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The Ramsey Numbers for Trees of Large Maximum Degree Versus the Wheel Graph W_8

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Abstract

The Ramsey numbers $R(T_n, W_8)$ are determined for each tree graph T_n of order $n \geq 7$ and maximum degree $\Delta(T_n)$ equal to either $n-4$ or $n-5$. These numbers indicate strong support for the conjecture, due to Chen, Zhang and Zhang and to Hafidh and Baskoro, that $R(T_n, W_m) = 2n-1$ for each tree graph T_n of order $n \geq m-1$ with $\Delta(T_n) \leq n-m+2$ when $m \geq 4$ is even.

Keywords Ramsey number · Tree · Wheel graph

Mathematics Subject Classification 05C55 · 05D10

1 Introduction

Let G and H be two simple graphs. The Ramsey number $R(G, H)$ is the smallest integer n such that, for any graph of order n, either it contains G or its complement contains H as a subgraph. Chvátal and Harary [7] proved that $R(G, H) \geq (c(G)-1)(\chi(H)-1)+1$ where $c(G)$ is the largest order of any connected component of G and where $\chi(H)$ is the chromatic number of H. For any tree graph $G = T_n$ of

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order n and the wheel graph $H = W_m$ of order $m + 1$ obtained by connecting a vertex to each vertex of the cycle graph C_m, the Chvátal-Harary bound implies that $R(T_n, W_m) \geq 2n - 1$ when m is even and $R(T_n, W_m) \geq 3n - 2$ when m is odd.

Chen et al. [12] and Zhang [23] showed that $R(P_n, W_m)$ achieves these Chvátal-Harary bounds for the path graph $T_n = P_n$ of order n when m is odd and $3 \leq m \leq n+1$ and when m is even and $4 \leq m \leq n+1$; see also [1, 21]. Baskoro et al. [3] and Surahmat and Baskoro [22] further proved that $R(T_n, W_m)$ achieves the Chvátal-Harary bounds for $m = 4, 5$ and all tree graphs T_n of order $n \geq 3$, except when $m = 4$ and T_n is the star graph S_n, in which case $R(S_n, W_4) = 2n + 1$. This led Baskoro et al. [3] to conjecture that $R(T_n, W_m) = 3m - 2$ for all tree graphs T_n of order n when $m \geq 5$ is odd. The conjecture is true for all sufficiently large n, according to a result of Burr et al. [5]. In contrast, the analogous equality $R(T_n, W_m) = 2n - 1$ for even $m \geq 4$ is false since the star graph $T_n = S_n$ does not achieve this bound, as the following combined result of Zhang [24] and Zhang et al. [25, 26] shows; see also [8, 15, 16, 18, 20].

Theorem 1.1 [24–26] *For $n \geq 5$,*

$$R(S_n, W_8) = \begin{cases} 2n + 1 & \text{if } n \text{ is odd;} \\ 2n + 2 & \text{if } n \text{ is even.} \end{cases}$$

Baskoro et al. [3] therefore conjectured that $R(T_n, W_m) = 2n - 1$ for all non-star tree graphs T_n of order n when $n \geq 4$ is even. This conjecture was disproved by Chen, Zhang and Zhang [9] who showed that $R(T_n, W_6) = 2n$ for certain non-star tree graphs T_n. Zhang [23] further proved the following theorem which shows that the conjecture is false when n is small, even for the path graph P_n; see also [2, 12, 19, 21].

Theorem 1.2 [23] *If m is even and $n + 2 \leq m \leq 2n$, then $R(P_n, W_m) = m + n - 2$.*

However, Chen, Zhang and Zhang [9] conjectured that $R(T_n, W_m) = 2n - 1$ for all tree graphs T_n of order $n \geq m - 1$ when m is even and the maximum degree $\Delta(T_n)$ "is not too large"; see also [10, 11, 13]. Hafidh and Baskoro [14] refined this conjecture by specifying the bound $\Delta(T_n) \leq n - m + 2$. When n is large compared to m, $\Delta(T_n)$ is not required to be small; indeed, the refined conjecture implies that, for each fixed even integer m, all but a vanishing proportion of the tree graphs $\{T_n : n \geq m - 1\}$ satisfy $R(T_n, W_m) = 2n - 1$.

For $m = 8$, the bound is $\Delta(T_n) \leq n - 6$. There is exactly one tree graph T_n of order n with maximum degree $\Delta(T_n) = n - 1$, namely the star graph S_n; see Theorem 1.1. There is exactly one tree graph T_n of order $n \geq m - 1$ with maximum degree $\Delta(T_n) = n - 2$: the graph $S_n(1, 1)$ obtained by subdividing an edge of S_{n-1}. More generally, let $S_n(\ell, m)$ be the tree graph of order n obtained by subdividing m times each of ℓ chosen edges of $S_{n-\ell m}$; see Fig. 1.

By Theorem 1.2, $R(P_4, W_8) = 10$. Hafidh and Baskoro [14] determined the Ramsey number $R\big(S_n(1, 1), W_8\big)$ as follows.

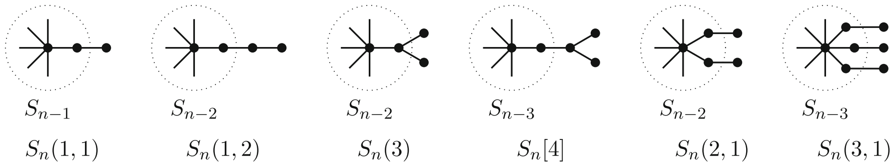


Fig. 1 Examples of $S_n(\ell, m)$, $S_n(\ell)$ and $S_n[\ell]$

Theorem 1.3 [14] *For $n \geq 5$,*

$$R\big(S_n(1,1), W_8\big) = \begin{cases} 2n+1 & \textit{if } n \textit{ is odd},\\ 2n & \textit{if } n \textit{ is even}. \end{cases}$$

There are exactly 3 tree graphs T_n of order n with maximum degree $n-3$, namely $S_n(1,2)$, $S_n(3)$ and $S_n(2,1)$, where $S_n(\ell)$ is the tree graph of order n obtained by adding an edge joining the centers of two star graphs S_ℓ and $S_{n-\ell}$; see Fig. 1. By Theorem 1.2, $R(P_5, W_8) = 11$. Hafidh and Baskoro [14] determined the Ramsey numbers for the three other graphs as follows.

Theorem 1.4 [14] *For $n \geq 6$,*

$$R(S_n(1,2), W_8) = \begin{cases} 2n+1 & \textit{if } n \equiv 3 \pmod 4;\\ 2n & \textit{otherwise} \end{cases}$$

$$R(S_n(3), W_8) = \begin{cases} 2n-1 & \textit{ifnis odd andn} \geq 9;\\ 2n & \textit{otherwise} \end{cases}$$

$$R(S_n(2,1), W_8) = \begin{cases} 2n-1 & \textit{if } n \textit{ is odd};\\ 2n & \textit{otherwise}. \end{cases}$$

The purpose of the present paper is to determine the Ramsey numbers $R(T_n, W_8)$ for all tree graphs T_n of order $n \geq 6$ with maximal degree $\Delta(T_n) \geq n-5$; see Theorems 2.1, 2.2 and 3.1 in Sects. 2 and 3. These Ramsey numbers show that the proportion of tree graphs T_n that satisfy the equality $R(T_n, W_8) = 2n-1$ quickly grows as the maximal degree $\Delta(T_n)$ decreases. When $\Delta(T_n) \geq n-2$, no tree graph T_n satisfies the equality. In contrast, when $\Delta(T_n) = n-3$, roughly one third of all tree graphs T_n satisfy the equality; see Theorem 1.4. When $\Delta(T_n) = n-4$, more than 85% of all tree graphs T_n satisfy the equality; see Theorems 2.1 and 2.2. And when $\Delta(T_n) = n-5$, roughly 94.7% of all tree graphs T_n satisfy the equality; see Theorem 3.1. These results thereby lend strong support for the conjecture described above by Chen, Zhang and Zhang [9] and Hafidh and Baskoro [14].

The contents of the present paper are as follows. Sections 2 and 3 present the main results, namely Theorems 2.1, 2.2 and 3.1 mentioned above. Section 4 provides useful auxiliary results that are used in the proofs of the main results. These proofs are presented in Sects. 5, 6 and 7, respectively.

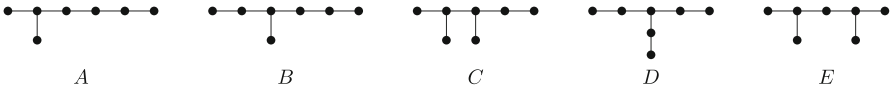


Fig. 2 Tree graphs of order 7 with $\Delta(T_n) = n - 4$

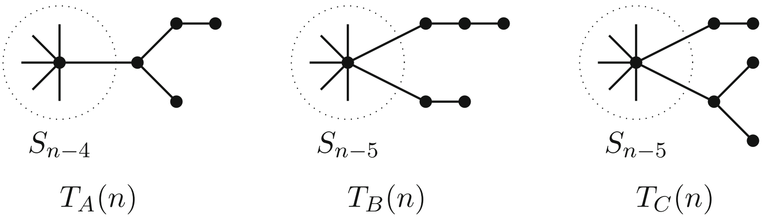


Fig. 3 Three tree graphs with $\Delta(T_n) = n - 4$

2 The Ramsey numbers $R(T_n, W_8)$ for $\Delta(T_n) = n - 4$

This section presents the Ramsey numbers $R(T_n, W_8)$ for all tree graphs T_n of order $n \geq 6$ with $\Delta(T_n) = n - 4$. For $n = 6$, there is just one such graph, namely the path graph $T_6 = P_6$. Theorem 1.2 provides the Ramsey number $R(P_6, W_8) = 12$. For $n = 7$, there are five tree graphs with $\Delta(T_n) = n - 4$, namely the graphs A, B, C, D and E shown in Fig. 2.

The Ramsey numbers $R(T_n, W_8)$ for these tree graphs are determined as follows.

Theorem 2.1 *$R(T, W_8) = 13$ for each $T \in \{A, B, C\}$, $R(D, W_8) = 14$ and $R(E, W_8) = 15$.*

For $n \geq 8$, there are 7 tree graphs T_n of order n with $\Delta(T_n) = n - 4$, namely the graphs $S_n(4)$, $S_n[4]$, $S_n(1, 3)$, $S_n(3, 1)$, $T_A(n)$, $T_B(n)$ and $T_C(n)$ shown in Figs. 1 and 3, where $S_n[\ell]$ is the tree graph of order n obtained by adding an edge joining the center of $S_{n-\ell}$ to a degree-one vertex of S_ℓ; see Fig. 1.

The Ramsey numbers $R(T_n, W_8)$ for these seven tree graphs are determined as follows.

Theorem 2.2 *If $n \geq 8$, then*

$$R(S_n(4), W_8) = \begin{cases} 2n - 1 & \text{if } n \geq 9\,; \\ 16 & \text{if } n = 8 \end{cases}$$

$$R(T_n, W_8) = \begin{cases} 2n - 1 & \text{if } n \not\equiv 0 \pmod 4\,; \\ 2n & \text{otherwise} \end{cases}$$

$$R(T'_n, W_8) = 2n - 1\,,$$

for each $T_n \in \{S_n[4], S_n(1, 3), T_A(n), T_B(n)\}$ and $T'_n \in \{T_C(n), S_n(3, 1)\}$.

Proofs of Theorems 2.1 and 2.2 are given in Sects. 5 and 6.

3 The Ramsey numbers $R(T_n, W_8)$ for $\Delta(T_n) = n - 5$

This section presents the Ramsey numbers $R(T_n, W_8)$ for all tree graphs T_n of order $n \geq 7$ with $\Delta(T_n) = n - 5$. For $n = 7$, there is just one such graph, namely the path graph $T_7 = P_7$. Theorem 1.2 provides the Ramsey number $R(P_7, W_8) = 13$. For $n = 8$, there are 16 tree graphs T_n of order n with $\Delta(T_n) = n - 5$, namely $S_n(1, 4)$, $S_n(2, 2)$ and the tree graphs shown in Fig. 4. For $n = 9$, there are 18 tree graphs T_n of order n with $\Delta(T_n) = n - 5$, namely $S_n(1, 4)$, $S_n[5]$, $S_n(2, 2)$, $S_n(4, 1)$ and the tree graphs shown in Fig. 4. For $n \geq 10$, there are 19 tree graphs T_n of order n with $\Delta(T_n) = n - 5$, namely $S_n(1, 4)$, $S_n(5)$, $S_n[5]$, $S_n(2, 2)$, $S_n(4, 1)$ and the tree graphs shown in Fig. 4.

The Ramsey numbers $R(T_n, W_8)$ for these tree graphs are determined as follows.

Theorem 3.1 *If $n \geq 8$, then $R(T_n, W_8) = 2n - 1$ for all*

$$T_n \in \{S_n(1, 4), S_n(2, 2), T_D(n), \ldots, T_S(n)\}$$

except when $T_n \in \{T_E(8), T_F(8), S_n(1, 4), S_n(2, 2), T_D(n), T_N(n)\}$ and $n \equiv 0 \pmod 4$, in which case $R(T_n, W_8) = 2n$.

Furthermore, if $n \geq 9$, then $R(T_n, W_8) = 2n - 1$ for each $T_n \in \{S_n[5], S_n(4, 1)\}$, and if $n \geq 10$, then $R(S_n(5), W_8) = 2n - 1$.

A proof of this theorem is given in Sect. 7.

4 Auxiliary results

To prove the main theorems, the following auxiliary results will be used. For any simple graph $G = (V, E)$, let $\delta(G)$ be the minimum degree of any vertex in G, and let $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$ be the complement of G.

Lemma 4.1 [4] *Let G be a graph of order n. If $\delta(G) \geq \frac{n}{2}$, then either G contains C_ℓ for all $3 \leq \ell \leq n$, or n is even and $G = K_{\frac{n}{2}, \frac{n}{2}}$.*

Lemma 4.2 [6] *Let G be a graph with $\delta(G) \geq n - 1$. Then G contains all tree graphs of order n.*

Observation 4.3 *If $G = H_1 \cup H_2$ is the disjoint union of graphs H_1 and H_2, where $\overline{H_1}$ contains S_5 and H_2 is a graph of order at least 4, then $\overline{G}$ contains W_8.*

Lemma 4.4 *Let H_1 be a graph whose complement $\overline{H_1}$ contains S_4, and let H_2 be a graph of order $m \geq 5$. If $G = H_1 \cup H_2$, then either $\overline{G}$ contains W_8, or H_2 is K_m or $K_m - e$, where e is an edge in K_m.*

Proof If $\overline{H_2}$ has at most one edge, then H_2 is the complete graph K_m or the graph $K_m - e$ obtained from removing an edge e from K_m. Suppose now that $\overline{H_2}$ has at least two edges. Consider a star S_4 in $\overline{H_1}$ and let v_0 be its center and v_1, v_2, v_3 its leaves. Note that each v_i is adjacent to each $a \in V(H_2)$ in $\overline{G}$. Choose 5 vertices $a, b, c, d, e \in V(H_2)$

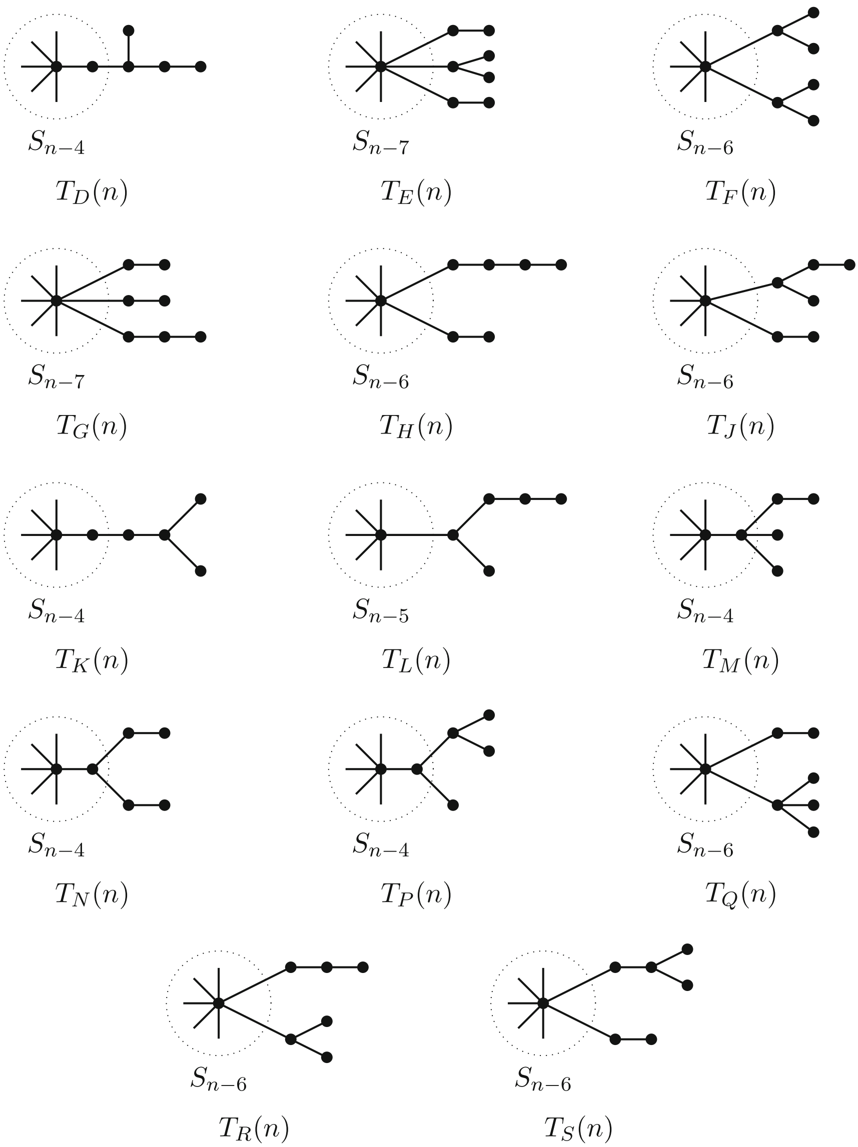


Fig. 4 Tree graphs T_n with $\Delta(T_n) = n - 5$

such that either ab and cd are independent edges, or abc is a path, in $\overline{H_2}$. In both cases, $\overline{G}$ contains W_8 with hub v_0. In the former case, $v_1abv_2cdv_3ev_1$ forms the C_8 rim; in the latter, $v_1abcv_2dv_3ev_1$ forms the C_8 rim. □

The neighbourhood $N_G(v)$ of a vertex v in G is the set of vertices that are adjacent to v in G and $d_G(v) = |N_G(v)|$ is the degree of the vertex v. For $X, Y \subseteq V$, $G[X]$ is the subgraph induced by X in G and $E_G(X, Y)$ is the set of edges in G with one endpoint in X and the other in Y. The following lemma provides sufficient conditions for a graph or its complement to contain C_8.

Lemma 4.5 *Suppose that $U = \{u_1, \dots, u_4\}$ and $V = \{v_1, \dots, v_4\}$ are two disjoint subsets of vertices of a graph G for which $|N_{G[V \cup \{u\}]}(u)| \leq 1$ for each $u \in U$ and $|N_{G[U \cup \{v\}]}(v)| \leq 2$ for each $v \in V$. Then $\overline{G}[U \cup V]$ contains C_8.*

Proof Suppose that $N_{G[U \cup \{v\}]}(v) \leq 1$ for each $v \in V$. Then $\overline{G}[U \cup V]$ contains a subgraph obtained by removing a matching from $K_{4,4}$ and therefore contains C_8. Suppose now that $N_{G[U \cup \{v_1\}]}(v_1) = \{u_1, u_2\}$, and assume without loss of generality that $v_3 \notin N_{G[V \cup \{u_3\}]}(u_3)$ and $v_4 \notin N_{G[V \cup \{u_4\}]}(u_4)$. Neither u_1 nor u_2 is adjacent to v_2, v_3 or v_4, so $v_1 u_3 v_3 u_1 v_2 u_2 v_4 u_4 v_1$ forms C_8 in $\overline{G}[U \cup V]$. □

Lemma 4.6 [17] *Let $G(u, v, k)$ be a simple bipartite graph with bipartition U and V, where $|U| = u \geq 2$ and $|V| = v \geq k$, and where each vertex of U has degree of at least k. If $u \leq k$ and $v \leq 2k - 2$, then $G(u, v, k)$ contains a cycle of length $2u$.*

Corollary 4.7 *Suppose that U and V are two disjoint subsets of vertices of a graph G for which $|N_{G[V \cup \{u\}]}(u)| \leq 2$ for each $u \in U$. If $|U| \geq 4$ and $|V| \geq 6$, then $\overline{G}[U \cup V]$ contains C_8.*

Proof Since $|U| \geq 4$ and $|V| \geq 6$, we can choose any 4 vertices from U to form U' and any 6 vertices from V to form V'. We have that $N_{G[V' \cup \{u\}]}(u) \leq 2$ for each $u \in U'$. Then each vertex of U' is adjacent to at least 4 vertices of V' in $\overline{G}$ and $\overline{G}[U' \cup V']$ must contain a graph with the properties of $G(4, 6, 4)$ in Lemma 4.6. Hence by that lemma, $\overline{G}[U \cup V]$ must contain C_8. □

We will also use the following corollary whose proof is almost identical to that of Corollary 4.7.

Corollary 4.8 *Suppose that U and V are two disjoint subsets of vertices of a graph G for which $|N_{G[V \cup \{u\}]}(u)| \leq 3$ for each $u \in U$. If $|U| \geq 4$ and $|V| \geq 8$, then $\overline{G}[U \cup V]$ contains C_8.*

5 Proof of Theorem 2.1

The proof of Theorem 2.1 is here proved as three theorems, the first of which is as follows.

Theorem 5.1 $R(T, W_8) = 13$ *for each* $T \in \{A, B, C\}$.

Proof Note that $G = 2K_6$ does not contain A, B or C and that $\overline{G}$ does not contain W_8. Therefore, $R(T, W_8) \geq 13$ for $T = A, B, C$.

Let G be a graph of order 13 whose complement $\overline{G}$ does not contain W_8. By Theorem 1.4, G has a subgraph $T = S_7(2, 1)$. Label $V(T)$ as in Fig. 5. Set $U = V(G) - V(T)$; then $|U| = 6$.

First, suppose that $A \nsubseteq G$. Then v_1 is not adjacent to v_2 or v_6. Similarly, v_2 and v_5 are not adjacent.

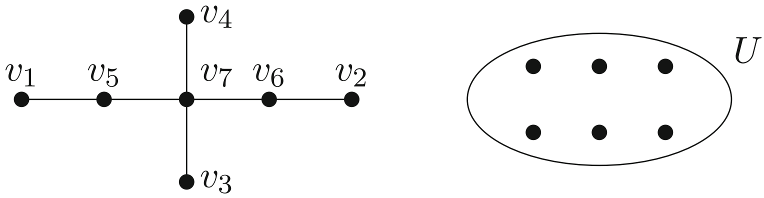


Fig. 5 $S_7(2,1)$ and U in G

Case 1a: There is a vertex in U, say u, that is adjacent to v_1.

