma, we get that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{Z}^3}(R_\alpha * F(u))F(u)\,d\mu \le \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{Z}^3}(R_\alpha * F(u_n))F(u_n)\,d\mu \\ &= \liminf_{n\to\infty} I(u_n), \end{aligned}$$

which implies that I is weakly lower semicontinuous.

(ii) Since $C_c(\mathbb{Z}^3)$ is dense in H, we only need to show that for any $\phi \in C_c(\mathbb{Z}^3)$,

$$\langle I'(u_n) - I'(u), \phi\rangle \to 0, \quad n \to \infty. \tag{9}$$

In fact, let $\text{supp}(\phi) \subset B_r$ with $r > 1$. A direct calculation yields that

$$\begin{aligned} \langle I'(u_n) - I'(u), \phi\rangle &= \int_{\mathbb{Z}^3}(R_\alpha * (F(u_n) - F(u))\,f(u)\phi\,d\mu \\ &\quad + \int_{\mathbb{Z}^3}(R_\alpha * F(u_n))\,(f(u_n) - f(u))\,\phi\,d\mu \\ &= T_1 + T_2. \end{aligned}$$

By (4) and (5), one gets easily that $\{F(u_n)\}$ is bounded in $\ell^{\frac{6}{3+\alpha}}(\mathbb{Z}^3)$. Then it follows from the HLS inequality (7) that $\{(R_\alpha * F(u_n))\}$ is bounded in $\ell^{\frac{6}{3-\alpha}}(\mathbb{Z}^3)$. Moreover, we have $F(u_n) \to F(u)$ pointwise in $\mathbb{Z}^3$. By passing to a subsequence, we have

$$(R_\alpha * F(u_n)) \rightharpoonup (R_\alpha * F(u)), \quad \text{in } \ell^{\frac{6}{3-\alpha}}(\mathbb{Z}^3).$$

Since $f(u)\phi \in \ell^{\frac{6}{3+\alpha}}(\mathbb{Z}^3)$, we get

$$T_1 \to 0, \quad n \to \infty.$$

By the HLS inequality (6) and (8), we obtain that

$$\begin{aligned}
|T_2| &\le C\|F(u_n)\|_{\frac{6}{3+\alpha}} \left(\int_{\mathbb{Z}^3} |(f(u_n)-f(u))\phi|^{\frac{6}{3+\alpha}}\, d\mu\right)^{\frac{3+\alpha}{6}} \\
&\le C\left(\int_{\mathbb{Z}^3} |(f(u_n)-f(u))\phi|^{\frac{6}{3+\alpha}}\, d\mu\right)^{\frac{3+\alpha}{6}} \\
&= C\left(\int_{B_r} |(f(u_n)-f(u))\phi|^{\frac{6}{3+\alpha}}\, d\mu\right)^{\frac{3+\alpha}{6}} \\
&\to 0, \quad n \to \infty.
\end{aligned}$$

Then (9) follows from $T_1, T_2 \to 0$ as $n \to \infty$. □

For any $u \in H\backslash\{0\}$, let

$$g(t) := I(tu) = \frac{1}{2}\int_{\mathbb{Z}^3} (R_\alpha * F(tu))\, F(tu)\, d\mu, \quad t \ge 0.$$

Lemma 2.3 *Let* (f_1)-(f_4) *hold. Then*

(i) *for* $t > 0$, $\left(\frac{1}{4}tg'(t) - g(t)\right)$ *is a positive and strictly increasing function;*
(ii) *for* $t \ge 1$, *we have* $g(t) \ge t^\theta g(1)$.

Proof (i) For $t > 0$, by (f_3), we get that

$$\begin{aligned}
g'(t) = \langle I'(tu), u\rangle &= \int_{\mathbb{Z}^3} (R_\alpha * F(tu))\, f(tu)u\, d\mu \\
&= \frac{1}{t}\int_{\mathbb{Z}^3} (R_\alpha * F(tu))\, f(tu)tu\, d\mu \\
&\ge \frac{\theta}{2t}\int_{\mathbb{Z}^3} (R_\alpha * F(tu))\, F(tu)\, d\mu \\
&= \frac{\theta}{t} g(t) \\
&> \frac{4}{t} g(t),
\end{aligned}$$

which implies that $\frac{1}{4}tg'(t) - g(t) > 0$.

By (f_4), one gets that $\frac{g'(t)}{t^3}$ is strictly increasing for $t > 0$. This means that

$$\frac{1}{4}tg'(t) - g(t) = \int_0^t \left(\frac{g'(t)}{t^3} - \frac{g'(s)}{s^3}\right) s^3\, ds$$

is strictly increasing for $t > 0$.

(ii) Clearly for $t = 1$, the result holds. From the proof of (i), one gets that

$$g'(s) \geq \frac{\theta}{s} g(s), \quad s > 0.$$

Integrating the above inequality from 1 to t with $t > 1$,

$$\int_1^t \frac{dg}{g} \geq \theta \int_1^t \frac{ds}{s}.$$

As a consequence, we get that

$$g(t) \geq t^\theta g(1).$$

□

Finally, we state some results about the compactness of H. The following one can be seen in [43]

Lemma 2.4 *Let (h_1) and (h_2) hold. Then for any $q \geq 2$, H is compactly embedded into $\ell^q(\mathbb{Z}^3)$. That is, there exists a constant C depending only on q such that, for any $u \in H$,*

$$\|u\|_q \leq C\|u\|.$$

Furthermore, for any bounded sequence $\{u_n\} \subset H$, there exists $u \in H$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H, \\ u_n \to u, & \text{pointwise in } \mathbb{Z}^3, \\ u_n \to u, & \text{in } \ell^q(\mathbb{Z}^3). \end{cases}$$

We also present a discrete Lions lemma, which denies a sequence $\{u_n\} \subset H$ to distribute itself over $\mathbb{Z}^3$.

