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Uniform Asymptotic Formulas of Ranks and Cranks for Cubic Partitions

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Abstract
In this paper, we establish uniform asymptotic formulas for the rank and crank statistics of cubic partitions. This partly improves upon the asymptotic results established by Kim–Kim–Nam in 2016.

Keywords Cubic partitions · Rank · Crank · Asymptotics

Mathematics Subject Classification Primary 11P82 · Secondary 05A17

1 Introduction and Statement of Results

A partition of an integer n is a sequence of non-increasing positive integers whose sum equals n. Let $p(n)$ be the number of partitions of n and let $p(0) := 1$. Euler discovered the generating function of $p(n)$:

$$\sum_{n\ge 0} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where we define $(a;q)_\infty = \prod_{k\ge 0}(1-aq^k)$ for any $a \in \mathbb{C}$ and $|q| < 1$. To explain Ramanujan's famous partition congruences with modulus 5, 7 and 11, the rank and

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crank statistic for integer partitions was introduced by Dyson [6], Andrews and Garvan [2, 7]. As a precise definition of rank and crank for integer partitions are not necessary for the rest of the paper, we do not give it here.

The cubic partition function $c(n)$ is defined by

$$\sum_{n\geq 0} c(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty},$$

which was introduced by Chan in a series of papers [3–5]. Chan [3] showed that $c(n)$ satisfies a Ramanujan-type congruence $c(3n+2) \equiv 0 \pmod 3$. He [4] further proved that $c(n)$ satisfies congruences modulo higher powers of 3. Motivated by cubic partition congruences [3, 4], Kim [8] introduced a cubic partition crank which explains infinitely many congruences for powers of 3 explicitly. As a precise definition is quite complicated and not necessary for the rest of the paper, we do not give it here. Let $C(m,n)$ be the number of cubic partitions of n with crank m. Kim [8] also established the generating function for $C(m,n)$ as follows:

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} C(m,n)z^m q^n = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty (zq^2;q^2)_\infty (z^{-1}q^2;q^2)_\infty}. \quad (1.2)$$

It is clear that $C(m,n) = 0$ for any $|m| > n$. On the other hand, in his thesis, Reti [10] defined a rank-like function which also explains the cubic partition congruence modulo 3. Let $R(m,n)$ be the number of cubic partitions of n with rank m, then

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} R(m,n)z^m q^n = \frac{1}{(q;q^2)_\infty (zq^2;q^2)_\infty (z^{-1}q^2;q^2)_\infty}. \quad (1.3)$$

It is clear that $R(m,n) = 0$ for $|m| > n/2$.

As we have two different partition statistics explaining cubic partition congruences and

$$c(n) = \sum_{m\in\mathbb{Z}} C(m,n) = \sum_{m\in\mathbb{Z}} R(m,n),$$

it is a natural question to ask how the crank and rank of cubic partitions are distributed. In 2016, Kim–Kim–Nam [9] established the following two-variable asymptotics for $C(m,n)$ and $R(m,n)$ by using a circle method.

Theorem 1 *(Kim–Kim–Nam [9, Theorems 1.1 and 1.2]) As $n \to \infty$,*

$$C(m,n) = \frac{\pi e^{\pi\sqrt{n}}}{16 n^{7/4}} \left(1 - \frac{\pi}{4}\right) \operatorname{sech}^2\left(\frac{\pi m}{2\sqrt{n}}\right)\left(1 + O\left(\frac{1+|m|^{1/3}}{n^{1/4}}\right)\right),$$

provided $|m| \leq n^{3/8}$,

and

$$R(m,n) = \frac{\pi e^{\pi\sqrt{n}}}{32n^{7/4}}\mathrm{sech}^2\left(\frac{\pi m}{2\sqrt{n}}\right)\left(1 + O\left(\frac{1+|m|^{1/3}}{n^{1/4}}\right)\right), \quad \textit{provided } |m| \leq \sqrt{n/2}.$$

In this paper, we establish uniform asymptotic formulas for $C(m,n)$ and $R(m,n)$ that hold for a wider range of m than those given in Theorem 1. This enables a deeper understanding of their distributions.

Throughout this paper, we set $\delta_n = \pi/\sqrt{4n}$. Our main results are as the follows.

Theorem 1.1 *Let m, n be integers. As $n \to +\infty$*

$$C(m,n) \sim \frac{1}{4}c(n)\delta_n \int_{\mathbb{R}} \mathrm{sech}^2(2t)\mathrm{sech}^2(t - m\delta_n/2)\mathrm{d}t,$$

and

$$R(m,n) \sim \frac{1}{2}c(n)\delta_n \mathrm{sech}^2(m\delta_n),$$

uniformly with respect to $m = o(n^{3/4})$.

