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Geometric Studies and the Bohr Radius for Certain Normalized Harmonic Mappings

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Abstract

Let $\mathcal{H}$ be the class of harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, where h and g are analytic in $\mathbb{D}$. In 2020, N. Ghosh and V. Allu introduced the class $\mathcal{P}^0_{\mathcal{H}}(M)$ of normalized harmonic mappings defined by $\mathcal{P}^0_{\mathcal{H}}(M) = \{f = h + \overline{g} \in \mathcal{H} : \mathrm{Re}(zh''(z)) > -M + |zg''(z)| \text{ with } M > 0, g'(0) = 0, z \in \mathbb{D}\}$. In this paper, we investigate various geometric properties such as starlikeness, convexity, convex combination and convolution for functions in the class $\mathcal{P}^0_{\mathcal{H}}(M)$. Furthermore, we determine the sharp Bohr–Rogosinski radius, improved Bohr radius and refined Bohr radius for the class $\mathcal{P}^0_{\mathcal{H}}(M)$.

Keywords Analytic · Univalent · Harmonic functions · Starlike · Convex · Close-to-convex functions · Coefficient estimate · Growth theorem · Bohr radius

Mathematics Subject Classification 30C45 · 30C50 · 30C80

1 Introduction and Preliminaries

The classical inequality of Bohr says that if f is an analytic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the following Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

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such that $|f(z)| < 1$ in $\mathbb{D}$. Then

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \text{ for } |z| = r \leq \frac{1}{3}. \tag{1.2}$$

Here $1/3$ is known as Bohr radius and it can't be improved, while the inequality (1.2) is known as Bohr inequality. In 1914, H. Bohr [10] obtained the inequality (1.2) for $r \leq 1/6$ but later Weiner, Riesz and Schur [12] independently improved it to $1/3$. An observation shows that the quantity $1 - |a_0|$ is equal to $d(f(0), \partial\mathbb{D})$, where d is the Euclidean distance and $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$. Therefore, the inequality (1.2) also can be written as

$$\sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0| = d(f(0), \partial\mathbb{D}) \text{ for } |z| = r \leq \frac{1}{3}. \tag{1.3}$$

It is important to note that the constant $1/3$ is independent of the coefficients of the Taylor series (1.1). This fact can be elucidated by saying that Bohr inequality occurs in the class $\mathcal{B}$ of analytic self maps of the unit disk $\mathbb{D}$. Analytic functions $f \in \mathcal{B}$ of the form (1.1) satisfying the inequality (1.2) for $|z| = r \leq 1/3$, are said to satisfy the classical Bohr phenomenon. The concept of Bohr phenomenon can be generalized to the class $\mathcal{A}$ consisting of analytic functions of the form (1.1) which map from $\mathbb{D}$ into a given domain $\Theta \subseteq \mathbb{C}$ such that $f(\mathbb{D}) \subseteq \Theta$. The class $\mathcal{A}$ is said to satisfy the Bohr phenomenon if $\exists$ largest radius $r_\Theta \in (0, 1)$ such that (1.3) holds for $|z| = r \leq r_\Theta$. Here r_Θ is known as Bohr radius for the class $\mathcal{A}$. The Bohr radius has been obtained for the class $\mathcal{A}$ when Θ is convex domain [4], simply connected domain [1], the exterior of the closed unit disk, the punctured unit disk, and concave wedge domain (see [5]). In 1997, Boas and Khavinson [9] generalized the Bohr inequality in several complex variables by finding multidimensional Bohr radius. In 2021, Liu and Ponnusamy [22] obtained multidimensional analogues of refined Bohr inequality.

There are many improved versions of Bohr's inequality (1.2) in various forms obtained by several authors. In 2020, Kayumov and Ponnusamy [20] obtained several interesting improved versions of Bohr inequality. For more results on this, we refer the reader to glance through the articles (see [16–21, 23, 24]). In 2017, Kayumov and Ponnusamy [18] introduced Bohr–Rogosinski radius motivated by Rogosinski radius for bounded analytic functions in $\mathbb{D}$. Rogosinski radius is defined as follows: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and its corresponding partial sum of f is defined by $S_N(z) := \sum_{n=0}^{N-1} a_n z^n$. Then, for every $N \geq 1$, we have $|\sum_{n=0}^{N-1} a_n z^n| < 1$ in the disk $|z| < 1/2$ and the radius $1/2$ is sharp. Motivated by Rogosinski radius, Kayumov and Ponnusamy have considered the Bohr–Rogosinski sum $R_N^f(z)$ which is defined by

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n. \tag{1.4}$$

It is worth to point out that $|S_N(z)| = \left|f(z) - \sum_{n=N}^{\infty} a_n z^n\right| \leq R_N^f(z)$. Therefore, it is easy to see that the validity of Bohr-type radius for $R_N^f(z)$, which is related to the classical Bohr sum (Majorant series) in which $f(0)$ is replaced by $f(z)$, gives Rogosinski radius in the case of bounded analytic functions in $\mathbb{D}$. We have the following interesting results by Kayumov and Ponnusamy [18].

Theorem A [18] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and $|f(z)| \leq 1$. Then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq 1 \tag{1.5}$$

for $|z| = r \leq R_N$, where R_N is the positive root of the equation $\psi_N(r) = 0$, $\psi_n(r) = 2(1+r)r^N - (1-r)^2$. The radius R_N is the best possible. Moreover,

$$|f(z)|^2 + \sum_{n=N}^{\infty} |a_n||z|^n \leq 1 \tag{1.6}$$

for R_N', where R_N' is the positive root of the equation $(1+r)r^N - (1-r)^2 = 0$. The radius R_N' is the best possible.

In 2020, Kayumov and Ponnusamy [20] have proved the following several improved versions of Bohr's inequality for analytic functions.

Theorem B [20] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$, $|f(z)| \leq 1$ and S_r denotes the area of the image of the subdisk $|z| < r$ under mapping f. Then*

$$B_1(r) := \sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9}\left(\frac{S_r}{\pi}\right) \leq 1 \text{ for } r \leq \frac{1}{3} \tag{1.7}$$

and the numbers $1/3$, $16/9$ cannot be improved. Moreover,

$$B_2(r) := |a_0|^2 + \sum_{n=1}^{\infty} a_n r^n + \frac{9}{8}\left(\frac{S_r}{\pi}\right) \leq 1 \text{ for } r \leq \frac{1}{2} \tag{1.8}$$

and the numbers $1/2$, $9/8$ cannot be improved.

In 2020, Ponnusamy et al. [23] established the following refined Bohr inequality by applying a refined version of the coefficient inequalities.

Theorem C [23] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and $|f(z)| \leq 1$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 1 \tag{1.9}$$

for $r \leq 1/(2+|a_0|)$ *and the numbers* $1/(1+|a_0|)$ *and* $1/(2+|a_0|)$ *cannot be improved. Moreover,*

$$|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 1$$

for $r \leq 1/2$ *and the numbers* $1/(1+|a_0|)$ *and* $1/2$ *cannot be improved.*

Bohr's phenomenon for the complex-valued harmonic mappings have been studied extensively by many authors (see [1, 2, 6, 7]). Improved Bohr inequality for locally univalent harmonic mappings have been discussed by Evdoridis et al. [13].

A complex-valued function $f = u + iv$ is harmonic if u and v are real-harmonic in $\mathbb{D}$. Every harmonic function f has the canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ known respectively as the analytic and co-analytic parts of f. A locally univalent harmonic function f is said to be sense-preserving if the Jacobian of f, defined by $J_f(z) := |h'(z)|^2 - |g'(z)|^2$, is positive in $\mathbb{D}$ and sense-reversing if $J_f(z)$ is negative in $\mathbb{D}$. Let $\mathcal{H}$ be the class of all complex- valued harmonic functions $f = h + \overline{g}$ defined in $\mathbb{D}$, where h and g are analytic in $\mathbb{D}$ such that $h(0) = h'(0) - 1 = 0$ and $g(0) = 0$. If the co-analytic part $g(z) \equiv 0$ in $\mathbb{D}$, then the class $\mathcal{H}$ reduces to the class $\mathcal{A}$ of analytic functions in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. A function $f \in \mathcal{H}$ is said to be in $\mathcal{H}_0$ if $g'(0) = 0$. Thus, every $f = h + \overline{g} \in \mathcal{H}_0$ has the following form

$$f(z) = h(z) + \overline{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}. \tag{1.10}$$

A domain Ω is called starlike with respect to a point $z_0 \in \Omega$ if the line segment joining z_0 to any point in Ω lies in Ω. In particular, if $z_0 = 0$, then Ω is simply called starlike. A complex-valued harmonic mapping $f \in \mathcal{H}$ is said to be starlike if $f(\mathbb{D})$ is starlike. We denote the class of harmonic starlike functions in $\mathbb{D}$ by $\mathcal{S}^*_{\mathcal{H}}$. A domain Ω is called convex if it is starlike with respect to every point in Ω. A function $f \in \mathcal{H}$ is said to be convex if $f(\mathbb{D})$ is convex. We denote $\mathcal{K}_{\mathcal{H}}$ by the class of harmonic convex mappings in $\mathbb{D}$.

Definition A *The polylogarithm* $Li_s(z)$*, also known as Jonquière's function, is a special function of order* s *and argument* z

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots$$

defined in the complex plane over the unit disk. The special case $s = 1$ *involves the ordinary natural logarithm,* $Li_1(z) = -\ln(1-z)$*, while the special cases* $s = 2$ *and* $s = 3$ *are called the dilogarithm (also known as Spence's function) and trilogarithm respectively.*

In 2020, Ghosh and Allu [14] considered the following class for $M > 0$,

$$\mathcal{P}_{\mathcal{H}}^{0}(M) = \{f = h + \overline{g} \in \mathcal{H} : \mathrm{Re}\left(zh''(z)\right) > -M + |zg''(z)| \text{ with } g'(0) = 0 \text{ for } z \in \mathbb{D}\}.$$

The organization of this paper is: In Sect. 3, we discuss some geometric properties for functions in the class $\mathcal{P}_{\mathcal{H}}^{0}(M)$. In Sect. 4, we obtain sharp Bohr–Rogosinski radius for the class $\mathcal{P}_{\mathcal{H}}^{0}(M)$. In Sect. 5, we establish interesting sharp improved Bohr radius $\mathcal{P}_{\mathcal{H}}^{0}(M)$. In Sect. 6, we prove sharp refined Bohr radius for the class $\mathcal{P}_{\mathcal{H}}^{0}(M)$. The rest sections are for lemmas and proofs of the main results.

2 Some Lemmas

We have the following lemmas related to coefficient bounds and growth estimates for the class $\mathcal{P}_{\mathcal{H}}^{0}(M)$.

Lemma 2.1 [14] *The harmonic map $f = h + \overline{g}$ belongs to $\mathcal{P}_{\mathcal{H}}^{0}(M)$ if and only if the function $F_{\epsilon} = h + \epsilon g$ belongs to $\mathcal{P}(M)$ for $|\epsilon| = 1$, where $\mathcal{P}(M)$ is defined by*

$$\mathcal{P}(M) := \left\{\phi \in \mathcal{A} : Re\left(z\phi''(z)\right) > -M \ \ for\ M > 0\ and\ z \in \mathbb{D}\right\}. \tag{2.1}$$

Lemma 2.2 [14] *Let $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ for $M > 0$ be given by* (1.10). *Then for each $n \geq 2$, we have $|b_n| \leq \frac{2M}{n(n-1)}$. The result is sharp with $f(z) = z - \frac{M}{n(n-1)}\overline{z^n}$ being extremal.*

Remark 2.1 We have found some typographical error in Lemma 2.2 of [14] and the correct statement is given below:

If $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ for $M > 0$ is given by (1.10). Then for each $n \geq 2$, we have $|b_n| \leq \frac{M}{n(n-1)}$. The result is sharp with $f(z) = z - \frac{M}{n(n-1)}\overline{z^n}$ being extremal.

