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Low Regularity for LS Type Equations on the Half Line

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Abstract

We study the initial-boundary vaule problem of the long and short wave equations posed on the half line with initial datas $(u_0, n_0) \in H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$ and boundary datas $(h, f) \in H^{\frac{2s_0+1}{4}}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$. We show the local well-posedness by giving the bilinear estimates of the coupling terms for the equations in suitable spaces of Bourgain type. Moreover, we consider results concerning ill-posedness for the system. Finally, the system is proved to be globally well-posed in Sobolev spaces $H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$.

Keywords Long wave-short wave equations · The Fourier restriction norm · Initial-boundary value problem · Bilinear estimates · Local well-posedness

Mathematics Subject Classification 35M33 · 35Q55

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1 Introduction

Consideration is given to the equations where u is an unknown complex function and n is an unknown real function:

$$\begin{cases} iu_t + u_{xx} = nu - |u|^2 u, \ (x,t) \in \mathbb{R}^+ = (0,\infty), \\ n_t + \delta n = -|u|_x^2, \ (x,t) \in \mathbb{R}^+ = (0,\infty), \\ u(x,0) = u_0(x), \ n(x,0) = n_0(x), \\ u(0,t) = h(t), \ n(0,t) = f(t), \end{cases} \tag{1}$$

where δ is a given positive constant, the initial conditions $(u_0, n_0) \in H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$ and the boundary conditions $(h, f) \in H^{\frac{2s_0+1}{4}}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$.

The long-short (LS) type equations depict the resonance interaction between the short wave and long wave, which were first deduced by Djordjevic and Redekopp [14]. The system has been well-studied in the past, where the short wave is often described by Schrödinger equation and the long wave is described by some sort of dispersive wave equation such as KdV equation, Benjamin-Ono equation so that the system came in various forms. The well-posedness of the Cauchy problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = uv, \ x \in M, \ t \in \mathbb{R}, \\ \sigma\partial_t v + v = \varepsilon|u|^2, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \end{cases}$$

was studied by Corcho and Matheus [12], which is called Schrödinger-Debye system. To apply contraction mapping principle to handle this problem, the authors started by decoupling the system to obtain the integral formulation:

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')(e^{-\frac{t'}{\sigma}} v_0 u(t') + \frac{\varepsilon}{\sigma} u(t') \int_0^{t'} e^{-\frac{(t'-\tau)}{\sigma}} |u(\tau)|^2 d\tau) dt',$$

where $U(t) = e^{it\Delta/2}$ is the Schrödinger linear unitary group. When M was the real line $\mathbb{R}$, this system was proved to be local well-posed in $H^k(\mathbb{R}) \times H^s(\mathbb{R})$ for $|k| - \frac{1}{2} \le s < \min\{k + \frac{1}{2}, 2k + \frac{1}{2}\}$ and $k > -\frac{1}{4}$. When M was the torus T, this system was proved to be local well-posed in $H^k(\mathbb{T}) \times H^s(\mathbb{T})$ for $0 \le s \le 2k$ and $|s-k| < 1$. Moreover, the authors proved the global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $-\frac{3}{14} < s < 0$. Corcho, Oliveira and Silva in [13] considered this system with initial data in the classic Sobolev spaces $H^k(\mathbb{M}) \times H^s(\mathbb{M})$, with k and s satisfying $\max(0, k-1) \le s \le \min(2k, k+1)$ when $M = R^n (n = 2, 3)$. In [1], Arbieto and Matheus slao considered the system when $M = T^n$. They proved that the system was locally and globally well-posed in $H^s(T) \times H^s(T)$ for $s \ge 0$ when $n = 1$.