Remark 1.1 We have established asymptotic formulas for $C(m, n+|m|)$ and $R(m, n+2|m|)$, which hold for all $n \to +\infty$ and uniformly with respect to $m \in \mathbb{Z}$. For details, see Theorems 3.2 and 3.3 in Sect. 3.

Throughout the paper, we use the Landau symbols O and the Vinogradov symbol $\ll$. We recall that the assertions $U = O(V)$ and $U \ll V$ (sometimes we write this also as $V \gg U$) are both equivalent to the inequality $|U| \leq cV$ with some constant $c > 0$, while $U = o(V)$ means that $U/V \to 0$. In this paper, the constants implied in the symbols o, O and $\ll$ are absolute and independent of any parameters.

2 Lemmas

We need some facts on the Andrews–Garvan–Dyson cranks of partitions. Let $M(m,n)$ (with a slight modification in the case that $n = 1$, where the values are instead $M(\pm 1, 1) = 1, M(0,1) = -1$) be the number of partitions of n with crank m, then we have

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{2.1}$$

It is clear that $M(m,m) = 1$ for any $m \geq 0$. We need the uniform asymptotics of $M(m,n)$, which can be find in [11, Proposition 2.1]:

Lemma 2.1 *Let* $g(x) = \frac{\pi}{12\sqrt{2}}\left(1+e^{-|x|}\right)^{-2}$. *As integer* $\ell \to +\infty$

$$M(k, |k|+\ell) \sim g\left(\pi k/\sqrt{6\ell}\right)\ell^{-3/2}e^{2\pi\sqrt{\ell/6}},$$

uniformly with respect to $k \in \mathbb{Z}$. *In particular, for any* $k \in \mathbb{Z}$ *and* $\ell \geq 0$ *we have*

$$M(k, |k|+\ell) \ll (1+\ell)^{-3/2}e^{2\pi\sqrt{\ell/6}}.$$

The following lemma gives the algebraic relations between partition cranks and cubic partition cranks and ranks.

Lemma 2.2 *Let* $m, n \geq 0$. *With* $A := n+m-2|k|-|m-k|$, *we have* $C(m, m) = 1$ *and*

$$C(m, n+m) = \sum_{\substack{k\in\mathbb{Z}\\ A\geq 0}}\sum_{\substack{\ell\geq 0\\ \ell\leq A/2}} M(k, |k|+\ell)M(m-k, |m-k|+A-2\ell), \quad \textit{for all } n \geq 1.$$

We have $R(m, 2m) = R(m, 2m+1) = 1$ *and*

$$R(m, n+2m) = \sum_{0\leq\ell\leq n/2} p(n-2\ell)M(m, m+\ell), \quad \textit{for all } n \geq 1.$$

Proof Using (2.1) and (1.1), the generating function (1.2) and (1.3) can be rewritten as

$$\begin{aligned}\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} C(m,n)z^m q^n =& \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}\frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}\\ &= \sum_{\substack{n_1\geq 0\\ m_1\in\mathbb{Z}}} M(m_1,n_1)z^{m_1}q^{n_1}\sum_{\substack{n_2\geq 0\\ m_2\in\mathbb{Z}}} M(m_2,n_2)z^{m_2}q^{2n_2}\\ &= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} z^m q^n \sum_{\substack{n_1,n_2\geq 0\\ n_1+2n_2=n}}\sum_{\substack{m_1,m_2\in\mathbb{Z}\\ m_1+m_2=m}} M(m_1,n_1)M(m_2,n_2)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} R(m,n)z^m q^n &= \frac{1}{(q;q)_\infty}\frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}\\ &= \sum_{n_1\geq 0} p(n_1)q^{n_1}\sum_{\substack{n_2\geq 0\\ m\in\mathbb{Z}}} M(m,n_2)z^m q^{2n_2}\\ &= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} z^m q^n \sum_{\substack{n_1,n_2\geq 0\\ n_1+2n_2=n}} p(n_1)M(m,n_2).\end{aligned}$$

Noting that $M(m,n)=0$ for all $|m|>n$, we have

$$
\begin{aligned}
C(m,n) &= \sum_{\substack{n_1+2n_2=n\\ n_1,n_2\geq 0}} \sum_{\substack{m_1+m_2=m\\ m_1,m_2\in\mathbb{Z}}} M(m_1,n_1)M(m_2,n_2)\\
&= \sum_{0\leq n_2\leq n/2} \sum_{m_2\in\mathbb{Z}} M(m_2,n_2)M(m-m_2,n-2n_2)\\
&= \sum_{0\leq n_2\leq n/2} \sum_{m_2\in\mathbb{Z}} M(m_2,|m_2|+(n_2-|m_2|))M(m-m_2,n-2|m_2|-2(n_2-|m_2|))\\
&= \sum_{k\in\mathbb{Z}} \sum_{0\leq \ell\leq n/2-|k|} M(k,|k|+\ell)M(m-k,n-2|k|-2\ell). \qquad (2.2)
\end{aligned}
$$