Since A is not contained in G, v_1 is not adjacent to v_3, v_4 or any vertex of U other than u. Let $W = \{v_2, v_3, v_4, v_6, u_1, \ldots, u_4\}$ for any 4 vertices $u_1, \ldots, u_4$ in U other than u. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 and, together with v_1 as hub, forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Note that $|N_{G[\{u_1,\ldots,u_4,v_i\}]}(v_i)| \leq 1$ for $i = 2, 3, 4, 6$ since G does not contain A. It is now straightforward to check that v_2, v_3, v_4 and v_6 cannot be the vertex with degree at least 4. Without loss of generality, assume that u_1 has degree at least 4 in $G[W]$. Then u_1 is adjacent to at least one of v_2, v_3, v_4, v_6, so G contains A, a contradiction.

Case 1b: v_1 is not adjacent to any vertices in U.

By arguments similar to those in Case 1a, v_2 is not adjacent to any vertex in U. Let $W = \{v_2, v_6\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which, with v_1 as hub, forms W_8 in $\overline{G}[W]$, a contradiction. Thus, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Since v_2 is not adjacent to any vertex in U, there are only three subcases to be considered.

Subcase 1b.1: $d_{G[W]}(v_6) \geq 4$.

Label $U = \{u_1, \ldots, u_6\}$ so that v_6 is adjacent to u_1, u_2 and u_3 in $G[W]$. Since G does not contain A, vertices u_1, u_2, u_3, v_2 are not adjacent to v_3 or v_4 in G. Note that by arguments as in Case 1a, u_1, u_2 and u_3 are isolated vertices in $G[U]$. Then $v_1u_4u_2v_3v_2u_5u_3u_6v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction.

Subcase 1b.2: $d_{G[W]}(v_6) \leq 3$ and v_6 is adjacent to a vertex $u \in U$ with $d_{G[W]}(u) \geq 4$.

The graph G contains A, with u as the vertex of degree 3 in A, a contradiction.

Subcase 1b.3: $d_{G[W]}(v_6) \leq 3$ and v_6 is not adjacent to any vertex $u \in U$ with $d_{G[W]}(u) \geq 4$.

Label $V(U) = \{u_1, \ldots, u_6\}$ so that u_6 is adjacent to u_2, u_3, u_4 and u_5 in G. Since $A \nsubseteq G$, none of $v_1, \ldots, v_7$ is adjacent in G to any of $u_2, \ldots, u_5$. If v_1 is not adjacent in G to any two of the vertices v_3, v_4, v_7, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Therefore, $N_{G[v_3,v_4,v_7]}(v_1) \geq 2$ and, similarly, $N_{G[v_3,v_4,v_7]}(v_2) \geq 2$. Hence, one of v_3, v_4, v_7 is adjacent in G to both v_1 and v_2. If v_3 or v_4 is adjacent to both v_1 and v_2, then G contains A, with v_7 as vertex of degree 3, a contradiction. Finally, if both v_1 and v_2 are adjacent in G to v_7 and each of them is adjacent to a different vertex in v_3 and v_4, then G also contains A, where either v_1 or v_2 is the vertex of degree 3, a contradiction.

Therefore, $R(A, W_8) \leq 13$, so $R(A, W_8) = 13$.

Now, suppose that $B \nsubseteq G$. Then v_1, v_2, v_5, v_6 are not adjacent to v_3 or v_4 in G, and v_1 and v_2 are not adjacent to U in G. Label the vertices $U = \{u_1, \ldots, u_6\}$ and let $W = \{v_3, v_4\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which, with

v_1 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. If v_3 or v_4 is adjacent to the vertex of degree at least 4 in $G[W]$, then B is contained in G, with v_7 as the vertex of degree 3. Hence, only two cases need to be considered.

Case 2a: v_3 or v_4 is the vertex of degree at least 4 in $G[W]$.

Without loss of generality, assume that v_3 is the vertex of degree at least 4 in $G[W]$. As previously shown, v_3 is not adjacent to v_4. Therefore, it may be assumed that v_3 is adjacent to u_1, u_2, u_3 and u_4 in G. Since $B \nsubseteq G$, $u_1, \ldots, u_4$ are independent in G and are not adjacent to $\{v_1, v_2, v_4, v_5, v_6\}$. Also, v_1 is not adjacent to v_6 and v_2 is not adjacent to v_5. Then $v_1v_6u_2v_2v_5u_3v_4u_4v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction.

Case 2b: One of the vertices in U, say u_1, is the vertex of degree at least 4 in $G[W]$.

As above, u_1 is not adjacent to v_3 or v_4 in G. It may then be assumed that u_1 is adjacent to u_2, u_3, u_4 and u_5. Since $B \nsubseteq G$, $v_1, \ldots, v_7$ are not adjacent to $\{u_2, \ldots, u_5\}$. Note that v_3 is not adjacent to $\{v_1, v_2, v_5, v_6\}$. By Observation 4.3, $\overline{G}$ contains W_8, a contradiction.

Therefore, $R(B, W_8) \leq 13$.

Lastly, suppose that $C \nsubseteq G$. Then v_5 and v_6 are not adjacent in G to each other or to v_3, v_4 or U. Furthermore, v_5 is not adjacent to v_2 and v_6 is not adjacent to v_1. Label the vertices $U = \{u_1, \ldots, u_6\}$ and let $W = \{v_3, v_4, v_6, u_1, \ldots, u_5\}$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which, with v_5 as hub, forms W_8, a contradiction. Then $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Note that v_6 is not adjacent to any other vertex in $G[W]$, v_6 is not the vertex of degree at least 4 in $G[W]$. If v_3 or v_4 is the vertex of degree 4, then G contains C, with v_3 or v_4 and v_7 as the vertices of degree 3. Thus, one of the vertices in U, say u_1, is the vertex of degree at least 4 in $G[W]$. Now, consider the following three cases.

Case 3a: Both v_3 and v_4 are adjacent to u_1 in $G[W]$.

Suppose that u_1 is also adjacent to u_2 and u_3 in $G[W]$. Since $C \nsubseteq G$, v_3 is not adjacent in G to v_4 and neither v_3 nor v_4 is adjacent to $\{v_1, v_2, v_5, v_6, u_2, \ldots, u_6\}$. Note that $|N_{G[\{v_1, v_2, u_i\}]}(u_i)| \leq 1$ for $i = 2, 3$ since $C \nsubseteq G$. If v_1 is adjacent to u_2 and u_3 in $\overline{G}$, then $v_1u_2v_5u_4v_3u_5v_6u_3v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction. Therefore, v_1 is adjacent in G to at least one of u_2 and u_3. Similarly, v_2 is adjacent to at least one of u_2 and u_3. Since $|N_{G[\{v_1, v_2, u_i\}]}(u_i)| \leq 1$ for $i = 2, 3$, v_1 is adjacent to u_2 and v_2 is adjacent to u_3, or vice versa. Then neither u_2 nor u_3 is adjacent in G to u_4, u_5, u_6, since $C \nsubseteq G$. Therefore, $v_1v_3v_2v_5u_2u_4u_3v_6v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction.

Case 3b: One of v_3 and v_4, say v_3, is adjacent to u_1 in $G[W]$.

Suppose that u_1 is adjacent to u_2, u_3 and u_4 in $G[W]$. Then $v_1, v_2, v_4, v_5, v_6, u_2, u_3, u_4 \notin N_G(v_3)$ and $|N_{G[\{v_4, u_2, u_3, u_4\}]}(v_4)| \leq 1$. Without loss of generality, assume that v_4 is not adjacent to u_2 or u_3 in G. Now, suppose that v_4 is adjacent to u_4 in G. Since $C \nsubseteq G$, u_4 is not adjacent to v_1 or v_2 in G. Then $v_1u_4v_2v_5u_2v_4u_3v_6v_1$ and v_3 form W_8 in $\overline{G}$, a contradiction. Otherwise, suppose that v_4 is not adjacent to u_4 in G. Then, $|N_{G[\{u_i, v_1, v_2\}]}(u_i)| \leq 1$ for $i = 2, 3, 4$ and at least two of u_2, u_3 and u_4 are not adjacent to v_1 or v_2 in G. Without loss of generality, assume that u_2 and u_3 are not adjacent to v_1 in G. In this case, $v_1u_2v_4u_4v_5u_5v_6u_3v_1$ and v_3 form W_8 in $\overline{G}$, again a contradiction.

Case 3c: v_3 and v_4 are both not adjacent in $G[W]$ to u_1.

Fig. 6 The graph H

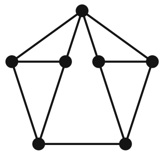
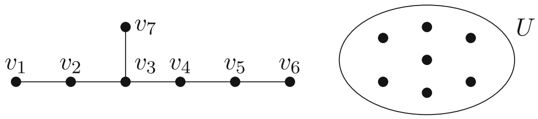


Fig. 7 $B \subseteq G$



Assume that u_1 is adjacent to each of $u_2, \ldots, u_5$ in $G[W]$. Since $C \nsubseteq G$, $|N_{G[\{v_1,\ldots,v_7,u_i\}]}(u_i)| \leq 1$ for $i = 2, \ldots, 5$, and $|N_{G[\{u_2,\ldots,u_5,v_j\}]}(v_j)| \leq 1$ for $j = 3, 4$. Since $|N_{G[\{v_1,v_2,u_i\}]}(u_i)| \leq 1$ for $i = 2, \ldots, 5$, one of v_1 and v_2, say v_1, satisfies $|N_{G[\{u_2,\ldots,u_5,v_1\}]}(v_1)| \leq 2$. By Lemma 4.5, $\overline{G}[v_1, v_3, v_4, v_5, u_2, \ldots, u_5]$ contains C_8 which, with hub v_6, forms W_8 in $\overline{G}$.

Therefore, $R(C, W_8) \leq 13$. This completes the proof of the theorem. □

Theorem 5.2 $R(D, W_8) = 14$.

Proof Let $G = K_6 \cup H$ where H is the graph shown in Fig. 6.

Since G does not contain D and $\overline{G}$ does not contain W_8, $R(D, W_8) \geq 14$.

Now, let G be any graph of order 14. Suppose neither G contains D as a subgraph, nor $\overline{G}$ contains W_8 as a subgraph. By Theorem 5.1, $B \subseteq G$. Label the vertices of B as shown in Fig. 7 and set $U = \{u_1, \ldots, u_7\} = V(G) - V(B)$. Since $D \nsubseteq G$, v_7 is non-adjacent to v_6 and U, and v_4 is non-adjacent to v_1 and v_2.

Let $W = \{v_6\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which, with v_7 as hub, forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Three cases will now be considered.

Case 1: v_6 is the vertex of degree at least 4 in $G[W]$.

Assume that v_6 is adjacent to u_1, u_2, u_3 and u_4 in $G[W]$. Then v_5 is adjacent to v_1 and v_2 in $\overline{G}$ and v_3 is adjacent in $\overline{G}$ to v_6, u_1, u_2, u_3 and u_4.

Subcase 1.1: $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$.

Without loss of generality, assume that u_1 is adjacent to u_5 in G. Since $D \nsubseteq G$, $\{u_2, u_3, u_4\}$ is independent in G and is adjacent to v_1, v_2, u_6 and u_7 in $\overline{G}$; v_6 is adjacent in $\overline{G}$ to v_1 and v_2; v_4 and v_5 are adjacent in $\overline{G}$ to u_1 and u_5; and v_3 is adjacent in $\overline{G}$ to u_5. If v_4 is adjacent to u_2 in G, then v_5 is adjacent in $\overline{G}$ to u_3 and u_4, so $v_1v_5v_2u_2u_6v_7u_7u_3v_1$ and u_4 form W_8 in $\overline{G}$, a contradiction. Thus, v_4 is adjacent to u_2 in $\overline{G}$, and $v_1v_4v_2u_4u_6v_7u_7u_3v_1$ and u_2 form W_8 in $\overline{G}$, again a contradiction.

Subcase 1.2: $\{u_1, \ldots, u_4\}$ is not adjacent to $\{u_5, u_6, u_7\}$ in $G[W]$.

Suppose that v_5 is adjacent in G to v_7; then v_7 is not adjacent to v_1 or v_2. If $|N_{G[\{u_1,\ldots,u_4,v_2\}]}(v_2)| \leq 2$, then $\overline{G}[u_1, \ldots, u_7, v_2]$ contains C_8 by Lemma 4.5 which with v_7 forms W_8 in $\overline{G}$, a contradiction. Thus, $|N_{G[\{u_1,\ldots,u_4,v_2\}]}(v_2)| \geq 3$, so v_1 is not adjacent to $u_1, \ldots, u_4$ in G. By Lemma 4.5, $\overline{G}[u_1, \ldots, u_7, v_1, v_7]$ contains W_8, a contradiction.

Hence, v_5 is not adjacent to v_7 in G. If $|N_{G[\{u_1,\ldots,u_4,v_5\}]}(v_5)| \leq 2$, then $\overline{G}[u_1, \ldots, u_7, v_5]$ contains C_8 by Lemma 4.5 which with v_7 forms W_8 in $\overline{G}$, a contradiction. Thus $|N_{G[\{u_1,\ldots,u_4,v_5\}]}(v_5)| \geq 3$, so v_4 is not adjacent to $\{u_1, \ldots, u_4\}$ in G, or else G will contain D with v_4 be the vertex of degree 3. By Lemma 4.5, $\overline{G}[u_1, \ldots, u_7, v_1]$ contains C_8. If v_4 is not adjacent to v_7 in G, then $\overline{G}$ contains W_8, a contradiction. Thus, v_4 is adjacent to v_7, and since $D \nsubseteq G$, v_1 is not adjacent to v_7. If $|N_{G[\{u_1,\ldots,u_4,v_1\}]}(v_1)| \leq 2$, then $\overline{G}[u_1, \ldots, u_7, v_1]$ contains C_8 by Lemma 4.5 which with v_7 forms W_8, a contradiction. Thus, $|N_{G[\{u_1,\ldots,u_4,v_1\}]}(v_1)| \geq 3$, so $|N_{G[\{u_1,\ldots,u_4,v_1\}]}(v_1) \cap N_{G[\{u_1,\ldots,u_4,v_5\}]}(v_5)| \geq 2$, and G contains D with v_5 as the vertex of degree 3, a contradiction.

Case 2: u_1 is the vertex of degree at least 4 in $G[W]$ and v_6 is adjacent to u_1.

Without loss of generality, suppose that u_1 is adjacent to u_2, u_3 and u_4 in $G[W]$. If v_5 is adjacent to u_1, then Case 1 applies with v_6 replaced by u_1. Suppose then that v_5 is not adjacent to u_1. Since $D \nsubseteq G$, v_1 and v_2 are not adjacent in G to v_4, v_5 or v_6; v_3 is not adjacent to $v_6, u_1, \ldots, u_4$; and v_4 is not adjacent to $u_1, \ldots, u_4$.

Subcase 2.1: $E_G(\{u_2, u_3, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$.

Without loss of generality, assume that u_2 is adjacent to u_5 in G. Then u_3 and u_4 are not adjacent to each other or to v_1, v_2, u_6, u_7. Also, u_1 is not adjacent to v_1 or v_2, and neither u_2 nor u_5 is adjacent to v_3, v_4, v_5, v_6.

Suppose that v_7 is adjacent to v_4 in G. If u_1 is adjacent to v_1, u_5, u_6 or u_7, then Case 1 can be applied through a slight adjustment of the vertex labelings. Suppose that u_1 is not adjacent to any of these vertices. Since $D \nsubseteq G$, v_7 is not adjacent to v_1. If v_6 is not adjacent to u_6, then $v_1u_1u_5v_6u_6u_3u_7u_4v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction. Similarly, $\overline{G}$ contains W_8 if v_6 is not adjacent to u_7, a contradiction. Therefore, v_6 is adjacent to both u_6 and u_7 in G. Since $D \nsubseteq G$, u_6 is not adjacent to u_7, and neither u_6 nor u_7 is adjacent to u_2. Then $v_1u_1u_5v_6u_2u_6u_7u_3v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction.

Suppose now that v_7 is not adjacent to v_4 in G. If v_7 is adjacent to v_5, then v_7 is not adjacent to v_1 or v_2, and v_4 is not adjacent to v_6, u_6 or u_7. Then $v_1u_1v_2u_3u_6v_4u_7u_4v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction. Therefore, v_7 is not adjacent to v_5 in G. If v_6 is not adjacent to u_3, then $u_3v_6u_2v_5u_5v_4u_4u_6u_3$ and v_7 form W_8 in $\overline{G}$, a contradiction. Similarly, $\overline{G}$ contains W_8 if v_6 is not adjacent to u_4, a contradiction. Then v_6 is adjacent to both u_3 and u_4 in G, so v_6 is not adjacent to u_6 and u_7, or else Case 1 applies. Hence, $v_4u_2v_5u_5v_6u_6u_3u_4v_4$ and v_7 form W_8 in $\overline{G}$, a contradiction.

Subcase 2.2: $\{u_2, u_3, u_4\}$ is not adjacent to $\{u_5, u_6, u_7\}$ in $G[W]$.

If $|N_{G[\{u_2,u_3,u_4,v_6\}]}(v_6)| \geq 3$ or $|N_{G[\{u_5,u_6,u_7,v_6\}]}(v_6)| \geq 3$, then Case 1 applies, so $|N_{G[\{u_2,u_3,u_4,v_6\}]}(v_6)| \leq 2$ and $|N_{G[\{u_5,u_6,u_7,v_6\}]}(v_6)| \leq 2$. Without loss of generality, assume that v_6 is not adjacent in G to u_2 or u_5.

Suppose that v_4 is not adjacent to v_7 in G. If u_5 is adjacent to u_6 or u_7, say u_6, then v_4 is not adjacent to u_5 or u_6, so $v_4u_2v_6u_5u_3u_7u_4u_6v_4$ and v_7 form W_8 in $\overline{G}$, a contradiction. If u_5 is not adjacent to u_6 or u_7, then $v_4u_2v_6u_5u_6u_3u_7u_4v_4$ and v_7 form W_8 in $\overline{G}$, a contradiction. Suppose that v_4 is adjacent to v_7 in G. By similar arguments to those in Subcase 2.1, u_1 is not adjacent to v_1, u_5, u_6 or u_7, and v_7 is not adjacent to v_1. Then $v_1v_6u_5u_2u_6u_3u_7u_1v_1$ and v_7 form W_8 in $\overline{G}$, a contradiction.

Case 3: u_1 is the vertex of degree at least 4 in $G[W]$ and v_6 is not adjacent to u_1.

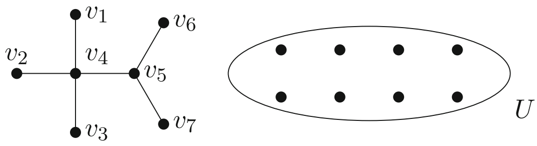


Fig. 8 $S_7(3)$ and U in G

Assume that u_1 is adjacent to u_2, u_3, u_4 and u_5 in $G[W]$. Since $D \nsubseteq G$, v_3 and v_4 are not adjacent to u_1, u_2, u_3, u_4 or u_5 in G. If either v_1 or v_5 are adjacent to u_1 in G, then Case 1 applies, so suppose that v_1 and v_5 are not adjacent to u_1. In addition, v_1 and v_5 are not adjacent to u_2, u_3, u_4 or u_5 in G, or else Case 2 applies.

Subcase 3.1: $N_{G[u_2,\ldots,u_5]}(v_6) \neq \emptyset$.

Assume that v_6 is adjacent to u_2 in G. Note that v_4 is not adjacent to v_6, v_7, u_6 or u_7 in G, and v_3 is not adjacent to v_5 in G, or else Case 2 applies by slight adjustment of vertex labels. Since $D \nsubseteq G$, v_1 and v_2 are not adjacent in G to v_5, v_6 or u_2, and v_3 is not adjacent to v_6 in G.

If u_2 and u_6 are not adjacent in G, then $v_1u_1v_6v_2u_2u_6v_7u_3v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction. A similar contradiction arises if u_2 and u_7 are not adjacent. Therefore, u_2 is adjacent to both u_6 and u_7 in G, and u_3, u_4 and u_5 are not adjacent to u_6 or u_7 in G since $D \nsubseteq G$. Then $v_1u_1v_6v_2u_2v_7u_6u_3v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction.

Subcase 3.2: $N_{G[u_2,\ldots,u_5]}(v_6) = \emptyset$.

Suppose that v_1 is adjacent to v_7 in G. Then v_2 is not adjacent to v_5, v_6 or U since $D \nsubseteq G$. If $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \leq 2$, then Lemma 4.5 implies that $\overline{G}[u_2, \ldots, u_5, v_4, v_5, v_6, u_6]$ contains C_8 in $\overline{G}$ which with v_2 forms W_8, a contradiction. Thus, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \geq 3$. Similarly, $|N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 3$. By the Inclusion-exclusion Principle, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6) \cap N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 2$. Without loss of generality, u_6 is adjacent to u_2, u_3 and u_4 in G, and u_7 is adjacent to u_3 and u_4, and $G[u_1, \ldots, u_7]$ contains D with u_3 or u_4 being the vertex of degree 3, a contradiction.

Now suppose that v_1 is not adjacent to v_7 in G. If v_7 is adjacent to v_4 in G, then v_2 is not adjacent to any of $u_1, \ldots, u_5$ in G, or else either Case 1 or 2 applies. Also, $|N_{G[\{v_2,v_5,v_7\}]}(v_7)| \leq 1$ since $D \subseteq G$. Assume that v_7 is not adjacent to v_2 in G. If $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \leq 2$, then Lemma 4.5 implies that $\overline{G}[u_2, \ldots, u_5, v_1, v_2, v_6, u_6]$ contains C_8 which with v_7 forms W_8, a contradiction. Thus, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \geq 3$. Similarly, $|N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 3$, so $|N_{G[\{u_2,\ldots,u_6\}]}(u_6) \cap N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 2$. By arguments similar to those in the previous paragraph, G will contain a subgraph D, a contradiction.

Thus, $R(D, W_8) \leq 14$ which completes the proof of the theorem. □

Theorem 5.3 $R(E, W_8) = 15$.

Proof The graph $G = K_6 \cup K_{4,4}$ does not contain E and $\overline{G}$ does not contain W_8. Thus, $R(E, W_8) \geq 15$. For the upper bound, let G be any graph of order 15. Suppose that G does not contain E and that $\overline{G}$ does not contain W_8. By Theorem 1.4, G contains a $T = S_7(3)$ subgraph. Label the vertices of this subgraph as in Fig. 8 and set $U = V(G) - V(T)$. Note that $|U| = 8$.

Case 1: Some vertex u in U is adjacent to v_6.