There are also many results regarding other types of long and short wave equations. Tsutsumi and Hatano [37] studied the Benney system:

$$\begin{cases} iu_t + u_{xx} = nu + \beta|u|^2u, \\ n_t + n_x = |u|_x^2, \end{cases}$$

the authors proved the well-posedness for initial data $(u_0, n_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R})$ with $\frac{1}{2} \le k < 1, l > k - \frac{1}{2}$ in the general case $\beta \ne 0$, and $(u_0, n_0) \in H^k(\mathbb{R}) \times H^{\frac{1}{k}}(\mathbb{R})$ with $0 < k < \frac{1}{2}$ in the special case $\beta = 0$. Guo and Wang [18] considered the following nonlinear equations with periodic boundary conditions:

$$\begin{cases} iu_t + u_{xx} - nu + i\alpha u + \beta g(|u|^2u) + h_1(x) = 0, \\ n_t + |u|_x^2 + \delta n + \gamma f(|u|^2) + h_2(x) = 0, \end{cases}$$

the authors showed the existence and uniqueness of the global solution of the generalized LS type equations. Huo [22] proved the Cauchy problem of the LS type equations where the long wave is described by Benjamin-Ono equation is locally well-posed in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ for $s > 0$. In [34], Pecher investigated the Cauchy problem for Schrödinger-Benjamin-Ono system and showed global well-posedness for data with infinite energy where $s > \frac{1}{3}$. Bekiranov, Ogawa and Ponce [2] proved that the Cauchy problem of the coupled Schrödinger-KdV system is locally well-posed in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ for $s > 0$. In [38], Wu improved the local results for the Schrödinger-KdV system and obtained some ill-posedness results at low regularity and global well-posedness results in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > \frac{1}{2}$. The LS type equations were also studied in many other works([3, 31–33, 39]), including (2+1)-dimensional LS type equations which were worth learning for researchers.

This paper is devoted to the local well-posedness of (1) in low regularity sapces and the global well-posedness in Sobolev spaces. Bourgain space was introduced by Bourgain [6] and [7] to research the low regularity for some equations, which are to dispersive equations as Sobolev spaces are to elliptic equations. The Bourgain method is also called the Fourier restriction norm method. In [27–29], Kenig, Ponce and Vega developed a bilinear estimate and applied it to these equations in Bourgain space, which simplifies the Fourier restriction norm method. This method was initially applied to Schrödinger equation and KdV equation and has improved significantly compared to previous results, see [20, 35]. To study the global well-posedness in low regularity, $[k; Z]$-multiplier norm method and the I-method were introduced in [10, 36]. So far, the Bourgain method has been applied to many other equations such as Boussinesq equation, Hirota equation, Ostrovsky equation and the fifth order shallow water equation, see [9, 11, 17, 19, 21, 23–25, 30].

Although the Cauchy problem for the LS type equations have been studied by many reseachers, the initial-boundary value problem(IBVP) of the LS type equations is discussed by few authors. How do we deal with the system with nonhomogeneous boundary data on the half line in low regularity? Erdoğan and Tzirakis's work in [15] is the motivation for our paper. In this work we investigate the local and global regularities of the LS type equations with initial conditions $(u_0, n_0) \in H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$ and boundary conditions $(h, f) \in H^{\frac{2s_0+1}{4}}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$. Recall that the Fourier restricted norm method has been used to many other equations and has got success, therefore we try to use this method to obtain the local well-posedness in low regularity. However, it seems that the Bourgain type spaces were always applied to dispersive equations. To find admissible s_0 and s_1, we try to apply the Bourgain space method for the system by special techniques where the second equation is not dispersive in this article. Then the existence and uniqueness of the solutions is obtained by the contraction mapping principle. For the completeness of the research, the ill-posedness is proved accordingly.

Moreover, the global well-posedness for the nonhomogeneous boundary-value problem is considered. Regarding the well-posedness, there is a difficulty in our case because of the presence of the boundary conditions. Keeping the difficulty in mind, we apply the work of Carroll [8] to overcome this, where the unknown boundary conditions are seen as the given undetermined coefficient.