Lemma 2.5 *Let $2 \leq s < \infty$. Assume that $\{u_n\}$ is bounded in H and*

$$\|u_n\|_\infty \to 0, \quad n \to \infty.$$

Then, for any $s < t < \infty$,

$$u_n \to 0, \quad \text{in } \ell^t(\mathbb{Z}^3).$$

Proof By (5), we get that $\{u_n\}$ is bounded in $\ell^s(\mathbb{Z}^3)$. Hence, for $s < t < \infty$, this result follows from an interpolation inequality

$$\|u_n\|_t^t \le \|u_n\|_s^s \, \|u_n\|_\infty^{t-s}.$$

□

3 Proof of Theorem 1.1

In this section, we prove the existence of nontrivial solutions to the Eq. (2) by the mountain pass theorem. First we show that the functional $J(u)$ satisfies the mountain pass geometry.

Lemma 3.1 *Let* (h_1) *and* (f_1)-(f_3) *hold. Then*

(i) *there exist* $\sigma, \rho > 0$ *such that* $J(u) \ge \sigma > 0$ *for* $\|u\| = \rho$*;*
(ii) *there exists* $e \in H$ *with* $\|e\| > \rho$ *such that* $J(e) < 0$.

Proof (i) By (4) and the HLS inequality (6), we get that

$$\begin{aligned}\int_{\mathbb{Z}^3} (R_\alpha * F(u))F(u)\,d\mu &\le C\left(\int_{\mathbb{Z}^3} |F(u_n)|^{\frac{6}{3+\alpha}}\,d\mu\right)^{\frac{3+\alpha}{3}}\\ &\le C\left(\int_{\mathbb{Z}^3} \left(\varepsilon|u|^2 + C_\varepsilon|u|^p\right)^{\frac{6}{3+\alpha}}\,d\mu\right)^{\frac{3+\alpha}{3}}\\ &\le \varepsilon\|u\|^4_{\frac{12}{3+\alpha}} + C_\varepsilon\|u\|^{2p}_{\frac{6p}{3+\alpha}}\\ &\le \varepsilon\|u\|^4 + C_\varepsilon\|u\|^{2p}.\end{aligned} \tag{10}$$

Then by (10), we have

$$\begin{aligned}J(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 - \frac{1}{2}\int_{\mathbb{Z}^3} (R_\alpha * F(u))F(u)\,d\mu\\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{Z}^3} (R_\alpha * F(u))F(u)\,d\mu\\ &\ge \frac{1}{2}\|u\|^2 - \varepsilon\|u\|^4 - C_\varepsilon\|u\|^{2p}.\end{aligned}$$

Note that $p > \frac{3+\alpha}{3} > 1$. Let $\varepsilon \to 0^+$, then there exist $\sigma, \rho > 0$ small enough such that $J(u) \ge \sigma > 0$ for $\|u\| = \rho$.

(ii) Let $u \in H\backslash\{0\}$ be fixed. Then it follows from Lemma 2.3 (ii), (10) and $\theta > 4$ that

$$\begin{aligned}\lim_{t\to\infty} J(tu) &= \lim_{t\to\infty}\left[\frac{t^2}{2}\|u\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{Z}^3}|\nabla u|^2 d\mu\right)^2 - \frac{1}{2}\int_{\mathbb{Z}^3}(R_\alpha * F(tu))F(tu)\,d\mu\right]\\ &\leq \lim_{t\to\infty}\left[\frac{t^2}{2}\|u\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{Z}^3}|\nabla u|^2 d\mu\right)^2 - \frac{t^\theta}{2}\int_{\mathbb{Z}^3}(R_\alpha * F(u))\,F(u)\,d\mu\right]\\ &\to -\infty. \end{aligned} \tag{11}$$

Hence, we can choose $t_0 > 0$ large enough such that $\|e\| > \rho$ with $e = t_0 u$ and $J(e) < 0$. □

In the following, we prove the compactness of Palais-Smale sequence. Recall that, for a given functional $\Phi \in C^1(X, \mathbb{R})$, a sequence $\{u_n\} \subset X$ is a Palais-Smale sequence at level $c \in \mathbb{R}$, $(PS)_c$ sequence for short, of the functional Φ, if it satisfies, as $n \to \infty$,

$$\Phi(u_n) \to c, \qquad \text{in } X, \qquad \text{and} \qquad \Phi'(u_n) \to 0, \qquad \text{in } X^*$$

where X is a Banach space and X^* is the dual space of X. Moreover, we say that Φ satisfies $(PS)_c$ condition, if any $(PS)_c$ sequence has a convergent subsequence.

Lemma 3.2 *Let (h_1), (h_2) and (f_1)-(f_3) hold. Then for any $c \in \mathbb{R}$, J satisfies the $(PS)_c$ condition.*

Proof For any $c \in \mathbb{R}$, let $\{u_n\}$ be a $(PS)_c$ sequence for $J(u)$,

$$J(u_n) = c + o_n(1), \quad \text{and} \quad J'(u_n) = o_n(1), \tag{12}$$

where $o_n(1) \to 0$ as $n \to \infty$.

Note that $\theta > 4$ and $b > 0$. By (12), we get that

$$\begin{aligned}\|u_n\|^2 &= \int_{\mathbb{Z}^3}(R_\alpha * F(u_n))F(u_n)\,d\mu - \frac{b}{2}\left(\int_{\mathbb{Z}^3}|\nabla u_n|^2\,d\mu\right)^2 + 2c + o_n(1)\\ &\leq \frac{2}{\theta}\int_{\mathbb{Z}^3}(R_\alpha * F(u_n))f(u_n)u_n\,d\mu - \frac{b}{2}\left(\int_{\mathbb{Z}^3}|\nabla u_n|^2\,d\mu\right)^2 + 2c + o_n(1)\\ &\leq \frac{1}{2}\left(\|u_n\|^2 + b\left(\int_{\mathbb{Z}^3}|\nabla u_n|^2\,d\mu\right)^2 + o_n(1)\|u_n\|\right)\\ &\quad -\frac{b}{2}\left(\int_{\mathbb{Z}^3}|\nabla u_n|^2\,d\mu\right)^2 + 2c + o_n(1)\\ &= \frac{1}{2}\|u_n\|^2 + o_n(1)\|u_n\| + 2c + o_n(1), \end{aligned} \tag{13}$$

which implies that $\{u_n\}$ is bounded in H. Then by Lemma 2.4, up to a subsequence, there exists $u \in H$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H, \\ u_n \to u, & \text{pointwise in } \mathbb{Z}^3, \\ u_n \to u, & \text{in } \ell^q(\mathbb{Z}^3), q \geq 2. \end{cases} \tag{14}$$