Since $E \nsubseteq G$, v_6 is not adjacent to v_1, v_2, v_3, v_7 or any vertex of U other than u. Let $W = \{v_1, v_2, v_3, v_7, u_1, \ldots, u_4\}$, for any vertices $u_1, \ldots, u_4$ in U other than u. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which with v_6 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) \leq 3$ and $\Delta(G[W]) \geq 4$. Since $E \nsubseteq G$, $N_{G[\{u_1,\ldots,u_4,v_1,v_7\}]}(v_7) \leq 1$ and $N_{G[\{u_1,\ldots,u_4,v_7,v_i\}]}(v_i) \leq 1$ for $i = 1, 2, 3$, so none of v_1, v_2, v_3, v_7 has degree at least 4. Without loss of generality, assume that u_1 has degree at least 4. If u_1 is adjacent to v_7, then G contains E with u_1 and v_5 as the vertices of degree 3, a contradiction. Similarly, if u_1 is adjacent to v_1, v_2 or v_3, then G contains E with u_1 and v_4 as the vertices of degree 3, a contradiction. Therefore, u_1 is not adjacent to v_1, v_2, v_3 or v_7. However, then u_1 has degree at most 3 in $G[W]$, a contradiction.

Case 2: v_6 is not adjacent to any vertices in U.

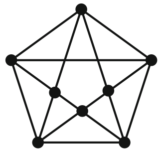
If v_7 is adjacent to some vertex in U, then Case 1 applies with v_7 replacing v_6, so suppose that v_7 is not adjacent to any vertex in U. Now, if $\delta(\overline{G}[U]) \geq 4$, then $\overline{G}[U]$ contains C_8 by Lemma 4.1 which with v_6 or v_7 forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) \leq 3$ and $\Delta(G[U]) \geq 4$. Let $V(U) = \{u_1, \ldots, u_8\}$. Without loss of generality, assume that u_1 is adjacent to u_2, u_3, u_4 and u_5. Since $E \nsubseteq G$, v_4 is not adjacent in G to any of $u_1, \ldots, u_5$; v_5 is not adjacent to any of $v_1, v_2, v_3, u_1, \ldots, u_5$; and u_1 is not adjacent to v_1, v_2 or v_3. Furthermore, $|N_{G[\{u_2,\ldots,u_5,v_i\}]}(v_i)| \leq 1$ for $i = 1, 2, 3$ and $|N_{G[\{v_1,v_2,v_3,u_j\}]}(u_j)| \leq 1$ for $j = 2, \ldots, 5$.

Suppose that $N_{G[\{v_5,u_6,u_7,u_8\}]}(v_5) = \emptyset$. If $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \leq 1$, then $\overline{G}[u_2, \ldots, u_5, v_1, v_2, v_3, u_6]$ contains C_8 by Lemma 4.5 which with v_5 forms W_8, a contradiction. Therefore, $|N_{G[\{u_2,\ldots,u_6\}]}(u_6)| \geq 2$. Similarly, $|N_{G[\{u_2,\ldots,u_5,u_7\}]}(u_7)| \geq 2$ and $|N_{G[\{u_2,\ldots,u_5,u_8\}]}(u_8)| \geq 2$. By the Inclusion-Exclusion Principle, u_2, u_3, u_4 or u_5 is adjacent in G to at least two of u_6, u_7, u_8. Without loss of generality, assume that u_2 is adjacent to u_6 and u_7. Then u_2 is not adjacent to u_3, u_4 or u_5, Therefore, Lemma 4.5 implies that $\overline{G}[u_1, u_3, u_4, u_5, v_1, v_2, v_3, u_2]$ contains C_8 which with v_5 forms W_8, a contradiction.

On the other hand, if $N_{G[u_6,u_7,u_8]}(v_5) \neq \emptyset$, then without loss of generality assume that u_6 is adjacent to v_5 in G. Since $E \nsubseteq G$, v_4 is not adjacent to v_6, v_7 or u_6 in G. Also, $\{v_1, v_2, v_3\}$ and $\{v_6, v_7, u_6\}$ are independent in G, and $v_1, v_2, v_3, v_6, v_7, u_6 \notin N_G(u_i)$ for $i = 1, \ldots, 5, 7, 8$, or else Case 1 applies with vertex label adjustments. Now, if u_1 is not adjacent to both u_7 and u_8 in G, then $v_1v_2v_3u_7v_6v_7u_6u_8v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction. Therefore, $N_{G[\{u_1,u_7,u_8\}]}(u_1) \neq \emptyset$. Without loss of generality, assume that u_1 is adjacent to u_7 in G. Note that for $E \nsubseteq G$, $|N_{G[\{v_4,v_5,u_8\}]}(u_8)| \leq 1$. Assume that u_8 is not adjacent to v_4 in G. If $|N_{G[\{u_2,\ldots,u_5,u_8\}]}(u_8)| \leq 3$, then assume without loss of generality that u_8 is not adjacent to u_2 or u_3 in G. Then $v_6u_4v_7u_5u_6u_2u_8u_3v_6$ and v_4 form W_8 in $\overline{G}$, a contradiction. Similar arguments work if u_8 is not adjacent to v_5 in G, by replacing v_4 with v_5 and v_6, v_7, u_6 with v_1, v_2, v_3, respectively. Hence, $|N_{G[\{u_2,\ldots,u_5,u_7,u_8\}]}(u_8)| \geq 4$. However, G then contains E with u_1 and u_8 of degree 3, a contradiction.

Thus, $R(E, W_8) \leq 15$. This completes the proof of the theorem. □

Fig. 9 The graphs H_8



6 Proof of Theorem 2.2

Consider the tree graphs T_n of order $n \geq 8$ with $\Delta(T_n) = n-4$, namely $S_n(4)$, $S_n[4]$, $S_n(1,3)$, $S_n(3,1)$, $T_A(n)$, $T_B(n)$ and $T_C(n)$; see Figs. 1 and 3.

Lemma 6.1 *Let* $n \geq 8$. *Then* $R(T_n, W_8) \geq 2n-1$ *for each* $T_n \in \{S_n(4), S_n(3,1), T_C(n)\}$. *Also for each* $T_n \in \{S_n[4], S_n(1,3), T_A(n), T_B(n)\}$, $R(T_n, W_8) \geq 2n-1$ *if* $n \not\equiv 0 \pmod 4$ *and* $R(T_n, W_8) \geq 2n$ *otherwise.*

Proof The graph $G = 2K_{n-1}$ clearly does not contain any tree graph of order n, and $\overline{G}$ does not contain W_8. Finally, if $n \equiv 0 \pmod 4$, then the graph $G = K_{n-1} \cup K_{4,\ldots,4}$ of order $2n-1$ does not contain $S_n[4]$, $S_n(1,3)$, $T_A(n)$ or $T_B(n)$; nor does the complement $\overline{G}$ contain W_8. □

Theorem 6.2 *If* $n \geq 8$, *then*

$$R(S_n(4), W_8) = \begin{cases} 2n-1 & \text{if } n \geq 9; \\ 16 & \text{if } n = 8. \end{cases}$$

Proof By Lemma 6.1, $R(S_n(4), W_8) \geq 2n-1$ for $n \geq 8$. For $n = 8$, observe that the graph $G = K_7 \cup H_8$, where H_8 is the graph of order 8 as shown in Fig. 9 does not contain $S_8(4)$ and its complement $\overline{G}$ does not contain W_8. Therefore, for $n = 8$, we have a better bound of $R(S_8(4), W_8) \geq 16$.

For the upper bound, let G be any graph of order $2n-1$ if $n \geq 9$, and of order 16 if $n = 8$. Assume that G does not contain $S_n(4)$ and that $\overline{G}$ does not contain W_8.

If $n \geq 9$ is odd or $n = 8$, then G has a subgraph $T = S_n(3)$ by Theorem 1.4. Let $V(T) = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1w_1, v_1w_2\}$. Also, let $V = \{v_2, \ldots, v_{n-3}\}$ and $U = V(G) - V(T)$; then $|V| = n-4 \geq 5$ and $|U| = n-1 \geq 8$ if n is odd, while $|U| = 8$ if $n = 8$. Since $S_n(4) \nsubseteq G$, v_1 is not adjacent to any vertex of $U \cup V$ in G. Furthermore, for each $2 \leq i \leq n-3$, v_i is adjacent to at most two vertices of U in G. By Corollary 4.7, $\overline{G}[U \cup V]$ contains C_8, and together with v_1, gives us W_8 in $\overline{G}$, a contradiction.

For the remaining case when $n \geq 10$ is even, $S_{n-1} \subseteq G$ by Theorem 1.1. Let v_0 be the center of S_{n-1} and set $L = N_{S_{n-1}}(v_0) = \{v_1, \ldots, v_{n-2}\}$ and $U = V(G) - V(S_{n-1})$. Then $|U| = n$. Since G does not contain $S_n(4)$, each vertex of L is adjacent to at most two vertices of U. We consider two cases.

Case 1: $E(L, U) = \emptyset$.

If $\Delta(\overline{G}[U]) \geq 4$, then some vertex u in U is adjacent to at least four vertices in $\overline{G}[U]$. These four vertices and any four vertices from L form C_8 in $\overline{G}$ which with u forms W_8, a contradiction. Therefore, $\Delta(\overline{G}[U]) \leq 3$ and $\delta(G[U]) \geq n-4$. Suppose $\delta(G[U]) =$

$n-4+l$ for some $l \geq 0$, and let u_0 be a vertex in U with minimum degree in $G[U]$. Label the remaining vertices in U as $u_1, \ldots, u_{n-1}$ such that $U_A = \{u_1, \ldots, u_{n-4}\} \subseteq N_G(u_0)$, and let $U_B = \{u_{n-3}, u_{n-2}, u_{n-1}\}$. Since $S_n(4) \nsubseteq G$, each vertex in U_A is adjacent to at most two vertices in U_B, and so $|E_G(U_A, U_B)| \leq 2(n-4)$. On the other hand, noting that u_0 is adjacent to exactly l vertices in U_B and letting $e_B \leq 3$ be the number of edges in $G[U_B]$, we see that $|E_G(U_A, U_B)| \geq 3\delta(G[U]) - l - 2e_B = 3(n-4+l) - l - 2e_B$. Therefore, $2(n-4) \geq |E_G(U_A, U_B)| \geq 3n - 12 + 2l - 2e_B$, implying that $n + 2l \leq 4 + 2e_B \leq 10$, which is only possible when $n = 10$, $l = 0$, $e_B = 3$, and $|E_G(U_A, U_B)| = 2(n-4) = 12$. For such scenario where $n = 10$, noting that u_0 was an arbitrary vertex with minimum degree in $G[U]$, it is straightforward to deduce that the only possible edge set of $G[U]$ (up to isomorphism) with $S_{10}(4) \nsubseteq G[U]$ is $\{u_0u_1, \ldots, u_0u_6\} \cup \{u_1u_7, \ldots, u_4u_7\} \cup \{u_1u_8, u_2u_8, u_5u_8, u_6u_8\} \cup \{u_3u_9, \ldots, u_6u_9\} \cup \{u_1u_2, u_3u_4, u_5u_6\} \cup \{u_1u_3, u_1u_5, u_3u_5\} \cup \{u_2u_4, u_2u_6, u_4u_6\} \cup \{u_7u_8, u_7u_9, u_8u_9\}$. Observe now that $\overline{G}[U]$ contains C_8 which forms W_8 in $\overline{G}$ with any vertex in L as hub, a contradiction.

Case 2: $E(L, U) \neq \emptyset$.

Without loss of generality, assume that v_1 is adjacent to u_1 in G. Since $S_n(4) \nsubseteq G$, v_1 is adjacent to at most one vertex of $U \cup L \backslash \{u_1\}$ in G. Therefore, we can find a 4-vertex set $V' \subseteq V \backslash \{v_1\}$ and an 8-vertex set $U' \subseteq U \backslash \{u_1\}$ such that v_1 is not adjacent in G to any vertex of $U' \cup V'$. Note that each vertex of V' is adjacent to at most two vertices of U' in G, so $|E(V', U')| \leq 8$. This implies that there are four vertices in U' that are each adjacent in G to at most one vertex of V', and so $\overline{G}$ contains C_8 by Lemma 4.5 which with v_1 forms W_8, a contradiction.

Thus, $R(S_n(4), W_8) \leq 2n-1$ when $n \geq 9$ and $R(S_n(4), W_8) \leq 16$ when $n = 8$. This completes the proof of the theorem. □

Lemma 6.3 *Let H be a graph of order $n \geq 8$ with minimum degree $\delta(H) \geq n-4$. Then either H contains $S_n[4]$ and $T_A(n)$, or $n \equiv 0 \pmod 4$ and $\overline{H}$ is the disjoint union of $\frac{n}{4}$ copies of K_4, i.e., $\overline{H} = \frac{n}{4} K_4$.*

Proof Let $V(H) = \{u_0, \ldots, u_{n-1}\}$. First, consider the case where H has a vertex of degree at least $n-3$, say u_0, and that $\{u_1, \ldots, u_{n-3}\} \subseteq N_H(u_0)$.

Suppose u_{n-2} is adjacent to u_{n-1} in H. Since $\delta(H) \geq n-4$, u_{n-2} is adjacent to at least $n-6 \geq 2$ vertices of $\{u_1, \ldots, u_{n-3}\}$, say u_1 and u_2, and so H contains $S_n[4]$. Furthermore by the minimum degree condition, u_1 is adjacent to at least $n-7 \geq 1$ vertices of $\{u_1, \ldots, u_{n-3}\}$, and so H contains $T_A(n)$.

Suppose now that u_{n-2} is not adjacent to u_{n-1} in H. Then by the minimum degree condition, there is a vertex in $\{u_1, \ldots, u_{n-3}\}$, say u_1, that is adjacent to both u_{n-2} and u_{n-1}. The vertices u_1 and u_{n-2} must also each be adjacent to a vertex of $\{u_2, \ldots, u_{n-3}\}$, and so H contains both $S_n[4]$ and $T_A(n)$.

For the remaining case, suppose that H is $(n-4)$-regular and that $N_H(u_0) = \{u_1, \ldots, u_{n-4}\}$. Let $U = \{u_{n-3}, u_{n-2}, u_{n-1}\}$ and suppose that $H[U]$ has an edge, say $u_{n-3}u_{n-2}$. Since u_{n-3} must be adjacent in H to some vertex of $N_H(u_0)$, it follows that H contains $S_n[4]$ if u_{n-3} or u_{n-2} is adjacent to u_{n-1}. Suppose then that neither u_{n-3} nor u_{n-2} is adjacent to u_{n-1}. Then u_{n-1} is adjacent to every vertex of $N_H(u_0)$. Note that $d_{H[N_H(u_0) \cup \{u_{n-3}\}]}(u_{n-3}) = n-5$ and let u be the vertex of $N_H(u_0)$ that is not adjacent in H to u_{n-3}. Since $d_H(u) = n-4$, u is adjacent in H to some vertex in

$N_H(u_{n-3})$, so H contains $S_n[4]$. Also, note that u_{n-3} is adjacent in H to at least $n-6$ vertices of $N_H(u_0)$. If u_{n-1} is adjacent to some vertex of $N_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3})$, then H contains $T_A(n)$. Note that this will always happen for $n \geq 9$. For $n = 8$, there is a case where $|N_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3})| = |N_{H[N_H(u_0)\cup\{u_{n-1}\}]}(u_{n-1})| = 2$ and $N_{H[N_H(u_0)\cup\{u_{n-3}\}]}(u_{n-3}) \cap N_{H[N_H(u_0)\cup\{u_{n-1}\}]}(u_{n-1}) = \emptyset$, so u_{n-1} is adjacent to u_{n-3} and u_{n-2}, giving $T_A(n)$ in H.

Now, suppose that $H[U]$ contains no edge. Then $U_1 = U \cup \{u_0\}$ is an independent set in H. Furthermore, $N_H(u) = \{u_1, \ldots, u_{n-4}\}$ for every $u \in U$, as every vertex has degree $n-4$. Therefore, $\overline{H}[U_1]$ is a K_4 component in $\overline{H}$. Repeating the above proof for each vertex u of H shows that either u is contained in a K_4 component of $\overline{H}$, or H contains both $S_n[4]$ or $T_A(n)$. In other words, either H contains both $S_n[4]$ and $T_A(n)$, or $\overline{H}$ is the disjoint union of $\frac{n}{4}$ copies of K_4, and so $n \equiv 0 \pmod 4$. ◻

Theorem 6.4 *If $n \geq 8$, then*

$$R(S_n[4], W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 6.1 provides the lower bounds, so it remains to prove the upper bounds. Now let G be a graph that does not contain $S_n[4]$ and assume that $\overline{G}$ does not contain W_8.

First, suppose that G has order $2n$ if $n \equiv 0 \pmod 4$ and G has order $2n-1$ if n is odd. By Theorem 1.4, G has a subgraph $T = S_n(3)$. Let $V(T) = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}\} \cup \{v_1w_1, v_1w_2\}$. Set $U = V(G) - V(T)$ and $V = \{v_2, \ldots, v_{n-3}\}$. Then $|U| = n - j$, for $j = 0$ if $n \equiv 0 \pmod 4$ and $j = 1$ if n is odd, and $|V| = n-4$. Since G does not contain $S_n[4]$, v_1 is not adjacent to any vertex of V in G, and each vertex of V is adjacent to at most $n-6$ vertices of $U \cup V$ in G. Noting also that w_1 and w_2 each is adjacent to at most one vertex of $\{w_1, w_2\} \cup U$ in G, we consider two cases.

Case 1: At least one of w_1 and w_2 is not an isolated vertex in $G[\{w_1, w_2\} \cup U]$.

Without loss of generality, assume that w_1 is adjacent to some vertex $u \in \{w_2\} \cup U$ in G. Let $Z = \big(V \cup U \cup \{w_2\}\big) \backslash \{u\}$ and note that $|Z| = 2n-4-j$. Since $S_n[4] \nsubseteq G$, w_1 is not adjacent to any vertex of Z in G. If $\delta(\overline{G}[Z]) \geq \lceil \frac{2n-4-j}{2} \rceil$, then $\overline{G}[Z]$ contains C_8 by Lemma 4.1 which with w_1, forms W_8 in $\overline{G}$, a contradiction. Therefore, $\delta(\overline{G}[Z]) \leq \lceil \frac{2n-4-j}{2} \rceil - 1$ and $\Delta(G[Z]) \geq \lfloor \frac{2n-4-j}{2} \rfloor = n-2-j$. Since each v of V is adjacent to at most $n-6$ vertices of $U \cup V$ in G, and w_2 is adjacent to at most one vertex of U in G, a vertex with maximum degree in $G[Z]$ must be a vertex of $U \backslash \{u\}$. So let u_2 be a vertex of U with $d_{G[Z]}(u_2) \geq n-2$. As $S_n[4] \nsubseteq G$, observe that $N_{G[Z]}(u_2) \subseteq U$; each vertex of V is adjacent to at most one vertex of $N_{G[Z]}(u_2)$ in G; and each vertex of $N_{G[Z]}(u_2)$ is adjacent to at most one vertex of V in G. Then by Lemma 4.5, any four vertices from V and any four vertices from $N_{G[Z]}(u_2)$ form C_8 in $\overline{G}$ which with w_1 forms W_8 in $\overline{G}$, a contradiction.

Case 2: w_1 and w_2 are isolated vertices in $G[\{w_1, w_2\} \cup U]$.

If $\delta(\overline{G}[U]) \geq \frac{n-j}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 4.1 which with w_1 forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) \leq \frac{n-j}{2} - 1$, and $\Delta(G[U]) \geq \frac{n-j}{2}$. Let u_1 be a vertex of U with $d_{G[U]} \geq \frac{n-j}{2}$. Since $S_n[4] \nsubseteq G$, v_0 is not adjacent to any vertex of $N_{G[U]}(u_1)$ in G. Now, if v_1 is adjacent to some vertex u of $N_{G[U]}(u_1)$ in G, then apply Case 1 with w_1 and u interchanged. So assume that v_1 is not adjacent to any vertex of $N_{G[U]}(u_1)$ in G.

If $E(V, N_{G[U]}(u_1)) = \emptyset$ in G, then any four vertices of V and any four vertices of $N_{G[U]}(u_1)$ form C_8 in $\overline{G}$, and with v_1, form W_8 in $\overline{G}$, a contradiction. So without loss of generality, assume that v_2 is adjacent to some vertex u_2 of $N_{G[U]}(u_1)$ in G. Since $S_n[4] \nsubseteq G$, u_2 is not adjacent to any vertex of $U \backslash \{u_1\}$. Then v_0, v_1, w_1, w_2 and any four vertices from $U \backslash \{u_1, u_2\}$, at least three of which are from $N_{G[U]}(u_1) \backslash \{u_2\}$, form C_8 in $\overline{G}$ and, with u_2, form W_8 in $\overline{G}$, a contradiction.

In either case, $R(S_n[4], W_8) \leq 2n$ for $n \equiv 0 \pmod 4$ and $R(S_n[4], W_8) \leq 2n-1$ for odd n.

Next, suppose that $n \equiv 2 \pmod 4$ and G has order $2n-1$. If G contains a subgraph $S_n(3)$, then the previous arguments show that $R(S_n[4], W_8) \leq 2n-1$. Hence, we only need to consider the case where G does not contain $S_n(3)$. Now, by Theorem 6.2, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Let $U = V(G) - V(T)$; then $|U| = n-1$. Since G does not contain $S_n(3)$ and $S_n[4]$, v_0 is not adjacent in G to w_1, w_2, w_3 or U. Now, set $U' = N_{G[U\cup\{w_1\}]}(w_1) \cup N_{G[U\cup\{w_2\}]}(w_2) \cup N_{G[U\cup\{w_3\}]}(w_3)$. Then $|U'| \leq 3$ and w_1, w_2 and w_3 are not adjacent in G to any vertex of $U \backslash U'$. By Lemma 4.4, $G[U \backslash U']$ is either $K_{n-1-|U'|}$ or $K_{n-1-|U'|} - e$. If $d_{\overline{G}[U\backslash U']}(u') \geq 2$ for some vertex u' in U', then at least two vertices of $U \backslash U'$ are not adjacent to u' in G. Let X be a set containing these two vertices and any other two vertices in $U \backslash U'$, and set $Y = \{w_1, w_2, w_3, u'\}$. Note that $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.5 which with v_0 forms W_8, a contradiction. Therefore, every vertex of U' is adjacent in G to at least $n-2-|U'|$ vertices of $U \backslash U'$. Hence, $\delta(G[U]) \geq n-5$, and since $S_n[4] \nsubseteq G$, $E_G(T, U) = \emptyset$. Now, if $\overline{G}[V(T)]$ contains S_5, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Therefore, $\delta(G[V(T)]) \geq n-4$. By Lemma 6.3, G contains $S_n[4]$, a contradiction. Thus, $R(S_n[4], W_8) \leq 2n-1$ for $n \equiv 2 \pmod 4$. ◻

Theorem 6.5 *If $n \geq 8$, then*

$$R(S_n(1,3), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 6.1 provides the lower bounds, so it remains to prove the upper bounds. Let G be any graph of order $2n$ if $n \equiv 0 \pmod 4$ and of order $2n-1$ if $n \not\equiv 0 \pmod 4$. Assume that G does not contain $S_n(1,3)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.4, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, w_1v_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$. Since $S_n(1,3) \nsubseteq G$, w_2 and w_3 are not adjacent to each other, or to any vertex in $U \cup V$. Since $C_8 \nsubseteq \overline{G}[U \cup V]$ as $W_8 \nsubseteq \overline{G}$, Lemma 4.1 implies that $G[U \cup V]$ has a vertex u of degree at least $n-3$ in $G[U \cup V]$. Since $S_n(1,3) \nsubseteq G$, $u \in U$ and u is not adjacent to any vertices in V. Furthermore, $E(V, N_{G[U]}(u)) = \emptyset$. Finally, note that w_3, any 3 vertices in V and any 4 vertices in $N_{G[U]}(u)$ form C_8 in $\overline{G}$ which, with w_2 as hub, form W_8, a contradiction. ◻

Theorem 6.6 *If* $n \geq 8$*, then*

$$R(T_A(n), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 6.1 provides the lower bounds, so it remains to prove the upper bounds. Let G be any graph of order $2n$ if $n \equiv 0 \pmod 4$ and of order $2n-1$ if $n \not\equiv 0 \pmod 4$. Assume that G does not contain $T_A(n)$ and that $\overline{G}$ does not contain W_8.