In this paper, we say (s_0, s_1) is admissible if s_0, s_1 satisfy

$$0 < s_0 - s_1 \leq 1,\ 2s_0 \geq s_1 + \frac{1}{2} \quad and \quad s_0 > 0,\ s_1 > -\frac{1}{2}.$$

Definition 1 The system (1) is locally well-posed in $H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$, if for any $u_0 \in H^{s_0}(\mathbb{R}^+)$, $h \in H^{\frac{2s_0+1}{4}}(R^+)$, $n_0 \in H^{s_1}(\mathbb{R}^+)$, and $f \in H^{s_1}(\mathbb{R}^+)$, the integral equation (4) below has a unique solution in

$$[X^{s_0,b} \cap C_t^0 H_x^{s_0} \cap C_x^0 H_t^{\frac{2s_0+1}{4}}] \times [Y^{s_1,b_1} \cap C_t^0 H_x^{s_1} \cap C_x^0 H_t^{s_1}],$$

where $b, b_1 < \frac{1}{2}$ and T is sufficiently small.

Theorem 1 For any admissible pair (s_0, s_1), if $(u_0, n_0) \in H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$, $(h, f) \in H^{\frac{2s_0+1}{4}}(R^+) \times H^{s_1}(\mathbb{R}^+)$, the equation (1) is locally well-posed in

$$[X^{s_0,b} \cap C_t^0 H_x^{s_0} \cap C_x^0 H_t^{\frac{2s_0+1}{4}}] \times [Y^{s_1,b_1} \cap C_t^0 H_x^{s_1} \cap C_x^0 H_t^{s_1}].$$

The introduction of $X^{s_0,b}$ and Y^{s_1,b_1} is presented in the next section.

In the next theorem, an ill-posedness result is stated for the LS type equations.

Theorem 2 If $2s_0 < s_1 + \frac{1}{2}$, the associated map data-solution $H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+) \to C_t^0([0, T]; H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+))$ is not C^2 at zero.

Theorem 3 If $(u_0, n_0) \in H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, $(h, f) \in H^1(R^+) \times L^2(\mathbb{R}^+)$, the equation (1) is globally well-posed in $H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$.

The paper is organized as follows: In Sect. 2, we give several notations and outline important results prepared for the later study. We use the Bourgain space method to present the bilinear estimates in Sect. 3. Section 4 is committed to the local results of the equations. In Sect. 5, the ill-posedness result is stated for the system. In Sect. 6, we obtain a priori bounds and show the global well-posedness of the LS type equations in Sobolev spaces.

2 Preliminaries

Now, we give several basic notations and important results in preparation for subsequent proof.

2.1 Notations and Function Spaces

Throughout the paper, C will signify an arbitrarily positive constant. In all this paper, for $x,\ y \in \mathbb{R}^{+}$, we denote $x \lesssim y$ by $x \leq Cy$. In addition, we denote $x \sim y$ by $x \lesssim y$ and $y \lesssim x$. Furthermore, we use $x \ll y$ to denote the statement $x \leq C^{-1}y$. The notation $x+$ means $x+\epsilon$, where ϵ is small enough. We denote $x-$ similarly.

We set

$$\langle \xi \rangle = \sqrt{1+|\xi|^2} \approx 1+|\xi|,$$

$D_0 f(t) = f(0,t)$, for a space time function f.

We fix a cut-off function $\phi(t) := \phi$, $\phi \in C_0^{\infty}(\mathbb{R})$ such that $\phi \equiv 1$ on $[0,1]$, $\phi \equiv 0$ for $|t|> 2$.

Let $s > -\frac{1}{2}$, we define

$$H^s(\mathbb{R}^+) = \{f \in \mathscr{D}(\mathbb{R}^+) : \exists \tilde{f} \in H^s(\mathbb{R}) \textit{ so that } \tilde{f}\chi_{(0,\infty)} = f\}$$

by the norm

$$\|f\|_{H^s(\mathbb{R}^+)} := \inf\{\|\tilde{f}\|_{H^s(\mathbb{R})} \colon \tilde{f}\chi_{(0,\infty)} = f\}.$$

where χ is the usual characteristic function.

We denote the linear Schrödinger group by

$$W_{\mathbb{R}} f(x,t) = e^{it\partial xx} f(x) = \mathcal{F}^{-1}(e^{itp(\cdot)}\widehat{f}(\cdot))(x),$$

where $f \in L^2(\mathbb{R})$ and $p(\xi) = -\xi^2$. Similarly, we use the notation $e^{-t\delta} f(x) = \mathcal{F}^{-1}(e^{it\phi(\cdot)}\widehat{f}(\cdot))(x)$ for the second linear equation. It should be pointed out that $\widehat{f}$ represnts the Fourier transform of f.