Since $|\nabla u(x)|^2 = \frac{1}{2}\sum\limits_{y\sim x}(u(y)-u(x))^2$, one gets easily that

$$\int_{\mathbb{Z}^3} |\nabla u|^2 \, d\mu \leq C\|u\|_2^2.$$

Hence by Hölder inequality, the boundedness of $\{u_n\}$ and (14), we get

$$\begin{aligned} \int_{\mathbb{Z}^3} |\nabla u_n||\nabla(u_n-u)| \, d\mu &\leq \left(\int_{\mathbb{Z}^3} |\nabla u_n|^2 \, d\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{Z}^3} |\nabla(u_n-u)|^2 \, d\mu\right)^{\frac{1}{2}} \\ &\leq C\|u_n\|\|u_n-u\|_2 \\ &\to 0, \quad n\to\infty. \end{aligned} \tag{15}$$

Moreover, by the HLS inequality (6), Hölder inequality, the boundedness of $\{u_n\}$ and (14), we have

$$\begin{aligned} |\langle \ I'(u_n), u_n-u\rangle| &\leq \int_{\mathbb{Z}^3} (R_\alpha * F(u_n))|f(u_n)(u_n-u)| \, d\mu \\ &\leq C\left(\int_{\mathbb{Z}^3} |F(u_n)|^{\frac{6}{3+\alpha}} \, d\mu\right)^{\frac{3+\alpha}{6}} \left(\int_{\mathbb{Z}^3} |f(u_n)(u_n-u)|^{\frac{6}{3+\alpha}} \, d\mu\right)^{\frac{3+\alpha}{6}} \\ &\leq C\left(\int_{\mathbb{Z}^3} (|u_n|^2+|u_n|^p)^{\frac{6}{3+\alpha}} \, d\mu\right)^{\frac{3+\alpha}{6}} \left(\int_{\mathbb{Z}^3} \left[(|u_n|+|u_n|^{p-1})|u_n-u|\right]^{\frac{6}{3+\alpha}} \, d\mu\right)^{\frac{3+\alpha}{6}} \\ &\leq C\left[\left(\int_{\mathbb{Z}^3} (|u_n||u_n-u|)^{\frac{6}{3+\alpha}} \, d\mu\right)^{\frac{3+\alpha}{6}} + \left(\int_{\mathbb{Z}^3} (|u_n|^{p-1}|u_n-u|)^{\frac{6}{3+\alpha}} \, d\mu\right)^{\frac{3+\alpha}{6}}\right] \\ &\leq C\|u_n\|_{\frac{12}{3+\alpha}}\|u_n-u\|_{\frac{12}{3+\alpha}} + C\|u_n\|_{\frac{6p}{3+\alpha}}^{p-1}\|u_n-u\|_{\frac{6p}{3+\alpha}} \\ &\leq C\|u_n-u\|_{\frac{12}{3+\alpha}} + C\|u_n-u\|_{\frac{6p}{3+\alpha}} \\ &\to 0, \quad n\to\infty. \end{aligned} \tag{16}$$

Then it follows from (12), (15) and (16) that

$$\begin{aligned} |(u_n, u_n-u)| &\leq |\langle J'(u_n), u_n-u\rangle| + b\int_{\mathbb{Z}^3} |\nabla u_n|^2 d\mu \int_{\mathbb{Z}^3} |\nabla u_n||\nabla(u_n-u)| \, d\mu \\ &\quad +|\langle I'(u_n), u_n-u\rangle| \\ &\leq o_n(1)\|u_n-u\| + Cb\|u_n\|^2 \int_{\mathbb{Z}^3} |\nabla u_n||\nabla(u_n-u)| \, d\mu + |\langle I'(u_n), u_n-u\rangle| \\ &\to 0, \quad n\to\infty. \end{aligned}$$

Furthermore, since $u_n \rightharpoonup u$ in H, we have

$$(u, u_n - u) \to 0, \quad n \to \infty.$$

Hence we obtain that

$$\|u_n - u\| \to 0, \quad n \to \infty.$$

Note that $u_n \to u$ pointwise in $\mathbb{Z}^3$, we get $u_n \to u$ in H.

□

Proof of Theorem 1.1. By Lemma 3.1, one sees that J satisfies the geometric structure of the mountain pass theorem. Hence for $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}$, there exists a $(PS)_c$ sequence. By Lemma 3.2, J satisfies the $(PS)_c$ condition. Then c is a critical value of J by the mountain pass theorem due to Ambrosetti–Rabinowitz [39]. In particular, there exists $u \in H$ such that $J(u) = c$. Since $J(u) = c \geq \sigma > 0$, we have $u \neq 0$. Hence the Eq. (2) possesses at least a nontrivial solution. □

4 Proof of Theorem 1.2

In this section, we prove the existence of ground state solutions to the Eq. (2) under the conditions (h_1) and (h_2) on V. Now we show some properties of J on the Nehari manifold $\mathcal{N}$ that are useful in our proofs.