Lemma 2.3 [14] *Let $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ for $M > 0$ be given by* (1.10). *Then for any $n \geq 2$, we have (i) $|a_n| + |b_n| \leq \frac{2M}{n(n-1)}$; (ii) $||a_n| - |b_n|| \leq \frac{2M}{n(n-1)}$; (iii) $|a_n| \leq \frac{2M}{n(n-1)}$. The result is sharp for the function f given by $f'(z) = 1 - 2M \ln(1-z)$.*

Lemma 2.4 [7, 14] *Let $f = h + \overline{g} \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ for $M > 0$ be given by* (1.10). *Then*

$$|z| + 2M \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n(n-1)}|z|^n \leq |f(z)| \leq |z| + 2M \sum_{n=2}^{\infty} \frac{1}{n(n-1)}|z|^n.$$

Both the inequalities are sharp for the function $f_M = z + 2M \sum_{n=2}^{\infty} \frac{1}{n(n-1)} z^n$.

To prove our convolution results, we need the following definitions and lemmas.

Definition 2.1 [11, 25] Let ψ_1 and ψ_2 be two analytic functions in $\mathbb{D}$ given by $\psi_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi_2(z) = \sum_{n=0}^{\infty} b_n z^n$. The convolution (or, Hadamard product) is defined by

$$(\psi_1 * \psi_2)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad \text{for } z \in \mathbb{D}.$$

Definition 2.2 [15] For harmonic functions $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ in $\mathcal{H}$, the convolution is defined as $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$.

Definition 2.3 [25] A sequence $\{a_n\}$ of non-negative numbers is said to be a convex null sequence if $a_n \to 0$ as $n \to \infty$ and $a_0 - a_1 \geq a_1 - a_2 \geq \cdots \geq a_{n-1} - a_n \geq \cdots \geq 0$.

Lemma 2.5 [25] *Let $\{a_n\}$ be a convex null sequence. Then the function p given by $p(z) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $Re(p(z)) > 0$, $z \in \mathbb{D}$.*

Lemma 2.6 [25] *Let the function p be analytic in $\mathbb{D}$ with $p(0) = 1$ and $Re\ p(z) > \frac{1}{2}$ in $\mathbb{D}$. Then for any analytic function f in $\mathbb{D}$, the function $p * f$ takes values in the convex hull of the image of $\mathbb{D}$ under f.*

Lemma 2.7 *Let $\mathcal{P}(M)$ be the subclass of $\mathcal{A}$ defined in (2.1). If $F \in \mathcal{P}(M)$ with $0 < M \leq \frac{3}{5}$. Then $Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$.*

Proof Let $F \in \mathcal{P}(M)$ be given by $F(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$. Then, we have

$$\begin{aligned}
\text{Re}\left(zF''(z)\right) > -M \Rightarrow \text{Re}\left(M + \sum_{n=2}^{\infty} n(n-1)A_n z^{n-1}\right) > 0 \\
\Rightarrow \text{Re}\left(1 + \frac{1}{2M}\sum_{n=2}^{\infty} n(n-1)A_n z^{n-1}\right) > \frac{1}{2} \quad \text{for } \ z \in \mathbb{D}.
\end{aligned}$$

Let $p(z) = 1 + \frac{1}{2M}\sum_{n=2}^{\infty} n(n-1)A_n z^{n-1}$. Then $p(0) = 1$ and $\text{Re}(p(z)) > \frac{1}{2}$ in $\mathbb{D}$. Now, we consider a sequence $\{c_n\}$ defined by $c_0 = 1$ and $c_{n-1} = \frac{2M}{n(n-1)}$ for $n \geq 2$. It is clear that $c_n \to 0$ as $n \to \infty$. Note that $c_0 - c_1 = 1 - M$ and $c_1 - c_2 = 2\,M/3$. So $c_0 - c_1 \geq c_1 - c_2 \geq \cdots \geq c_{n-1} - c_n \geq \cdots \geq 0$ is possible only when $0 < M \leq 3/5$. Thus $\{c_n\}$ is a convex null sequence. In view of Lemma 2.5, the function

$$q(z) = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2M}{n(n-1)} z^{n-1}$$

is analytic in $\mathbb{D}$ with $\text{Re}(q(z)) > 0$. Now,

$$\begin{aligned}
\frac{F(z)}{z} = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} &= p(z) * \left(1 + \sum_{n=2}^{\infty} \frac{2M}{n(n-1)} z^{n-1}\right) \\
&= p(z) * \left(q(z) + \frac{1}{2}\right). \qquad (2.2)
\end{aligned}$$

In view of Lemma 2.6 and (2.2), we have $\mathrm{Re}\left(\frac{F(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. This completes the proof. □

Lemma 2.8 *Let $F_1, F_2 \in \mathcal{P}(M)$ with $0 < M \leq \frac{3}{5}$, where $\mathcal{P}(M)$ is defined in (2.1). Then $F_1 * F_2 \in \mathcal{P}(M)$.*

Proof Let $F_1(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $F_2(z) = z + \sum_{n=2}^{\infty} B_n z^n$. Then the convolution of F_1 and F_2 is given by

$$F(z) = F_1(z) * F_2(z) = z + \sum_{n=2}^{\infty} A_n B_n z^n.$$

Now,

$$zF''(z) = \sum_{n=2}^{\infty} n(n-1) A_n B_n z^{n-1} = \left(\frac{F_2(z)}{z}\right) * \left(zF_1''(z)\right). \tag{2.3}$$

Since $F_1, F_2 \in \mathcal{P}(M)$, so $\mathrm{Re}(zF_1''(z)) > -M$ and in view of Lemma 2.7, we have $\mathrm{Re}\left(\frac{F_2(z)}{z}\right) > \frac{1}{2}$. In view of Lemma 2.6 and (2.3), we have $\mathrm{Re}(zF''(z)) > -M$ in $\mathbb{D}$. Therefore $F = F_1 * F_2 \in \mathcal{P}(M)$. This completes the proof. □

Lemma 2.9 [8] *Let $f = h + \overline{g}$ be given by (1.10).*

(i) *If $\sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right) \leq 1$, then f is starlike in $\mathbb{D}$;*
(ii) *If $\sum_{n=2}^{\infty} n^2\left(|a_n| + |b_n|\right) \leq 1$, then f is convex in $\mathbb{D}$.*

3 Convex Combinations and Convolutions

In this section, we will show that $\mathcal{P}_{\mathcal{H}}^{0}(M)$ is closed under convex combinations and convolutions.

Theorem 3.1 *The class $\mathcal{P}_{\mathcal{H}}^{0}(M)$ is closed under convex combinations.*

Proof Let $f_i = h_i + \overline{g_i} \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} t_i = 1$, where $0 \leq t_i \leq 1$ for each i. Then, we have

$$\mathrm{Re}\left(zh_i''(z)\right) > -M + |zg_i''(z)| \text{ with } h_i(0) = g_i(0) = h_i'(0) - 1 = g_i'(0) = 0 \text{ for } 1 \leq i \leq n.$$

The convex combination of the f_i's can be written as

$$f(z) = \sum_{i=1}^{n} t_i f_i(z) = h(z) + \overline{g(z)},$$

where $h(z)=\sum_{i=1}^{n} t_i h_i(z)$ and $g(z)=\sum_{i=1}^{n} t_i g_i(z)$. Then both h and g are analytic in $\mathbb{D}$ with $h(0)=g(0)=h'(0)-1=g'(0)=0$. Now,

$$\begin{aligned}\operatorname{Re}\left(zh''(z)\right) &= \operatorname{Re}\left(z\sum_{i=1}^{n} t_i h_i''\right)=\sum_{i=1}^{n} t_i \operatorname{Re}\left(zh_i''\right)\\ &> \sum_{i=1}^{n} t_i\left(-M+|zg_i''(z)|\right)=-M+\sum_{i=1}^{n} t_i\left|zg_i''(z)\right|\\ &\geq -M+\left|z\left(\sum_{i=1}^{n} t_i g_i''(z)\right)\right|=-M+|zg''(z)|.\end{aligned}$$

This shows that $f\in\mathcal{P}_{\mathcal{H}}^{0}(M)$. This completes the proof. □

Theorem 3.2 *Let $F_1, F_2\in\mathcal{P}_{\mathcal{H}}^{0}(M)$ with $0<M\leq\frac{3}{5}$. Then $F_1*F_2\in\mathcal{P}_{\mathcal{H}}^{0}(M)$.*

Proof Let $F_1=h_1+\overline{g_1}$ and $F_2=h_2+\overline{g_2}$ be two functions in $\mathcal{P}_{\mathcal{H}}^{0}(M)$. Then the convolution of F_1 and F_2 is given by $F_1*F_2=h_1*h_2+\overline{g_1*g_2}$. To show that $F_1*F_2\in\mathcal{P}_{\mathcal{H}}^{0}(M)$, it is sufficient to show that $F=h_1*h_2+\epsilon\,(g_1*g_2)\in\mathcal{P}(M)$ for each ϵ with $|\epsilon|=1$. By Lemma 2.1, we have $h_1+\epsilon g_1, h_2+\epsilon g_2\in\mathcal{P}(M)$ for each ϵ with $|\epsilon|=1$. Thus, we deduce that

$$F=h_1*h_2+\epsilon\,(g_1*g_2)=\frac{1}{2}\left((h_1-g_1)*(h_2-\epsilon g_2)\right)+\frac{1}{2}\left((h_1+g_1)*(h_2+\epsilon g_2)\right).$$

In view of Lemma 2.8, we have $(h_1-g_1)*(h_2-\epsilon g_2)$, $(h_1+g_1)*(h_2+\epsilon g_2)\in\mathcal{P}(M)$. Then in view of Theorem 3.1, we get $F\in\mathcal{P}(M)$. Hence $\mathcal{P}_{\mathcal{H}}^{0}(M)$ is closed under convolution. This completes the proof. □

In 2002, Goodloe [15] considered the Hadamard product of a harmonic function with an analytic function as follows.

$$f\tilde{*}\phi=h*\phi+\overline{g*\phi},$$

where $f=h+\overline{g}$ is harmonic and ϕ is an analytic function in $\mathbb{D}$.