We give a defineition of the function space $X^{s,b}$:

$$\|u\|_{X^{s,b}} = \|\langle \xi \rangle^s \langle \tau+\xi^2 \rangle^b \widehat{u}(\xi,\tau)\|_{L^2_\tau L^2_\xi}.$$

We also give a defineition of the space $Y^{s,b}$:

$$\|n\|_{Y^{s,b}}= \|\langle\xi\rangle^s\langle\tau-\delta\rangle^b\widehat{n}(\xi,\tau)\|_{L^2_\tau L^2_\xi}.$$

2.2 Notion of a Solution

To begin with, it is necessary to consider the linear problem on $\mathbb{R}^+$ to explore the solutions of the IBVP (1). Considering the extension of functions, the following lemma need to be stated.

Lemma 1 ([16]) For $g\in H^s(\mathbb{R}^+)$.

(1) Suppose $-\frac{1}{2}<s<\frac{1}{2}$, it holds that $\|\chi_{(0,\infty)}g\|_{H^s(\mathbb{R})}\lesssim\|g\|_{H^s(\mathbb{R}^+)}$.

(2) Suppose $\frac{1}{2}<s<\frac{3}{2}$ and $g(0)=0$, it holds that $\|\chi_{(0,\infty)}g\|_{H^s(\mathbb{R})}\lesssim\|g\|_{H^s(\mathbb{R}^+)}$.

Then the idea is to write the solutions of two linear equations respectively. Concerning the linear Schrödinger equation, the solution

$$\begin{cases} iu_t+u_{xx}=0,\ (x,t)\in\mathbb{R}^+=(0,\infty),\\ u(x,0)=u_0(x),\ u(0,t)=h(t)\end{cases}\tag{2}$$

can be written as

$$W_0^t(\tilde{u}_0,h)=W_{\mathbb{R}}(t)\tilde{u}_0+W_0^t(0,h-p),$$

where $p(t)=\phi(t)D_0[W_{\mathbb{R}}(t)\tilde{u}_0]$, which is in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ and $\tilde{u}_0$ is the extension of u_0. Note that $W_0^t(0,h)$ stands for the boundary operator which has been given in [4].

The porperties of the second linear problem is also considered:

$$\begin{cases} n_t+\delta n=0,\ (x,t)\in\mathbb{R}^+=(0,\infty),\\ n(x,0)=n_0(x),\ n(0,t)=f(t),\end{cases}\tag{3}$$

where $f(t)=e^{-\delta t}g(0)$ and $g(x)$ is a arbitrary function with respect to x. Observe that when $n_0=0$,

$$V_0^t(0,f)(x)=e^{-\delta t}g(x)$$

on $\mathbb{R}^+$ is the solution of (3).

Let $\tilde{n}_0$ be extension of n_0 and $V_0^t(\tilde{n}_0,f)$ be written as

$$V_0^t(\tilde{n}_0,f)=e^{-\delta t}\tilde{n}_0(x)+V_0^t(0,f-r)(x),$$

where $g(x)$ is taken as $n_0(x)$ and $r(t)=\phi(t)D_0(e^{-\delta t}\tilde{n}_0)=\phi(t)[e^{-\delta t}\tilde{n}_0]|_{x=0}$. Note that the restrictionof $V_0^t(\tilde{n}_0,f)$ on $\mathbb{R}^+\times[0,1]$ is the solution of (3).

The LS type equations (1) can be rewritten as the equivalent integral formulations:

$$\begin{cases} u(t) = \phi(t)W_0^t(\tilde{u}_0, h) - i\phi(t)\int_0^t W_{\mathbb{R}}(t-\tau)F(u,n)d\tau + i\phi(t)W_0^t(0,q), \\ n(t) = \phi(t)e^{-\delta t}\tilde{n}_0(x) + \phi(t)\int_0^t e^{-\delta(t-\tau)}G(u,n)d\tau + \phi(t)V_0^t(0,z), \end{cases} \tag{4}$$

where $F(u,n) = \phi(t/T)(nu - |u|^2 u)$, $q(t) = \phi(t)[\int_0^t W_{\mathbb{R}}(t-\tau)F(u,n)d\tau]|_{x=0}$ and $G(u,n) = -\phi(t/T)|u|_x^2$, $z(t) = \phi(t)[\int_0^t e^{-\delta(t-\tau)}G(u,n)d\tau]|_{x=0}$.