Lemma 4.1 *Let (h_1) and (f_1)-(f_4) hold. Then*

(i) *for any $u \in H\backslash\{0\}$, there exists a unique $s_u > 0$ such that $s_u u \in \mathcal{N}$ and $J(s_u u) = \max_{s>0} J(su)$;*

(ii) *there exists $\eta > 0$ such that $\|u\| \geq \eta$ for $u \in \mathcal{N}$;*

(iii) *J is bounded from below on $\mathcal{N}$ by a positive constant.*

Proof (i) For any $u \in H\backslash\{0\}$ and $s > 0$, similar to (10), we get that

$$\int_{\mathbb{Z}^3} (R_\alpha * F(su))F(su)\, d\mu \leq \varepsilon s^4 \|u\|^4 + C_\varepsilon s^{2p} \|u\|^{2p}.$$

Then we have

$$\begin{aligned} J(su) &= \frac{s^2}{2} \int_{\mathbb{Z}^3} \left(a|\nabla u|^2 + V(x)u^2\right) d\mu + \frac{bs^4}{4} \left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 \\ &\quad - \frac{1}{2} \int_{\mathbb{Z}^3} (R_\alpha * F(su))F(su)\, d\mu \\ &= \frac{s^2}{2}\|u\|^2 + \frac{bs^4}{4} \left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 - \frac{1}{2} \int_{\mathbb{Z}^3} (R_\alpha * F(su))F(su)\, d\mu \\ &\geq \frac{s^2}{2}\|u\|^2 - \varepsilon s^4 \|u\|^4 - C_\varepsilon s^{2p} \|u\|^{2p}. \end{aligned} \tag{17}$$

Since $p > \frac{3+\alpha}{3} > 1$, let $\varepsilon \to 0^+$, we get easily that $J(su) > 0$ for $s > 0$ small enough.

On the other hand, similar to (11), we get that

$$J(su) \to -\infty, \quad s \to \infty.$$

Therefore, $\max_{s>0} J(su)$ is achieved at some $s_u > 0$ with $s_u u \in \mathcal{N}$.

Now we show the uniqueness of s_u. By contradiction, suppose that there exist $s_u' > s_u > 0$ such that $s_u' u, s_u u \in \mathcal{N}$. Then we have

$$\begin{aligned} \frac{1}{\left(s_u'\right)^2}\|u\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 &= \int_{\mathbb{Z}^3} \frac{(R_\alpha * F(s_u' u)) f(s_u' u) u}{\left(s_u'\right)^3} d\mu, \\ \frac{1}{(s_u)^2}\|u\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 &= \int_{\mathbb{Z}^3} \frac{(R_\alpha * F(s_u u)) f(s_u u) u}{(s_u)^3} d\mu. \end{aligned}$$

As a consequence, we get

$$\begin{aligned} \left(\frac{1}{\left(s_u'\right)^2} - \frac{1}{(s_u)^2}\right)\|u\|^2 = &\int_{\mathbb{Z}^3} \frac{(R_\alpha * F(s_u' u)) f(s_u' u) u}{\left(s_u'\right)^3} d\mu \\ &- \int_{\mathbb{Z}^3} \frac{(R_\alpha * F(s_u u)) f(s_u u) u}{(s_u)^3} d\mu, \end{aligned}$$

which is a contradiction in view of (f_4).

(ii) By the HLS inequality (6), we have

$$\begin{aligned} \int_{\mathbb{Z}^3} (R_\alpha * F(u)) f(u) u \, d\mu &\le C\left(\int_{\mathbb{Z}^3} |F(u)|^{\frac{6}{3+\alpha}} d\mu\right)^{\frac{3+\alpha}{6}} \left(\int_{\mathbb{Z}^3} |f(u)u|^{\frac{6}{3+\alpha}} d\mu\right)^{\frac{3+\alpha}{6}} \\ &\le C\left(\int_{\mathbb{Z}^3} \left(\varepsilon|u|^2 + C_\varepsilon|u|^p\right)^{\frac{6}{3+\alpha}} d\mu\right)^{\frac{3+\alpha}{3}} \\ &\le \varepsilon\|u\|^4_{\frac{12}{3+\alpha}} + C_\varepsilon\|u\|^{2p}_{\frac{6p}{3+\alpha}} \\ &\le \varepsilon\|u\|^4 + C_\varepsilon\|u\|^{2p}. \end{aligned} \tag{18}$$

Let $u \in \mathcal{N}$. Then we have

$$\begin{aligned} 0 &= \left\langle J'(u), u \right\rangle \\ &= \|u\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 - \int_{\mathbb{Z}^3} (R_\alpha * F(u)) f(u) u \, d\mu \\ &\ge \|u\|^2 - \varepsilon\|u\|^4 - C_\varepsilon\|u\|^{2p}. \end{aligned}$$

Since $p > 1$, we get easily that there exists a constant $\eta > 0$ such that $\|u\| \ge \eta > 0$.

(iii) For any $u \in \mathcal{N}$, by (f_3) and (ii), we derive that

$$
\begin{aligned}
J(u) &= J(u) - \frac{1}{\theta}\langle J'(u), u\rangle \\
&= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u\|^2 + b\left(\frac{1}{4}-\frac{1}{\theta}\right)\left(\int_{\mathbb{Z}^3}|\nabla u|^2 d\mu\right)^2 \\
&\quad + \frac{1}{2}\int_{\mathbb{Z}^3}(R_\alpha * F(u))\left(\frac{2}{\theta}f(u)u - F(u)\right)d\mu \\
&\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u\|^2 \\
&\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\eta^2 \\
&> 0.
\end{aligned}
$$

□

In the following, we establish a homeomorphic map between the unit sphere $S \subset H$ and the Nehari manifold $\mathcal{N}$.