Thus

$$
C(m,m)=\sum_{k\in\mathbb{Z}} \sum_{0\leq \ell\leq m/2-|k|} M(k,|k|+\ell)M(m-k,m-2|k|-2\ell)=M(0,0)M(m,m)=1.
$$

Replacing n by $n+m$ and letting $A=n+m-2|k|-|m-k|$ in (2.2), then we have

$$
C(m,n+m)=\sum_{\substack{k\in\mathbb{Z}\\ A\geq 0}} \sum_{\substack{\ell\geq 0\\ \ell\leq A/2}} M(k,|k|+\ell)M(m-k,|m-k|+A-2\ell),
$$

which completes the proof for $C(m,n+m)$. Similarly,

$$
\begin{aligned}
R(m,n) &= \sum_{\substack{n_1+2n_2=n\\ n_1,n_2\geq 0}} p(n_1)M(m,n_2)\\
&= \sum_{0\leq n_2\leq n/2} p(n-2n_2)M(m,n_2)\\
&= \sum_{0\leq \ell\leq n/2-m} p(n-2m-2\ell)M(m,m+\ell).
\end{aligned}
$$

Replacing n by $n+m$ in above, we have

$$
R(m,n+2m)=\sum_{0\leq \ell\leq n/2} p(n-2\ell)M(m,m+\ell).
$$

From this we see that $R(m,1+2m) = p(1)M(m,m) = 1$ and $R(m,2m) = p(0)M(m,m)=1$, which completes the proof of Lemma 2.2. □

We need the following auxiliary lemmas.

Lemma 2.3 *For* $x \in [0, 1]$, *define*

$$f(x) = \sqrt{1-x} + \sqrt{x/2}.$$

Then $f(x)$ *is increasing on* $[0, 1/3]$ *and decreasing on* $[1/3, 1]$. *Moreover,*

$$f(1/3 + t) = \sqrt{3/2} - \kappa t^2 + O(|t|^3),$$

as $t \to 0$, *where* $\kappa := 2^{-9/2} \cdot 3^{5/2}$.

Proof The proof of this lemma is a direct calculation and we shall omit it. □

Lemma 2.4 *Let* $g(x)$ *be defined as in Lemma 2.1. For any* $x_0 \in \mathbb{R} \cup \{\infty\}$. *If* $y \sim x$ *as* $x \to x_0$, *then* $g(y) \sim g(x)$ *as* $x \to x_0$.

Proof Recall that

$$g(x) = \frac{\pi}{12\sqrt{2}} \left(1 + e^{-|x|}\right)^{-2}.$$

We have

$$\left|\sqrt{g(y)/g(x)} - 1\right| = \left|\frac{1 + e^{-|x|}}{1 + e^{-|y|}} - 1\right| = \frac{|e^{-|x|} - e^{-|y|}|}{1 + e^{-|y|}} \le |e^{-|x|} - e^{-|y|}| \to 0,$$

whenever $y \sim x$ and $x \to x_0$ with $x_0 \in \mathbb{R} \cup \{\infty\}$. The proof follows. □

In this paper, the Euler-Maclaurin summation formula we use is always stated as follows.

Lemma 2.5 *Let* $a, b \in \mathbb{Z}$ *with* $a \le b$, $h \in \mathcal{C}^1([a, b])$. *The we have*

$$\sum_{a \le \ell \le b} h(\ell\varepsilon) = \frac{1}{\varepsilon} \int_{a\varepsilon}^{b\varepsilon} h(u)\,du + \frac{h(a\varepsilon) + h(b\varepsilon)}{2} + O\left(\int_{a\varepsilon}^{b\varepsilon} |h'(u)|\,du\right),$$

for any $\varepsilon \in (0, 1)$, *where the implied constant is absolute.*

3 The Proofs of the Main Results

In view of $C(m, n) = C(|m|, n)$ and $R(m, n) = R(|m|, n)$, $C(m, |m|) = R(m, 2|m|) = 1$ for all $m \in \mathbb{Z}$, and as well as $C(m, n + |m|) = R(m, n + 2|m|) = 0$ for all $n < 0$ and $m \in \mathbb{Z}$, this section will only consider the cases for $C(m, n+m)$ and $R(m, n + 2m)$ with $n \ge 1$ and $m \ge 0$. We assume that the function $f(x)$ is always defined by Lemma 2.3.