Suppose first that G has a subgraph $T = S_n(3)$. Let $V(T) = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1w_1, v_1w_2\}$. Set $V = \{v_2, \ldots, v_{n-3}\}$ and $U = V(G) - V(T)$. Since G does not contain $T_A(n)$, w_1 and w_2 are not adjacent to any vertex of $U \cup V$ in G. Let V' be the set of any $n-5$ vertices in V, and U' be the set of any $n-1$ vertices in U. If $\delta(\overline{G}[U' \cup V']) \geq n-3$, then $\overline{G}[U' \cup V']$ contains C_8 by Lemma 4.1 which, with w_1 as hub, form W_8, a contradiction. Therefore, $\delta(\overline{G}[U' \cup V']) \leq n-4$ and $\Delta(G[U' \cup V']) \geq n-3$. Since $T_A(n) \nsubseteq G$, $d_{G[U' \cup V']}(v) \leq n-6$ for each $v \in V'$. Hence, some vertex $u \in U'$ satisfies $d_{G[U' \cup V']}(u) \geq n-3$, which also implies that u is adjacent to at least two vertices of U.

Since $T_A(n) \nsubseteq G$, each vertex of V is adjacent to at most one vertex of $N_{G[U]}(u)$. If $|N_{G[U]}(u)| \geq n-4$, then each vertex of $N_{G[U]}(u)$ is adjacent to at most one vertex of V, and so $\overline{G}[V \cup N_{G[U]}(u)]$ contains C_8 by Lemma 4.1 which with w_1 forms W_8, a contradiction. Thus, at least three vertices of V' (and so of V), say v_2, v_3, v_4, are adjacent to u in G. Let a and b be any two vertices in $N_{G[U]}(u)$. As $T_A(n) \nsubseteq G$, each of v_2, v_3, v_4 is not adjacent to any vertex of $V(G) \backslash \{u, v_0\}$. Then $w_1v_5w_2v_3av_1bv_4w_1$ and v_2 form W_8 in $\overline{G}$, a contradiction.

By Theorem 1.4, $R(S_n(3), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$. So now assume that G has order $2n-1$ with $n \not\equiv 0 \pmod 4$ and that G does not contain $S_n(3)$. By Theorem 6.2, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Then $U = V(G) - V(T)$ and $|U| = n-1$. Since $T_A(n) \nsubseteq G$, w_1, w_2, w_3 are not adjacent to each other in G or to any vertex of U. Since $S_3(n) \nsubseteq G$, v_0 is not adjacent to any vertex of $U \cup \{w_1, w_2, w_3\}$. By Lemma 4.4, $G[U]$ is K_{n-1} or $K_{n-1} - e$. Since $T_A(n) \nsubseteq G$, each vertex of T is not adjacent to any vertex of U in G, and so $\delta(G[V(T)]) \geq n-4$ by Observation 4.3, which in turn implies that $G[V(T)]$ contains $T_A(n)$ by Lemma 6.3, a contradiction.

This completes the proof of the theorem. □

Theorem 6.7 *If* $n \geq 8$*, then*

$$R(T_B(n), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 6.1 provides the lower bounds, so it remains to prove the upper bounds. Let G be a graph with no $T_B(n)$ subgraph whose complement $\overline{G}$ does not contain W_8.

Suppose that $n \equiv 0 \pmod 4$ and that G has order $2n$. By Theorem 6.4, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) -$

$V(T)$; then $|V| = n-5$ and $|U| = n$. Since $T_B(n) \nsubseteq G$, $E_G(U, V) = \emptyset$ and neither w_2 nor w_3 is adjacent in G to V. Suppose that $n \geq 12$. If w_2 is non-adjacent to some 4 vertices from U, then these 4 vertices and any 4 vertices from V form C_8 in $\overline{G}$ that with w_2 forms W_8, a contradiction. Otherwise, w_2 must be adjacent to at least $n-3$ vertices of U in G. Since $T_B(n) \nsubseteq G$, w_3 must not be adjacent to these $n-3$ vertices; then any 4 vertices from these $n-3$ vertices and 4 vertices from V form C_8 in $\overline{G}$ and with w_3 forms W_8, again a contradiction. For $n = 8$, $|V| = 3$ and $|U| = 8$. If w_2 is not adjacent to any vertex of U in G, then by Lemma 4.4, $G[U]$ is K_8 or $K_8 - e$ which contains $T_B(8)$, a contradiction. Otherwise, suppose that w_2 is adjacent to $u \in U$. Since $T_B(8) \nsubseteq G$, w_1 must not be adjacent to $(U \cup V)\backslash\{u\}$ in G. Now, if w_3 is not adjacent to v_0 in G, then by Observation 4.3, $\overline{G}$ contains W_8, a contradiction. Otherwise, u is not adjacent to $V \cup \{w_3\}$, and again by Observation 4.3, $\overline{G}$ contains W_8, another contradiction. Thus, $R(T_B(n), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$.

Next, suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 6.4, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_B(n) \nsubseteq G$, $E_G(U, V) = \emptyset$ and neither w_2 nor w_3 is adjacent in G to V. For $n \geq 9$, if w_2 is non-adjacent to some 4 vertices from U, then these 4 vertices and any 4 vertices from V form C_8 in $\overline{G}$ and with w_2 form W_8, a contradiction. Otherwise, w_2 is adjacent to at least $n-4$ vertices of U in G. Since $T_B(n) \nsubseteq G$, w_3 is not adjacent to these $n-4$ vertices, so any 4 vertices from these $n-4$ vertices and 4 vertices from V form C_8 in $\overline{G}$ which with w_3 form W_8, again a contradiction. Therefore, $R(T_B(n), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$.

This completes the proof. □

Theorem 6.8 *For* $n \geq 8$, $R(T_C(n), W_8) = 2n-1$.

Proof Lemma 6.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$ and assume that G does not contain $T_C(n)$ and that $\overline{G}$ does not contain W_8.

Suppose first that there is a subset $X \subseteq V(G)$ of size n with $\delta(G[X]) \geq n-4$. If $\delta(G[X]) = n-4$, then let $x \in X$ be such that $d_{G[X]}(x) = n-4$, and set $Y = X\backslash(\{x\} \cup N_{G[X]}(x))$ where $|Y| = 3$. Noting that $3(n-6) > n-4$ for $n \geq 8$, there must be two vertices of Y that are adjacent to a common vertex of $N_{G[X]}(x)$ in G, say to $x' \in N_{G[X]}(x)$. Then the remaining vertex of Y is not adjacent to any vertex of $N_{G[X]}(x)\backslash\{x'\}$, as $T_C(n) \nsubseteq G$, contradicting $\delta(G[X]) \geq n-4$. So $\delta(G[X]) \geq n-3$. Pick any vertex $x \in X$ and any subset $X' \subseteq N_{G[X]}(x)$ of size $n-3$. Set $Y = X\backslash(\{x\} \cup X')$ where $|Y| = 2$. As $2(n-5) > n-3$ for $n \geq 8$, the two vertices of Y must be adjacent to a common vertex of X' in G, say x'. Then $G[X'\backslash\{x'\}]$ is an empty graph as $T_C(n) \nsubseteq G$, contradicting $\delta(G[X]) \geq n-3$.

Now assume that $\delta(G[X]) \leq n-5$ whenever $X \subseteq V(G)$ is of size n. By Theorem 1.4, G has a subgraph $T = S_{n-1}(3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2\}$. Set $V = \{v_2, v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n$. Since $T_C(n) \nsubseteq G$, $E_G(U, V) = \emptyset$.

For the case $n = 8$ such that v_1 is not adjacent to any vertex of U in G, or the case $n \geq 9$, there are four vertices of $V(T)$ that are not adjacent to any vertex of U in G.

Since $\delta(G[U]) \leq n-5$, $\overline{G}[U]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

For the final case $n = 8$ with v_1 adjacent to some vertex u of U in G, observe that since $T_C(8) \nsubseteq G$, the vertex u is not adjacent to any vertex of $\{v_2, v_3, v_4\} \cup U$. By Lemma 4.4, $G[U\backslash\{u\}]$ is K_7 or $K_7 - e$, which implies that no vertex of $V(T) \cup \{u\}$ is adjacent to any vertex of $U\backslash\{u\}$ in G, as $T_C(8) \nsubseteq G$. Since $\delta(G[V(T) \cup \{u\}]) \leq n-5$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

This completes the proof of the theorem. □

Theorem 6.9 *For $n \geq 8$, $R(S_n(3, 1), W_8) = 2n - 1$.*

Proof Lemma 6.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n - 1$. Assume that G does not contain $S_n(3, 1)$ and that $\overline{G}$ does not contain W_8.

Suppose first that there is a subset $X \subseteq V(G)$ of size n with $\delta(G[X]) \geq n - 4$. Let x_0 be any vertex of X, and pick a subset $X' \subseteq N_{G[X]}(x_0)$ of size $n - 4$. Set $Y = X\backslash(\{x_0\} \cup X')$, and so $|Y| = 3$. Since $\delta(G[X]) \geq n - 4$, each vertex of Y is adjacent to at least $n - 7$ vertices of X' in G. For $n \geq 10$, it is straightforward to see that there is a matching from Y to X' in G; hence, G contains $S_n(3, 1)$, a contradiction. For $n = 9$, if $d_{G[X]}(x_0) = n - 4 = 5$, then we can similarly deduce the contradiction that G contains $S_9(3, 1)$, since in this case, each vertex of Y is adjacent to at least $n - 6 = 3$ vertices of X' in G. As x_0 was arbitrary, we may assume for the case when $n = 9$ that $\delta(G[X]) \geq n - 3 = 6$, which again leads to the contradiction that G contains $S_9(3, 1)$.

Now for $n = 8$, suppose $d_{G[X]}(x_0) = 4$. Let $X' = \{x_1, x_2, x_3, x_4\}$ and $Y = \{x_5, x_6, x_7\}$. Since $\delta(G[X]) \geq n - 4$ and $S_8(3, 1) \nsubseteq G$, $G[Y]$ is K_3; all three vertices of Y are adjacent to exactly two common vertices of X' in G, say to x_1 and x_2; and neither x_3 nor x_4 are adjacent to any vertex of Y in G. By the minimum degree condition, x_3 and x_4 are then adjacent in G, and each is also adjacent to both x_1 and x_2. This implies that G contains $S_8(3, 1)$, with x_1 being the vertex with degree four, a contradiction. As x_0 was arbitrary, assume for the case when $n = 8$ that $\delta(G[X]) \geq 5$, which again leads to the contradiction that G contains $S_8(3, 1)$.

Now assume that $\delta(G[X]) \leq n - 5$ whenever $X \subseteq V(G)$ is of size n. Recall that G has order $2n - 1$, and so by Theorem 1.4, G has a subgraph $T = S_{n-1}(2, 1)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2\}$. Set $V = \{v_3, v_4, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n - 6$ and $|U| = n$. Since $S_n(3, 1) \nsubseteq G$, $E_G(U, V) = \emptyset$. Now as $\delta(G[U]) \leq n - 5$, $\overline{G}[U]$ contains S_5, and so for $n \geq 10$, $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

For $n = 9$, Theorem 1.4 shows that G has a subgraph $T = S_9(2, 1)$, so without loss of generality assume that v_0 is adjacent to some vertex u in U. Since $S_9(3, 1) \nsubseteq G$, $G[V \cup \{u\}]$ is an empty graph and u is not adjacent to any vertex of U in G. By Lemma 4.4, $G[U\backslash\{u\}]$ is K_8 or $K_8 - e$, which implies that no vertex of $V(T) \cup \{u\}$ is adjacent to any vertex of $U\backslash\{u\}$ in G, as $S_9(3, 1) \nsubseteq G$. Since $\delta(G[V(T) \cup \{u\}]) \leq n-5$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Finally for $n = 8$, recall that G has order 15, and so G has a subgraph $T' = S_7$ by Theorem 1.1. Let $V(T') = \{v'_0, \ldots, v'_6\}$ and $E(T') = \{v'_0v'_1, \ldots, v'_0v'_6\}$. Set $V' =$

$\{v_1', \dots, v_6'\}$ and $U' = V(G) - V(T')$, then $|U'| = 8$. Suppose that v_2' and v_3' are adjacent to a common vertex u of U' in G, while v_1' is adjacent to another vertex $u' \neq u$ of U' in G. Then as $S_8(3, 1) \nsubseteq G$, no vertex of $\{v_4', v_5', v_6'\} \cup (U' \setminus \{u, u'\})$ is adjacent to any vertex of $V' \setminus \{v_1'\}$ in G. Now $G[V' \setminus \{v_1'\}]$ contains S_5 and $|U' \setminus \{u, u'\}| = 6$, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Similar arguments lead to the same contradiction when the roles of v_1', v_2', and v_3' are replaced by any three vertices of V'. So assume that it is not the case that two vertices of V' are adjacent to a common vertex of U' in G while a third vertex of V' is adjacent to another vertex of U' in G.

For $1 \leq i \leq 6$, let $d_i = |E_G(\{v_i'\}, U)|$ be the number of vertices of U' that are adjacent to v_i'. Without loss of generality, assume that $d_1 \geq d_2 \geq \cdots \geq d_6$. Recalling that $\delta(G[U']) \leq 3$ and so $S_5 \subseteq \overline{G}[U']$, Observation 4.3 implies that $d_3 \geq 1$. If $d_1 \geq 3$ and $d_2 \geq 2$, then it is trivial that G contains $S_8(3, 1)$, a contradiction. By our assumption on the adjacencies of vertices in V' to vertices of U' in G, it is clear that when (d_1, d_2, d_3) is of the form $(\geq 3, 1, 1)$, $(2, 2, 2)$, or $(2, 2, 1)$, there is a matching from $\{v_1', v_2', v_3'\}$ to U' in G, as v_2' and v_3' are adjacent to different vertices of U' in G. This implies that G contains $S_8(3, 1)$, a contradiction. If $(d_1, d_2, d_3) = (2, 1, 1)$, then, similarly, v_2' and v_3' are adjacent to different vertices of U' in G, say to u and u', respectively, which in turn implies that v_1' is adjacent to two vertices in $U' \setminus \{u, u'\}$. So G contains $S_8(3, 1)$, again a contradiction.

For the final case when $d_1 = d_2 = d_3 = 1$, our assumption implies that v_1', v_2' and v_3' must be adjacent to a common vertex u of U' in G to avoid a matching from $\{v_1', v_2', v_3'\}$ to U' in G. Furthermore, no vertex of $\{v_4', v_5', v_6'\}$ is adjacent to any vertex of $U' \setminus \{u\}$ in G. Now if $S_5 \subseteq \overline{G}[V']$, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. So $\delta(G[V']) \geq 2$, and in particular, v_4' is adjacent to some vertex of V' in G. Without loss of generality, v_4 is adjacent to either v_1 or v_5 in G. Since $S_8(3, 1) \nsubseteq G$, $\overline{G}[\{v_5', v_2', v_3', v_6'\}]$ contains S_4 if v_4' is adjacent to v_1' in G, while $\overline{G}[\{v_6', v_1', v_2', v_3'\}]$ contains S_4 if v_4' is adjacent to v_5' in G. By Lemma 4.4, $G[U' \setminus \{u\}]$ is K_7 or $K_7 - e$, which implies that no vertex of $V(T') \cup \{u\}$ is adjacent to any vertex of $U' \setminus \{u\}$ in G, as $S_8(3, 1) \nsubseteq G$. Since $\delta(G[V(T') \cup \{u\}]) \leq 3$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Thus, $R(S_n(3, 1), W_8) \leq 2n - 1$ for $n \geq 8$ which completes the proof. □

7 Proof of Theorem 3.1

Lemma 7.1 *Let $n \geq 8$. If the tree graph T_n exists, then $R(T_n, W_8) \geq 2n - 1$ for each*

$$T_n \in \{S_n(1, 4), S_n(5), S_n[5], S_n(2, 2), S_n(4, 1), T_D(n), \dots, T_S(n)\}.$$

Also, $R(T_n, W_8) \geq 2n$ if $n \equiv 0 \pmod 4$ and $T_n \in \{S_n(1, 4), S_n(2, 2), T_D(n), T_N(n)\}$ or if $T_n \in \{T_E(8), T_F(8)\}$.

Proof The graph $G = 2K_{n-1}$ clearly does not contain any tree graph of order n, and $\overline{G}$ does not contain W_8. Furthermore, if $n \equiv 0 \pmod 4$, then the graph $G = K_{n-1} \cup K_{4,\dots,4}$ of order $2n - 1$ does not contain $S_n(1, 4)$, $T_D(n)$ or $S_n(2, 2)$; nor does the complement $\overline{G}$ contain W_8. Finally, the graph $G = K_7 \cup K_{4,4}$ does not contain $T_E(8)$ or $T_F(8)$ and $\overline{G}$ does not contain W_8. □

Theorem 7.2 *If $n \geq 8$, then*

$$R(S_n(1,4), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $S_n(1,4)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that G has order $2n$ if $n \equiv 0 \pmod 4$ and that G has order $2n-1$ if $n \not\equiv 0 \pmod 4$. By Theorem 6.5, G has a subgraph $T = S_n(1,3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_2w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = j$ where $j = n$ if $n \equiv 0 \pmod 4$ and $j = n-1$ if $n \not\equiv 0 \pmod 4$. Since $S_n(1,4) \nsubseteq G$, w_3 is not adjacent in G to any vertex of $U \cup V$ and $d_{G[U\cup V]}(v_i) \leq n-7$ for each $v_i \in V$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{n-5+j}{2} \rceil \geq \frac{n-5+j}{2}$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 4.1 and thus W_8 with w_3 as hub, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{n-5+j}{2} \rceil - 1$ and $\Delta(G[U \cup V]) \geq n-5+j-\lceil \frac{n-5+j}{2} \rceil = \lfloor \frac{n-5+j}{2} \rfloor \geq n-3$. Since $d_{G[U\cup V]}(v_i) \leq n-7$ for each $v_i \in V$, $d_{G[U\cup V]}(u) \geq n-3$ for some vertex $u \in U$. Since $S_n(1,4) \nsubseteq G$, no vertex of V is adjacent to $\{u\} \cup N_{G[U\cup V]}(u)$ in G.

For $n \geq 9$, any 4 vertices from V and any 4 vertices from $\{u\} \cup N_{G[U\cup V]}(u)$ form C_8 in $\overline{G}$ and, with w_3 as hub, form W_8, a contradiction. Suppose that $n = 8$; then $V = \{v_2, v_3, v_4\}$. Let $\{u_1, \ldots, u_4\}$ be 4 vertices in $N_{G[U\cup V]}(u)$. Since $S_8(1,4) \nsubseteq G$, w_1 is not adjacent to $N_{G[U\cup V]}(u)$. If w_1 is not adjacent to w_3, then $w_1u_1v_2u_2v_3u_3v_4u_4w_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. Therefore, w_1 is adjacent to w_3 in G. Then w_2 is not adjacent to any vertex of $U \cup V$ in G. Since $d_{G[V]}(v_i) \leq 1$ for $i = 2, 3, 4$, one of the vertices of V, say v_2, is not adjacent to the other two vertices of V. Then $u_1w_2u_2w_3u_3v_3u_4v_4u_1$ and v_2 form W_8 in $\overline{G}$, a contradiction. Thus, $R(S_n(1,4), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$ and $R(S_n(1,4), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$.

This completes the proof. □

Theorem 7.3 *If $n \geq 10$, then $R(S_n(5), W_8) = 2n-1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n(5)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.2, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Set $V = \{v_2, v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $S_n(5) \nsubseteq G$, v_1 is not adjacent to any vertex of $U \cup V$ in G. Furthermore, for each v_i in V, v_i is adjacent to at most three vertices of U in G.

For $n \geq 10$, $|V| \geq 5 > 4$ and $|U| \geq 9 > 8$. By Corollary 4.8, $\overline{G}[U \cup V]$ contains C_8 which together with v_1 gives W_8 in $\overline{G}$, a contradiction. Thus, $R(S_n(5), W_8) \leq 2n-1$ which completes the proof. □

Theorem 7.4 *If $n \geq 9$, then $R(S_n[5], W_8) = 2n-1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n[5]$ and that $\overline{G}$ does not contain W_8. By Theorem 7.3, G has a subgraph $T = S_n(5)$. Let $V(T) = \{v_0, \dots, v_{n-5}, w_1, \dots, w_4\}$ and $E(T) = \{v_0v_1, \dots, v_0v_{n-5}, v_1w_1, \dots, v_1w_4\}$. Set $V = \{v_2, \dots, v_{n-5}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n-1$. Since $S_n[5] \nsubseteq G$, v_0 is not adjacent to $w_1, \dots, w_4$ in G and $w_1, \dots, w_4$ are each adjacent to at most two vertices of U in G. Now, suppose that v_0 is non-adjacent to at least six vertices of U in G. By Corollary 4.7, six of these vertices together with $w_1, \dots, w_4$ contain C_8 in $\overline{G}$ which with v_0 gives W_8 in $\overline{G}$, a contradiction. Then, suppose that v_0 is adjacent to at least $n-6$ vertices of U in G. Choose a set U' of $n-6$ of these vertices. Since $S_n[5] \nsubseteq G$, v_1 is not adjacent to any vertex of $V \cup U'$ in G. If $\delta(\overline{G}[V \cup U']) \geq n-6$, then by Lemma 4.1, $\overline{G}[V \cup U']$ contains C_8 which with v_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, $\delta(\overline{G}[V \cup U']) \leq n-7$ and $\Delta(G[V \cup U']) \geq n-6$. However, this gives $S_n[5]$ in G with u and v_1 as the center of S_{n-5} and S_5, respectively, where u is a vertex in $V \cup U'$ with $d_{G[V \cup U']}(u) \geq n-6$, a contradiction. Thus, $R(S_n[5], W_8) \leq 2n-1$ which completes the proof. □

Theorem 7.5 *If* $n \geq 8$, *then*

$$R(S_n(2,2), W_8) = \begin{cases} 2n-1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Assume that G is a graph with no $S_n(2,2)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n \equiv 0 \pmod 4$ and that G has order $2n$. By Theorem 6.7, G has a subgraph $T = T_B(n)$. Let $V(T) = \{v_0, \dots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \dots, v_0v_{n-4}, v_1w_1, w_1w_2, v_2w_3\}$. Set $V = \{v_3, \dots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n$. Since $S_n(2,2) \nsubseteq G$, w_3 is not adjacent in G to $U \cup V$ and v_2 is not adjacent to V. If $\delta(\overline{G}[U \cup V]) \geq \frac{2n-6}{2} = n-3$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 4.1 which with w_2 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq n-4$, and $\Delta(G[U \cup V]) \geq n-3$. Now, there are two cases to be considered.