Theorem 3.3 *Let $f\in\mathcal{P}_{\mathcal{H}}^{0}(M)$ and $\phi\in\mathcal{A}$ be such that Re $\left(\frac{\phi(z)}{z}\right)>\frac{1}{2}$ for $z\in\mathbb{D}$. Then $f\tilde{*}\phi\in\mathcal{P}_{\mathcal{H}}^{0}(M)$.*

Proof Let $f=h+\overline{g}\in\mathcal{P}_{\mathcal{H}}^{0}(M)$. In view of Lemma 2.1, we have $f_1=h+\epsilon g\in\mathcal{P}(M)$ for each ϵ with $|\epsilon|=1$. To prove that $f\tilde{*}\phi=h*\phi+\overline{g*\phi}\in\mathcal{P}_{\mathcal{H}}^{0}(M)$, it is sufficient to show that $F(z)=h*\phi+\epsilon(g*\phi)\in\mathcal{P}(M)$ for each $\epsilon(|\epsilon|=1)$. Since $f_1\in\mathcal{P}(M)$ and $\phi\in\mathcal{A}$, so we assume that $f_1(z)=z+\sum_{n=2}^{\infty} A_n z^n$ and $\phi(z)=z+\sum_{n=2}^{\infty} B_n z^n$. Then, we deduce that

$$F=f_1*\phi=z+\sum_{n=2}^{\infty} A_n B_n z^n$$

$$\text{and } zF'' = \sum_{n=2}^{\infty} n(n-1)A_n B_n z^{n-1} = \left(\frac{\phi(z)}{z}\right) * (zf_1''(z)). \tag{3.1}$$

Since $\text{Re}\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ and $f_1 \in \mathcal{P}(M)$, $\text{Re}(zf_1''(z)) > -M$, so in view of Lemma 2.6 and (3.1), we have $\text{Re}(zF''(z)) > -M$ in $\mathbb{D}$. Hence $F \in \mathcal{P}(M)$. This completes the proof. □

Corollary 3.1 *Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ and $\phi \in \mathcal{K}$, where $\mathcal{K}$ denotes the family of all convex functions in $\mathbb{D}$. Then $f \tilde{*} \phi \in \mathcal{P}_{\mathcal{H}}^0(M)$.*

Proof Since $\phi \in \mathcal{K}$, so $\text{Re}\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. The result immediately follows from Theorem 3.3. □

Theorem 3.4 *Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ and $\phi \in \mathcal{A}$ be such that $Re\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. Then $f * \left(\phi + \beta\overline{\phi}\right) \in \mathcal{P}_{\mathcal{H}}^0(M)$, where $|\beta| = 1$.*

Proof Let $f = h + \overline{g} \in \mathcal{P}_{\mathcal{H}}^0(M)$. Then $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = \sum_{n=2}^{\infty} b_n z^n$. Now,

$$f * \left(\phi + \beta\overline{\phi}\right) = (h + \overline{g}) * \left(\phi + \overline{\overline{\beta}\phi}\right) = h * \phi + \overline{\overline{\beta}(g * \phi)}.$$

To prove that $f * \left(\phi + \beta\overline{\phi}\right) \in \mathcal{P}_{\mathcal{H}}^0(M)$, it is sufficient to show that $f_\epsilon = h * \phi + \epsilon\overline{\beta}(g * \phi) \in \mathcal{P}(M)$ for each $\epsilon (|\epsilon| = 1)$. Let $\phi(z) = z + \sum_{n=2}^{\infty} C_n z^n$. For each $|\epsilon| = 1$, we have

$$zf_\epsilon''(z) = \left(\frac{\phi(z)}{z}\right) * \left(z\left(h(z) + \epsilon\overline{\beta}g(z)\right)''\right). \tag{3.2}$$

Since $f = h + \overline{g} \in \mathcal{P}_{\mathcal{H}}^0(M)$, so in view of Lemma 2.1, we have $h + \epsilon\overline{\beta}g \in \mathcal{P}(M)$ for ϵ, β with $|\epsilon\overline{\beta}| = 1$, i.e., $|\beta| = 1$. Thus, we have

$$\text{Re}\left(z\left(h(z) + \epsilon\overline{\beta}g(z)\right)''\right) > -M \text{ for } z \in \mathbb{D}.$$

Since $\text{Re}\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$, so in view of Lemma 2.6 and (3.2), we have

$$\text{Re}\left(zf_\epsilon''(z)\right) > -M \text{ for } z \in \mathbb{D}.$$

Hence $f_\epsilon \in \mathcal{P}(M)$. This completes the proof. □

Corollary 3.2 *Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ and $\phi \in \mathcal{K}$, where $\mathcal{K}$ denotes the family of all convex functions in $\mathbb{D}$. Then $f * \left(\phi + \beta\overline{\phi}\right) \in \mathcal{P}_{\mathcal{H}}^0(M)$, where $|\beta| = 1$.*

Proof Since $\phi \in \mathcal{K}$, so $\text{Re}\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. The result immediately follows from Theorem 3.4. □

By Lemmas 2.3 and 2.9, it is possible to show that each $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ is convex (resp. starlike) in some disk D, i.e., $f(D)$ is a convex domain (resp. $f(D)$ is a domain starlike with respect to the origin).

Theorem 3.5 *Let $f = h + \overline{g} \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ be given by (1.10). Then f is starlike in $|z| < 1 - e^{-\frac{1}{2M}} = r^*$ and convex in $|z| < r_c$, where $r_c \in (0, 1)$ is the smallest root of the equation $\frac{r}{1-r} - \ln(1 - r) - \frac{1}{2M} = 0$.*

Proof Let $0 < r < 1$ and $f_r(z) = \frac{1}{r} f(rz) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}$ for $z \in \mathbb{D}$. For convenience, we let $S_1 = \sum_{n=2}^{\infty} n\,(|a_n| + |b_n|)\, r^{n-1}$ and $S_2 = \sum_{n=2}^{\infty} n^2\,(|a_n| + |b_n|)\, r^{n-1}$. According to Lemma 2.9, it suffices to show that $S_1 \le 1$ for $|z| = r < r^*$ and $S_2 \le 1$ for $|z| = r < r_c$. In view of Lemma 2.3, we have

$$S_1 \le 2M \sum_{n=1}^{\infty} \frac{r^n}{n} = -2M \ln(1 - r).$$

Thus, $S_1 \le 1$ if $r < 1 - e^{-\frac{1}{2M}} = r^*$. Again

$$S_2 \le 2M \sum_{n=1}^{\infty} \left(r^n + \frac{r^n}{n} \right) = 2M \left(\frac{r}{1 - r} - \ln(1 - r) \right).$$

Thus, $S_2 \le 1$ if $r < r_c$, where $r_c \in (0, 1)$ is the smallest root of the equation $\frac{r}{1-r} - \ln(1 - r) - \frac{1}{2M} = 0$. This completes the proof. ◻

4 Bohr–Rogosinski Radius for the Class $\mathcal{P}_{\mathcal{H}}^{0}(M)$

In 2023, Ahamed et al. [3] obtained the following results regarding Bohr–Rogosinski radius for the class $\mathcal{P}_{\mathcal{H}}^{0}(M)$.

Theorem D [3] *Let $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ be given by (1.10). Then*

$$|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\, |z|^n \le d\,(f(0), \partial f(\mathbb{D})) \tag{4.1}$$

for $|z| = r \le r_N(M)$ with $N \ge 2$, where $r_N(M) \in (0, 1)$ is the smallest root of the equation

$$r - 1 + 2M \left(2r - 1 + 2(1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n - 1)} + \ln 4 \right) = 0.$$

The constant $r_N(M)$ is the best possible.

Theorem E [3] *Let* $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ *be given by* (1.10). *Then*

$$|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \leq d\,(f(0), \partial f(\mathbb{D})) \tag{4.2}$$

for $|z| = r \leq r_N(M)$ *with* $N \geq 2$, *where* $r_N(M) \in (0,1)$ *is the smallest root of the equation*

$$\begin{aligned}&(r + 2M\,(r + (1-r)\ln(1-r)))^2\\ &+2M\left(r - 1 + (1-r)\ln(1-r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + \ln 4\right) = 1.\end{aligned}$$

The constant $r_N(M)$ *is the best possible.*

Theorem F [3] *Let* $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ *be given by* (1.10). *Then*

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d\,(f(0), \partial f(\mathbb{D})) \tag{4.3}$$

for $|z| = r \leq r_{m,N}(M)$ *with* $N \geq 2$, *where* $r_{m,N}(M) \in (0,1)$ *is the smallest root of the equation*

$$\begin{aligned}&r^m - 2M\\ &\left(r^m + r - 1 + \left(1 - r^m\right)\ln\left(1 - r^m\right) + (1-r)\ln(1-r) + \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + \ln 4\right) = 1.\end{aligned}$$

The constant $r_{m,N}(M)$ *is the best possible.*

Note that, $0 < M < \frac{1}{2(\ln 4 - 1)}$ in Theorems D-F. Now we focus on the following question.

Question 4.1 *Can we further reduce the Bohr–Rogosinski radius for the class* $\mathcal{P}_{\mathcal{H}}^{0}(M)$ *in Theorems D and E?*

Corresponding to the question above, we first prove the following Bohr–Rogosinski radius for the class $\mathcal{P}_{\mathcal{H}}^{0}(M)$.

Theorem 4.1 *Let* $f \in \mathcal{P}_{\mathcal{H}}^{0}(M)$ *for* $0 < M < \frac{1}{2(\ln 4 - 1)}$ *be given by* (1.10). *Then*

$$|f(z)| + \sum_{n=N_1}^{\infty} |a_n||z|^n + \sum_{n=N_2}^{\infty} |b_n||z|^n \leq d\,(f(0), \partial f(\mathbb{D})) \tag{4.4}$$

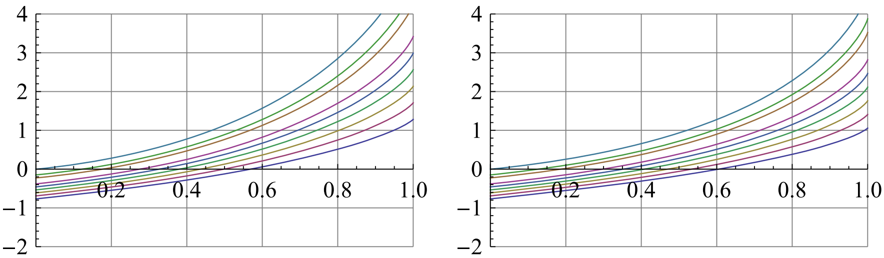


Fig. 1 The graphs of $r_{3,2}(M)$ and $r_{5,3}(M)$ in (0, 1)

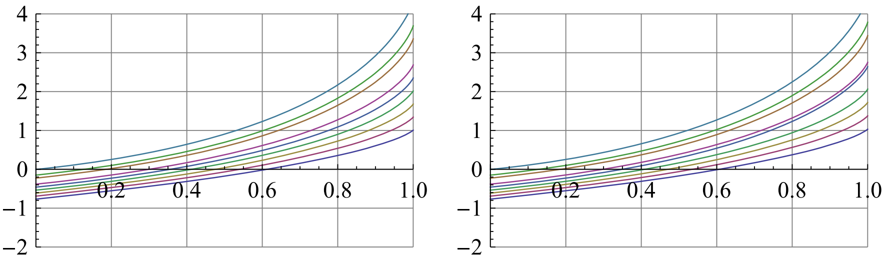


Fig. 2 The graphs of $r_{5,4}(M)$ and $r_{7,3}(M)$ in (0, 1)

for $|z| = r \leq r_{N_1,N_2}(M)$ *with* $N_1 \geq N_2 \geq 2$, *where* $r_{N_1,N_2}(M) \in (0, 1)$ *is the smallest root of the equation*

$$r - 1 + 2M\left(2r - 1 + 2\ln 2 + 2(1-r)\ln(1-r) - \sum_{n=2}^{N_1-1} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)}\right) = 0.$$

The constant $r_{N_1,N_2}(M)$ *is the best possible (Figs. 1 and 2).*

Remark 4.1 Clearly Theorem 4.1 holds for the small Bohr–Rogosinski radius than the radius in Theorem D. It can be checked from the Table 1, e.g., when $N = 3$, then $r_3(0.4) = 0.527$ [3] and $r_{3,2}(0.4) = 0.497629$.