3.1 Unform Asymptotic Formulas for $C(m, n+m)$

For simplify our writing, we denote $A := A_{m,n,k} = n + m - 2|k| - |m - k|$ and $S_A = \{k \in \mathbb{Z} : A \geq 0\} \times \{\ell \in \mathbb{Z} : 0 \leq \ell \leq A/2\}$. Then one can check that:

$$A \leq n \quad \text{and} \quad \#S_A \ll n^2.$$

We split that $S_A = S_0 \cup S_1 \cup S_2$ with $S_0 := \left\{(k, \ell) \in S_A : A \leq n^{0.5}\right\}$,

$$S_1 := \left\{(k, \ell) \in S_A : A > n^{0.5},\ |2\ell/A - 1/3| \leq A^{-0.2}\right\},$$

and

$$S_2 := \left\{(k, \ell) \in S_A : A > n^{0.5},\ |2\ell/A - 1/3| > A^{-0.2}\right\}.$$

Therefore, using Lemma 2.2 we can rewrite the formula for $C(m, n+m)$ as:

$$C(m, n+m) = \sum_{0 \leq j \leq 2} C_{S_j}(m, n),$$

where

$$C_{S_j}(m, n) := \sum_{(k,\ell) \in S_j} M(k, |k| + \ell) M(m - k, |m - k| + A - 2\ell). \tag{3.1}$$

From Lemma 2.1, for any $k \in \mathbb{Z}$, $\ell \geq 0$

$$M(k, |k| + \ell) \ll (1 + \ell)^{-3/2} e^{2\pi\sqrt{\ell/6}}.$$

Thus for $(k, \ell) \in S_A$, we have

$$M(k, |k| + \ell) M(m - k, |m - k| + A - 2\ell) \ll e^{\frac{2\pi}{\sqrt{6}}(\sqrt{\ell} + \sqrt{A - 2\ell})}.$$

For $(k, \ell) \in S_0$, we have

$$\sqrt{\ell} + \sqrt{A - 2\ell} \leq 2n^{0.25}.$$

For $(k, \ell) \in S_2$, using Lemma 2.3 we have

$$\begin{aligned} \sqrt{A} f(2\ell/A) &\leq \sqrt{A} \max\left(f(1/3 - A^{-0.2}), f(1/3 + A^{-0.2})\right) \\ &= \sqrt{A}\left(f(1/3) - \kappa A^{-0.4} + O(A^{-0.6})\right) \\ &\leq \sqrt{3n/2} - \kappa n^{0.1} + O(1). \end{aligned}$$

Therefore, using $\#S_A \le n^2$ and above estimates we have

$$\sum_{j\in\{0,2\}} C_{S_j}(m,n) \ll e^{\frac{2\pi}{\sqrt{6}}(2n^{0.25})} \sum_{(k,\ell)\in S_0} 1 + e^{\frac{2\pi}{\sqrt{6}}(\sqrt{3n/2}-\kappa n^{0.1}+O(1))} \sum_{(k,\ell)\in S_2} 1$$
$$\ll e^{n^{1/3}} n^2 + n^2 e^{\pi\sqrt{n}-n^{1/11}} \ll e^{\pi\sqrt{n}-n^{1/12}}. \tag{3.2}$$

We will now prove that the main contribution of the summation for $C(m, m+n)$ comes from $C_{S_1}(m,n)$, as defined by equation (3.1).

Lemma 3.1 *Let $g(x)$ be defined as in Lemma 2.4. As $n \to +\infty$*

$$C(m, m+n) \sim 18 \sum_{\substack{k\in\mathbb{Z} \\ A>n^{0.5}}} g\left(\frac{\pi k}{\sqrt{A}}\right) g\left(\frac{\pi(m-k)}{\sqrt{4A}}\right) A^{-9/4} e^{\pi\sqrt{A}},$$

uniformly with respect to $m \ge 0$.