Case 1a: One of the vertices of V, say v_3, is a vertex of degree at least $n-3$ in $G[U \cup V]$.

Note that in this case, there are at least 4 vertices from U, say $u_1, \dots, u_4$, that are adjacent to v_3 in G. Since $S_n(2,2) \nsubseteq G$, these 4 vertices are independent and are not adjacent to any other vertices of U. Since $n \geq 8$, U contains at least 4 other vertices, say $u_5, \dots, u_8$, so $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_3 forms W_8 in $\overline{G}$, a contradiction.

Case 1b: Some vertex $u \in U$ has degree at least $n-3$ in $G[U \cup V]$.

Since $S_n(2,2) \nsubseteq G$, u is not adjacent to any vertex of V in G. Therefore, u must be adjacent to at least $n-3$ vertices of U in G. Without loss of generality, suppose that $u_1, \dots, u_{n-3} \in N_{G[U]}(u)$. Note that V is not adjacent to $N_{G[U]}(u)$, or else there will be $S_n(2,2)$ in G, a contradiction. If $n \geq 12$, then any 4 vertices from $N_{G[U]}(u)$ and any 4 vertices from V form C_8 in $\overline{G}$ which, with w_3 as hub, forms W_8, a contradiction. Suppose that $n = 8$ and let the remaining two vertices be u_6

and u_7. If $|N_{G[\{u_1,\ldots,u_5,u_i\}]}(u_i)| \leq 1$ for $i = 6, 7$, then let $X = \{u_1, \ldots, u_4\}$ and $Y = \{v_3, v_4, u_6, u_7\}$. By Lemma 4.5, $\overline{G}[X \cup Y]$ contains C_8 and, with w_3 as hub, forms W_8 in $\overline{G}$, a contradiction. Therefore, one of u_6 and u_7, say u_6, is adjacent to at least two of $u_1, \ldots, u_5$, say u_1 and u_2. Since $S_8(2,2) \nsubseteq G$, u_7 is adjacent in $\overline{G}$ to at least two of u_3, u_4, u_5, say u_3 and u_4, and $v_0, \ldots, v_4, w_1$ are not adjacent in G to $u, u_1, \ldots, u_6$. Now, if w_3 is not adjacent to some vertex $a \in \{v_0, v_1, w_1\}$, then $u_1v_3u_2v_4u_3u_7u_4au_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. Hence, w_3 is adjacent to v_0, v_1 and w_1 in G. Similarly, v_2 is not adjacent to u_7 and v_2 is adjacent to v_1 and w_1. Since $S_8(2,2) \nsubseteq G$, w_2 is not adjacent to $U \cup V$, and w_1 is not adjacent to V. Then $u_1v_2u_2w_1u_3w_2u_4w_3u_1$ and v_3 forms W_8 in $\overline{G}$, a contradiction.

In either case, $R(S_n(2,2), W_8) \leq 2n$.

Suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 6.7, G has a subgraph $T = T_B(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n-1$. Since $S_n(2,2) \nsubseteq G$, w_3 is not adjacent in G to $U \cup V$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{2n-5}{2} \rceil$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 4.1 which with w_3 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{2n-5}{2} \rceil - 1 = n-3$, and $\Delta(G[U \cup V]) \geq n-3$. Again, there are two cases to be considered.

Case 2a: A vertex of V, say v_3, has degree at least $n-3$ in $G[U \cup V]$.

There must be at least 4 vertices from U, say $u_1, \ldots, u_4$ that are adjacent to v_3 in G. Since $S_n(2,2) \nsubseteq G$, $u_1, \ldots, u_4$ are independent and are not adjacent to any other vertex of U. Since $n \geq 9$, there are at least 4 other vertices of U, say $u_5, \ldots, u_8$, and $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_3 form W_8 in $\overline{G}$, a contradiction.

Case 2b: A vertex $u \in U$ has degree at least $n-3$ in $G[U \cup V]$.

Since $S_n(2,2) \nsubseteq G$, no vertex of V is adjacent to u or to $N_{G[U]}(u)$. Then u is adjacent to at least $n-3$ vertices of U in G; suppose without loss of generality that $u_1, \ldots, u_{n-3} \subseteq N_{G[U]}(u)$. If $n \geq 10$, then any 4 vertices from $N_{G[U]}(u)$, any 4 vertices from V and w_3 form W_8 in $\overline{G}$, a contradiction. Suppose that $n = 9$ and let u_7 be the vertex in $U \backslash \{u, u_1, \ldots, u_{n-3}\}$. If u_7 is adjacent in $\overline{G}$ to at least two of $u_1, \ldots, u_6$, say u_1 and u_2, then $u_1u_7u_2v_3u_3v_4u_4v_5u_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. Therefore, u_7 is adjacent in G to at least 5 of the vertices $u_1, \ldots, u_6$, say $u_1, \ldots, u_5$. Since $S_9(2,2) \nsubseteq G$, U is not adjacent in G to $\{v_0, v_1, v_2, w_1\} \cup V$ and w_2 is not adjacent to u or u_7. If w_3 is not adjacent to some vertex $a \in \{v_0, v_1, w_1, w_2\}$, then $uv_3u_1v_4u_2v_5u_7au$ and w_3 form W_8 in $\overline{G}$, a contradiction. Hence, w_3 is adjacent to v_0, v_1, w_1 and w_2 in G. Similarly, v_2 is adjacent to v_1, w_1 and w_2. Since $S_9(2,2) \nsubseteq G$, w_2 is non-adjacent to at least one of v_3, v_4, v_5, say v_3 without loss of generality. If v_1 is also not adjacent to v_3, then $uw_2u_7v_1u_1v_2u_2w_3u$ and w_3 form W_8 in $\overline{G}$, a contradiction. Thus, v_1 is adjacent to v_3, then v_3 is not adjacent to both v_4 and v_5, or else G contains $S_9(2,2)$. Without loss of generality, assume that v_3 is not adjacent to v_4 in G. Then $uw_2u_7v_4u_1v_2u_2w_3u$ and w_3 form W_8 in $\overline{G}$, a contradiction.

In either case, $R(S_n(2,2), W_8) \leq 2n-1$ for $n \not\equiv 0 \pmod 4$, which completes the proof. □

Theorem 7.6 *If $n \geq 9$, then $R(S_n(4,1), W_8) = 2n-1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $S_n(4,1)$ and that $\overline{G}$ does not contain W_8.

Suppose first that there is a subset $X \subseteq V(G)$ of size n with $\delta(G[X]) \geq n-4$. Let x_0 be any vertex of X, and pick a subset $X' \subseteq N_{G[X]}(x_0)$ of size $n-5$. Set $Y = X \backslash (\{x_0\} \cup X')$, and so $|Y| = 4$. Since $\delta(G[X]) \geq n-4$, each vertex of Y is adjacent to at least $n-8$ vertices of X' in G and each vertex of X' is adjacent to at least one vertex of Y in G. Hence, for $n \geq 11$, it is straightforward to see that there is a matching from Y to X' in G; hence, G contains $S_n(4,1)$, a contradiction.

For $n = 10$ and $\delta(G[X]) \geq n-4 = 6$, let $X = \{x_0, \ldots, x_9\}$ and $\{x_1, \ldots, x_6\} \subseteq N_{G[X]}(x_0)$. Since $\delta(G[X]) \geq 6$, vertices x_7, x_8 and x_9 must each be adjacent to at least 3 vertices of $x_1, \ldots, x_6$. It is straightforward to see that there is a matching from $\{x_7, x_8, x_9\}$ to $\{x_1, \ldots, x_6\}$ in G; without loss of generality, assume that x_i is adjacent to x_{i+6} in G for $i = 1, 2, 3$. Now, if there is any edge in $G[\{x_4, x_5, x_6\}]$, then $S_{10}(4,1) \subseteq G$, a contradiction. Otherwise, $G[\{x_4, x_5, x_6\}]$ must be independent and each of x_4, x_5, x_6 must be adjacent to at least two vertices of x_7, x_8, x_9 in G. Without loss of generality, assume that x_4 is adjacent to x_7 and x_8 in G. Since $S_{10}(4,1) \nsubseteq G$, x_5 cannot be adjacent to x_1 and x_2 in G, but this is impossible since $\delta(G[X]) \geq 6$.

Now for $n = 9$, suppose that $d_{G[X]}(x_0) = n-4 = 5$. Let $N_{G[X]}(x_0) = \{x_1, \ldots, x_5\}$ and $Y = \{x_6, x_7, x_8\}$. Then, three vertices of Y are each adjacent to at least $n-6 = 3$ vertices of $N_{G[X]}(x_0)$ in G. Without loss of generality, assume that x_1 is adjacent to x_6, x_2 is adjacent to x_7 and x_3 is adjacent to x_8, respectively. Now, if x_4 is adjacent to x_5, then G contains $S_9(4,1)$, a contradiction. Otherwise, x_4 and x_5 must each be adjacent to at least one of x_6, x_7 and x_8. Assume that x_4 is adjacent to x_6. Then x_5 is not adjacent to x_1 and x_4 in G, or else G contains $S_9(4,1)$. If x_5 is adjacent to x_6, then x_1, x_4, x_5 must be independent in G, and they are each adjacent to x_7 or x_8 in G; assume that x_1 is adjacent to x_7. Then, x_4 and x_5 are not adjacent to x_2 in G, and since $\delta(G[X]) \geq 5$, they are adjacent to x_7 and x_8 in G, and G contains $S_9(4,1)$, a contradiction. If x_5 is not adjacent to x_6, then since $d_{G[X]}(v_0) \geq 5$, x_5 is adjacent to x_2, x_3, x_7 and x_8 in G. Then, x_4 is not adjacent to x_2 and x_3 in G, and x_4 is adjacent to x_1, x_6, x_7 and x_8 in G, and this gives us $S_9(4,1)$ in G, a contradiction. As x_0 was arbitrary, assume for the case when $n = 9$ that $\delta(G[X]) \geq n-3 = 6$, which again leads to the contradiction that G contains $S_9(4,1)$.

Now assume that $\delta(G[X]) \leq n-5$ whenever $X \subseteq V(G)$ is of size n. Recall that G has order $2n-1$, and so by Theorem 6.9, G has a subgraph $S_n(3,1)$ and thus a subgraph $T = S_{n-1}(3,1)$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, v_2w_2, v_3w_3\}$. Set $V = \{v_4, \ldots, v_{n-5}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_n\}$; then $|V| = n-8$ and $|U| = n$. Since $S_n(4,1) \nsubseteq G$, V is not adjacent to any vertex of U in G. Now as $\delta(G[U]) \leq n-5$, $\overline{G}[U]$ contains S_5, and so for $n \geq 12$, $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Suppose that $n = 11$. If v_0 is not adjacent to any vertex of U in G, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Assume that v_0 is adjacent to some vertex $u \in U$. Since $S_{11}(4,1) \nsubseteq G$, $G[V \cup \{u\}]$ is an empty graph and u is not adjacent to any vertex of U in G. By Lemma 4.4, $G[U \backslash \{u\}]$ is K_{10} or $K_{10}-e$, so no vertex of $V(T) \cup \{u\}$

is adjacent to any vertex of $U\backslash\{u\}$ in G, as $S_{11}(4, 1) \nsubseteq G$. Since $\delta(G[V(T) \cup \{u\}]) \leq n-5$, $\overline{G}[V(T) \cup \{u\}]$ contains S_5, so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Now, suppose that $n = 10$. Then G has order 19, and by Theorem 2.2, G has a subgraph $T' = S_{10}(3, 1)$. Let $V(T') = \{v_0', \dots, v_6', w_1', w_2', w_3'\}$ and $E(T') = \{v_0'v_1', \dots, v_0'v_6', v_1'w_1', v_2'w_2', v_3'w_3'\}$. Set $V' = \{v_4', v_5', v_6'\}$ and $U' = V(G) - V(T') = \{u_1', \dots, u_9'\}$. Since $S_{10}(4, 1) \nsubseteq G$, V' must be independent in G and is not adjacent to any vertex of U' in G. If v_0' is adjacent to some vertices in U' in G, say u_1'. Since $S_{10}(4, 1) \nsubseteq G$, u_1' is not adjacent to any vertex of V' or $U'\backslash\{u_1'\}$ in G. Then, by Lemma 4.4, $G[U'\backslash\{u_1'\}]$ is K_8 or $K_8 - e$, so no vertex of $V(T')$ is adjacent to any vertex of $U'\backslash\{u_1'\}$ in G, as $S_{10}(4, 1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq 5$, $\overline{G}[V(T')]$ contains S_5, so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Now, suppose that v_0' is not adjacent to any vertex of U' in G. Note that $|U' \cup \{w_1'\}| = n$; therefore, $\delta(G[U' \cup \{w_1'\}]) \leq 5$, and so $\overline{G}[U' \cup \{w_1'\}]$ contains S_5. If w_1' is not adjacent to any vertex from $V' \cup \{v_0'\}$, then by Observation 4.3, $\overline{G}$ contains W_8, a contradiction. Otherwise, there are two cases to be considered.

Case 1a: w_1' is adjacent to some vertices of V' in G.

Without loss of generality, assume that w_1' is adjacent to v_4' in G. In this case, v_1' is not adjacent to $U' \cup \{v_5', v_6'\}$. Then by Lemma 4.4, $G[U']$ is K_9 or $K_9 - e$, so no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_{10}(4, 1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq 5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Case 1b: w_1' is non-adjacent to each vertex of V' in G.

In this case, w_1' is adjacent to v_0' in G. Note that w_1' is not adjacent to U', since this would revert to the case where v_0' is adjacent to some vertex of U'. Then again by Lemma 4.4, $G[U']$ is K_9 or $K_9 - e$, so no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_{10}(4, 1) \nsubseteq G$. Since $\delta(G[V(T')]) \leq 5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Finally, suppose that $n = 9$. Then G has order 17, and so G has a subgraph $T' = S_9(2, 1)$ by Theorem 1.4. Let $V(T') = \{v_0', \dots, v_6', w_1', w_2'\}$ and $E(T') = \{v_0'v_1', \dots, v_0'v_6', v_1'w_1', v_2'w_2'\}$. Set $V' = \{v_3', \dots, v_6'\}$ and $U' = V(G) - V(T') = \{u_1', \dots, u_8'\}$.

Now, suppose that $E_G(V', U') \neq \emptyset$. Without loss of generality, assume that v_3' is adjacent to u_1' in G. Since $S_9(4, 1) \nsubseteq G$, v_4', v_5', v_6' are independent and not adjacent to any vertex of $U'\backslash\{u_1'\}$ in G.

Suppose that v_0' is adjacent to some vertex of $U'\backslash\{u_1'\}$, say u_2'. Then u_2' is non-adjacent to $\{v_4', v_5', v_6'\} \cup U'\backslash\{u_1', u_2'\}$ in G. Since $\delta(G[\{w_1', w_2'\} \cup U'\backslash\{u_2'\}]) \leq n-5$, $\overline{G}[\{w_1', w_2'\} \cup U'\backslash\{u_2'\}]$ contains S_5. If v_4', v_5', v_6' and u_2' are not adjacent to w_1', w_2' or u_1' in G, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Assume that v_4' is adjacent to w_1' in G. In this case, v_1' is not adjacent to $\{v_5', v_6'\} \cup U'\backslash\{u_1'\}$ in G, and $v_1'u_3'v_4'u_4'v_6'u_7'u_2'u_8'v_1'$ and v_5' form W_8 in $\overline{G}$, a contradiction. Similar contradictions occur if we assume that v_5', v_6' or u_2' are adjacent to w_1', w_2' or u_1' in G.

Thus, v_0' is not adjacent to any vertex of $U'\backslash\{u_1'\}$ in G. Since $\delta(G[\{w_1', w_2'\} \cup U'\backslash\{u_1'\}]) \leq n - 5$, $\overline{G}[\{w_1', w_2'\} \cup U'\backslash\{u_1'\}]$ contains S_5. If v_0', v_4', v_5' and v_6' are not adjacent to w_1' or w_2' in G, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. There are two cases to be considered.

Case 2a: v_0' is adjacent to w_1' or w_2' in G.

Without loss of generality, assume that v_0' is adjacent to w_1' in G. Note that v_1' and w_1' are not adjacent to $U'\backslash\{u_1'\}$, since this would revert to the case where v_0' is adjacent to some vertex of $U'\backslash\{u_1'\}$. Again, since $\delta(G[\{w_2'\} \cup U']) \le n-5$, $\overline{G}[\{w_2'\} \cup U'\}]$ contains S_5. If v_1', v_4', v_5' and v_6' are not adjacent to w_2' and u_1' in G, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Suppose that v_1' is adjacent to w_2' or u_1', say w_2', in G. If w_1' is not adjacent to v_4', v_5' or v_6', then by Lemma 4.4, $G[U'\backslash\{u_1'\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of $U'\backslash\{u_1'\}$ in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \le n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Otherwise, w_1' is adjacent to at least one of v_4', v_5', v_6' in G, say v_4'. Then, v_2' is not adjacent to $\{v_5', v_6'\} \cup U'\backslash\{u_1'\}$, since G does not contain $S_9(4,1)$. Similarly, by Lemma 4.4, $G[U'\backslash\{u_1'\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of $U'\backslash\{u_1'\}$ in G, as $S_9(4,1) \nsubseteq G$. Again, since $\delta(G[V(T')]) \le n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Now suppose that v_1' is non-adjacent to both w_2' and u_1' in G. Then, one of v_4', v_5', v_6' is adjacent to w_2' or u_1' in G. Without loss of generality, assume that v_4' is adjacent to w_2' in G. In this case, v_2' is not adjacent to $\{v_5', v_6'\} \cup U'\backslash\{u_1'\}$. Then, again, by Lemma 4.4, $G[U'\backslash\{u_1'\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of $U'\backslash\{u_1'\}$ in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \le n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Case 2b: v_0' is non-adjacent to both w_1' and w_2' in G.

In this case, one of v_4', v_5', v_6' is adjacent to w_1' or w_2' in G, say v_4' to w_1' in G. Since $S_9(4,1) \nsubseteq G$, v_1' is not adjacent to $\{v_5', v_6'\} \cup U'\backslash\{u_1'\}$ in G. By Lemma 4.4, $G[U'\backslash\{u_1'\}]$ is K_7 or $K_7 - e$, so no vertex of $V(T') \cup \{u_1'\}$ is adjacent to any vertex of $U\backslash\{u_1'\}$ in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \le n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Now suppose that $E_G(V', U') = \emptyset$. If $\delta(G[V']) = 0$, then by Lemma 4.4, $G[U']$ is K_8 or $K_8 - e$, and no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_9(4,1) \nsubseteq G$. Since $\delta(G[V(T')]) \le n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Hence, $\delta(G[V']) \ge 1$, and since $S_9(4,1) \nsubseteq G$, one of the vertices in V' is adjacent to other three in G. Without loss of generality, assume that v_3' is adjacent to v_4', v_5' and v_6' in G. Since G does not contain $S_9(4,1)$, v_4', v_5', v_6' are independent in G. Furthermore, v_0' is not adjacent to U' in G or else this reverts to the case where v_3' is adjacent to u_1' and v_0' is adjacent to any vertex of $U'\backslash\{u_1'\}$. Since $\delta(G[\{w_1'\} \cup U']) \le n-5$, $\overline{G}[\{w_1'\} \cup U']$ contains S_5. If v_0', v_4', v_5' and v_6' are non-adjacent to w_1' in G, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction. Again, there are two cases to be considered.

Case 3a: v_0' is adjacent to w_1' in G.

Note that v_1' and w_1' are not adjacent to U', or else this reverts to the case where v_3' is adjacent to u_1' and v_0' is adjacent to any vertex of $U'\backslash\{u_1'\}$. Now, since $\delta(G[\{w_2'\} \cup$

$U'])\leq n-5$, $\overline{G}[\{w_2'\}\cup U'\}]$ contains S_5. If v_0', v_4', v_5' and v_6' are non-adjacent to w_2' in G, then $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Suppose that v_0' is adjacent to w_2' in G. Again, v_2' and w_2' are non-adjacent to U', or else else this reverts to the case where v_3' is adjacent to u_1' and v_0' is adjacent to any vertex of $U'\backslash\{u_1'\}$. Now, $E_G(V(T'),U')=\emptyset$, and since $\delta(G[V(T')])\leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Therefore, w_2' is adjacent to at least one of v_4', v_5' and v_6' in G, say v_4'. Then, v_2' is not adjacent to v_5', v_6' or U', as $S_9(4,1)\nsubseteq G$, a contradiction. By Lemma 4.4, $G[U']$ is K_8 or K_8-e, so no vertex of $V(T')$ is adjacent to any vertex of U' in G, as $S_9(4,1)\nsubseteq G$. Again, since $\delta(G[V(T')])\leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Case 3b: v_0' is not adjacent to w_1' in G.

In this case, one of v_4', v_5', v_6' is adjacent to w_1' in G, say v_4'. Since $S_9(4,1)\nsubseteq G$, v_1' is not adjacent to v_5', v_6' or U' in G. By Lemma 4.4, $G[U']$ is K_8 or K_8-e, so no vertex of $V(T')\cup\{u_1'\}$ is adjacent to any vertex of U' in G, as $S_9(4,1)\nsubseteq G$. Since $\delta(G[V(T')])\leq n-5$, $\overline{G}[V(T')]$ contains S_5, and so $\overline{G}$ contains W_8 by Observation 4.3, a contradiction.

Thus, $R(S_n(4,1),W_8)\leq 2n-1$ for $n\geq 9$ which completes the proof. □

Theorem 7.7 *If $n\geq 8$, then*

$$R(T_D(n),W_8)=\begin{cases}2n-1 & \text{if } n\not\equiv 0 \pmod 4;\\ 2n & \text{otherwise.}\end{cases}$$

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $T_D(n)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n\equiv 0 \pmod 4$ and that G has order $2n$. By Theorem 6.2, G has a subgraph $T=S_n[4]$. Let $V(T)=\{v_0,\dots,v_{n-4},w_1,w_2,w_3\}$ and $E(T)=\{v_0v_1,\dots,v_0v_{n-4},v_1w_1,w_1w_2,w_1w_3\}$. Set $V=\{v_2,\dots,v_{n-4}\}$ and $U=V(G)-V(T)$; then $|V|=n-5$ and $|U|=n$. Since $T_D(n)\nsubseteq G$, neither w_2 nor w_3 is adjacent in G to $U\cup V$.

Suppose that $n=8$. Since G does not contain $T_D(n)$, V must be independent and non-adjacent to U in G. Then for any vertices $u_1,\dots,u_4$ in U, $v_3u_1v_4u_2w_2u_3w_3u_4v_3$ and v_2 form W_8 in $\overline{G}$, a contradiction. Suppose that $n\geq 12$. Then $|U\cup V|=2n-5$. If $\delta(\overline{G}[U\cup V])\geq\lceil\frac{2n-5}{2}\rceil$, then $\overline{G}[U\cup V]$ contains C_8 by Lemma 4.1 which with w_2 forms W_8, a contradiction. Thus, $\delta(\overline{G}[U\cup V])\leq\lceil\frac{2n-5}{2}\rceil-1=n-3$, and $\Delta(G[U\cup V])\geq n-3$. Now, there are two cases to consider.