Theorem 4.2 *Let* $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ *for* $0 < M < \frac{1}{2(\ln 4 - 1)}$ *be given by* (1.10). *Then*

$$|f(z^m)|^l + \sum_{n=N_1}^{\infty} |a_n||z^n + \sum_{n=N_2}^{\infty} |b_n||z|^n \leq d\left(f(0), \partial f(\mathbb{D})\right) \tag{4.5}$$

for $|z| = r \leq r_{l,m,N_1,N_2}(M)$ *with* $N_1 \geq N_2 \geq 2$ *and* $l, m \in \mathbb{N}$, *where* $r_{l,m,N_1,N_2}(M) \in (0, 1)$ *is the smallest root of the equation*

Table 1 This table shows the values of the roots $r_{N_1,N_2}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $N_1 = 3, 5, 7$ and $N_2 = 2, 3, 4$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5	0.4
$r_{3,2}(M)$	0.00333894	0.123509	0.176597	0.276795	0.327135	0.379523	0.435604	0.497629
$r_{5,3}(M)$	0.00334608	0.130202	0.188264	0.298156	0.352903	0.409263	0.468732	0.533305
$r_{5,4}(M)$	0.00334609	0.130512	0.189046	0.300392	0.356047	0.413385	0.473856	0.539378
$r_{7,3}(M)$	0.00334608	0.130204	0.188273	0.298226	0.349395	0.40952	0.469165	0.533991
$r_{7,4}(M)$	0.0.00334609	0.130513	0.189056	0.300467	0.356199	0.413663	0.474326	0.540125

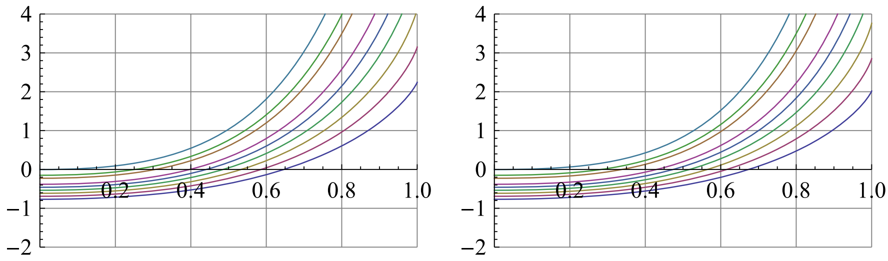


Fig. 3 The graphs of $r_{2,1,3,2}(M)$ and $r_{2,1,5,3}(M)$ in (0, 1)

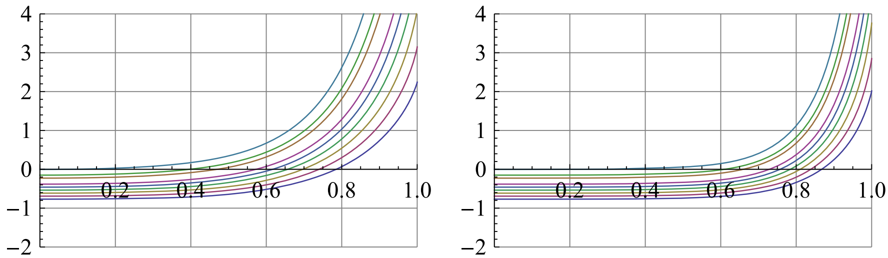


Fig. 4 The graphs of $r_{2,2,3,2}(M)$ and $r_{3,2,5,3}(M)$ in (0, 1)

$$\left[r^m + 2M\left\{r^m + (1-r^m)\ln(1-r^m)\right\}\right]^l + 2M\left((1-r)\ln(1-r) + (1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n}\right.$$
$$\left. -1 + 2\ln 2 + \frac{r^{N_1-1}}{N_1-1} + \sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right) - 1 = 0.$$

The constant $r_{l,m,N_1,N_2}(M)$ is the best possible (Figs. 3 and 4).

Remark 4.2 Clearly Theorem 4.2 holds for the small Bohr–Rogosinski radius than the radius in Theorem E. It can be checked from the Table 2, e.g., when $N = 3$, then $r_3(1.29) = 0.053$ [3], and $r_{2,1,3,2}(1.29) = 0.0434376$.

Theorem 4.3 *Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $0 < M < \frac{1}{2(\ln 4-1)}$ be given by (1.10). Then*

$$|f(z^m)|^l + \sum_{n=N}^{\infty}|a_n b_n||z|^n \le d\left(f(0), \partial f(\mathbb{D})\right) \tag{4.6}$$

for $|z| = r \le r_{l,m,N}(M)$ with $N \ge 2$ and $l, m \in \mathbb{N}$, where $r_{l,m,N}(M) \in (0,1)$ is the smallest root of the equation

Table 2 This table shows the values of the roots $r_{l,m,N_1,N_2}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $l = 2, 3$, $m = 1, 2, 3$, $N_1 = 3, 5, 7$ and $N_2 = 2, 3$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5	0.4
$r_{2,1,3,2}(M)$	0.0434376	0.252353	0.30618	0.397684	0.441418	0.48625	0.533875	0.586386
$r_{2,1,5,3}(M)$	0.0538602	0.283094	0.338234	0.430387	0.473916	0.518243	0.565006	0.616163
$r_{2,2,3,2}(M)$	0.070232	0.396725	0.467366	0.571559	0.61531	0.656869	0.69794	0.740149
$r_{3,2,5,3}(M)$	0.234132	0.607511	0.658316	0.728891	0.757746	0.784914	0.811575	0.838786
$r_{3,3,7,3}(M)$	0.239233	0.663559	0.708153	0.766548	0.790146	0.812514	0.834716	0.857697

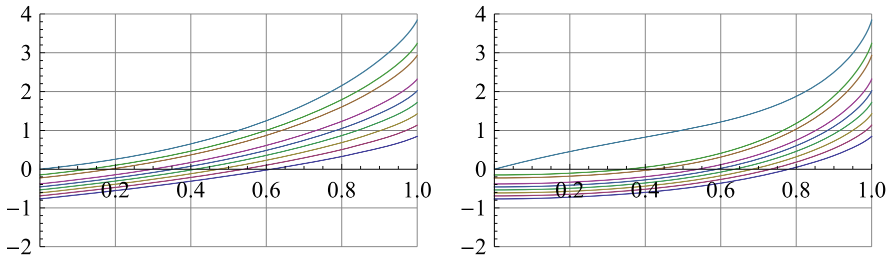


Fig. 5 The graphs of $r_{1,1,3}(M)$ and $r_{1,2,3}(M)$ in $(0, 1)$

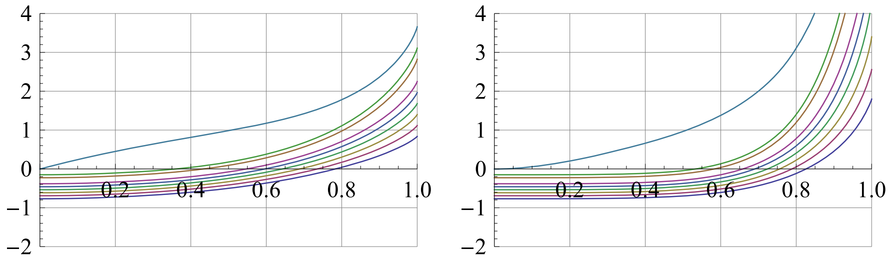


Fig. 6 The graphs of $r_{1,2,4}(M)$ and $r_{2,2,4}(M)$ in $(0, 1)$

$$\left[r^m + 2M\left\{r^m + (1-r^m)\ln(1-r^m)\right\}\right]^l - 1 + 8M^2(r-1)\left(\ln(1-r) + \sum_{n=1}^{N-2}\frac{r^n}{n}\right)$$
$$+4M^2(1+r)\left(Li_2(r) - \sum_{n=1}^{N-2}\frac{r^n}{n^2}\right) - 8M^2\frac{r^{N-1}}{N-1} - 4M^2\frac{r^{N-1}}{(N-1)^2} - 2M(1-2\ln 2) = 0,$$

where $Li_2(r)$ is the dilogarithm function. The constant $r_{l,m,N}(M)$ is the best possible (Figs. 5 and 6; Table 3).

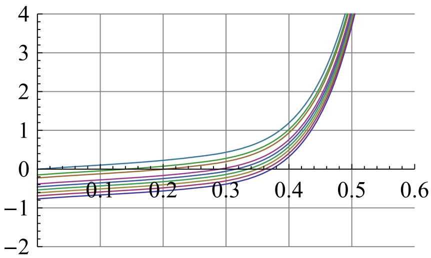
Theorem 4.4 *Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $0 < M < \frac{1}{2(\ln 4-1)}$ be given by (1.10). Then*

$$r + k_1|h(r)|^p + k_2|g(r)|^q + \sum_{n=N_1}^{\infty}|a_n|r^n + \sum_{n=N_2}^{\infty}|b_n|r^n \le d\left(f(0), \partial f(\mathbb{D})\right) \quad (4.7)$$

for $r \le r_{p,k_1,q,k_2,N_1,N_2}(M)$ with $N_1 \ge N_2 \ge 2$, $p, q \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{R}$, where $r_{p,k_1,q,k_2,N_1,N_2}(M) \in (0, 1)$ is the smallest root of the equation

Table 3 This table shows the values of the roots $r_{l,m,N}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $N = 3, 4, 5$, $m = 1, 2$ and $l = 1, 2$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5	0.4
$r_{1,1,3}(M)$	0.00334608	0.1303	0.188589	0.299575	0.355289	0.412964	0.474151	0.540884
$r_{1,2,3}(M)$	0.00130358	0.354225	0.42689	0.540807	0.590473	0.638112	0.685193	0.733128
$r_{1,2,4}(M)$	0.00130358	0.360601	0.433948	0.547366	0.596299	0.643041	0.689136	0.736057
$r_{1,1,5}(M)$	0.00334609	0.130535	0.189139	0.300882	0.356933	0.414857	0.476164	0.542849
$r_{2,2,4}(M)$	0.0227869	0.533189	0.583555	0.65933	0.692263	0.724249	0.756497	0.790203



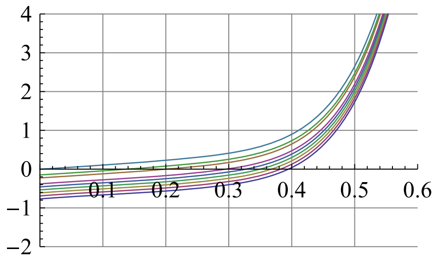


Fig. 7 The graphs of $r_{25,100,8,1000,3,2}(M)$ and $r_{7,200,9,200,5,2}(M)$ in $(0, 1)$

$$r + k_1 r^p + k_2 r^q - 1 + 2M\left((1-r)\ln(1-r) - 1 + 2\ln 2 + (1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n} + \frac{r^{N_1-1}}{N_1-1} + \sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right) = 0.$$

The radius $r_{p,k_1,q,k_2,N_1,N_2}(M)$ is the best possible (Figs. 7; Table 4).