Proof Notice the fact that as $\ell \to +\infty$

$$M(k, |k|+\ell) \sim g\left(\frac{\pi k}{\sqrt{6\ell}}\right) \frac{e^{2\pi\sqrt{\ell/6}}}{\ell^{3/2}},$$

uniformly with respect to $k \in \mathbb{Z}$, see Lemma 2.1. For $(k,\ell) \in S_1$, since $A > n^{0.5} \to +\infty$,

$$\ell \sim A/6,\ \frac{\pi k}{\sqrt{6\ell}} \sim \frac{\pi k}{\sqrt{A}},\quad A-2\ell \sim 2A/3,\quad \frac{\pi(m-k)}{\sqrt{6(A-2\ell)}} \sim \frac{\pi(m-k)}{\sqrt{4A}},$$

using the above estimates and Lemma 2.4, we have

$$M(k,|k|+\ell)M(m-k,|m-k|+A-2\ell) \sim g\left(\frac{\pi k}{\sqrt{A}}\right) g$$
$$\left(\frac{\pi(m-k)}{\sqrt{4A}}\right) \frac{e^{\frac{2\pi}{\sqrt{6}}(\sqrt{\ell}+\sqrt{A-2\ell})}}{(A/6)^{3/2}(2A/3)^{3/2}}.$$

Moreover, using Lemma 2.3, we have

$$\begin{aligned}\frac{2\pi}{\sqrt{6}}(\sqrt{\ell}+\sqrt{A-2\ell}) &= \frac{2\pi}{\sqrt{6}}\sqrt{A} f(2\ell/A) \\ &= \frac{2\pi}{\sqrt{6}}\sqrt{A}\left(\sqrt{3/2} - \kappa(2\ell/A-1/3)^2 + O(|2\ell/A-1/3|^3)\right) \\ &= \pi\sqrt{A} - \frac{8\pi\kappa}{\sqrt{6}A^{3/2}}(\ell - A/6)^2 + O(A^{-0.1}).\end{aligned}$$

Therefore, further simplifications yields

$$M(k,|k|+\ell)M(m-k,|m-k|+A-2\ell)\sim g\left(\frac{\pi k}{\sqrt{A}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4A}}\right)\frac{e^{\pi\sqrt{A}-\frac{8\pi\kappa}{\sqrt{6}A^{3/2}}(\ell-A/6)^2}}{(A/3)^3}.$$

Hence using (3.1) yields

$$C_{S_1}(m,n)\sim\sum_{(k,\ell)\in S_1}g\left(\frac{\pi k}{\sqrt{A}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4A}}\right)\frac{e^{\pi\sqrt{A}-\frac{8\pi\kappa}{\sqrt{6}A^{3/2}}(\ell-A/6)^2}}{(A/3)^3}$$
$$=\sum_{\substack{k\in\mathbb{Z}\\A>n^{0.5}}}\frac{g\left(\frac{\pi k}{\sqrt{A}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4A}}\right)e^{\pi\sqrt{A}}}{(A/3)^3}\sum_{\substack{0\le\ell\le A/2\\|\ell-A/6|\le 0.5A^{0.8}}}e^{-\frac{8\pi\kappa}{\sqrt{6}A^{3/2}}(\ell-A/6)^2}.$$

Notice that $(0.5A^{0.8})^2/A^{3/2}=0.25A^{0.1}\to+\infty$, the inner summation above is asymptotically equivalent to the following Gauss integral:

$$\int_{\mathbb{R}}e^{-\frac{8\pi\kappa}{\sqrt{6}A^{3/2}}(u-A/6)^2}\mathrm{d}u=\sqrt{\frac{\sqrt{6}A^{3/2}}{8\kappa}},$$

by using the Euler–Maclaurin summation formula. Therefore, by noting that $\kappa=2^{-9/2}\cdot 3^{5/2}$, we have

$$C_{S_1}(m,n)\sim 18\sum_{\substack{k\in\mathbb{Z}\\A>n^{0.5}}}g\left(\frac{\pi k}{\sqrt{A}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4A}}\right)A^{-9/4}e^{\pi\sqrt{A}}.$$

Notice that $A=n+m-2|k|-|m-k|$, we pick out the term $k=0$ from the sum above yields

$$C_{S_1}(m,n)\gg n^{-9/4}e^{\pi\sqrt{n}}.$$

While considering estimate (3.2), we see that

$$C_{S_0}(m,n)+C_{S_2}(m,n)\ll e^{\pi\sqrt{n}-n^{1/12}},$$

which completes the proof. □

We now evaluate the summation in Lemma 3.1. Note that $A=n+k-2|k|$ for $k\le m$, and $A=n+2m-3k$ for $k>m$. Therefore, the summation in Lemma 3.1

can be rewritten as

$$\begin{aligned}
&(1+o(1))C(m,n+m)\\
&=18\sum_{\substack{k\le m\\ n-(2|k|-k)>n^{1/2}}}\frac{g\left(\frac{\pi k}{\sqrt{n-(2|k|-k)}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4(n-(2|k|-k))}}\right)}{(n-(2|k|-k))^{9/4}}e^{\pi\sqrt{n-(2|k|-k)}}\\
&\quad+18\sum_{\substack{k>m\\ n-(3k-2m)>n^{1/2}}}\frac{g\left(\frac{\pi k}{\sqrt{n-(3k-2m)}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4(n-(3k-2m))}}\right)}{(n-(3k-2m))^{9/4}}e^{\pi\sqrt{n-(3k-2m)}},
\end{aligned}$$