Lemma 4.2 *Let (h_1) and (f_1)-(f_4) hold. Define the maps $s : H\backslash\{0\} \to (0, \infty)$, $u \mapsto s_u$ and*

$$
\begin{aligned}
\widehat{m} : H\backslash\{0\} &\to \mathcal{N}, \\
u &\mapsto \widehat{m}(u) = s_u u.
\end{aligned}
$$

Then

(i) *the maps s and $\widehat{m}$ are continuous;*
(ii) *the map $m := \widehat{m}\,|_S$ is a homeomorphism between S and $\mathcal{N}$, and the inverse of m is given by*

$$
m^{-1}(u) = \frac{u}{\|u\|}.
$$

Proof (i) Let $u_n \to u$ in $H\backslash\{0\}$. Denote $s_n = s_{u_n}$, then $\widehat{m}(u_n) = s_n u_n \in \mathcal{N}$. Since, for any $s > 0$, $\widehat{m}(su) = \widehat{m}(u)$, without loss of generality, we may assume that $\{u_n\} \subset S$. By Lemma 4.1 (ii), we get that

$$
s_n = \|s_n u_n\| \ge \eta > 0.
$$

We claim that $\{s_n\}$ is bounded. Otherwise, $s_n \to \infty$ as $n \to \infty$. By Lemma 2.3 (ii), we have that

$$
\int_{\mathbb{Z}^3}(R_\alpha * F(s_n u_n))F(s_n u_n)\,d\mu \ge s_n^\theta \int_{\mathbb{Z}^3}(R_\alpha * F(u_n))\,F(u_n)\,d\mu.
$$

Moreover, since $\|u_n\| = 1$, one gets easily that

$$\int_{\mathbb{Z}^3} (R_\alpha * F(u_n)) F(u_n)\, d\mu \le C.$$

Then it follows from (i) and (iii) of Lemma 4.1 and $\theta > 4$ that

$$\begin{aligned} 0 &< \frac{J(s_n u_n)}{\|s_n u_n\|^4} \\ &= \frac{1}{2\|s_n u_n\|^2} + \frac{b\left(\int_{\mathbb{Z}^3} |\nabla(s_n u_n)|^2\, dx\right)^2}{4\|s_n u_n\|^4} - \frac{\int_{\mathbb{Z}^3} (R_\alpha * F(s_n u_n)) F(s_n u_n)\, d\mu}{\|s_n u_n\|^4} \\ &\le \frac{1}{2s_n^2} + \frac{b}{4} - s_n^{\theta-4} \int_{\mathbb{Z}^3} (R_\alpha * F(u_n)) F(u_n)\, d\mu \\ &\to -\infty, \end{aligned}$$

which is a contradiction. Hence $\{s_n\}$ is bounded. By the boundedness of $\{s_n\}$, up to a subsequence, there exists $s_0 > 0$ such that $s_n \to s_0$ and $\widehat{m}(u_n) \to s_0 u$. Since $\mathcal{N}$ is closed, $s_0 u \in \mathcal{N}$. This implies that $s_0 = s_u$. As a consequence,

$$s_{u_n} \to s_u$$

and

$$\widehat{m}(u_n) \to s_0 u = s_u u = \widehat{m}(u).$$

Hence the maps s and $\widehat{m}$ are continuous.

(ii) Clearly, m is continuous. For any $u \in \mathcal{N}$, let $\bar{u} = \frac{u}{\|u\|}$, then $\bar{u} \in S$. Since $u = \|u\|\bar{u}$ and $s_{\bar{u}}$ is unique, we get $s_{\bar{u}} = \|u\|$. Hence $m(\bar{u}) = s_{\bar{u}}\bar{u} = u \in \mathcal{N}$, which means that m is surjective. Next, we prove m is injective. Let $u_1, u_2 \in S$ and $m(u_1) = m(u_2)$. Then $s_1 u_1 = s_2 u_2$ implies that $s_1 = s_2$, and hence $u_1 = u_2$. Therefore m has an inverse mapping $m^{-1} : \mathcal{N} \to S$ with $m^{-1}(u) = \frac{u}{\|u\|}$. Then for any $u \in S$, $m^{-1}(m(u)) = u = \mathrm{id}(u)$. □

Now we set

$$c := \inf_{\mathcal{N}} J > 0,$$
$$c_1 := \inf_{u \in H \setminus \{0\}} \max_{s > 0} J(su),$$

and

$$c_2 := \inf_{\gamma \in \Gamma_2} \max_{s \in [0,1]} J(\gamma(s)),$$

where

$$\Gamma_2 = \{\gamma \in C([0,1], H) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Lemma 4.3 *Let* (h_1) *and* (f_1)*-*(f_4) *hold. Then* $c_1 = c_2 = c > 0$.

Proof We first prove $c_1 = c$. By Lemma 4.1 (i), there exists a unique $s_u > 0$ such that $J(s_u u) = \max_{s>0} J(su)$. Then

$$c_1 = \inf_{u\in H\setminus\{0\}} \max_{s>0} J(su) = \inf_{u\in H\setminus\{0\}} J(s_u u) = \inf_{u\in\mathcal{N}} J(u) = c.$$

Next we prove $c_1 \geq c_2$. By (11), for any $u \in H\setminus\{0\}$, there exists a large $s_0 > 0$ such that $J(s_0 u) < 0$. Define

$$\begin{aligned} \gamma_0 : [0,1] &\to H, \\ s &\mapsto s s_0 u. \end{aligned}$$

Since $\gamma_0(0) = 0$ and $J(\gamma_0(1)) < 0$, we have $\gamma_0 \in \Gamma_2$. Then for any $u \in H\setminus\{0\}$,

$$\max_{s>0} J(su) \geq \max_{s\in[0,1]} J(s s_0 u) = \max_{s\in[0,1]} J(\gamma_0(s)) \geq \inf_{\gamma\in\Gamma_2} \max_{s\in[0,1]} J(\gamma(s)),$$

which implies that $c_1 \geq c_2$.

Now we prove $c_2 \geq c$. By Lemma 4.1 (i), for any $u \in H\setminus\{0\}$, there exists a unique $s_u > 0$ such that $s_u u \in \mathcal{N}$. Then we can separate H into two components $H = H_1 \cup H_2$, where $H_1 = \{u \in H : s_u \geq 1\}$ and $H_2 = \{u \in H : s_u < 1\}$.