5 Proof of the Theorems 4.1–4.4

Proof of Theorem 4.1 Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ be as (1.10). Using Lemma 2.4, we get

$$|z| + 2M\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}|z|^n \leq |f(z)|, \quad \text{where } \ |z| < 1. \tag{5.1}$$

Since $f(0) = 0$, so the Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$ is $d\,(f(0), \partial f(\mathbb{D})) := \liminf_{|z|\to 1}|f(z) - f(0)| = \liminf_{|z|\to 1}|f(z)|$. Thus from (5.1), we get

$$1 + 2M(1 - 2\ln 2) \leq d\,(f(0), \partial f(\mathbb{D}))\,. \tag{5.2}$$

In view of Lemmas 2.2, 2.3 and 2.4, we now deduce for $N_1 \geq N_2 \geq 2$ that

$$\begin{aligned}
&|f(z)| + \sum_{n=N_1}^{\infty}|a_n|r^n + \sum_{n=N_2}^{\infty}|b_n|r^n \\
&= |f(z)| + \sum_{n=N_1}^{\infty}|a_n|r^n + \sum_{n=N_1}^{\infty}|b_n|r^n + \sum_{n=N_2}^{N_1-1}|b_n|r^n \\
&\leq r + 2M\left(\sum_{n=2}^{\infty}\frac{r^n}{n(n-1)} + \sum_{n=N_1}^{\infty}\frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right)
\end{aligned}$$

Table 4 This table shows the values of the roots $r_{p,k_1,q,k_2,N_1,N_2}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $p = 7, 25, 70$, $k_1 = 100, 200, 2000$, $q = 8, 9, 90$, $k_2 = 200, 1000, 2000$, $N_1 = 3, 5$ and $N_2 = 2, 3$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5
$r_{25,100,8,1000,3,2}(M)$	0.00335328	0.138432	0.201391	0.290227	0.316002	0.33501	0.349867
$r_{7,200,9,200,5,2}(M)$	0.00335329	0.13882	0.202453	0.29744	0.327362	0.349954	0.367799
$r_{70,2000,90,2000,5,3}(M)$	0.00336054	0.149485	0.225224	0.372949	0.395035	0.515784	0.58616

$$= r + 2M \left(2r + 2(1-r)\ln(1-r) - \sum_{n=2}^{N_1-1} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)} \right). \tag{5.3}$$

Now, we deduce that

$$r + 2M \left(2r + 2(1-r)\ln(1-r) - \sum_{n=2}^{N_1-1} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)} \right) \\ \le 1 + 2M(1 - 2\ln 2) \tag{5.4}$$

for $0 < r \le r_{N_1,N_2}(M) < 1$, where $r_{N_1,N_2}(M)$ is the smallest root of the equation

$$r - 1 + 2M \left(2r - 1 + 2\ln 2 + 2(1-r)\ln(1-r) - \sum_{n=2}^{N_1-1} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)} \right) = 0.$$

To ensure about the existence of a root $r_{N_1,N_2}(M)$, we construct the function $\mathcal{G}_1 : [0, 1) \to \mathbb{R}$ such that

$$\mathcal{G}_1(r) := r - 1 + 2M \\ \left(2r - 1 + 2\ln 2 + 2(1-r)\ln(1-r) - \sum_{n=2}^{N_1-1} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)} \right).$$

It is clear that (i) $\mathcal{G}_1$ is continuous on $[0, 1)$, (ii) $\mathcal{G}_1(0) = -1 + 2M(2\ln 2 - 1) < 0$ and (iii) $\lim_{r\to 1} \mathcal{G}_1(r) > 0$, since $\lim_{r\to 1}(1-r)\ln(1-r) = 0$. Thus the claim follows from Intermediate value theorem. Thus, we have

$$r_{N_1,N_2}(M) - 1 + 2M \left(2r_{N_1,N_2}(M) - 1 + 2\ln 2 + 2(1 - r_{N_1,N_2}(M))\ln(1 - r_{N_1,N_2}(M)) \right. \\ \left. - \sum_{n=2}^{N_1-1} \frac{r^n_{N_1,N_2}(M)}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n_{N_1,N_2}(M)}{2n(n-1)} \right) = 0. \tag{5.5}$$

Combining (5.2), (5.3) and (5.4) for $|z| = r \le r_{N_1,N_2}(M)$, we deduce that

$$|f(z)| + \sum_{n=N_1}^{\infty} |a_n| r^n + \sum_{n=N_2}^{\infty} |b_n| r^n \le d\left(f(0), \partial f(\mathbb{D})\right).$$

Now we show that the radius $r_{N_1,N_2}(M)$ is the best possible. We set

$$f := f_M(z) = z + 2M \sum_{n=2}^{\infty} \frac{z^n}{n(n-1)}. \tag{5.6}$$

Note that $f_M(0) = 0$, $f_M \in \mathcal{P}^0_{\mathcal{H}}(M)$. For $z = r$, we have

$$|f_M(r) - f_M(0)| = |f_M(r)| = \left| r + 2M \sum_{n=2}^{\infty} \frac{(r)^n}{n(n-1)} \right| = r + 2M \sum_{n=2}^{\infty} \frac{r^n}{n(n-1)}$$

$$\text{and} \quad \liminf_{r \to 1^-} |f_M(r)| = 1 + 2M. \tag{5.7}$$

and for $z = -r$, we have

$$|f_M(-r) - f_M(0)| = \left| -r + 2M \sum_{n=2}^{\infty} \frac{(-r)^n}{n(n-1)} \right| = r + 2M \sum_{n=2}^{\infty} \frac{(-1)^{n-1} r^n}{n(n-1)}$$

$$\text{and} \quad \liminf_{r \to 1^-} |f_M(-r)| = 1 + 2M(1 - 2\ln 2). \tag{5.8}$$

From (5.7) and (5.8), we have

$$d\left(f_M(0), \partial f_M(\mathbb{D})\right) = \liminf_{|z|=r \to 1^-} |f_M(z) - f_M(0)| = 1 + 2M(1 - 2\ln 2). \tag{5.9}$$

For $|z| = r_{N_1,N_2}(M)$, it follows from (5.5) and (5.9) that

$$\begin{aligned} &|f(z)| + \sum_{n=N_1}^{\infty} |a_n| r^n_{N_{1,2}}(M) + \sum_{n=N_2}^{\infty} |b_n| r^n_{N_{1,2}}(M) \\ &= r_{N_1,N_2}(M) + 2M \Big(2r_{N_1,N_2}(M) + 2\left(1 - r_{N_1,N_2}(M)\right) \ln\left(1 - r_{N_1,N_2}(M)\right) \\ &\quad - \sum_{n=2}^{N_1-1} \frac{r^n_{N_1,N_2}(M)}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n_{N_1,N_2}(M)}{2n(n-1)} \Big) = 1 + 2M(1 - 2\ln 2) = d\left(f_M(0), \partial f_M(\mathbb{D})\right). \end{aligned}$$

Thus the result follows. □

Proof of Theorem 4.2 Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ be given by (1.10). Note that $N_1 \geq N_2 \geq 2$ and $l, m \in \mathbb{N}$. In view of Lemmas 2.2, 2.3 and 2.4, we get

$$\begin{aligned} &|f(z^m)|^l + \sum_{n=N_1}^{\infty} |a_n||z|^n + \sum_{n=N_2}^{\infty} |b_n||z|^n \\ &\leq \left(r^m + 2M \sum_{n=2}^{\infty} \frac{r^{mn}}{n(n-1)}\right)^l + 2M \left(\sum_{n=N_1}^{\infty} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)}\right) \\ &= \left[r^m + 2M\left\{r^m + (1 - r^m)\ln(1 - r^m)\right\}\right]^l + 2M \Bigg((1 - r)\ln(1 - r) + (1 - r) \sum_{n=1}^{N_1-2} \frac{r^n}{n} \\ &\quad + \frac{r^{N_1-1}}{N_1 - 1} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)}\Bigg) \end{aligned} \tag{5.10}$$

Similarly as the proof of Theorem 4.1, we get (5.2) and

$$\left[r^m + 2M\left\{r^m + (1 - r^m)\ln(1 - r^m)\right\}\right]^l + 2M \Bigg((1 - r)\ln(1 - r) + (1 - r) \sum_{n=1}^{N_1-2} \frac{r^n}{n}$$

$$+\frac{r^{N_1-1}}{N_1-1}+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\Bigg)\leq 1+2M(1-2\ln 2) \tag{5.11}$$

for $0<r\leq r_{l,m,N_1,N_2}(M)$, where $r_{l,m,N_1,N_2}(M)$ is the smallest root of $\mathcal{G}_2(r)=0$ in $(0,1)$, where $\mathcal{G}_2:[0,1)\to\mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{G}_2(r)=&\left[r^m+2M\left\{r^m+(1-r^m)\ln(1-r^m)\right\}\right]^l\\&+2M\Bigg((1-r)\ln(1-r)+(1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n}\\&-1+2\ln 2+\frac{r^{N_1-1}}{N_1-1}+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\Bigg)-1.\end{aligned}$$

Similarly as the proof of Theorem 4.1, we have
$\mathcal{G}_2(r_{l,m,N_1,N_2}(M))=0$, i.e.,

$$\begin{aligned}&\left[r^m_{l,m,N_1,N_2}(M)+2M\left\{r^m_{l,m,N_1,N_2}(M)+(1-r^m_{l,m,N_1,N_2}(M))\ln(1-r^m_{l,mN_1,N_2}(M))\right\}\right]^l\\&+2M\Big((1-r_{l,m,N_1,N_2}(M))\ln(1-r_{l,m,N_1,N_2}(M))+(1-r_{l,m,N_1,N_2}(M))\\&\sum_{n=1}^{N_1-2}\frac{r^n_{l,m,N_1,N_2}(M)}{n}\\&-1+2\ln 2+\frac{r^{N_1-1}_{l,m,N_1,N_2}(M)}{N_1-1}+\sum_{n=N_2}^{N_1-1}\frac{r^n_{l,m,N_1,N_2}(M)}{2n(n-1)}\Bigg)-1=0.\end{aligned} \tag{5.12}$$

Combining (5.2), (5.10) and (5.11), we obtain for $|z|=r\leq r_{l,m,N_1,N_2}(M)$

$$|f(z^m)|^l+\sum_{n=N_1}^{\infty}|a_n||z^n+\sum_{n=N_2}^{\infty}|b_n||z|^n\leq d\left(f(0),\partial f(\mathbb{D})\right).$$

In order to show that $r_{l,m,N_1,N_2}(M)$ is the best possible, we consider the function $f=f_M$ defined by (5.6) and we again get (5.9). For $|z|=r_{l,m,N_1,N_2}(M)$, it follows from (5.9) and (5.12) that

$$\begin{aligned}&|f(z^m)|^l+\sum_{n=N_1}^{\infty}|a_n||z^n+\sum_{n=N_2}^{\infty}|b_n||z|^n\\&=\left[r^m_{l,m,N_1,N_2}(M)+2M\left\{r^m_{l,m,N_1,N_2}(M)+(1-r^m_{l,m,N_1,N_2}(M))\ln(1-r^m_{l,m,N_1,N_2}(M))\right\}\right]^l\\&+2M(1-r_{l,m,N_1,N_2}(M))\ln(1-r_{l,m,N_1,N_2}(M))\\&+2M\left((1-r_{l,m,N_1,N_2}(M))\sum_{n=1}^{N_1-2}\frac{r^n_{l,m,N_1,N_2}(M)}{n}+\frac{r^{N_1-1}_{l,m,N_1,N_2}(M)}{N_1-1}+\sum_{n=N_2}^{N_1-1}\frac{r^n_{l,m,N_1,N_2}(M)}{2n(n-1)}\right)\\&=1+2M(1-2\ln 2)=d\left(f_M(0),\partial f_M(\mathbb{D})\right).\end{aligned}$$

Therefore, the radius $r_{l,m,N_1,N_2}(M)$ is the best possible. Thus the result follows. □

Proof of Theorem 4.3 Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ be given by (1.10). Note that $N \geq 2$ and $l, m \in \mathbb{N}$. In view of Lemmas 2.2, 2.3 and 2.4, we obtain

$$
\begin{aligned}
&|f(z^m)|^l + \sum_{n=N}^{\infty} |a_n b_n||z|^n \\
&\leq \left(r^m + 2M \sum_{n=2}^{\infty} \frac{r^{mn}}{n(n-1)} \right)^l + 4M^2 \sum_{n=N}^{\infty} \frac{r^n}{n^2(n-1)^2}.
\end{aligned} \tag{5.13}
$$