as $n\to+\infty$. We write

$$\begin{aligned}
C_I(m,n):=&18\sum_{\substack{k\le m\\ (2|k|-k)\le n^{5/8}}}\frac{g\left(\frac{\pi k}{\sqrt{n-(2|k|-k)}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4(n-(2|k|-k))}}\right)}{(n-(2|k|-k))^{9/4}}e^{\pi\sqrt{n-(2|k|-k)}}\\
&+18\sum_{\substack{k>m\\ (3k-2m)\le n^{5/8}}}\frac{g\left(\frac{\pi k}{\sqrt{n-(3k-2m)}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4(n-(3k-2m))}}\right)}{(n-(3k-2m))^{9/4}}e^{\pi\sqrt{n-(3k-2m)}},
\end{aligned}$$

for replacing the above summation of $(1+o(1))C(m,n+m)$, then the error term is

$$\begin{aligned}
&(1+o(1))C(m,n+m)-C_I(m,n)\\
&\ll\sum_{\substack{k\le m\\ 2|k|-k>n^{5/8}\\ n+k-2|k|>\sqrt{n}}}e^{\pi\sqrt{n+k-2|k|}}+\sum_{\substack{k>m\\ 3k-2m>n^{5/8}\\ n+2m-3k>\sqrt{n}}}e^{\pi\sqrt{n+2m-3k}}\\
&\ll ne^{\pi\sqrt{n-n^{5/8}}}+ne^{\pi\sqrt{n-n^{5/8}}}\ll e^{\pi\sqrt{n}-n^{1/8}}.
\end{aligned}$$

Moreover, using Lemma 2.4 for $g(x)$, and the fact that $e^{\pi\sqrt{x-r}}\sim e^{\pi\sqrt{x}-\pi r/\sqrt{4x}}$ for all $r=o(x^{3/4})$ as $x\to+\infty$, one can find that

$$\begin{aligned}
&C_I(m,n)\sim\\
&\frac{18e^{\pi\sqrt{n}}}{n^{9/4}}\Bigg(\sum_{\substack{k\le m\\ (2|k|-k)\le n^{5/8}}}e^{-\frac{\pi(2|k|-k)}{2\sqrt{n}}}+\sum_{\substack{k>m\\ (3k-2m)\le n^{5/8}}}e^{-\frac{\pi(3k-2m)}{2\sqrt{n}}}\Bigg)g\left(\frac{\pi k}{\sqrt{n}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4n}}\right)\\
&=\frac{18e^{\pi\sqrt{n}}}{n^{9/4}}\left(\sum_{k\le m}e^{-\frac{\pi(2|k|-k)}{2\sqrt{n}}}+\sum_{k>m}e^{-\frac{\pi(3k-2m)}{2\sqrt{n}}}\right)g\left(\frac{\pi k}{\sqrt{n}}\right)g\left(\frac{\pi(m-k)}{\sqrt{4n}}\right)+O(e^{\pi\sqrt{n}-n^{1/8}}).
\end{aligned}$$

Therefore, using the definition of $g(x)$:

$$g(x) = \frac{\pi}{12\sqrt{2}} \frac{1}{(1+e^{-|x|})^2} = \frac{\pi}{48\sqrt{2}} e^{|x|} \operatorname{sech}^2(x/2),$$

and with $\delta_n = \pi/\sqrt{4n}$, by a straightforward calculation, the main term in the above formula can be evaluated in the following form:

$$C(m, n+m) \sim \frac{\pi^2 e^{\pi\sqrt{n}+m\delta_n}}{16^2 n^{9/4}} \sum_{k\in\mathbb{Z}} \operatorname{sech}^2(k\delta_n) \operatorname{sech}^2\left(2^{-1}(m-k)\delta_n\right).$$

Note that for all $t \in \mathbb{R}$, using the Euler–Maclaurin summation formula implies:

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \operatorname{sech}^2(k\delta_n)\operatorname{sech}^2((t-k\delta_n)/2)\\ &\qquad = \frac{1}{\delta_n}\int_{\mathbb{R}} \operatorname{sech}^2(x)\operatorname{sech}^2\left((t-x)/2\right) \mathrm{d}x + O\left(\int_{\mathbb{R}} \left|\partial_x \left(\operatorname{sech}^2(x)\operatorname{sech}^2\left((tx)/2\right)\right)\right|\right).\end{aligned}$$