Case 1: One of the vertices of V, say v_2, is a vertex of degree at least $n-3$ in $G[U\cup V]$.

Since $T_D(n)\nsubseteq G$, v_1 is not adjacent in G to w_2, w_3 or $U\cup V\backslash\{v_2\}$. Let $U'=\{w_2,w_3\}\cup U\cup V\backslash\{v_2\}$; then $|U'|=2n-4$. Now, if $\delta(\overline{G}[U'])\geq\frac{2n-4}{2}=n-2$, then $\overline{G}[U']$ contains C_8 by Lemma 4.1 which with v_1 forms W_8, a contradiction. Hence, $\delta(\overline{G}[U'])\leq n-3$, and $\Delta(G[U'])\geq n-2$. Note that neither w_2 nor w_3 have degree $\Delta(G[U'])$. Therefore, $d_{G[U']}(u')\geq n-2$ for some vertex $u'\in U\cup V\backslash\{v_2\}$. By the Inclusion–Exclusion Principle, some vertex $a\in U\cup V\backslash\{v_2\}$ is adjacent in G to both

u' and v_2. Then G has a subgraph $T_D(n)$ in which u' is the vertex of degree $n-5$ and v_2 is the vertex of degree 3, a contradiction.
Case 2: Some vertex $u \in U$ has degree at least $n-3$ in $G[U \cup V]$.

Suppose that there is at least one vertex in V that is adjacent to u in G, say v_2. Then G has a subgraph $T_D(n)$ in which u is the vertex of degree $n-5$ and v_0 is the vertex of degree 3, a contradiction. Similarly, no other vertex of V is adjacent to u. Now, since $T_D(n) \nsubseteq G$, $d_{G[N_{G[U]}(u) \cup \{v\}]}(v) \leq 1$ and $d_{G[V \cup \{x\}]}(x) \leq 1$, for any $v \in V$ and $x \in N_{G[U]}(u)$. Then, by Lemma 4.5, $\overline{G}[V \cup N_{G[U]}(u)]$ must contain C_8, which with w_2 as hub, forms W_8 in $\overline{G}$, a contradiction.

Now, suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 6.4, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \dots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \dots, v_0v_{n-4}, v_1w_1, w_1w_2, w_1w_3\}$. Set $V = \{v_2, \dots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_D(n) \nsubseteq G$, neither w_2 nor w_3 is adjacent to $U \cup V$ in G. If $\delta(\overline{G}[U \cup V]) \geq \frac{2n-6}{2} = n-3$, then $\overline{G}[U \cup V]$ contains C_8, by Lemma 4.1, which with w_2 forms W_8 in $\overline{G}$, a contradiction. Thus, $\delta(\overline{G}[U \cup V]) \leq n-4$, and $\Delta(G[U \cup V]) \geq n-3$. The arguments of the preceding cases then lead to contradictions.

Thus, $R(T_D(n), W_8) \leq 2n$, which completes the proof. □

Lemma 7.8 *Each graph H of order $n \geq 8$ with minimal degree at least $n-4$ contains $T_E(n)$ unless $n = 8$ and $H = K_{4,4}$.*

Proof Let $V(H) = \{u_0, \dots, u_{n-1}\}$. First, suppose that $\Delta(H) \geq n-3$ and assume without loss of generality that $u_1, \dots, u_{n-3} \in N_H(u_0)$. Suppose that u_{n-2} and u_{n-1} are adjacent in H. Since $\delta(H) \geq n-4$, $N_H(u_0) \cap N_H(u_{n-2}) \neq \emptyset$, so assume without loss of generality that u_1 is adjacent to u_{n-2} in H. Furthermore, u_1 must be adjacent to at least $n-7$ vertices from $\{u_2, \dots, u_{n-3}\}$ in H. Without loss of generality, assume that u_1 is adjacent to $u_2, \dots, u_{n-6}$ in H. Now, if any vertex of $\{u_2, \dots, u_{n-6}\}$ is adjacent to u_{n-5}, u_{n-4} or u_{n-3} in H, then we have $T_E(n)$ in H. Suppose that is not the case; then each vertex of $\{u_2, \dots, u_{n-6}\}$ must be adjacent to each other and to u_0, u_1, u_{n-2} and u_{n-1} in H. Since $d_H(u_{n-3}) \geq n-4$, u_{n-3} is adjacent to at least one of u_1, u_{n-2} and u_{n-1} in H, so H contains $T_E(n)$, a contradiction.

Suppose that u_{n-2} is not adjacent to u_{n-1} in H. Since $\delta(H) \geq n-4$, u_{n-2} and u_{n-1} are each adjacent to at least $n-5$ vertices in $N_H(u_0)$, so at least one vertex of $N_H(u_0)$, say u_1, is adjacent in H to both u_{n-2} and u_{n-1}. If $H[\{u_2, \dots, u_{n-3}\}]$ contains subgraph $2K_2$, then H contains subgraph $T_E(n)$. Note that this will always happens for $n \geq 11$, since $\delta(H) \geq n-4$.

Suppose that $n = 10$. Since $\delta(H) \geq 6$, u_2 must be adjacent in H to at least two vertices of $u_3, \dots, u_7$, without loss of generality say u_3 and u_4. If $H[\{u_4, \dots, u_7\}]$ contains any edge, then H contains $T_E(10)$. Otherwise, $\{u_4, \dots, u_7\}$ must be independent in H and each of these vertices must be adjacent to u_0, u_1, u_2, u_3, u_8 and u_9; this also gives a subgraph $T_E(10)$ in H.

Similarly, for $n = 9$, u_2 must be adjacent to at least one of $u_3, \dots, u_6$, say u_3, in H. If $H[\{u_4, u_5, u_6\}]$ contains any edge, then H contains $T_E(9)$. Otherwise, $\{u_4, u_5, u_6\}$ is independent in H and since $\delta(H) \geq 5$, u_4 is adjacent to at least one of u_2 and u_3, and u_5 is adjacent to at least one of u_7 and u_8. Again, this gives a subgraph $T_E(9)$ in H.

For $n = 8$, if $u_2, \dots, u_5$ are independent in H, then they are each adjacent to u_0, u_1, u_6 and u_7 in H, which gives $T_E(8)$ in H. Otherwise, we can assume that u_2 is adjacent to u_3 in H. If u_4 is adjacent to u_5 in H, we will have $T_E(8)$ in H; otherwise, assume that u_4 is not adjacent to u_5. Now, suppose that u_4 is adjacent to u_2 or u_3 in H. If u_5 is adjacent to u_6 or u_7 in H, then H contains $T_E(8)$. Otherwise, u_5 must be adjacent to u_0, u_1, u_2 and u_3 since $\delta(H) \geq 4$. However, this also gives $T_E(8)$ in H. On the other hand, suppose that u_4 is adjacent to neither u_2 nor u_3 in H. Similarly, u_5 is not adjacent to u_2 or to u_3 in H. Since $\delta(H) \geq 4$, both u_4 and u_5 must be adjacent to u_0, u_1, u_6 and u_7 in H, and this also gives $T_E(8)$ in H.

Suppose that H is $(n-4)$-regular and that $N_H(u_0) = \{u_1, \dots, u_{n-4}\}$. By the Handshaking Lemma, this only happens when n is even.

Suppose that $n \geq 10$. Note that u_{n-3}, u_{n-2} and u_{n-1} are each adjacent to at least $n-6$ vertices of $N_H(u_0)$ in H. By the Inclusion–Exclusion Principle, at least one of $u_1, \dots, u_{n-4}$ is adjacent to two of $u_{n-3}, u_{n-2}, u_{n-1}$ in H, say u_1 to u_{n-3} and u_{n-2}, and there must be another vertex, say u_2, that is adjacent to u_{n-1} in H. Now, if there is any edge in $H[\{u_3, \dots, u_{n-4}\}]$, then $T_E(n) \subseteq H$, and this always happens for $n \geq 12$. For $n = 10$, since $d_H(u_1) = 6$, u_1 is non-adjacent in H to at least one of $u_3, \dots, u_6$, say u_3. Since $d_H(u_3) = 6$, u_3 is adjacent to one of u_4, u_5, u_6, giving $T_E(10)$ in H.

Now suppose that $n = 8$. If u_5, u_6 and u_7 are independent in H, then $H = K_{4,4}$. Otherwise, we can assume that u_5 is adjacent to u_6 in H. If u_5 is also adjacent to u_7 in H, then u_5 is adjacent in H to two vertices of $N_H(u_0)$, say u_1 and u_2. Suppose that u_6 is adjacent to u_1 or u_2, say u_1, in H. Since $d_H(u_6) = 4$, u_6 is also adjacent to at least one of u_2, u_3, u_4, u_7, so $T_E(8) \subseteq H$. Otherwise, suppose that neither u_6 nor u_7 is adjacent to u_1 or u_2 in H. Since H is a 4-regular graph, u_6 and u_7 are both adjacent to u_3 and u_4 in H, and u_1 is adjacent to at least one of u_3 and u_4 in H. This gives $T_E(8)$ in H. On the other hand, suppose that u_5 is not adjacent to u_7 in H. Then, similarly, u_6 is not adjacent to u_7 in H, so u_7 is adjacent to u_1, u_2, u_3 and u_4 in H, and H contains $T_E(8)$. □

Theorem 7.9 *For $n \geq 8$,*

$$R(T_E(n), W_8) = \begin{cases} 2n-1 & \text{if } n \geq 9; \\ 16 & \text{if } n = 8. \end{cases}$$

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$ if $n \geq 9$ and of order 16 if $n = 8$. Assume that G does not contain $T_E(n)$ and that $\overline{G}$ does not contain W_8.

By Theorem 6.9, G has a subgraph $T = S_n(3, 1)$. Let $V(T) = \{v_0, \dots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \dots, v_0v_{n-4}, v_1w_1, v_2w_2, v_3w_3\}$. Set $V = \{v_4, \dots, v_{n-4}\}$ and $U = V(G) - V(T)$. Then $|V| = n-7$ and $|U| \geq n-1$. Since $T_E(n) \nsubseteq G$, each of v_1, v_2, v_3 is not adjacent to any vertex of $V \cup U$ in G, and each vertex of V is adjacent to at most one vertex of U in G. Let W be a set of $n-2$ vertices of U that are not adjacent to v_4 in G. By Lemma 4.4, $G[W]$ is K_{n-2}

or $K_{n-2} - e$. Since $T_E(n) \nsubseteq G$, no vertex of T is adjacent to any vertex of W, and so $\delta(G[V(T)]) \geq n-4$ by Observation 4.3.

Lemma 7.8 implies that $G[V(T)]$ contains $T_E(n)$ if $n \geq 9$, a contradiction, and so $n = 8$ and $G[V(T)] = K_{4,4}$. Note that $|U| = 8$, and as $T_E(8) \nsubseteq G$, no vertex of U is adjacent to any vertex of $G[V(T)]$. By Lemma 4.4, $G[U]$ is K_8 or $K_8 - e$, and thus contains $T_E(8)$, a contradiction.

Therefore, $R(T_E(n), W_8) \leq 2n-1$ when $n \geq 9$ and $R(T_E(n), W_8) \leq 16$ when $n = 8$. □

Lemma 7.10 *Each graph H of order $n \geq 8$ with minimal degree at least $n-4$ contains $T_F(n)$ unless $n = 8$ and $H = K_{4,4}$.*

Proof Let $V(H) = \{u_0, u_1 \ldots, u_{n-1}\}$ so that $d(u_0) = \delta(H)$ and $V = \{u_1, \ldots, u_{n-4}\} \subseteq N(u_0)$. Set $U = \{u_{n-3}, u_{n-2}, u_{n-1}\}$. By the minimum degree condition, every vertex of U is adjacent to at least $n-6$ vertices of V. It is straightforward to see that some pair of vertices in U have a common neighbour in V. Moreover, for $n \geq 9$, every pair of vertices in U has a common neighbour in V.

Assume without loss of generality that u_1 is adjacent to both u_{n-3} and u_{n-2}, and that u_2 is adjacent to u_{n-1}. If u_2 is adjacent to a vertex of $V \backslash \{u_1\}$, which is the case when $n \geq 10$, then H contains $T_F(n)$. Assume now that $n \leq 9$ and that u_2 is not adjacent to any vertex of $V \backslash \{u_1\}$.

For the case when $n = 9$, u_{n-1} is adjacent to at least $n-6 = 3$ vertices of V, and so it is adjacent to another vertex, say to u_3. As above, assume that u_3 is not adjacent to any vertex of $V \backslash \{u_1\}$. By the minimum degree condition, each of u_2 and u_3 is adjacent to every vertex of $\{u_1\} \cup U$, giving $T_F(9)$ in H.

For the final case when $n = 8$, the minimum degree condition implies that u_2 is adjacent to at least two vertices of $\{u_1, u_5, u_6\}$. If u_2 is adjacent to u_1, then H contains $T_F(8)$. Remaining is the case when u_2 is not adjacent to u_1 but is adjacent to both u_5 and u_6. Exchanging the roles of u_1 and u_2, we may assume that u_1 is adjacent to u_7 but not adjacent to any vertex of V. From the minimum degree condition on u_3 and u_4, it is easy to see that either H contains $T_F(8)$ or $H = K_{4,4}$. □

Theorem 7.11 *For $n \geq 8$,*

$$R(T_F(n), W_8) = \begin{cases} 2n-1 & \text{if } n \geq 9; \\ 16 & \text{if } n = 8. \end{cases}$$

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $T_F(n)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n = 8$ and that G has order 16. By Theorem 6.8, G has a subgraph $T = T_C(8)$. Let $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_4, v_1w_1, v_2w_2, v_2w_3\}$. Set $U = V(G) - V(T) = \{u_1, \ldots, u_8\}$; then $|U| = 8$. Since $T_F(8) \nsubseteq G$, v_1 is not adjacent in G to v_2, v_3, v_4 or any vertex of U, and $d_{G[U]}(v) \leq 1$ for $v = v_3, v_4, w_2, w_3$.

Suppose that v_1 is adjacent to w_2 or w_3, without loss of generality say w_2. Since $T_F(8) \nsubseteq G$, v_2 is not adjacent to $\{v_3, v_4\} \cup U$. If neither v_3 nor v_4 are adjacent to U, then by Lemma 4.4, $G[U]$ is K_8 or $K_8 - e$, so $G[U]$ contains $T_F(8)$, a contradiction. Suppose that only one of the vertices v_3 and v_4 is adjacent to U in G, say v_3. By Lemma 4.4, $G[U \backslash \{u_1\}]$ is K_7 or $K_7 - e$, and $G[V(T) \cup \{u_1\}]$ is not adjacent to $G[U \backslash \{u_1\}]$. By Observation 4.3, $\delta(G[V(T) \cup \{u_1\}]) \geq 5$, and by Lemma 7.10, $G[V(T) \cup \{u_1\}]$ contains $T_F(9)$ and hence $T_F(8)$, a contradiction. Suppose that both v_3 and v_4 are adjacent to U in G and assume that v_3 is adjacent to u_1 and that v_4 is adjacent to u_2. By Lemma 4.4, $G[U \backslash \{u_1, u_2\}]$ is K_6 or $K_6 - e$. At most one vertex from $G[V(T) \cup \{u_1, u_2\}]$ is adjacent to $G[U \backslash \{u_1, u_2\}]$ or else G contains $T_F(8)$. Therefore, 9 vertices from $G[V(T) \cup \{u_1, u_2\}]$ form a vertex set W that is not adjacent to $U \backslash \{u_1, u_2\}$. By Observation 4.3, $\delta(G[W]) \geq 5$, and by Lemma 7.10, $G[W]$ contains $T_F(9)$ and hence $T_F(8)$, a contradiction.

Suppose then that v_1 is not adjacent to w_2 or w_3. Since $d_{G[U]}(v) \leq 1$ for $v = v_3, v_4, w_2, w_3$, there are 4 vertices from U that are not adjacent to $\{v_3, v_4, w_2, w_3\}$. These 8 vertices form C_8 in $\overline{G}$ and thus, with v_1 as hub, W_8, a contradiction.

Thus, $R(T_F(8), W_8) \leq 16$.

Now, suppose that $n \geq 9$ and that G has order $2n - 1$. By Theorem 6.8, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|V| = n - 6$ and $|U| = n - 1$. Since $T_F(n) \nsubseteq G$, v_1 is not adjacent in G to any vertex of $U \cup V$, and $d_{G[U]}(v) \leq 1$ for $v \in V$. Since $n \geq 10$, there are 4 vertices from U, 4 vertices from V and v_1 that form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_F(n), W_8) \leq 2n - 1$ for $n \geq 10$.

Suppose that $n = 9$ and let m be the number of vertices of U that are adjacent in G to at least one vertex of V. Since $d_{G[U]}(v) \leq 1$ for $v \in V$, $0 \leq m \leq 3$. If $m = 0$, then $G[U]$ is K_8 or $K_8 - e$ by Lemma 4.4, so $G[V(T)]$ is not adjacent to $G[U]$. By Observation 4.3, $\delta(G[V(T)]) \geq 5$, and $G[V(T)]$ contains $T_F(9)$ by Lemma 7.10, a contradiction. Suppose that $m = 1$. Assume without loss of generality that u_1 is adjacent to some vertex of V, and that $E_G(V, U \backslash \{u_1\}) = \emptyset$. By Lemma 4.4, $G[U \backslash \{u_1\}]$ is K_7 or $K_7 - e$, and at most one vertex from $G[V(T) \cup \{u_1\}]$ is adjacent to $G[U \backslash \{u_1\}]$ or else G contains $T_F(9)$. There are then 9 vertices from $G[V(T) \cup \{u_1\}]$ that form a vertex set W_1 that is not adjacent to $U \backslash \{u_1\}$. By Observation 4.3, $\delta(G[W_1]) \geq 5$, and $G[W_1]$ contains $T_F(9)$ by Lemma 7.10, a contradiction. Suppose that $m = 2$. Assume that u_1 and u_2 are adjacent to some vertices of V and that $E_G(V, U \backslash \{u_1, u_2\}) = \emptyset$. By Lemma 4.4, $G[U \backslash \{u_1, u_2\}]$ is K_6 or $K_6 - e$. If at least three vertices in $U \backslash \{u_1, u_2\}$ are adjacent to $V(T) \cup \{u_1\}$, then $T_F(9) \subseteq G$. If at most two vertices in $U \backslash \{u_1, u_2\}$ are adjacent to $V(T) \cup \{u_1\}$, then there are 4 vertices in $U \backslash \{u_1, u_2\}$ that are not adjacent to $V(T)$. Then Observation 4.3 gives $\delta(G[V(T)]) \geq 5$, and $G[V(T)]$ contains $T_F(9)$ by Lemma 7.10, a contradiction. Suppose that $m = 3$. Assume that u_1, u_2, u_3 are each adjacent to some vertex of V and that $E_G(V, U \backslash \{u_1, u_2, u_3\}) = \emptyset$. Without loss of generality, assume that u_i is adjacent to v_{i+2} for $i = 1, 2, 3$. By Lemma 4.4, $G[U \backslash \{u_1, u_2, u_3\}]$ is K_5 or $K_5 - e$. Since $T_F(9) \nsubseteq G$, $\{v_1, v_3, v_4, v_5\}$ is independent and $V(T) \backslash \{w_1\}$ is not adjacent to $U \backslash \{u_1, u_2, u_3\}$. Then by Observation 4.3, $\delta(G[V(T) \backslash \{w_1\}]) \geq 4$, and v_1, v_3, v_4 and v_5 are each adjacent to v_2, w_2 and w_3 in G. This gives $T_F(9)$ in G. Therefore, $T_F(9) \leq 17 = 2n - 1$. □

Theorem 7.12 *If $n \geq 8$, then $R(T_G(n), W_8) = 2n - 1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_G(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.9, G has a subgraph $T = S_n(3, 1)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_3w_3\}$. Set $V = \{v_4, v_5, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n - 7$ and $|U| = n - 1$. Since $T_G(n) \nsubseteq G$, w_1, w_2, w_3 are not adjacent to $U \cup V$ in G, and v_1, v_2, v_3 are not adjacent to V.

Suppose that $n \geq 9$; then $|U| \geq 8$. If $\delta(\overline{G}[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 4.1 which, with w_2 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U \cup V]) \geq \frac{n-1}{2} \geq 4$. Therefore, some vertex $u \in U$ satisfies $|N_{G[U]}(u)| \geq 4$. Since $T_G(n) \nsubseteq G$, $N_{G[U]}(u)$ is not adjacent in G to $N_{G[V(T)]}(v_0)$. Hence, 4 vertices from $N_{G[U]}(u)$, v_1, v_2, v_3, w_1 and any vertex from V form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_G(n), W_8) \leq 2n - 1$ for $n \geq 9$.

Suppose that $n = 8$ and let $U = \{u_1, \ldots, u_7\}$ and $W = \{v_4\} \cup U$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}$ contains C_8 by Lemma 4.1 and thus W_8, with w_1 as hub, a contradiction. Therefore, $\delta(\overline{G}[W]) \leq 3$, and $\Delta(G[W]) \geq 4$. Now, suppose that $d_{G[W]}(v_4) \geq 4$. Then without loss of generality, assume that $u_1, \ldots, u_4 \in N_G(v_4)$. Then $u_1, \ldots, u_4, w_1, w_2, w_3$ are independent and are not adjacent to u_5, u_6 or u_7, giving W_8, a contradiction. On the other hand, suppose that some vertex in U, say u_1, satisfies $d_{G[W]}(u_1) \geq 4$. Then v_4 is not adjacent to u_1; therefore, assume that $u_2, \ldots, u_5 \in N_G(u_1)$. Then $v_1, \ldots, v_4$ are not adjacent to $\{u_1, \ldots, u_5\}$, so $v_1u_1v_2u_2v_3u_3w_1u_4v_1$ and v_4 form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_L(8), W_8) \leq 15$. □

Lemma 7.13 *Each graph H of order $n \geq 8$ with minimal degree at least $n-4$ contains $T_H(n)$, $T_K(n)$ and $T_L(n)$.*

Proof Let $V(H) = \{u_0, \ldots, u_{n-1}\}$ where $u_1, \ldots, u_{n-4} \in N_H(u_0)$. Suppose that u_{n-3}, u_{n-2} or u_{n-1}, say u_{n-3}, is adjacent in H to the two others.

Since $\delta(H) \geq n-4$, u_{n-3} is adjacent to at least one of $u_1, \ldots, u_{n-4}$, say u_1. If u_1 is adjacent to another vertex in $\{u_2, \ldots, u_{n-4}\}$, then H contains $T_K(n)$. Note that this always happens for $n \geq 9$. Suppose that $n = 8$ and that u_1 is not adjacent to any of u_2, u_3, u_4. Then u_1 is adjacent to u_6 and u_7. Since $\delta(H) \geq n - 4$, u_2 is adjacent to at least one of u_5, u_6, u_7, giving $T_K(n)$ in H.