Similarly as the proof of Theorem 4.1, we get (5.2) and

$$
\begin{aligned}
&\left(r^m + 2M \sum_{n=2}^{\infty} \frac{r^{mn}}{n(n-1)} \right)^l + 4M^2 \sum_{n=N}^{\infty} \frac{r^n}{n^2(n-1)^2} \\
&\leq \left[r^m + 2M \left\{ r^m + (1-r^m)\ln(1-r^m) \right\} \right]^l + 8M^2(r-1) \left(\ln(1-r) + \sum_{n=1}^{N-2} \frac{r^n}{n} \right) \\
&\quad + 4M^2(1+r) \left(Li_2(r) - \sum_{n=1}^{N-2} \frac{r^n}{n^2} \right) - 8M^2 \frac{r^{N-1}}{N-1} - 4M^2 \frac{r^{N-1}}{(N-1)^2} \\
&\leq 1 + 2M(1 - 2\ln 2)
\end{aligned} \tag{5.14}
$$

for $0 < r \leq r_{l,m,N}(M)$, where $r_{l,m,N}(M)$ is the smallest root of $\mathcal{G}_3(r) = 0$ in $(0, 1)$ and $\mathcal{G}_3 : [0, 1) \to \mathbb{R}$ is defined by

$$
\begin{aligned}
\mathcal{G}_3(r) &:= \left[r^m + 2M \left\{ r^m + (1-r^m)\ln(1-r^m) \right\} \right]^l - 1 + 8M^2(r-1) \\
&\left(\ln(1-r) + \sum_{n=1}^{N-2} \frac{r^n}{n} \right) + 4M^2(1+r) \left(Li_2(r) - \sum_{n=1}^{N-2} \frac{r^n}{n^2} \right) - 8M^2 \frac{r^{N-1}}{N-1} - 4M^2 \frac{r^{N-1}}{(N-1)^2} - 2M(1 - 2\ln 2).
\end{aligned}
$$

Similarly as the proof of Theorem 4.1, we have $\mathcal{G}_3\left(r_{l,m,N}(M)\right) = 0$, i.e.,

$$
\begin{aligned}
&\left[r^m_{l,m,N}(M) + 2M \left\{ r^m_{l,m,N}(M) + (1 - r^m_{l,m,N}(M)) \ln(1 - r^m_{l,m,N}(M)) \right\} \right]^l \\
&-1 - 2M(1 - 2\ln 2) \\
&+8M^2(r_{l,m,N}(M) - 1) \left(\ln(1 - r_{l,m,N}(M)) + \sum_{n=1}^{N-2} \frac{r^n_{l,m,N}(M)}{n} \right) - 8M^2 \frac{r^{N-1}_{l,m,N}(M)}{N-1} \\
&+4M^2(1 + r_{l,m,N}(M)) \left(Li_2(r_{l,m,N}(M)) - \sum_{n=1}^{N-2} \frac{r^n_{l,m,N}(M)}{n^2} \right) - 4M^2 \frac{r^{N-1}_{l,m,N}(M)}{(N-1)^2} = 0.
\end{aligned} \tag{5.15}
$$

Combining (5.2), (5.13) and (5.14), we obtain for $|z| = r \leq r_{l,m,N}(M)$

$$|f(z^m)|^l + \sum_{n=N}^{\infty} |a_n b_n||z|^n \leq d\left(f(0), \partial f(\mathbb{D})\right).$$

In order to show that $r_{l,m,N}(M)$ is the best possible, we consider the function $f = f_M$ defined by (5.6) and we again get (5.9). For $|z| = r_{l,m,N}(M)$, it follows from (5.9) and (5.15) that

$$\begin{aligned}
&|f(z^m)|^l + \sum_{n=N}^{\infty} |a_n b_n| r_{l,m,N}^n(M) \\
&= \left[r_{l,m,N}^m(M) + 2M\left\{r_{l,m,N}^m(M) + (1 - r_{l,m,N}^m(M))\ln(1 - r_{l,m,N}^m(M))\right\}\right]^l \\
&\quad + 8M^2(r_{l,m,N}(M) - 1)\left(\ln(1 - r_{l,m,N}(M)) + \sum_{n=1}^{N-2} \frac{r_{l,m,N}^n(M)}{n}\right) - 8M^2 \frac{r_{l,m,N}^{N-1}(M)}{N-1} \\
&\quad + 4M^2(1 + r_{l,m,N}(M))\left(Li_2(r_{l,m,N}(M)) - \sum_{n=1}^{N-2} \frac{r_{l,m,N}^n(M)}{n^2}\right) - 4M^2 \frac{r_{l,m,N}^{N-1}(M)}{(N-1)^2} \\
&= 1 + 2M(1 - 2\ln 2) = d\left(f_M(0), \partial f_M(\mathbb{D})\right).
\end{aligned}$$

Therefore, the radius $r_{l,m,N}(M)$ is the best possible. Thus the results follows. □

Proof of Theorem 4.4 Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ be given by (1.10). We know that, if ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, $\forall z \in \mathbb{D}$, then by Schwarz Lemma, we have $|\phi(z)| \leq |z|$. Note that $N_1 \geq N_2 \geq 2$, $p, q \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{R}$. In view of Lemmas 2.2, 2.3 and 2.4, we obtain

$$\begin{aligned}
&r + k_1|h(r)|^p + k_2|g(r)|^q + \sum_{n=N_1}^{\infty} |a_n| r^n + \sum_{n=N_2}^{\infty} |b_n| r^n \\
&\quad \leq r + k_1 r^p + k_2 r^q + \sum_{n=N_1}^{\infty} \frac{2Mr^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{Mr^n}{n(n-1)}. \qquad (5.16)
\end{aligned}$$

Now, we deduce that

$$\begin{aligned}
&r + k_1 r^p + k_2 r^q + \sum_{n=N_1}^{\infty} \frac{2Mr^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{Mr^n}{n(n-1)} \\
&= r + k_1 r^p + k_2 r^q + 2M\left(\sum_{n=N_1}^{\infty} \frac{r^n}{n(n-1)} + \sum_{n=N_2}^{N_1-1} \frac{r^n}{2n(n-1)}\right) \\
&= r + k_1 r^p + k_2 r^q + 2M\left((1-r)\ln(1-r) + (1-r)\sum_{n=1}^{N_1-2} \frac{r^n}{n} + \frac{r^{N_1-1}}{N_1-1}\right.
\end{aligned}$$

$$+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\Bigg)\leq 1+2M(1-2\ln 2) \tag{5.17}$$

for $0<r\leq r_{p,k_1,q,k_2,N_1,N_2}(M)$, where $r_{p,k_1,q,k_2,N_1,N_2}(M)$ is the smallest root of $\mathcal{G}_4(r)=0$ in $(0,1)$ and $\mathcal{G}_4:[0,1)\to\mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{G}_4(r) &:= r+k_1r^p+k_2r^q-1+2M\\ &\left((1-r)\ln(1-r)-1+2\ln 2+(1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n}\right.\\ &\left.+\frac{r^{N_1-1}}{N_1-1}+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right).\end{aligned}$$

Similarly as the proof of Theorem 4.1, we have $\mathcal{G}_4\left(r_{p,k_1,q,k_2,N_1,N_2}(M)\right)=0$, i.e.,

$$\begin{aligned}& r_{p,k_1,q,k_2,N_1,N_2}(M)+k_1r^p_{p,k_1,q,k_2,N_1,N_2}(M)+k_2r^q_{p,k_1,q,k_2,N_1,N_2}(M)-1\\ &+2M\left((1-r_{p,k_1,q,k_2,N_1,N_2}(M))\ln(1-r_{p,k_1,q,k_2,N_1,N_2}(M))-1+2\ln 2\right.\\ &+\frac{r^{N_1-1}_{p,k_1,q,k_2,N_1,N_2}(M)}{N_1-1}\\ &\left.+(1-r_{p,k_1,q,k_2,N_1,N_2}(M))\sum_{n=1}^{N_1-2}\frac{r^n_{p,k_1,q,k_2,N_1,N_2}(M)}{n}+\sum_{n=N_2}^{N_1-1}\frac{2r^n_{p,k_1,q,k_2,N_1,N_2}(M)}{2n(n-1)}\right)=0.\end{aligned} \tag{5.18}$$

Combining (5.2), (5.16) and (5.17), we obtain for $|z|=r\leq r_{p,k_1,q,k_2,N_1,N_2}(M)$

$$r+k_1|h(r)|^p+k_2|g(r)|^q+\sum_{n=N_1}^{\infty}|a_n|r^n+\sum_{n=N_2}^{\infty}|b_n|r^n\leq d\left(f(0),\partial f(\mathbb{D})\right).$$

In order to show that $r_{p,k_1,q,k_2,N_1,N_2}(M)$ is the best possible, we consider the function $f=f_M$ defined by (5.6) and we again get (5.9). For $|z|=r_{p,k_1,q,k_2,N_1,N_2}(M)$, it follows from (5.9) and (5.18) that

$$\begin{aligned}& r_{p,k_1,q,k_2,N_1,N_2}(M)+k_1|h(r_{p,k_1,q,k_2,N_1,N_2}(M))|^p+k_2|g(r_{p,k_1,q,k_2,N_1,N_2}(M))|^q\\ &\quad+\sum_{n=N_1}^{\infty}|a_n|r^n_{p,k_1,q,k_2,N_1,N_2}(M)+\sum_{n=N_2}^{\infty}|b_n|r^n_{p,k_1,q,k_2,N_1,N_2}(M)\\ &=r_{p,k_1,q,k_2,N_1,N_2}(M)+k_1r^p_{p,k_1,q,k_2,N_1,N_2}(M)+k_2r^q_{p,k_1,q,k_2,N_1,N_2}(M)+2M\times\\ &\left((1-r_{p,k_1,q,k_2,N_1,N_2}(M))\ln(1-r_{p,k_1,q,k_2,N_1,N_2}(M))+\frac{r^{N_1-1}_{p,k_1,q,k_2,N_1,N_2}(M)}{N_1-1}\right.\end{aligned}$$

$$+(1-r_{p,k_1,q,k_2,N_1,N_2}(M))\sum_{n=1}^{N_1-2}\frac{r^n_{p,k_1,q,k_2,N_1,N_2}(M)}{n}+\sum_{n=N_2}^{N_1-1}\frac{r^n_{p,k_1,q,k_2,N_1,N_2}(M)}{2n(n-1)}\Bigg)$$
$$=1+2M(1-2\ln 2)=d\left(f_M(0),\partial f_M(\mathbb{D})\right).$$

Therefore, the radius $r_{p,k_1,q,k_2,N_1,N_2}(M)$ is the best possible. Thus the result follows. □

6 Improved Bohr Radius for the Class $\mathcal{P}^0_{\mathcal{H}}(M)$

In 2023, Ahamed et al. [3] generalized the harmonic versions of Theorem B for the class $\mathcal{P}^0_{\mathcal{H}}(M)$ and obtained the following result.

Theorem G [3] *Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ be given by (1.10). Then*

$$r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^n+P\left(\frac{S_r}{\pi}\right)\leq d\left(f(0),\partial f(\mathbb{D})\right) \tag{6.1}$$

for $r \leq r_N(M)$, where $P(\omega)=\omega^N+\omega^{N-1}+\cdots+\omega$ and $r_N(M)\in(0,1)$ is the smallest root of the equation

$$r-1-2M(r-1+(1-r)\ln(1-r)+2\ln 2)+P\left(r^2+4M^2G(r)\right)=0, \tag{6.2}$$

where $G(r)$ is defined by $G(r):=r^2\left(Li_2(r^2)-1\right)+(1-r^2)\ln(1-r^2)$. The constant $r_N(M)$ is the best possible.

In order to generalize Theorem G, we consider a N-th degree polynomial of the form

$$P(\omega)=c_N(\omega)^N+c_{N-1}(\omega)^{N-1}+c_{N-2}(\omega)^{N-2}+\cdots+c_1\omega, \tag{6.3}$$

where $c_i \in \mathbb{R}$ $(1 \leq i \leq N)$ with $c_N \neq 0$. Concerning improved Bohr radius for the class $\mathcal{P}^0_{\mathcal{H}}(M)$, we have obtain the following results.