Note that $e^{-2|x|} \ll \operatorname{sech}^2(x) \ll e^{-2|x|}$ and $\partial_x \operatorname{sech}^2(x) \ll e^{-2|x|}$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}} \left|\partial_x \left(\operatorname{sech}^2(x)\operatorname{sech}^2\left((t-x)/2\right)\right)\right| \mathrm{d}x &\ll \int_{\mathbb{R}} e^{-2|x|-|t-x|}\mathrm{d}x\\ &\ll \int_{\mathbb{R}} \operatorname{sech}^2(x)\operatorname{sech}^2\left((t-x)/2\right) \mathrm{d}x.\end{aligned}$$

Moreover, note that

$$\frac{1}{8}\int_{\mathbb{R}} \operatorname{sech}^2(x)\operatorname{sech}^2\left((t-x)/2\right) \mathrm{d}x = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sech}^2(2x)\operatorname{sech}^2\left(x-t/2\right) \mathrm{d}x,$$

is an even function for $t \in \mathbb{R}$, and $C(-m, n+|m|) = C(m, n+|m|)$ for all $m \in \mathbb{Z}$. We conclude the above with the following theorem.

Theorem 3.2 *As $n \to +\infty$*

$$C(m, n+|m|) \sim \frac{\pi e^{\pi\sqrt{n}+|m|\delta_n}}{64 n^{7/4}} \int_{\mathbb{R}} \operatorname{sech}^2(2x)\operatorname{sech}^2\left(x - m\delta_n/2\right) \mathrm{d}x,$$

uniformly with respect to $m \in \mathbb{Z}$.

3.2 Uniform Asymptotic Formulas of $R(m, n)$

From Lemma 2.2, we can rewrite the formula for $R(m, n+2m)$ as:

$$\begin{aligned} R(m, n+2m) &= \sum_{0\le \ell \le n/2} p(n-2\ell)M(m, m+\ell) \\ &= \left(\sum_{\substack{0\le \ell\le n/2 \\ |2\ell/n-1/3|\le n^{-0.2}}} + \sum_{\substack{0\le \ell\le n/2 \\ |2\ell/n-1/3|> n^{-0.2}}} \right) p(n-2\ell)M(m, m+\ell) \\ &=: R_M(m, n) + R_E(m, n). \end{aligned}$$

We claim that the main contribution of $R(m, n+2m)$ arises from $R_M(m, n)$, while the $R_E(m, n)$ is an error term. In fact, by use of the Hardy–Ramanujan asymptotic formula:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{2\pi\sqrt{n/6}},$$

as $n \to +\infty$, Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} R_E(m, n) &= \sum_{\substack{0\le \ell\le n/2 \\ |2\ell/n-1/3|> n^{-0.2}}} p(n-2\ell)M(m, m+\ell) \\ &\ll \sum_{\substack{0\le \ell\le n/2 \\ |2\ell/n-1/3|> n^{-0.2}}} e^{2\pi\sqrt{(n-2\ell)/6}+2\pi\sqrt{\ell/6}} = \sum_{\substack{0\le \ell\le n/2 \\ |2\ell/n-1/3|> n^{-0.2}}} e^{\frac{2\pi}{\sqrt{6}}\sqrt{n} f(2\ell/n)} \\ &\ll n \exp\left(\frac{2\pi\sqrt{n}}{\sqrt{6}} \sup_{\substack{0\le x\le 1 \\ |x-1/3|> n^{-0.2}}} f(x) \right) \\ &\ll n \exp\left(\frac{2\pi\sqrt{n}}{\sqrt{6}} \left(\sqrt{3/2} - \kappa n^{-0.4} + O(n^{-0.6}) \right) \right) \ll e^{\pi\sqrt{n} - n^{1/11}}. \end{aligned}$$

Moreover, since $n \to +\infty$, $|2\ell/n - 1/3| \le n^{-0.2}$, we have $\ell \sim n/6$ and $n - 2\ell \sim 2n/3$. Using Lemmas 2.1 and 2.4 implies:

$$\begin{aligned} p(n-2\ell)M(m, m+\ell) &\sim g\left(\frac{\pi m}{\sqrt{6\ell}}\right) \frac{e^{\frac{2\pi}{\sqrt{6}}\left(\sqrt{n-2\ell}+\sqrt{\ell}\right)}}{4\sqrt{3}(n-2\ell)\ell^{3/2}} \\ &\sim g\left(\frac{\pi m}{\sqrt{n}}\right) \frac{e^{\frac{2\pi\sqrt{n}}{\sqrt{6}} f(2\ell/n)}}{4\sqrt{3}(2n/3)(n/6)^{3/2}} \end{aligned}$$

$$= g\left(\frac{\pi m}{\sqrt{n}}\right)\frac{9}{2^{3/2}n^{5/2}}e^{\frac{2\pi\sqrt{n}}{\sqrt{6}}\left(\sqrt{3/2}-\kappa(2\ell/n-1/3)^2+O(|2\ell/n-1/3|^3)\right)}$$
$$\sim \frac{3\pi}{16n^{5/2}}\left(1+e^{-\pi m/\sqrt{n}}\right)^{-2}e^{\pi\sqrt{n}-\frac{8\pi\kappa(\ell-n/6)^2}{\sqrt{6}n^{3/2}}}.$$