We claim that each $\gamma \in \Gamma_2$ has to cross $\mathcal{N}$. In fact, one gets easily that $\gamma(t)$ and 0 belong to H_1 for s small enough. We only need to prove $\gamma(1) \in H_2$. Let

$$G(s) = J(s\gamma(1)), \quad s \geq 0.$$

Clearly $G(0) = 0$ and $G(1) < 0$. By similar arguments to (17), we get that $G(t) > 0$ for $s > 0$ small enough. Hence there exists $s_{\gamma(1)} \in (0,1)$ such that $\max_{s\geq 0} G(s) = J(s_{\gamma(1)}\gamma(1))$, and hence $\gamma(1) \in H_2$. By the continuity of s in Lemma 4.2 (i), we obtain that each $\gamma \in \Gamma_2$ has to cross $\mathcal{N}$. The claim is completed.

Then for any $\gamma \in \Gamma_2$, there exists $t_0 \in (0,1)$ such that $\gamma(t_0) \in \mathcal{N}$. As a consequence,

$$\inf_{u\in\mathcal{N}} J(u) \leq J(\gamma(t_0)) \leq \max_{s\in[0,1]} J(\gamma(s)),$$

which implies that $c \leq c_2$. Therefore, we have $c_1 = c_2 = c$.

□

Proof of Theorem 1.2 By Lemma 3.1 and Lemma 3.2, one sees that J satisfies the geometric structure and $(PS)_{c_2}$ condition. Then by the mountain pass theorem, there exists $u \in H$ such that $J(u) = c_2$ and $J'(u) = 0$. Then it follows from Lemma 4.3 that $c_2 = c > 0$. Hence $u \neq 0$ and $u \in \mathcal{N}$. The proof is completed. □

5 Proof of Theorem 1.3

In this section, we prove the existence of ground state solutions to the Eq. (2) under the conditions (h_1) and (h_3) on V.

As we see, the condition (h_2) ensures a compact embedding, see Lemma 2.4, while the condition (h_3) leads to the lack of compactness. Moreover, since we only assume that f is continuous, $\mathcal{N}$ is not a C^1-manifold. Hence we cannot use the Ekeland variational principle on $\mathcal{N}$ directly. Note that Lemma 4.1 and Lemma 4.2 still hold, we shall follow the lines of Hua and Xu [18] to prove this theorem.

We show that Ψ (see below) is of class C^1 and there is a one-to-one correspondence between critical points of Ψ and nontrivial critical points of J. The proof of the lemma is similar to that as in [18, 35]. For completeness, we present the proof in the context.

Lemma 5.1 *Let (h_1) and (f_1)-(f_4) hold. Define the functional*

$$\begin{aligned} \Psi : S &\to \mathbb{R}, \\ w &\mapsto \Psi(w) = J(m(w)). \end{aligned}$$

Then

(i) $\Psi(w) \in C^1(S, \mathbb{R})$ *and*

$$\langle \Psi'(w), z\rangle = \|m(w)\| \left\langle J'(m(w)), z\right\rangle, \quad z \in T_w(S) = \{v \in H : (w, v) = 0\};$$

(ii) *$\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{m(w_n)\}$ is a Palais-Smale sequence for J;*

(iii) *$w \in S$ is a critical point of Ψ if and only if $m(w) \in \mathcal{N}$ is a nontrivial critical point of J. Moreover, the corresponding critical values of Ψ and J coincide and $\inf\limits_S \Psi = \inf\limits_{\mathcal{N}} J$.*

Proof (i) Define the functional

$$\begin{aligned} \widehat{\Psi} : H\backslash\{0\} &\to \mathbb{R}, \\ w &\mapsto \widehat{\Psi}(w) = J(\hat{m}(w)). \end{aligned}$$

Since $J \in C^1(H, \mathbb{R})$ and $\widehat{m}(w) = s_w w$ is a continuous map, we have

$$\begin{aligned} \langle \widehat{\Psi}'(w), z\rangle &= \frac{d}{dt} \mid_{t=0} \widehat{\Psi}(w + tz) \\ &= \frac{d}{dt} \mid_{t=0} J(\widehat{m}(w + tz)) \\ &= J'(\widehat{m}(w + tz)) \mid_{t=0} \cdot \frac{d}{dt} \mid_{t=0} \widehat{m}(w + tz) \\ &= J'(\widehat{m}(w)) s_w z \\ &= s_w \langle J'(\widehat{m}(w)), z\rangle \\ &= \frac{\|\widehat{m}(w)\|}{\|w\|} \langle J'(\widehat{m}(w)), z\rangle. \end{aligned}$$

Note that $\Psi = \widehat{\Psi}\ |_S$ and $m = \widehat{m}\ |_S$. Hence the result follows from the above equality.

(ii) Denote

$$\psi(u) = \frac{1}{2}\|u\|^2, \quad u \in H.$$

Clearly, $\psi \in C^1(H, \mathbb{R})$, and for any $v \in H$,

$$\langle \psi'(u), v \rangle = (u, v).$$

Hence ψ' is bounded on finite sets and $\langle \psi'(w), w \rangle = 1$ for all $w \in S$. Then we have $H = T_w(S) \oplus \mathbb{R}w$ for all $w \in S$, and the projection

$$H \to T_w(S) : z + tw \mapsto z$$

has uniformly bounded norm with respect to $w \in S$. In fact, ψ' is bounded on finite sets and $\langle \psi'(w), (z + tw) \rangle = t$, so if $\|z + tw\| = 1$, then $|t| \le C$. Hence

$$\|z\| \le |t| + \|z + tw\| \le (1 + C)\|z + tw\|, \quad w \in S, z \in T_w(S) \text{ and } t \in \mathbb{R}. \tag{19}$$

Let $u := m(w)$. On one hand, by (i), we have

$$\|\Psi'(w)\| = \sup_{\substack{z \in T_w(S), \\ \|z\|=1}} \langle \Psi'(w), z \rangle = \|u\| \sup_{\substack{z \in T_w(S), \\ \|z\|=1}} \langle J'(u), z \rangle. \tag{20}$$