Similarly, since $\delta(H) \geq n - 4$, u_{n-2} is adjacent to at least $n - 7$ vertices of $\{u_1, \ldots, u_{n-4}\}$. Suppose that u_{n-2} is adjacent to u_1. If $n \geq 10$, then at least two of $u_2, \ldots, u_{n-4}$ are adjacent, so H contains $T_H(n)$. If $n \geq 9$, then u_1 is adjacent to at least one of $u_2, \ldots, u_{n-4}$, so H contains $T_L(n)$. Now suppose that $n = 9$. If any of $u_2, \ldots, u_5$ are adjacent to each other, then H contains $T_H(9)$. Otherwise, $u_2, \ldots, u_5$ are each adjacent to u_6, u_7 and u_8, and so H contains $T_H(9)$. Finally, suppose that $n = 8$. If any two of u_2, u_3, u_4 are adjacent, then H contains $T_H(8)$; otherwise, they are each adjacent to u_6 or u_7. Now, if u_1 is adjacent to any of u_2, u_3, u_4, then H contains $T_H(8)$. Otherwise, $u_1, \ldots, u_4$ are each adjacent to u_5, u_6 and u_7, and H also contains $T_H(8)$. Furthermore, if u_1 is adjacent to u_2, u_3 or u_4, then H contains $T_L(8)$. If u_1 is not adjacent to u_2, u_3 or u_4, then u_6, u_7, u_8 are adjacent to u_2, u_3, u_4, and then

H contains $T_L(8)$. Now if u_{n-2} is adjacent to some $u_2, \ldots, u_{n-4}$, say u_2, then similar arguments apply by interchanging u_1 and u_2.

Suppose now that neither u_{n-3}, u_{n-2} nor u_{n-1} is adjacent to both of the others. Then one of these, say u_{n-3}, is adjacent to neither of the others. Since $\delta(H) \geq n-4$, u_{n-3} is adjacent to at least $n-5$ of the vertices $u_1, \ldots, u_{n-4}$. Without loss of generality, assume that $u_1, \ldots, u_{n-5} \in N_H(u_{n-3})$. Then u_{n-2} is adjacent to at least $n-7$ of the vertices $u_1, \ldots, u_{n-5}$ including, without loss of generality, the vertex u_1. Also, u_{n-1} is adjacent to at least one of $u_2, \ldots, u_{n-4}$, so H contains $T_H(n)$. If u_{n-2} is adjacent to u_{n-1}, then H also contains $T_L(n)$. If u_{n-2} is not adjacent to u_{n-1}, then u_{n-2} is adjacent to at least $n-6$ vertices of $u_1, \ldots, u_{n-5}$, so H contains $T_L(n)$. Now, suppose that $n \geq 9$. Then u_{n-2} and u_{n-1} are each adjacent to at least 3 of $u_1, \ldots, u_5$, and one of those vertices must be adjacent to both u_{n-2} and u_{n-1}; thus, H contains $T_K(n)$. Finally, suppose that $n = 8$. If u_6 and u_7 are each adjacent to at least two of the vertices u_1, u_2, u_3, then one of those vertices must be adjacent to both u_6 and u_7; thus, H contains $T_K(8)$. Otherwise, u_6 or u_7, say u_6, is non-adjacent to at least two of u_1, u_2, u_3, say u_1 and u_2. Then u_6 is adjacent to u_0, u_3, u_4 and u_7, and so H contains $T_K(8)$. □

Theorem 7.14 *If $n \geq 8$, then $R(T_H(n), W_8) = 2n - 1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$ and assume that G does not contain $T_H(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 7.12, G has a subgraph $T = T_G(n)$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, v_2w_2, v_3w_3, w_3w_4\}$. Set $U = \{u_1, \ldots, u_{n-1}\} = V(G) - V(T)$; then $|U| = n - 1$. Since $T_G(n) \nsubseteq G$, $E_G(\{w_1, w_2\}, \{w_3, w_4\}) = \emptyset$ and w_4 is not adjacent to U. Now, let $W = \{w_1\} \cup U$; then $|W| = n$. If $\delta(\overline{G}[W]) \geq \frac{n}{2}$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which, with w_4 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) < \frac{n}{2}$, and $\Delta(G[W]) \geq \lfloor \frac{n}{2} \rfloor \geq 4$.

First, suppose that w_1 is a vertex with degree at least $\frac{n}{2}$ in $G[W]$. Assume without loss of generality that $u_1, \ldots, u_4 \in N_{G[W]}(w_1)$. Since $T_H(n) \nsubseteq G$, $u_1, \ldots, u_4$ are independent and are not adjacent to $\{w_2, u_5, \ldots, u_{n-1}\}$ in G. Then $w_2, u_1, \ldots, u_4, w_4$ and any 3 vertices from $\{u_5, \ldots, u_{n-1}\}$ form W_8 in $\overline{G}$, a contradiction. Hence, $d_{G[W]}(u') \geq \frac{n}{2}$ for some vertex $u' \in U$, say $u' = u_1$. Note that w_1 is not adjacent to u_1, or else G contains $T_H(n)$. Without loss of generality, suppose that $u_2, \ldots, u_5 \in N_{G[W]}(u_1)$. Since $T_H(n) \nsubseteq G, u_2, \ldots, u_5$ are not adjacent to $V(T) \backslash \{v_0\}$ in G. Now, if v_0 is not adjacent to $\{u_2, \ldots, u_5\}$ in G, then by Observation 4.3, $\delta(G[V(T)]) \geq n-4$, or else $\overline{G}$ contains W_8. By Lemma 7.13, $G[V(T)]$ contains $T_H(n)$, a contradiction. On the other hand, suppose that v_0 is adjacent to at least one of $u_2, \ldots, u_5$, say u_2. Then u_3, u_4, u_5 are independent in G and are not adjacent to u_6 and u_7 in G. Furthermore, w_4 is not adjacent to v_1 or v_2. Then $v_1u_3v_2u_4u_6w_1u_7u_5v_1$ and w_4 form W_8 in $\overline{G}$, a contradiction. Thus, $R(T_H(n), W_8) \leq 2n - 1$. □

Theorem 7.15 *If $n \geq 8$, then $R(T_J(n), W_8) = 2n - 1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$ and assume that G does not contain $T_J(n)$ and that

$\overline{G}$ does not contain W_8. By Theorem 6.8, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|U| = n - 1$. Let $U = \{u_1, \ldots, u_{n-1}\}$. Since $T_J(n) \nsubseteq G$, neither w_1 nor w_2 is adjacent in G to any vertex from $U \cup V$.

Let $W = \{v_3\} \cup U$; then $|W| = n$. If $\delta(\overline{G}[W]) \geq \lceil \frac{n}{2} \rceil \geq \frac{n}{2}$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which with w_1 forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) < \lceil \frac{n}{2} \rceil$, and $\Delta(G[W]) \geq \lfloor \frac{n}{2} \rfloor \geq 4$.

Suppose that $d_{G[W]}(v_3) \geq \lfloor \frac{n}{2} \rfloor \geq 4$. Without loss of generality, assume that $u_1, \ldots, u_4 \in N_G(v_3)$. Since $T_J(n) \nsubseteq G$, $u_1, \ldots, u_4$ is independent in G and is not adjacent to any remaining vertices from U in G. Then $u_2w_1u_3u_5u_4u_6w_2u_7u_2$ and u_1 form W_8 in $\overline{G}$, a contradiction. Hence, there is a vertex in U, say u_1, such that $d_{G[W]}(u_1) \geq \lfloor \frac{n}{2} \rfloor \geq 4$.

Now, suppose that v_3 is adjacent to u_1 in $G[W]$. Then u_1 is adjacent to at least 3 other vertices of U in G, say u_2, u_3 and u_4. Since $T_J(n) \nsubseteq G$, v_3 is not adjacent to $v_1, v_2, v_4, \ldots, v_{n-4}, w_1, w_2, w_3, u_2, u_3, u_4$ and neither v_1 nor v_2 is adjacent to u_2, u_3 or u_4 in G. Then $v_2u_2v_1u_3w_1v_4w_2u_4v_2$ and v_3 form W_8 in $\overline{G}$, a contradiction.

Thus, v_3 is not adjacent to u_1 in G. Note that u_1 is not adjacent to any other vertices of V in G or else previous arguments apply. Similarly, v_0 is not adjacent to $N_{G[W]}(u_1)$ in G. Since $T_J(n) \nsubseteq G$, neither v_1 nor v_2 is adjacent to u_1 or $N_{G[W]}(u_1)$ in G, and so $d_{N_{G[W]}(u_1)}(v) \leq 1$ for all $v \in V$.

Suppose that $n \geq 10$; then $|V| \geq 4$ and $|N_{G[W]}(u_1)| \geq 5$. If $d_{G[V]}(u) \leq 2$ for each $u \in N_{G[W]}(u_1)$, then $\overline{G}[V \cup N_{G[W]}(u_1)]$ contains C_8 by Lemma 4.5 which, with w_1 as hub, forms W_8 in $\overline{G}$, a contradiction. Thus, $d_V(u') \geq 3$ for some vertex $u' \in N_{G[W]}(u_1)$. Then any 4 vertices from V, of which at least 3 are in $N_{G[V]}(u')$, and any 4 vertices from $N_{G[W]}(u_1) \backslash \{u'\}$ satisfy the condition in Lemma 4.5, so $\overline{G}[V \cup N_{G[W]}(u_1)]$ contains C_8 which with w_1 forms W_8, a contradiction.

Suppose that $n = 9$; then $V = \{v_3, v_4, v_5\}$. Assume that $u_2, \ldots, u_5 \in N_{G[W]}(u_1)$. Suppose that w_1 is not adjacent to w_2 in G. Let $X = \{v_3, v_4, v_5, w_2\}$ and $Y = \{u_2, \ldots, u_5\}$ and note that $d_{G[Y]}(x) \leq 1$ for each $x \in X$. If $d_{G[X]}(y) \leq 2$ for each $y \in Y$, then $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.5 which, with w_1 as hub, forms W_8, a contradiction. Thus, $d_{G[X]}(u') \geq 3$ for some $u' \in Y$, say $u' = u_2$, so X is not adjacent to $Y \backslash \{u_2\}$. Hence, $v_3u_1v_4u_3v_5u_4w_2u_5v_3$ and w_1 form W_8 in $\overline{G}$, a contradiction.

Thus, w_1 is adjacent to w_2 in G. Then v_1 is not adjacent to $\{v_3, v_4, v_5\} \cup U$. Suppose that v_1 is not adjacent to v_2. Then set $X = \{v_2, \ldots, v_5\}$ and $Y = \{u_2, \ldots, u_5\}$. If $d_{G[X]}(y) \leq 2$ for each $y \in Y$, then $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.5 which, with v_1 as hub, forms W_8, a contradiction. Thus, $d_{G[X]}(u') \geq 3$ for some $u' \in Y$, say $u' = u_2$, so X is not adjacent to $Y \backslash \{u_2\}$, and $v_2u_1v_3u_3v_4u_4v_5u_5v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus, v_1 is adjacent to v_2 in G. Then V is independent and is not adjacent to U in G. Since $W_8 \nsubseteq \overline{G}$, $G[U]$ is K_{n-1} or $K_{n-1} - e$ by Lemma 4.4. Since $T_J(9) \nsubseteq G$, T is not adjacent to U and, by Observation 4.3, $\delta(G[V(T)]) \geq 5$. However, this is impossible since V is independent and is not adjacent to v_1, w_1 or w_2.

Finally, suppose that $n = 8$; then $V = \{v_3, v_4\}$. Assume that $u_2, \ldots, u_5 \in N_{G[W]}(u_1)$. If v_3 is adjacent to any vertex of $\{u_2, \ldots, u_5\}$, say u_2, then v_3 is not adjacent to $\{v_1, v_2, v_4, w_3\} \cup U \backslash \{u_2\}$, so $v_1u_1v_2u_3w_1u_4w_2u_5v_1$ and v_3 form W_8 in $\overline{G}$, a contradiction. Thus, v_3 is not adjacent to $\{u_2, \ldots, u_5\}$. Similarly, v_4 is not adjacent to $\{u_2, \ldots, u_5\}$. Now, if w_3 is adjacent to any of the vertices $u_2, \ldots, u_5$, say u_2, then v_2 is not adjacent to $\{w_1, w_2, v_3, v_4\}$, so $v_3u_1v_4u_2w_1u_3w_2u_4v_3$ and v_2 form W_8 in $\overline{G}$, a contradiction. Thus, w_3 is not adjacent to $\{u_2, \ldots, u_5\}$. By Observation 4.3, $\delta(G[V(T)]) \geq 4$. Suppose that v_2 is adjacent to w_1. Since $T_J(8) \nsubseteq G$, neither v_3 nor v_4 is adjacent to w_3. Since $\delta(G[V(T)]) \geq 4$, v_3 and v_4 are adjacent to v_1 and v_2, and $\{w_1, w_2, w_3\}$ is not independent. However, then $T_J(8) \subseteq G[V(T)]$, a contradiction. Thus, v_2 is not adjacent to w_1 and, similarly, v_2 is not adjacent to w_2. Since $\delta(G[V(T)]) \geq 4$, w_1 and w_2 are adjacent to each other and to w_3. Since $T_J(8) \nsubseteq G$, neither v_3 nor v_4 is adjacent to v_1 or v_2; however, this contradicts $\delta(G[V(T)]) \geq 4$.

In each case, $R(T_J(8), W_8) \leq 2n - 1$, which completes the proof of the theorem. □

Theorem 7.16 *If $n \geq 8$, then $R(T_K(n), W_8) = 2n - 1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be a graph of order $2n - 1$ and assume that G does not contain $T_K(n)$ and that $\overline{G}$ does not contain W_8.

Suppose that $n \not\equiv 0 \pmod 4$. By Theorem 6.5, G has a subgraph $T = S_n(1, 3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_2w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_K(n) \nsubseteq G$, w_2 is not adjacent in G to any vertex of $U \cup V$. Now, if $\delta(G[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 4.1 which, with v_1 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U]) \geq \lfloor \frac{n-1}{2} \rfloor$. Let $U = \{u_1, \ldots, u_{n-1}\}$ and assume without loss of generality that $d_{G[U]}(u_1) \geq \lfloor \frac{n-1}{2} \rfloor \geq 4$. Since $T_K(n) \nsubseteq G$, $E_G(V, N_{G[U]}(u_1)) = \emptyset$, so any 4 vertices from V, any 4 vertices from $N_{G[U]}(u_1)$ and w_2 form W_8 in $\overline{G}$, a contradiction. Therefore, $R(T_K(n), W_8) \leq 2n - 1$ for $n \not\equiv 0 \pmod 4$.

Let $n = 8$. By Theorem 7.14, G has a subgraph $T = T_H(8)$. Let $V(T) = \{v_0, v_1, v_2, v_3, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_3, v_1w_1, w_1w_2, w_2w_3, v_2w_4\}$. Set $U = V(G) - V(T) = \{u_1, \ldots, u_7\}$; then $|U| = 7$. Since $T_K(8) \nsubseteq G$, w_2 is not adjacent to $\{w_4\} \cup U$. Let $W = \{w_4\} \cup U$. Then $|W| = 8$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which, with w_2 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[W]) < 3$, and $\Delta(G[W]) \geq 4$.

Now, suppose that $d_{G[W]}(w_4) \geq 4$ and assume without loss of generality that w_4 is adjacent to u_1, u_2, u_3 and u_4. Then v_1 is not adjacent to $\{v_3, w_2, w_3\} \cup U$ and neither v_2 nor v_3 is adjacent to $\{u_1, \ldots, u_4\}$, since $T_K(8) \nsubseteq G$. Now, suppose that $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$ and assume that u_1 is adjacent to u_5. Then u_1 is not adjacent to $\{w_1, w_2, w_3, u_2, \ldots, u_7\}$ in G, and $v_1u_2v_2u_3v_3u_4w_2u_6v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction. Thus, $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) = \emptyset$, so $u_1u_5u_2u_6u_3u_7u_4v_3u_1$ and v_1 form W_8 in $\overline{G}$, a contradiction.

Now suppose that $d_{G[W]}(u') \geq 4$ for some vertex $u' \in U$, say $u' = u_1$. Since, $T_K(8) \nsubseteq G$, w_4 is not adjacent to u_1. Then without loss of generality, suppose that $u_2, \ldots, u_5 \in N_G(u_1)$. Since $T_K(8) \nsubseteq G$, $E_G(\{v_1, v_2, v_3\}, \{u_2, \ldots, u_5\}) = \emptyset$. If u_2 is adjacent to w_1, then u_2 is not adjacent to $\{u_3, \ldots, u_7\}$ and v_1 is not adjacent to u_6.

Then $w_2u_3v_2u_4v_3u_5v_1u_6w_2$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_1. Similarly, u_3, u_4 and u_5 are not adjacent to w_1. If u_2 is adjacent to v_0, then v_2 is not adjacent to $\{v_1, v_3, w_1, w_2, w_3, u_2, \ldots, u_7\}$, and $v_1u_2v_3u_3w_1u_4w_2u_5v_1$ and v_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to v_0. Similarly, u_3, u_4 and u_5 are not adjacent to v_0. By similar arguments, u_3, u_4 and u_5 are not adjacent to w_3 or w_4.

Hence, $u_2, \ldots, u_5$ are not adjacent to $V(T)$ in G, so $\delta(G[V(T)]) \geq 4$ by Observation 4.3. By Lemma 7.13, $G[V(T)]$ contains $T_K(8)$, a contradiction. Thus, $R(T_K(8), W_8) \leq 15$.

Now suppose that $n \equiv 0 \pmod 4$ and that $n \geq 12$. If G has an $S_n(1,3)$ subgraph, then the arguments above lead to contradictions. Thus, G does not contain $S_n(1,3)$ as a subgraph. Now, by Theorem 7.14, G has a subgraph $T = T_H(n)$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, w_1w_2, w_2w_3, v_2w_4\}$. Set $V = \{v_3, \ldots, v_{n-5}\}$ and let $U = V(G)-V(T) = \{u_1, \ldots, u_{n-1}\}$. Then $|V| = n-7$ and $|U| = n-1$. Since $T_K(n) \nsubseteq G$, w_2 is not adjacent in G to $\{w_4\} \cup U$. Since $S_n(1,3) \nsubseteq G$, v_0 is not adjacent to $\{w_4\} \cup U$.

If $\delta(\overline{G}[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 4.1 which, with w_2, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U]) \geq \lfloor \frac{n-1}{2} \rfloor \geq 5$. Without loss of generality, assume that $u_2, \ldots, u_6 \in N_G(u_1)$. Since $T_K(n) \nsubseteq G$, v_1, v_2 and V are not adjacent to $\{u_2, \ldots, u_6\}$, and w_1 and w_2 are not adjacent to u_1.

Now, if u_2 is adjacent to w_1, then u_2 is not adjacent to $\{w_3, w_4\} \cup U \backslash \{u_1\}$, since $T_K(n) \nsubseteq G$, so $v_0u_3v_1u_4v_2u_5v_3u_6v_0$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_1. Similarly, $u_3, \ldots, u_6$ are not adjacent to w_1. If u_2 is adjacent to w_3 in G, then v_0 is not adjacent to w_1, w_2, w_3, and $d_{G[U\backslash\{u_1,u_2\}]}(u_i) \leq n-6$ for $i = 3, \ldots, 6$, since $S_n(1,3) \nsubseteq G$. Since $T_K(n) \nsubseteq G$, w_3 is not adjacent to w_1 or w_4. Since $d_{G[U\backslash\{u_1,u_2\}]}(u_3) \leq n-6$ and $d_{G[U\backslash\{u_1,u_2\}]}(u_4) \leq n-6$, u_3 and u_4 are adjacent in $\overline{G}$ to at least 2 vertices in $\{u_7, \ldots, u_{n-1}\}$. Without loss of generality, assume that u_3 is adjacent in $\overline{G}$ to u_7 and that u_4 is adjacent to u_8. Then $u_3u_7w_2u_8u_4w_1w_3w_4u_3$ and v_0 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_3. Similarly, $u_3, \ldots, u_6$ are not adjacent to w_4.

Hence, $u_2, \ldots, u_6$ are not adjacent to $V(T)$. By Observation 4.3, $\delta(G[V(T)]) \geq 4$, so $G[V(T)]$ contains $T_K(n)$ by Lemma 7.13, a contradiction. Thus, $R(T_K(n), W_8) \leq 2n-1$ for $n \equiv 0 \pmod 4$. This completes the proof. □

Theorem 7.17 *If* $n \geq 8$, *then* $R(T_L(n), W_8) = 2n-1$.

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be a graph with no $T_L(n)$ subgraph whose complement $\overline{G}$ does not contain W_8. Suppose that $n \not\equiv 0 \pmod 4$ and that G has order $2n-1$. By Theorem 6.5, G has a subgraph $T = S_n(1,3)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, w_1w_2, w_2w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_L(n) \nsubseteq G$, v_1 is not adjacent to $U \cup V$, and $d_{G[U]}(v_i) \leq n-7$ for each $v_i \in V$. Now, if $\delta(G[U]) \geq \frac{n-1}{2}$, then $\overline{G}[U]$ contains C_8 by Lemma 4.1 which, with v_1, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U]) < \frac{n-1}{2}$, and $\Delta(G[U]) \geq \lfloor \frac{n-1}{2} \rfloor$.

Let $U = \{u_1, \ldots, u_{n-1}\}$ and without loss of generality assume that $d_{G[U]}(u_1) \geq \lfloor\frac{n-1}{2}\rfloor \geq 4$ and that $u_2, \ldots, u_5 \in N_{G[U]}(u_1)$. Now if $E_G(V, N_{G[U]}(u_1)) = \emptyset$, then 4 vertices from V, 4 vertices from $N_{G[U]}(u_1)$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus, $E_G(V, N_{G[U]}(u_1)) \neq \emptyset$. Assume without loss of generality that v_2 is adjacent to u_2. Since $T_L(n) \nsubseteq G$, v_2 is not adjacent to $U\backslash\{u_1, u_2\}$. Since $d_{G[U]}(v_i) \leq n - 7$ for each $v_i \in V$, v_5 is non-adjacent to at least one of $u_6, \ldots, u_{n-1}$, say u_6. Now if $E_G(\{v_3, v_4, v_5\}, \{u_3, u_4, u_5\}) = \emptyset$, then $v_2u_3v_3u_4v_4u_5v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus assume, say, that v_3 is adjacent to u_3 in G; then v_3 is not adjacent to $U\backslash\{u_1, u_3\}$. Again, if $E_G(\{v_4, v_5\}, \{u_4, u_5\}) = \emptyset$, then $v_2u_7v_3u_4v_4u_5v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus assume, say, that v_4 is adjacent to u_4, then v_4 is not adjacent to $U\backslash\{u_1, u_4\}$. If v_5 is not adjacent to u_5, then $v_2u_7v_3u_2v_4u_5v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction. Thus, v_5 is adjacent to u_5, so v_5 is not adjacent to $U\backslash\{u_1, u_5\}$, and $v_2u_7v_3u_2v_4u_3v_5u_6v_2$ and v_1 form W_8 in $\overline{G}$, a contradiction.

Hence, $R(T_L(n), W_8) \leq 2n - 1$ for $n \not\equiv 0 \pmod 4$.