Therefore,

$$R_M(m,n)\sim\frac{3\pi}{16n^{5/2}}\left(1+e^{-\pi m/\sqrt{n}}\right)^{-2}\sum_{\substack{0\le\ell\le n/2\\ |2\ell/n-1/3|\le n^{-0.2}}}e^{\pi\sqrt{n}-\frac{8\pi\kappa(\ell-n/6)^2}{\sqrt{6}n^{3/2}}}.$$

Noting that $\kappa=2^{-9/2}\cdot 3^{5/2}$ and using similar arguments to $C(m,n+m)$, we have

$$R_M(m,n)\sim\frac{3\pi e^{\pi\sqrt{n}}}{16n^{5/2}}\left(1+e^{-\pi m/\sqrt{n}}\right)^{-2}\sqrt{\frac{\sqrt{6}n^{3/2}}{8\kappa}}=\frac{\pi e^{\pi\sqrt{n}}}{8n^{7/4}}\left(1+e^{-\pi m/\sqrt{n}}\right)^{-2}.$$

By combining this with the previous estimate for $R_E(m,n)$, $\delta_n=\pi/\sqrt{4n}$, and as well as $R(-m,n+2|m|)=R(m,n+2|m|)$ holds for all $m\in\mathbb{Z}$. This leads to the following theorem.

Theorem 3.3 *As $n\to+\infty$*

$$R(m,n+2|m|)\sim\frac{\pi e^{\pi\sqrt{n}+2|m|\delta_n}}{32n^{7/4}}\operatorname{sech}^2(m\delta_n),$$

uniformly with respect to $m\in\mathbb{Z}$.

3.3 The Proof of Theorems 1.1

We use Theorems 3.2 and 3.3 to prove Theorems 1.1.

Proof of Theorem 1.1 Notice that $\delta_n=\pi/\sqrt{4n}$ and note that for $m=o(n^{3/4})$,

$$\pi\sqrt{n-|m|}+|m|\delta_{n-|m|}=\pi\sqrt{n}-|m|\delta_n+|m|\delta_n+O(m^2n^{-3/2})=\pi\sqrt{n}+o(1),$$

and

$$\operatorname{sech}\left(x-2^{-1}m\delta_{n-|m|}\right)=\operatorname{sech}(x-2^{-1}m\delta_n+O(m^2n^{-3/2}))\sim\operatorname{sech}(x-2^{-1}m\delta_n),$$

uniformly with respect to $x\in\mathbb{R}$. Therefore, using Theorem 3.2 implies

$$C(m,n)\sim\frac{\pi e^{\pi\sqrt{n-|m|}+|m|\delta_n}}{16n^{7/4}}\cdot\frac{1}{4}\int_{\mathbb{R}}\operatorname{sech}^2(2x)\operatorname{sech}^2\left(x-2^{-1}m\delta_{n-|m|}\right)\mathrm{d}x$$
$$\sim\frac{\pi e^{\pi\sqrt{n}}}{16n^{7/4}}\cdot\frac{1}{4}\int_{\mathbb{R}}\operatorname{sech}^2(2x)\operatorname{sech}^2\left(x-m\delta_n/2\right)\mathrm{d}x. \tag{3.3}$$

Similarly, for $m = o(n^{3/4})$, using Theorem 3.3 implies

$$R(m,n) \sim \frac{\pi e^{\pi\sqrt{n-2|m|}+2m\delta_{n-2|m|}}}{32n^{7/4}}\operatorname{sech}^2(2m\delta_{n-2|m|}) \sim \frac{\pi e^{\pi\sqrt{n}}}{32n^{7/4}}\operatorname{sech}^2(m\delta_n). \quad (3.4)$$

Therefore, the proof of Theorem 1.1 will follows from (3.3),(3.4) and the fact that

$$c(n) \sim \frac{1}{8}n^{-5/4}e^{\pi\sqrt{n}},$$

see [9, Equation (1.5)]. □

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Data availability We declare that Data sharing not applicable to the present paper as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest There are no Conflict of interest. This paper is original, and it has not been submitted elsewhere.

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