Theorem 6.1 *Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $0<M<\frac{1}{2(\ln 4-1)}$ be given by (1.10). Then*

$$r+\sum_{n=N_1}^{\infty}|a_n||z|^n+\sum_{n=N_2}^{\infty}|b_n||z|^n+P\left(\frac{S_r}{\pi}\right)\leq d\left(f(0),\partial f(\mathbb{D})\right) \tag{6.4}$$

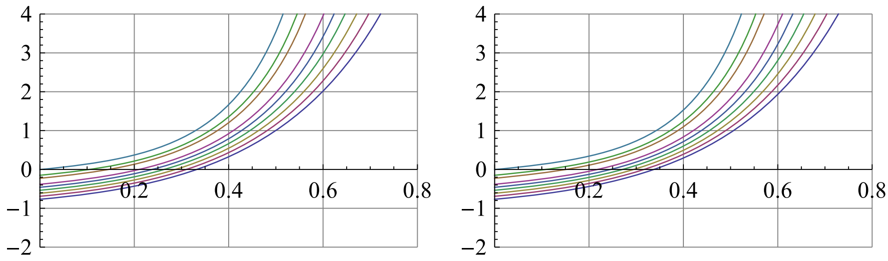


Fig. 8 The graphs of $r_{3,2}(M)$ and $r_{10,5}(M)$ in (0, 1) when $P(r) = r^3 + 6r^2 + 3r$

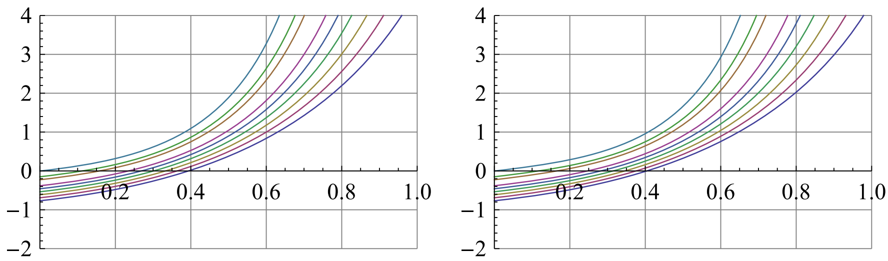


Fig. 9 The graphs of $r_{3,2}(M)$ and $r_{10,4}(M)$ in (0, 1) when $P(r) = r^2 + 2r$

for $r \leq r_{N_1,N_2}(M)$ *with* $N_1 \geq N_2 \geq 2$, *where* $P(\omega)$ *is defined in (6.3) and* $r_{N_1,N_2}(M) \in (0, 1)$ *is the smallest root of the equation*

$$r - 1 + 2M\left((1-r)\ln(1-r) - 1 + 2\ln 2 + (1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n} + \frac{r^{N_1-1}}{N_1-1} + \sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right) + P\left(r^2 + 4M^2\left((r^2-1)\ln(1-r^2) + r^2(Li_2(r^2)-1)\right)\right) = 0.$$

The constant $r_{N_1,N_2}(M)$ *is the best possible (Fig. 8 and 9; Tables 5 and 6).*

As a consequence of Theorem 6.1, we obtain the following corollary.

Corollary 6.1 *Let* $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ *for* $0 < M < \frac{1}{2(\ln 4 - 1)}$ *be given by* (1.10). *Then,*

$$r + \sum_{n=N_1}^{\infty}|a_n|r^n + \sum_{n=N_2}^{\infty}|b_n|r^n + \frac{S_r}{\pi} \leq d\,(f(0), \partial f(\mathbb{D})) \tag{6.5}$$

Table 5 This table shows the values of the roots $r_{N_1,N_2}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $N_1 = 3, 5, 10$, $N_2 = 2, 3, 5$ and $P(r) = r^3 + 6r^2 + 3r$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5	0.4
$r_{3,2}(M)$	0.00332035	0.107135	0.145988	0.2095	0.236943	0.262557	0.286798	0.309948
$r_{5,3}(M)$	0.00332733	0.110914	0.151394	0.216434	0.243994	0.269398	0.293145	0.315535
$r_{10,5}(M)$	0.00332733	0.111067	0.151693	0.216986	0.244627	0.270076	0.293828	0.316181

Table 6 This table shows the values of the roots $r_{N_1,N_2}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $N_1 = 3, 10$, $N_2 = 2, 4$ and $P(r) = r^2 + 2r$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5	0.4
$r_{3,2}(M)$	0.00333118	0.114879	0.159251	0.23461	0.268403	0.300733	0.332137	0.362981
$r_{10,4}(M)$	0.00333826	0.120068	0.167321	0.246397	0.281078	0.313708	0.34482	0.374739

for $r \leq r_{N_1,N_2}(M)$, *where* $r_{N_1,N_2}(M) \in (0,1)$ *is the smallest root of the equation*

$$r-1+2M\left((1-r)\ln(1-r)-1+2\ln 2+(1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n}+\frac{r^{N_1-1}}{N_1-1}\right.$$
$$\left.+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right)+r^2+4M^2\left((r^2-1)\ln(1-r^2)+r^2(Li_2(r^2)-1)\right)=0.$$

The constant $r_{N_1,N_2}(M)$ *is the best possible.*

7 Proof of the Theorem 6.1

Proof of Theorem 6.1 Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ be given by (1.10). For the analytic functions h and g, the area S_r of the disk $|z| < r$ under the harmonic map f is given by

$$S_r = \iint\limits_{|z|<r}\left(|h'(z)|^2-|g'(z)|^2\right)dxdy, \tag{7.1}$$

$$\text{where } \frac{1}{\pi}\iint\limits_{|z|<r}|h'(z)|^2 dxdy = \sum_{n=1}^{\infty} n|a_n|^2 r^{2n} \tag{7.2}$$

$$\text{and } \frac{1}{\pi}\iint\limits_{|z|<r}|g'(z)|^2 dxdy = \sum_{n=2}^{\infty} n|b_n|^2 r^{2n}. \tag{7.3}$$

Combining (7.1), (7.2), (7.3) and Lemma 2.3, we obtain

$$\begin{aligned}
\frac{S_r}{\pi} &= \frac{1}{\pi}\iint\limits_{|z|<r}\left(|h'(z)|^2-|g'(z)|^2\right)dxdy \\
&= r^2+\sum_{n=2}^{\infty} n|a_n|^2 r^{2n}-\sum_{n=2}^{\infty} n|b_n|^2 r^{2n} \\
&= r^2+\sum_{n=2}^{\infty} n\left(|a_n|+|b_n|\right)\left(|a_n|-|b_n|\right)r^{2n} \\
&\leq r^2+\sum_{n=2}^{\infty}\frac{4M^2 r^{2n}}{n(n-1)^2} \\
&= r^2+4M^2\left((r^2-1)\ln(1-r^2)+r^2(Li_2(r^2)-1)\right).
\end{aligned} \tag{7.4}$$

Note that, $N_1 \geq N_2 \geq 2$. In view of Lemmas 2.2, 2.3 and 2.4, we obtain

$$\begin{aligned}
& r+\sum_{n=N_1}^{\infty}|a_n|r^n+\sum_{n=N_2}^{\infty}|b_n|r^n+P\left(\frac{S_r}{\pi}\right)\\
&=r+\sum_{n=N_1}^{\infty}\left(|a_n|+|b_n|\right)r^n+\sum_{n=N_2}^{N_1-1}|b_n|r^n+P\left(\frac{S_r}{\pi}\right)\\
&\leq r+\sum_{n=N_1}^{\infty}\frac{2Mr^n}{n(n-1)}+\sum_{n=N_2}^{N_1-1}\frac{Mr^n}{n(n-1)}+P\left(r^2+4M^2\left((r^2-1)\ln(1-r^2)\right.\right.\\
&\left.\left.+r^2(Li_2(r^2)-1)\right)\right)\\
&=r+2M\left((1-r)\ln(1-r)+(1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n}+\frac{r^{N_1-1}}{N_1-1}+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right)\\
&+P\left(r^2+4M^2\left((r^2-1)\ln(1-r^2)+r^2(Li_2(r^2)-1)\right)\right)\\
&\leq 1+2M(1-2\ln 2)
\end{aligned} \tag{7.5}$$

for $0<r\leq r_{N_1,N_2}(M)$, where $r_{N_1,N_2}(M)$ is the smallest root of $\mathcal{G}_5(r)=0$ in $(0,1)$ and $\mathcal{G}_5:[0,1)\to\mathbb{R}$ be defined by

$$\begin{aligned}
\mathcal{G}_5(r)&:=r-1+2M\left((1-r)\ln(1-r)-1+2\ln 2+(1-r)\sum_{n=1}^{N_1-2}\frac{r^n}{n}+\frac{r^{N_1-1}}{N_1-1}\right.\\
&\left.+\sum_{n=N_2}^{N_1-1}\frac{r^n}{2n(n-1)}\right)+P\left(r^2+4M^2\left((r^2-1)\ln(1-r^2)+r^2(Li_2(r^2)-1)\right)\right).
\end{aligned}$$

Similarly as the proof of Theorem 4.1, we have $\mathcal{G}_5\left(r_{N_1,N_2}(M)\right)=0$, i.e.,

$$\begin{aligned}
& r_{N_1,N_2}(M)-1+2M\left((1-r_{N_1,N_2}(M))\ln(1-r_{N_1,N_2}(M))-1+2\ln 2\right.\\
&+\sum_{n=N_2}^{N_1-1}\frac{r^n_{N_1,N_2}(M)}{2n(n-1)}\\
&\left.+(1-r_{N_1,N_2}(M))\sum_{n=1}^{N_1-2}\frac{r^n_{N_1,N_2}(M)}{n}+\frac{r^{N_1-1}_{N_1,N_2}(M)}{N_1-1}\right)+P\left(r^2_{N_1,N_2}(M)+4M^2\times\right.\\
&\left.\left(\left(r^2_{N_1,N_2}(M)-1\right)\ln\left(1-r^2_{N_1,N_2}(M)\right)+r^2_{N_1,N_2}(M)\left(Li_2(r^2_{N_1,N_2}(M))-1\right)\right)\right)=0.
\end{aligned} \tag{7.6}$$

Combining (5.2), (7.5) and (7.6), we obtain for $|z| = r \leq r_{N_1,N_2}(M)$

$$r + \sum_{n=N_1}^{\infty} |a_n| r^n + \sum_{n=N_2}^{\infty} |b_n| r^n + P\left(\frac{S_r}{\pi}\right) \leq d\left(f(0), \partial f(\mathbb{D})\right).$$

In order to show that $r_{N_1,N_2}(M)$ is the best possible, we consider the function $f = f_M$ defined by (5.6) and we again get (5.9). For $|z| = r_{N_1,N_2}(M)$, it follows from (5.9) and (7.6) that

$$\begin{aligned}
&r_{N_1,N_2}(M) + \sum_{n=N_1}^{\infty} |a_n| r^n_{N_1,N_2}(M) + \sum_{n=N_2}^{\infty} |b_n| r^n_{N_1,N_2}(M) + P\left(\frac{S_r}{\pi}\right) \\
&= r_{N_1,N_2}(M) + 2M\left((1 - r_{N_1,N_2}(M)) \ln(1 - r_{N_1,N_2}(M)) + \sum_{n=N_2}^{N_1-1} \frac{r^n_{N_1,N_2}(M)}{2n(n-1)}\right. \\
&\left.+(1 - r_{N_1,N_2}(M)) \sum_{n=1}^{N_1-2} \frac{r^n_{N_1,N_2}(M)}{n} + \frac{r^{N_1-1}_{N_1,N_2}(M)}{N_1 - 1}\right) + P\left(r^2_{N_1,N_2}(M) + 4M^2 \times\right. \\
&\left.\left(\left(r^2_{N_1,N_2}(M) - 1\right) \ln\left(1 - r^2_{N_1,N_2}(M)\right) + r^2_{N_1,N_2}(M)\left(Li_2(r^2_{N_1,N_2}(M)) - 1\right)\right)\right) \\
&= 1 + 2M(1 - 2\ln 2) = d\left(f_M(0), \partial f_M(\mathbb{D})\right).
\end{aligned}$$

Hence the radius $r_{N_1,N_2}(M)$ is the best possible. This completes the proof. □

8 Refined Bohr Radius for the Class $\mathcal{P}^0_{\mathcal{H}}(M)$

In this section, we establish some sharp results on the harmonic analogue of Theorem C for the class $\mathcal{P}^0_{\mathcal{H}}(M)$ as follows.