For the dispersive equation, we collect the following basic results that are classical in previous studies. (see [15]).

For any s and b, we obtain

$$\|\phi(t)W_{\mathbb{R}}g\|_{X^{s,b}} \lesssim \|g\|_{H^s}. \tag{5}$$

For $s \in \mathbb{R}$, $0 \le b_1 < \frac{1}{2}$ and $0 \le b_2 < 1 - b_1$, we obtain

$$\|\phi(t)\int_0^t W_{\mathbb{R}}(t-\tau)F(\tau)d\tau\|_{X^{s,b_2}} \lesssim \|F\|_{X^{s,-b_1}}. \tag{6}$$

What's more, for $T < 1$ and $-\frac{1}{2} < b_1 < b_2 < \frac{1}{2}$, we obtain

$$\|\phi(t/T)F\|_{X^{s,b_1}} \lesssim T^{b_2-b_1}\|F\|_{X^{s,b_2}}. \tag{7}$$

There have been extensive results for the dispersive equations in Bourgain spaces. One of the features of this work is that we give similiar estimates in Bourgain type spaces for the second equation which is not dispersive.

Proposition 4 For any s and b, it holds that

$$\|\phi(t)e^{-\delta t}g\|_{Y^{s,b}} \lesssim \|g\|_{H^s}. \tag{8}$$

Proof According to the introduction of the space $Y^{s,b}$, we can see that

$$\begin{aligned} \|\phi(t)e^{-\delta t}g\|_{Y^{s,b}} &= \|\langle\xi\rangle^s\langle\tau-\delta\rangle^b\mathscr{F}_t(\phi(t)e^{-\delta t}\widehat{g}(\xi))(\tau)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|\langle\xi\rangle^s\widehat{g}(\xi)\|\langle\tau-\delta\rangle^b\mathscr{F}_t(\phi(t)e^{-\delta t})(\tau)\|_{L^2_\tau(\mathbb{R})}\|_{L^2_\xi(\mathbb{R})} \\ &\lesssim \|\langle\xi\rangle^s\widehat{g}(\xi)\|_{L^2_\xi(\mathbb{R})}. \end{aligned}$$

This completes the proof. □

Proposition 5 Let $0 \le b_1 < \frac{1}{2}$, $b_2 \ge 0$, with $b_1 + b_2 \le 1$, $s \in \mathbb{R}$. Define $k_\xi(t)$ by

$$k_\xi(t) = \phi(t)\int_{\mathbb{R}}\frac{e^{it\tau_1} - e^{-\delta t}}{i\tau_1 + \delta}\widehat{F}(\tau_1,\xi)d\tau_1.$$

Then, there exists

$$\|\langle\tau-\delta\rangle^{b_2}\widehat{k}_\xi(\tau)\|_{L^2_\tau}\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}.$$

Proof First, the estimates are given for some integrations. If $|\tau|\geq 1$, there exists

$$\int_{|\tau|\geq 1}\frac{\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2}d\tau\lesssim 1.$$

If $|\tau|\leq 1$, we have the estimates as follows:

$$\int_{|\tau|\leq 1}\frac{\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2}d\tau\lesssim\int_{|\tau|\leq 1}\langle\tau-\delta\rangle^{2b_1}d\tau\lesssim 1.$$