Now, suppose that $n \equiv 0 \pmod 4$ and that G has order $2n - 1$. Suppose first that $n = 8$. By Theorem 7.14, G has a subgraph $T = T_H(8)$. Let $V(T) = \{v_0, \ldots, v_3, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_3, v_1w_1, w_1w_2, w_2w_3, v_2w_4\}$. Set $U = V(G) - V(T) = \{u_1, \ldots, u_7\}$; then $|U| = 7$. Since $T_L(8) \nsubseteq G$, neither v_1 nor v_2 are adjacent to U, and $d_{G[U]}(v_3) \leq 1$. Furthermore, v_1 is not adjacent to w_4, and v_2 is not adjacent to w_1 or w_3. Let $W = \{w_4\} \cup U$; then $|W| = 8$. If $\delta(\overline{G}[W]) \geq 4$, then $\overline{G}[W]$ contains C_8 by Lemma 4.1 which with v_1 forms W_8, a contradiction. Thus, $\delta(\overline{G}[W]) < 3$ and $\Delta(G[W]) \geq 4$.

Now, suppose that $d_{G[W]}(w_4) \geq 4$ and assume without loss of generality that $u_1, \ldots, u_4 \in N_G(w_4)$. Then v_2 is not adjacent to v_1, v_3, w_1, w_2 and $d_{G[U]}(u_i) \leq 1$ for $1 \leq i \leq 4$, since $T_L(8) \nsubseteq G$. Since $d_{G[U]}(v_3) \leq 1$, assume without loss of generality that v_3 is not adjacent to u_3 or u_4. Now, suppose that $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$ and assume, say, that u_1 is adjacent to u_5. Then u_1 is not adjacent to $\{v_3, w_1, w_2, w_3, u_2, \ldots, u_7\}$. Since $T_L(8) \nsubseteq G$, at least one of w_1 and w_2 is adjacent in $\overline{G}$ to u_2, u_3 and u_4, say w_1, so $v_1u_2w_1u_3v_3u_4v_2u_6v_1$ and u_1 form W_8 in $\overline{G}$, a contradiction. Thus, $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\}) = \emptyset$. Then $u_1u_5u_2u_6u_3u_7u_4v_2u_1$ and v_1 form W_8 in $\overline{G}$, a contradiction. Therefore, $d_{G[W]}(u') \geq 4$ for some vertex of $u' \in U$, say $u' = u_1$.

Suppose that w_4 is adjacent to u_1. Then without loss of generality, assume that u_1 is adjacent to u_2, u_3 and u_4. Since $T_L(8) \nsubseteq G$, neither v_0 nor w_4 is adjacent to w_1 or w_2, and w_4 is not adjacent to $\{v_1, v_3\} \cup U\backslash\{u_1\}$. If $E_G(\{u_2, u_3, u_4\}, \{u_5, u_6, u_7\}) \neq \emptyset$, then, say, u_2 is adjacent to u_5 and is thus not adjacent to $\{v_0, v_3, w_1, w_2, w_3, u_3, u_4, u_6, u_7\}$, so $w_1v_0w_2w_4u_3v_1u_4v_2w_1$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus $E_G(\{u_1, \ldots, u_4\}, \{u_5, u_6, u_7\} = \emptyset$. Let $X = \{v_1, u_2, u_3, u_4\}$ and $Y = \{v_3, u_5, u_6, u_7\}$. Since $d_{G[U]}(v_3) \leq 1$, $\overline{G}[X \cup Y]$ contains C_8 by Lemma 4.5 which, with w_4, forms W_8, a contradiction.

Thus, u_1 is not adjacent to w_4 so assume without loss of generality that $u_2, \ldots, u_5 \in N_G(u_1)$. Since G does not contain $T_L(8)$, $d_{G[V(T)]}(u_i) \leq 1$ for $2 \leq i \leq 5$. If u_2 is adjacent to w_4, then u_2 is not adjacent to $V(G)\backslash\{u_1, w_4\}$ in G. Since $d_{G[U]}(v_3) \leq 1$, that v_3 is not adjacent to, say, u_3 or u_4. Since $d_{G[V(T)]}(u_i) \leq 1$ for $2 \leq i \leq 5$, u_4 and u_5 are each adjacent in $\overline{G}$ to at least 2 of w_1, w_2, w_3, so some $w_i \in \{w_1, w_2, w_3\}$

is adjacent in $\overline{G}$ to both u_4 and u_5. Therefore, $u_3v_3u_4w_iu_5v_2u_6v_1u_3$ and u_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to w_4. Similarly, u_3, u_4, u_5 are not adjacent to w_4. Similar arguments show that $u_2, \ldots, u_5$ are not adjacent to w_1 or w_2.

Now, if u_2 is adjacent to any other vertex of $V(T)$, then u_2 is not adjacent to $\{u_3, u_4, u_5\}$, so $u_3w_1u_4w_4u_5v_2u_6v_1u_3$ and u_2 form W_8 in $\overline{G}$, a contradiction. Hence, u_2 is not adjacent to $V(T)$ and, similarly, u_3, u_4, u_5 are not adjacent to $V(T)$. Therefore, by Observation 4.3, $\delta(G[V(T)]) \geq 4$. By Lemma 7.13, $G[V(T)]$ contains $T_L(8)$, a contradiction. Thus, $R(T_L(8), W_8) \leq 15$.

Now suppose that $n \geq 12$. If G contains $S_n(1, 3)$, then the previous arguments above lead to contradictions. Thus, G does not contain $S_n(1, 3)$. By Theorem 6.8, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_2w_3\}$. Set $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|U| = n - 1$.

Suppose that w_2 is not adjacent to U. If $\delta(\overline{G}[U]) \geq \frac{n-1}{2}$, then G contains C_8 by Lemma 4.1 and, with w_2 as hub, forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U]) < \frac{n-1}{2}$ and so $\Delta(G[U]) \geq \lfloor \frac{n-1}{2} \rfloor \geq 5$. Without loss of generality, assume that $u_2, \ldots, u_6 \in N_G(u_1)$. Since $S_n(1, 3) \nsubseteq G$, $u_2, \ldots, u_6$ are not adjacent to $V(T)\backslash\{v_0\}$. If u_2 is adjacent to v_0, then since $S_n(1, 3) \nsubseteq G$, $u_3, \ldots, u_6$ are not adjacent to $\{u_7, \ldots, u_{n-1}\}$, so $u_3u_7u_4u_8u_5u_9u_6u_{10}u_3$ and w_2 form W_8 in $\overline{G}$, a contradiction. Thus, u_2 is not adjacent to v_0 and, similarly, $u_3, \ldots, u_6$ are also not adjacent to v_0. Hence, $u_2, \ldots, u_6$ are not adjacent to $V(T)$. Therefore, by Observation 4.3, $\delta(G[V(T)]) \geq n - 4$, so $G[V(T)]$ contains $T_L(n)$ by Lemma 7.13, a contradiction.

Thus some vertex of U, say u_{n-1}, is adjacent to w_2. Set $U' = U\backslash\{u_{n-1}\}$; then $|U'| = n-2$. Since $T_L(n) \nsubseteq G$, u_{n-1} is not adjacent to U' in G. Now, if $\delta(\overline{G}[U']) \geq \frac{n-2}{2}$, then $\overline{G}[U']$ contains C_8 by Lemma 4.1 which, with u_{n-1}, forms W_8, a contradiction. Thus, $\delta(\overline{G}[U']) \leq \frac{n-2}{2} - 1$, and $\Delta(G[U']) \geq \frac{n-2}{2} \geq 5$. Without loss of generality, assume that $u_2, \ldots, u_6 \in N_G(u_1)$ and repeat the above arguments to prove that $u_2, \ldots, u_6$ are not adjacent to $V(T)$. Therefore, $\delta(G[V(T)]) \geq n - 4$ by Observation 4.3, so $G[V(T)]$ contains $T_L(n)$ by Lemma 7.13, a contradiction.

Therefore, $R(T_L(n), W_8) \leq 2n-1$ for $n \equiv 0 \pmod 4$, which completes the proof. □

Theorem 7.18 *If $n \geq 9$, then $R(T_M(n), W_8) = 2n - 1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n - 1$. Assume that G does not contain $T_M(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.4, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Set $V = \{v_2, v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|V| = n-5$ and $|U| = n - 1$. Since $T_M(n) \nsubseteq G$, w_1, w_2 and w_3 are not adjacent to any vertex of $U \cup V$ in G.

Now, suppose that some vertex in V is adjacent to at least 4 vertices of U in G, say v_2 to $u_1, \ldots, u_4$. Then $u_1, \ldots, u_4$ are not adjacent to other vertices in U. Then $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_1 form W_8 in $\overline{G}$, a contradiction. Therefore, each vertex in V is adjacent to at most three vertices of U in G. Choose any 8 vertices of U.

By Corollary 4.8, $\overline{G}[U \cup V]$ contains C_8 which together with w_1 gives W_8 in $\overline{G}$, a contradiction.

Thus, $R(T_M(n), W_8) \leq 2n - 1$ for $n \geq 9$. This completes the proof. □

Theorem 7.19 *If $n \geq 9$, then*

$$R(T_N(n), W_8) = \begin{cases} 2n - 1 & \text{if } n \not\equiv 0 \pmod 4; \\ 2n & \text{otherwise.} \end{cases}$$

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n$ if $n \equiv 0 \pmod 4$ and of order $2n - 1$ if $n \not\equiv 0 \pmod 4$. Assume that G does not contain $T_N(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.6, G has a subgraph $T = T_A(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, w_1w_3\}$. Set $V = \{v_2, v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_j\}$, where $j = n - 1$ if $n \not\equiv 0 \pmod 4$ and $j = n$ otherwise. Since $T_N(n) \nsubseteq G$, w_2 is not adjacent to $U \cup V$ in G. If each $v_i \in V$ is adjacent to at most three vertices of U in G, then by Corollary 4.8, $\overline{G}[U \cup V]$ contains C_8 which with w_2 gives W_8 in $\overline{G}$, a contradiction. Therefore, some vertex in V, say v_2, is adjacent to at least four vertices of U in G, say $u_1, \ldots, u_4$. If none of these is adjacent to other vertices of U in G, then $u_1u_5u_2u_6u_3u_7u_4u_8u_1$ and w_2 form W_8 in $\overline{G}$, a contradiction.

Therefore, assume that u_1 is adjacent to u_5 in G. Since $T_N(n) \nsubseteq G$, u_2, u_3, u_4 are not adjacent to $\{u_6, \ldots, u_j\}$ in G. For $n = 9$ and $n = 10$, $\{v_3, \ldots, v_{n-4}\}$ is not adjacent to $\{u_5, \ldots, u_{n-1}\}$ or else G will contain $T_N(n)$ with v_2 and v_0 being the vertices of degree $n - 5$ and 3, respectively. However, $v_3u_5v_4u_6u_2u_7u_3u_8v_3$ and w_2 form W_8 in $\overline{G}$, a contradiction. For $n \geq 11$, if v_2 is not adjacent to $\{u_6, \ldots, u_j\}$ in G, then $v_2u_6u_2u_7u_3u_8u_4u_9v_2$ and w_2 form W_8 in $\overline{G}$, a contradiction. Therefore, assume that v_2 is adjacent to u_6 in G. Then u_6 is not adjacent to $\{u_7, \ldots, u_j\}$ in G, and $u_2u_7u_3u_8u_4u_9u_6u_{10}u_2$ and w_2 form W_8 in $\overline{G}$, again a contradiction.

Thus, $R(T_N(n), W_8) \leq 2n$ for $n \equiv 0 \pmod 4$ and $R(T_N(n), W_8) \leq 2n - 1$ for $n \not\equiv 0 \pmod 4$. □

Theorem 7.20 *If $n \geq 9$, then $R(T_P(n), W_8) = 2n - 1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n - 1$. Assume that G does not contain $T_P(n)$ and that $\overline{G}$ does not contain W_8. Suppose $n \not\equiv 0 \pmod 4$. By Theorem 6.6, G has a subgraph $T = T_A(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, w_1w_3\}$. Set $V = \{v_2, v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n - 5$ and $|U| = n - 1$. Since $T_P(n) \nsubseteq G$, w_1 is not adjacent to any vertex of $U \cup V$ in G. If each v_i in V is adjacent to at most three vertices of U in G, then by Corollary 4.8, $\overline{G}[U \cup V]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, some vertex in V, say v_2, is adjacent to at least four vertices of U in G, say $u_1, \ldots, u_4$. For $n = 9$ and $n = 10$, G contains $T_P(9)$ and $T_P(10)$ with edge set $\{u_1v_2, u_2v_2, u_3v_2, v_2v_0, v_0v_1, v_0v_3, v_1w_1, v_1w_2\}$ and $\{u_1v_2, u_2v_2, u_3v_2, u_4v_2, v_2v_0, v_0v_1, v_0v_3, v_1w_1, v_1w_2\}$, respectively. For $n \geq 11$, each of $u_1, \ldots, u_4$ is adjacent to at most two remaining vertices in U. Then by Corollary 4.7, $\overline{G}[U]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction.

Suppose that $n \equiv 0 \pmod 4$. By Theorem 7.18, G contains a subgraph $T = T_M(n)$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-5}, v_1w_1, v_1w_2, v_1w_3, w_1w_4\}$. Let $V = \{v_2, v_3, \ldots, v_{n-5}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n-1$. Since $T_P(n) \nsubseteq G$, w_1 is not adjacent to $\{v_0, w_2, w_3\} \cup U$ in G, and so $d_{G[U]}(w_2) \leq 1$, $d_{G[U]}(w_3) \leq 1$ and $d_{G[U]}(v) \leq n-7$ for any vertex $v \in V$. Now, if G contains a subgraph $T_A(n)$, then arguments similar to those used for the case $n \not\equiv 0 \pmod 4$ above can be used. Therefore, G contains no $T_A(n)$. Then v_0 is not adjacent to $\{w_2, w_3\} \cup U$ in G.

Suppose that some vertex $v \in V$ is not adjacent to w_1 in G. Let X be any four vertices in U that are not adjacent to v in G and set $Y = \{v, v_0, w_2, w_3\}$. By Lemma 4.5, $\overline{G}[X \cup Y]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, each vertex of V is adjacent to w_1 in G. Since $T_P(n) \nsubseteq G$, w_4 is adjacent to at most $n-7$ vertices of U in G. Since $T_A(n) \nsubseteq G$, w_2 and w_3 are not adjacent in G. Now, if w_4 is adjacent to both w_2 and w_3 in G, then w_4 is not adjacent to v_0 in G since $T_P(n) \nsubseteq G$. Let X be any four vertices of U that are not adjacent to w_4 in G and let $V = \{w_1, \ldots, w_4\}$. By Lemma 4.5, $\overline{G}[X \cup Y]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, a contradiction. Therefore, w_4 is non-adjacent to either w_2 or w_3 in G, say w_2. Since $d_{G[U]}(w_2) \leq 1$ and $d_{G[U]}(w_4) \leq n-7$, there is a set X of four vertices in U that are not adjacent to both w_2 and w_4 in G. Let $Y = \{v_0, w_1, w_3, w_4\}$. By Lemma 4.5, $\overline{G}[X \cup Y]$ contains C_8 which with w_1 gives W_8 in $\overline{G}$, again a contradiction.

In either case, $R(T_P(n), W_8) \leq 2n-1$ for $n \geq 9$ and this completes the proof. □

Theorem 7.21 *If* $n \geq 9$, *then* $R(T_Q(n), W_8) = 2n - 1$.

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_Q(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.4, G has a subgraph $T = S_n(4)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_1w_2, v_1w_3\}$. Set $V = \{v_2, v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_Q(n) \nsubseteq G$, $G[V]$ are independent vertices and not adjacent to U.

Suppose that $n \geq 10$. Then $|V| \geq 5$ and $|U| \geq 9$, so by Observation 4.3, $\overline{G}$ contains W_8, a contradiction. If $n = 9$, then $|V| = 4$ and $|U| = 8$. By Lemma 4.4, $G[U]$ is K_8 or $K_8 - e$. Since $T_Q(9) \nsubseteq G$, T is not adjacent to U, and $\delta(G[V(T)] \geq 5$. As $v_2, \ldots, v_5$ are independent in G, they are each adjacent to all other vertices in $G[V(T)]$. Therefore, $G[V(T)]$ contains $T_Q(9)$ with v_2 and v_0 as the vertices of degree 4, a contradiction.

Thus, $R(T_Q(n), W_8) \leq 2n-1$ for $n \geq 9$, which completes the proof. □

Theorem 7.22 *If* $n \geq 9$, *then* $R(T_R(n), W_8) = 2n - 1$.

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_R(n)$ and that $\overline{G}$ does not contain W_8. By Theorem 6.8, G has a subgraph $T = T_C(n)$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_1, \ldots, v_0v_{n-4}, v_1w_1, v_2w_2, v_2w_3\}$. Set $V = \{v_3, \ldots, v_{n-4}\}$ and $U = V(G) - V(T) = \{u_1, \ldots, u_{n-1}\}$; then $|V| = n-6$

and $|U| = n-1$. Since $T_R(n) \nsubseteq G$, w_1 is not adjacent in G to any vertex of $U \cup V$. If $\delta(\overline{G}[U \cup V]) \geq \lceil \frac{2n-7}{2} \rceil$, then $\overline{G}[U \cup V]$ contains C_8 by Lemma 4.1 which with w_3 forms W_8, a contradiction. Therefore, $\delta(\overline{G}[U \cup V]) \leq \lceil \frac{2n-7}{2} \rceil - 1$, and $\Delta(G[U \cup V]) \geq \lfloor \frac{2n-7}{2} \rfloor = n-4$. Now, there are two cases to be considered.

Case 1: One of the vertices of V, say v_3, is a vertex of degree at least $n-4$ in $G[U \cup V]$.

Note that in this case, there are at least 3 vertices from U, say u_1, u_2, u_3, that are adjacent to v_3 in G. Suppose that v_3 is also adjacent to a in G, where a is a vertex in $U \cup V$. Since $T_R(n) \nsubseteq G$, these 4 vertices are independent and are not adjacent to any other vertices of U. Since $n \geq 9$, U contains at least 4 other vertices, say $u_5, \ldots, u_8$, so $u_1 u_5 u_2 u_6 u_3 u_7 a u_8 u_1$ and w_3 form W_8 in $\overline{G}$, a contradiction.

Case 2: Some vertex $u \in U$ has degree at least $n-4$ in $G[U \cup V]$.

Since $T_R(n) \nsubseteq G$, u is not adjacent to any vertex of V in G. Therefore, u must be adjacent to at least $n-4$ vertices of U in G. Without loss of generality, suppose that $u_1, \ldots, u_{n-4} \in N_{G[U]}(u)$. Note that V is not adjacent to $N_{G[U]}(u)$, or else it will form $T_R(n)$ in G, a contradiction. If $n \geq 10$, then any 4 vertices from $N_{G[U]}(u)$ and any 4 vertices from V form C_8 in $\overline{G}$ which with w_3 forms W_8, a contradiction. Suppose that $n = 9$ and let the remaining two vertices be u_6 and u_7. If either u_6 or u_7 is non-adjacent to any two vertices of $\{u_1, \ldots, u_5\}$ in G, say u_6 is not adjacent to u_1 and u_2 in G, then $u_1 u_6 u_2 v_3 u_3 v_4 u_4 v_5 u_1$ and w_3 form W_8 in $\overline{G}$, a contradiction. So, both u_6 and u_7 are adjacent to at least 4 vertices of $\{u_1, \ldots, u_5\}$ in G. Since $T_R(9) \nsubseteq G$, T cannot be adjacent to U, and $\delta(G[V(T)] \geq 5$. As both v_2 and w_3 are not adjacent to v_3, v_4 or v_5 in G, they are adjacent to all other vertices in $G[V(T)]$. Similarly, since v_3 is not adjacent to v_2 or w_3 in G, v_3 is adjacent to w_1 or w_2 in G. Without loss of generality, assume that v_3 is adjacent to w_1. Then $G[V(T)]$ contains $T_R(9)$ with edge set $\{v_2 w_2, v_2 v_1, v_2 v_0, v_0 v_4, v_0 v_5, v_2 w_3, v_2 w_1, w_1 v_3\}$, a contradiction.

In either case, $R(T_R(n), W_8) \leq 2n-1$. □

Theorem 7.23 *If $n \geq 9$, then $R(T_S(n), W_8) = 2n-1$.*

Proof Lemma 7.1 provides the lower bound, so it remains to prove the upper bound. Let G be any graph of order $2n-1$. Assume that G does not contain $T_S(n)$ and that $\overline{G}$ does not contain W_8. Suppose $n \not\equiv 0 \pmod 4$. By Theorem 6.4, G has a subgraph $T = S_n[4]$. Let $V(T) = \{v_0, \ldots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0 v_1, \ldots, v_0 v_{n-4}, v_1 w_1, w_1 w_2, w_1 w_3\}$. Set $V = \{v_2, \ldots, v_{n-4}\}$ and $U = V(G) - V(T)$; then $|V| = n-5$ and $|U| = n-1$. Since $T_S(n) \nsubseteq G$, $G[V]$ are independent vertices and are not adjacent to U. If $n \geq 10$, then $|V| \geq 5$ and $|U| \geq 9$, so by Observation 4.3, $\overline{G}$ contains W_8, a contradiction. Suppose that $n = 9$. Then $|V| = 4$ and $|U| = 8$. By Lemma 4.4, $G[U]$ is K_8 or $K_8 - e$. Since $T_S(9) \nsubseteq G$, T is not adjacent to U, and $\delta(G[V(T)] \geq 5$. As $v_2, \ldots, v_5$ are independent in G, they are adjacent to all other vertices in $G[V(T)]$, and so $G[V(T)]$ contains $T_S(9)$ with edge set $\{v_0 v_1, v_0 v_2, v_1 v_4, v_1 v_5, v_2 w_1, v_2 w_2, v_2 w_3, v_3 w_1\}$.

Now suppose that $n \equiv 0 \pmod 4$. By Theorem 6.4, G has a subgraph $T = S_{n-1}[4]$. Let $V(T) = \{v_0, \ldots, v_{n-5}, w_1, w_2, w_3\}$ and $E(T) = \{v_0 v_1, \ldots, v_0 v_{n-5}, v_1 w_1, w_1 w_2, w_1 w_3\}$. Set $V = \{v_2, \ldots, v_{n-5}\}$ and $U = V(G) - V(T)$; then $|V| = n-6$ and $|U| = n$. Since $T_S(n) \nsubseteq G$, $G[V]$ is not adjacent to U. Since $|V| = n-6 > 4$, by Observation 4.3, $\Delta(\overline{G}[U]) \leq 3$ and $\delta(G[U]) \geq n-4$ since

$W_8 \nsubseteq \overline{G}$. By Lemma 6.3, either $G[U]$ is $K_{4,\ldots,4}$ and contains $T_S(n)$ or $G[U]$ contains $S_n[4]$ and the arguments from the $n \not\equiv 0 \pmod 4$ case above lead to a contradiction.

Thus, $R(T_S(n), W_8) \leq 2n-1$ for $n \geq 9$, which completes the proof. □

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Declarations

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