Theorem 8.1 *Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ for $0 < M < \frac{1}{2(\ln 4 - 1)}$ be given by (1.10). Then*

$$\begin{aligned}
&|f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\, r^n + \frac{r^q}{1 - r^q} \sum_{n=N}^{\infty} (n(n-1))^{q-1} (|a_n| + |b_n|)^q\, r^{qn} \\
&\leq d\left(f(0), \partial f(\mathbb{D})\right)
\end{aligned}$$

for $r \leq r_{p,q,N}(M)$ with $p, q (\neq 1) \in \mathbb{N}$ and $N \geq 2$, where $r_{p,q,N}(M) \in (0, 1)$ is the smallest root of the equation

$$\begin{aligned}
&[r + 2M\{r + (1-r)\ln(1-r)\}]^p - 1 + 2M\left((1-r)\ln(1-r) + (1-r)\sum_{n=1}^{N-2} \frac{r^n}{n}\right. \qquad (8.1) \\
&\left.-1 + 2\ln 2 + \frac{r^{N-1}}{N-1}\right) + \frac{(2rM)^q}{1 - r^q}\left((1 - r^q)\ln(1 - r^q) + (1 - r^q)\sum_{n=1}^{N-2} \frac{r^{nq}}{n} + \frac{r^{q(N-1)}}{N-1}\right) = 0.
\end{aligned}$$

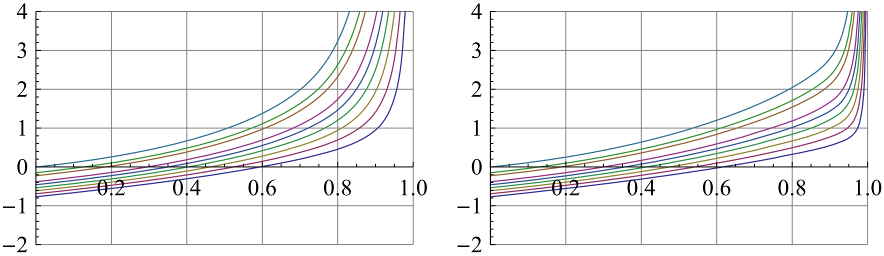


Fig. 10 The graphs of $r_{1,2,3}(M)$ and $r_{1,2,10}(M)$ in $(0, 1)$

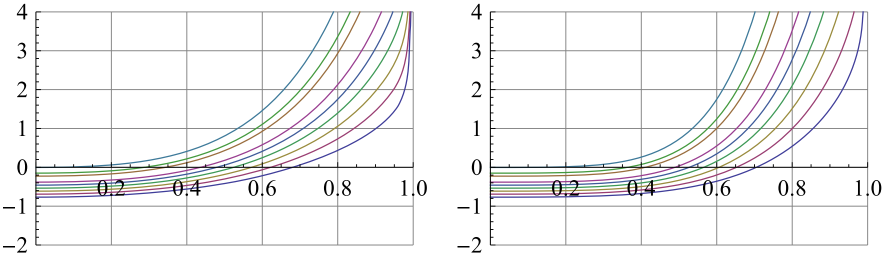


Fig. 11 The graphs of $r_{2,3,5}(M)$ and $r_{3,3,5}(M)$ in $(0, 1)$

The constant $r_{p,q,N}(M)$ is the best possible (Figs. 10 and 11; Table 7).

9 Proof of the Theorem 8.1

Proof of Theorem 8.1 Let $f \in \mathcal{P}^0_{\mathcal{H}}(M)$ be given by (1.10). Note that $p, q(\neq 1) \in \mathbb{N}$ and $N \geq 2$. In view of Lemmas 2.3 and 2.4, we obtain

$$
\begin{aligned}
&|f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\, r^n + \frac{r^q}{1-r^q} \sum_{n=N}^{\infty} (n(n-1))^{q-1} (|a_n| + |b_n|)^q\, r^{qn} \\
&\leq \left(r + 2M \sum_{n=2}^{\infty} \frac{r^n}{n(n-1)} \right)^p + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)} + \frac{r^q}{1-r^q} \sum_{n=N}^{\infty} \frac{2^q M^q r^{qn}}{n(n-1)}. \qquad (9.1)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left(r + 2M \sum_{n=2}^{\infty} \frac{r^n}{n(n-1)} \right)^p + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)} + \frac{r^q}{1-r^q} \sum_{n=N}^{\infty} \frac{2^q M^q r^{qn}}{n(n-1)} \\
&= [r + 2M \{r + (1-r)\ln(1-r)\}]^p \\
&+2M \left((1-r)\ln(1-r) + (1-r) \sum_{n=1}^{N-2} \frac{r^n}{n} + \frac{r^{N-1}}{N-1} \right)
\end{aligned}
$$

Table 7 This table shows the values of the roots $r_{p,q,N}(M)$ for different values of M in $\left(0, \frac{1}{2(\ln 4-1)}\right)$ with $p = 1, 2, 3$, $q = 2, 3$ and $N = 3, 5, 10$

M	1.29	1.1	1.0	0.8	0.7	0.6	0.5	0.4
$r_{1,2,3}(M)$	0.00334607	0.129878	0.187427	0.295681	0.349344	0.404458	0.462527	0.525558
$r_{1,2,10}(M)$	0.00334609	0.130536	0.189141	0.300894	0.356954	0.414888	0.476205	0.542889
$r_{2,2,3}(M)$	0.0536052	0.279711	0.334063	0.42494	0.467878	0.511613	0.557777	0.608336
$r_{2,3,5}(M)$	0.0541211	0.286639	0.342568	0.435872	0.479858	0.524577	0.571659	0.623024
$r_{3,3,5}(M)$	0.127715	0.363414	0.410746	0.489439	0.52697	0.565578	0.606763	0.652324

$$
\begin{aligned}
&+\frac{(2rM)^q}{1-r^q}\left((1-r^q)\ln(1-r^q)+(1-r^q)\sum_{n=1}^{N-2}\frac{r^{nq}}{n}+\frac{r^{q(N-1)}}{N-1}\right)\\
&\le 1+2M(1-2\ln 2)
\end{aligned}
$$

for $0 < r \le r_{p,q,N}(M)$, where $r_{p,q,N}(M)$ is the smallest root of $\mathcal{G}_7(r) = 0$ in $(0, 1)$ and $\mathcal{G}_7 : [0, 1) \to \mathbb{R}$ be defined by

$$
\begin{aligned}
\mathcal{G}_7(r) &:= [r+2M\{r+(1-r)\ln(1-r)\}]^p - 1 + 2M\left((1-r)\ln(1-r)+(1-r)\sum_{n=1}^{N-2}\frac{r^n}{n}\right.\\
&\left.-1+2\ln 2+\frac{r^{N-1}}{N-1}\right)+\frac{(2rM)^q}{1-r^q}\left((1-r^q)\ln(1-r^q)+(1-r^q)\sum_{n=1}^{N-2}\frac{r^{nq}}{n}+\frac{r^{q(N-1)}}{N-1}\right).
\end{aligned}
$$

Similarly as the proof of Theorem 4.1, we have $\mathcal{G}_7\left(r_{p,q,N}(M)\right) = 0$, i.e.,

$$
\begin{aligned}
&\left[r_{p,q,N}(M)+2M\left\{r_{p,q,N}(M)+(1-r_{p,q,N}(M))\ln(1-r_{p,q,N}(M))\right\}\right]^p\\
&+2M\left((1-r_{p,q,N}(M))\times\right.\\
&\left.\ln(1-r_{p,q,N}(M))+(1-r_{p,q,N}(M))\sum_{n=1}^{N-2}\frac{r^n_{p,q,N}(M)}{n}-1+2\ln 2+\frac{r^{N-1}_{p,q,N}(M)}{N-1}\right)-1\\
&+\frac{(2r_{p,q,N}(M)M)^q}{1-r^q_{p,q,N}(M)}\left((1-r^q_{p,q,N}(M))\ln(1-r^q_{p,q,N}(M))+(1-r^q_{p,q,N}(M))\sum_{n=1}^{N-2}\frac{r^{nq}_{p,q,N}(M)}{n}\right.\\
&\left.+\frac{r^{q(N-1)}_{p,q,N}(M)}{N-1}\right)=0.
\end{aligned}
\tag{9.2}
$$

Combining (5.2), (9.1) and (9.2), we obtain for $|z| = r \le r_{p,q,N}(M)$

$$
\begin{aligned}
&|f(z)|^p+\sum_{n=N}^{\infty}(|a_n|+|b_n|)\,r^n+\frac{r^q}{1-r^q}\sum_{n=N}^{\infty}(n(n-1))^{q-1}\,(|a_n|+|b_n|)^q\,r^{qn}\\
&\le d\left(f(0),\partial f(\mathbb{D})\right).
\end{aligned}
$$

In order to show that $r_{p,q,N}(M)$ is the best possible, we consider the function $f = f_M$ defined by (5.6) and we again get (5.9). For $|z| = r_{p,q,N}(M)$, it follows from (5.9) and (9.2) that

$$
\begin{aligned}
&|f(z)|^p+\sum_{n=N}^{\infty}(|a_n|+|b_n|)\,r^n_{p,q,N}(M)+\frac{r^q_{p,q,N}(M)}{1-r^q_{p,q,N}(M)}\sum_{n=N}^{\infty}(n(n-1))^{q-1}\\
&(|a_n|+|b_n|)^q\,r^{qn}_{p,q,N}(M)\\
&\quad=\left[r_{p,q,N}(M)+2M\left\{r_{p,q,N}(M)+(1-r_{p,q,N}(M))\ln(1-r_{p,q,N}(M))\right\}\right]^p\\
&\quad+2M\left((1-r_{p,q,N}(M))\times\right.\\
&\left.\ln(1-r_{p,q,N}(M))+(1-r_{p,q,N}(M))\sum_{n=1}^{N-2}\frac{r^n_{p,q,N}(M)}{n}+\frac{r^{N-1}_{p,q,N}(M)}{N-1}\right)+\frac{(2r_{p,q,N}(M)M)^q}{1-r^q_{p,q,N}(M)}\times
\end{aligned}
$$

$$\left((1-r^q_{p,q,N}(M))\ln(1-r^q_{p,q,N}(M))+(1-r^q_{p,q,N}(M))\sum_{n=1}^{N-2}\frac{r^{nq}_{p,q,N}(M)}{n}+\frac{r^{q(N-1)}_{p,q,N}(M)}{N-1}\right)$$
$$=1+2M(1-2\ln 2)=d\left(f_M(0),\partial f_M(\mathbb{D})\right).$$

Hence the radius $r_{p,q,N}(M)$ is the best possible. Thus the result follows. □

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Declarations

Conflict of interest Authors declare that they have no Conflict of interest.

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