On the other hand, since $u \in \mathcal{N}$, we have $\langle J'(u), w \rangle = \frac{1}{\|u\|}\langle J'(u), u \rangle = 0$. Then it follows from (19), (20) and (i) that

$$\begin{aligned} \|\Psi'(w)\| &\le \|u\| \|J'(u)\| \\ &= \|u\| \sup_{\substack{z \in T_w(S), t \in \mathbb{R}, \\ z + tw \ne 0}} \frac{\langle J'(u), (z + tw) \rangle}{\|z + tw\|} \\ &\le (1 + C) \sup_{z \in T_w(S) \setminus \{0\}} \frac{\|u\| \langle J'(u), z \rangle}{\|z\|} \\ &= (1 + C) \sup_{z \in T_w(S) \setminus \{0\}} \frac{\langle \Psi'(w), z \rangle}{\|z\|} \\ &= (1 + C)\|\Psi'(w)\|. \end{aligned}$$

By Lemma 4.1 (ii), we have $\|u\| \ge \eta > 0$ for all $u \in \mathcal{N}$. Then the result follows from the previous estimate and the fact $J(u) = \Psi(w)$.

(iii) By (20), $\Psi'(w) = 0$ if and only if $J'(u) = 0$. The rest is clear. □

Proof of Theorem 1.3. Note that $c = \inf\limits_S \Psi$. Let $\{w_n\} \subset S$ be a minimizing sequence such that $\Psi(w_n) \to c$. By Ekeland's variational principle, we may assume that $\Psi'(w_n) \to 0$ as $n \to \infty$. Hence $\{w_n\}$ is a $(PS)_c$ sequence for Ψ.

Let $u_n = m(w_n) \in \mathcal{N}$. Then it follows from Lemma 5.1 that

$$J(u_n) \to c, \quad \text{and} \quad J'(u_n) \to 0, \quad n \to \infty.$$

By (13), one gets that $\{u_n\}$ is bounded in H. Hence there exists $u \in H$ such that

$$u_n \rightharpoonup u, \quad \text{in } H, \quad \text{and} \quad u_n \to u, \quad \text{pointwise in } \mathbb{Z}^3.$$

If

$$\|u_n\|_\infty \to 0, \quad n \to \infty, \tag{21}$$

then by Lemma 2.5, we have that $u_n \to 0$ in $\ell^t(\mathbb{Z}^3)$ with $t > 2$. Hence

$$\begin{aligned}\int_{\mathbb{Z}^3} (R_\alpha * F(u_n)) f(u_n) u_n \, d\mu &\le C\left(\|u_n\|^4_{\frac{12}{3+\alpha}} + \|u_n\|^{2p}_{\frac{6p}{3+\alpha}}\right) \\ &\to 0, \quad n \to \infty.\end{aligned}$$

Namely

$$\int_{\mathbb{Z}^3} (R_\alpha * F(u_n)) f(u_n) u_n \, d\mu = o_n(1).$$

Then

$$\begin{aligned}0 = \langle J'(u_n), u_n \rangle &= \|u_n\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u_n|^2 \, d\mu\right)^2 - \int_{\mathbb{Z}^3} (R_\alpha * F(u_n)) f(u_n) u_n \, d\mu \\ &\ge \|u_n\|^2 + o_n(1),\end{aligned}$$

which implies that $\|u_n\| \to 0$ as $n \to \infty$. This contradicts $\|u_n\| \ge \eta > 0$ in Lemma 4.1 (ii). Hence (21) does not hold, and hence there exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \|u_n\|_\infty \ge \delta > 0, \tag{22}$$

which implies that $u \neq 0$. Therefore, there exists a sequence $\{y_n\} \subset \mathbb{Z}^3$ such that

$$|u_n(y_n)| \ge \frac{\delta}{2}.$$

Let $k_n \in \mathbb{Z}^3$ satisfy $\{y_n - k_n\tau\} \subset \Omega$, where $\Omega = [0, \tau)^3$. By translations, let $v_n(y) := u_n(y + k_n\tau)$. Then for any v_n,

$$\|v_n\|_{l^\infty(\Omega)} \ge |v_n(y_n - k_n\tau)| = |u_n(y_n)| \ge \frac{\delta}{2} > 0.$$

Since V is τ-periodic, J and $\mathcal{N}$ are invariant under the translation, we obtain that $\{v_n\}$ is also a $(PS)_c$ sequence for J and bounded in H. Then there exists $v \in H$ with $v \neq 0$ such that

$$v_n \rightharpoonup v, \quad \text{in } H, \qquad \text{and} \qquad v_n \to v, \quad \text{pointwise in } \mathbb{Z}^3.$$

We prove that v is a critical point of J. Let $A \geq 0$ be a constant such that $\int_{\mathbb{Z}^3} |\nabla v_n|^2 d\mu \to A$ as $n \to \infty$. Note that

$$\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{Z}^3} |\nabla v_n|^2 \, d\mu = A.$$

We claim that

$$\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu = A.$$

Arguing by contradiction, we assume that $\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu < A$. For any $\phi \in C_c(\mathbb{Z}^3)$, we have $\langle J'(v_n), \phi\rangle = o_n(1)$, namely

$$\int_{\mathbb{Z}^3} (a\nabla v_n \nabla\varphi + V(x) v_n \varphi) \, d\mu + b \int_{\mathbb{Z}^3} |\nabla v_n|^2 \, d\mu \int_{\mathbb{Z}^3} \nabla v_n \nabla \varphi \, d\mu \\ - \int_{\mathbb{Z}^3} (R_\alpha * F(v_n)) f(v_n) \phi \, d\mu = o_n(1). \tag{23}$$

Let $n \to \infty$ in (23) and by Lemma 2.2, we get that

$$\int_{\mathbb{Z}^3} (a\nabla v \nabla\varphi + V(x) v\varphi) \, d\mu + bA \int_{\mathbb{Z}^3} \nabla v \nabla \varphi \, d\mu - \int_{\mathbb{Z}^3} (R_\alpha * F(v)) f(v) \phi \, d\mu = 0. \tag{24}$$

Since $C_c(\mathbb{Z}^3)$ is dense in H, (24) holds for any $\phi \in H$. Let $\phi = v$ in (24), then we have

$$\begin{aligned}\langle J'(v), v\rangle &= \int_{\mathbb{Z}^3} \left(a|\nabla v|^2 + V(x)v^2\right) d\mu + b \left(\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu\right)^2 \\ &\quad - \int_{\mathbb{Z}^3} (R_\alpha * F(v)) f(v) v \, d\mu \\ &< \int_{\mathbb{Z}^3} \left(a|\nabla v|^2 + V(x)v^2\right) d\mu + bA \int_{\mathbb{Z}^3} |\nabla v|^2 d\mu \\ &\quad - \int_{\mathbb{Z}^3} (R_\alpha * F(v)) f(v) v \, d\mu \\ &= 0.\end{aligned}$$

Let

$$h(s) = \left\langle J'(sv), sv \right\rangle, \quad s > 0.$$

Then $h(1) = \left\langle J'(v), v \right\rangle < 0$.

By (18), we get that

$$\int_{\mathbb{Z}^3} (R_\alpha * F(sv)) f(sv) sv \, d\mu \leq \varepsilon s^4 \|v\|^4 + C_\varepsilon s^{2p} \|v\|^{2p}.$$

Then for $s > 0$ small enough,

$$\begin{aligned} h(s) &= \left\langle J'(sv), sv \right\rangle \\ &= s^2 \|v\|^2 + s^4 b \left(\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu \right)^2 - \int_{\mathbb{Z}^3} (R_\alpha * F(sv)) f(sv) sv \, d\mu \\ &\geq s^2 \|v\|^2 - \varepsilon s^4 \|v\|^4 - C_\varepsilon s^{2p} \|v\|^{2p} \\ &> 0. \end{aligned} \tag{25}$$

Hence, there exists $s_0 \in (0, 1)$ such that $h(s_0) = 0$, i.e. $\left\langle J'(s_0 v), s_0 v \right\rangle = 0$. This means that $s_0 v \in \mathcal{N}$, and hence $J(s_0 v) \geq c$. By Lemma 2.3, we get that

$$\frac{1}{4} \int_{\mathbb{Z}^3} (R_\alpha * F(sv)) f(sv) sv \, d\mu - \frac{1}{2} \int_{\mathbb{Z}^3} (R_\alpha * F(sv)) F(sv) \, d\mu = \frac{1}{4} s g'(s) - g(s) > 0,$$

and is strictly increasing with respect to $s > 0$. By (f_3), one has that

$$\left(\frac{1}{4} f(v) v - \frac{1}{2} F(v) \right) > \frac{1}{2} \left(\frac{2}{\theta} f(v) v - F(v) \right) \geq 0.$$

Then by Fatou's lemma, we obtain that

$$\begin{aligned} c &\leq J(s_0 v) = J(s_0 v) - \frac{1}{4} \left\langle J'(s_0 v), s_0 v \right\rangle \\ &= \frac{s_0^2}{4} \|v\|^2 + \frac{1}{4} \int_{\mathbb{Z}^3} (R_\alpha * F(s_0 v)) f(s_0 v) s_0 v \, d\mu - \frac{1}{2} \int_{\mathbb{Z}^3} (R_\alpha * F(s_0 v)) F(s_0 v) \, d\mu \\ &< \frac{1}{4} \|v\|^2 + \frac{1}{4} \int_{\mathbb{Z}^3} (R_\alpha * F(v)) f(v) v \, d\mu - \frac{1}{2} \int_{\mathbb{Z}^3} (R_\alpha * F(v)) F(v) \, d\mu \\ &\leq \liminf_{n \to \infty} \left[\frac{1}{4} \|v_n\|^2 + \frac{1}{4} \int_{\mathbb{Z}^3} (R_\alpha * F(v_n)) f(v_n) v_n \, d\mu - \frac{1}{2} \int_{\mathbb{Z}^3} (R_\alpha * F(v_n)) F(v_n) \, d\mu \right] \\ &= \liminf_{n \to \infty} \left[J(v_n) - \frac{1}{4} \left\langle J'(v_n), v_n \right\rangle \right] \\ &= c. \end{aligned}$$

This is a contradiction. Hence,

$$\int_{\mathbb{Z}^3} |\nabla v_n|^2 \, d\mu \to \int_{\mathbb{Z}^3} |\nabla v|^2 d\mu = A.$$

The claim is completed. Then by (23) and (24), we get that $J'(v) = 0$, i.e. $v \in \mathcal{N}$. It remains to prove that $J(v) = c$. In fact, by Fatou's lemma, we obtain that

$$\begin{aligned}
c &\le J(v) - \frac{1}{4}\left\langle J'(v), v\right\rangle \\
&= \frac{1}{4}\|v\|^2 + \frac{1}{4}\int_{\mathbb{Z}^3} (R_\alpha * F(v)) f(v) v \, d\mu - \frac{1}{2}\int_{\mathbb{Z}^3} (R_\alpha * F(v)) F(v) \, d\mu \\
&\le \liminf_{n\to\infty} \left[\frac{1}{4}\|v_n\|^2 + \frac{1}{4}\int_{\mathbb{Z}^3} (R_\alpha * F(v_n)) f(v_n) v_n \, d\mu - \frac{1}{2}\int_{\mathbb{Z}^3} (R_\alpha * F(v_n)) F(v_n) \, d\mu\right] \\
&= \liminf_{n\to\infty} \left[J(v_n) - \frac{1}{4}\left\langle J'(v_n), v_n\right\rangle\right] \\
&= c.
\end{aligned}$$

Hence $J(v) = c$. □

Declarations

Conflict of interest The author declares that there are no Conflict of interest regarding the publication of this paper.

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