Then, we separate the integrand as follows:

$$k_\xi(t):=k_1-k_2+k_3+k_4,$$

where

$$k_1=\phi(t)\int_{|\tau_1|\geq 1}\frac{e^{it\tau_1}}{i\tau_1+\delta}\widehat{F}(\tau_1,\xi)d\tau_1,\ k_2=\phi(t)\int_{|\tau_1|\geq 1}\frac{e^{-\delta t}}{i\tau_1+\delta}\widehat{F}(\tau_1,\xi)d\tau_1,$$

and

$$k_3=\phi(t)\int_{|\tau_1|\leq 1}\frac{1-e^{-\delta t}}{i\tau_1+\delta}\widehat{F}(\tau_1,\xi)d\tau_1,\ k_4=\phi(t)\int_{|\tau_1|\leq 1}\frac{e^{it\tau_1}-1}{i\tau_1+\delta}\widehat{F}(\tau_1,\xi)d\tau_1.$$

For k_1, we obtain

$$\begin{aligned}
&\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(k_1)(\tau)\|_{L^2_\tau}\\
&=\|\langle\tau-\delta\rangle^{b_2}(\widehat{\phi}*\frac{\widehat{F}(\cdot,\xi)\chi_{|\cdot|\geq 1}}{i\cdot+\delta})(\tau)\|_{L^2_\tau}\\
&\lesssim\|(\langle\cdot\rangle^{b_2}\widehat{\phi})*\frac{\widehat{F}(\cdot,\xi)\chi_{|\cdot|\geq 1}}{i\cdot+\delta}(\tau)\|_{L^2_\tau}\\
&\lesssim\|\langle\cdot\rangle^{b_2}\widehat{\phi}\|_{L^1_\tau}\|\frac{\widehat{F}(\cdot,\xi)\chi_{|\cdot|\geq 1}}{i\cdot+\delta}(\tau)\|_{L^2_\tau}\\
&\lesssim\|\frac{\widehat{F}(\cdot,\xi)\chi_{|\cdot|\geq 1}}{\langle\tau-\delta\rangle^{b_1}}\frac{\langle\tau-\delta\rangle^{b_1}}{|i\tau+\delta|}\|_{L^2_\tau}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}(\sup_{|\tau|\geq 1}\frac{\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2})^{\frac{1}{2}}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}.
\end{aligned}$$

For k_2, we obtain

$$
\begin{aligned}
&\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(k_2)(\tau)\|_{L^2_\tau}\\
&=\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)e^{-\delta t})\int_{|\tau|\geq 1}\frac{\widehat{F}(\tau_1,\xi)}{i\tau_1+\delta}d\tau_1\|_{L^2_\tau}\\
&\lesssim\int_{|\tau_1|\geq 1}\frac{\widehat{F}(\tau_1,\xi)}{i\tau_1+\delta}d\tau_1\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)e^{-\delta t})\|_{L^2_\tau}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}\left(\int_{|\tau|\geq 1}\frac{\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2}d\tau\right)^{\frac{1}{2}}\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)e^{-\delta t})\|_{L^2_\tau}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}.
\end{aligned}
$$

For k_3, we obtain

$$
\begin{aligned}
&\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(k_3)(\tau)\|_{L^2_\tau}\\
&=\int_{|\tau_1|\leq 1}\frac{\widehat{F}(\tau_1,\xi)}{i\tau_1+\delta}d\tau_1\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)(1-e^{-\delta t}))(\tau)\|_{L^2_\tau}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}\left(\int_{|\tau|\leq 1}\frac{\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2}d\tau\right)^{\frac{1}{2}}\cdot\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)(1-e^{-\delta t}))\|_{L^2_\tau}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}.
\end{aligned}
$$

For k_4, we obtain

$$
\begin{aligned}
&\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(k_4)(\tau)\|_{L^2_\tau}\\
&=\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}\sum_{n=1}^{\infty}\frac{\|\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(|t|^n\phi(t))\|_{L^2_\tau}}{n!}\left(\int_{|\tau|\leq 1}\frac{|\tau|^{2n}\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2}d\tau\right)^{\frac{1}{2}}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}\sum_{n=1}^{\infty}\frac{\|t^n\phi(t)\|_{H^{b_2}_\tau}}{n!}\left(\int_{|\tau|\leq 1}\frac{|\tau|^{2n}\langle\tau-\delta\rangle^{2b_1}}{|i\tau+\delta|^2}d\tau\right)^{\frac{1}{2}}\\
&\lesssim\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}.
\end{aligned}
$$

The proof is completed. □

Proposition 6 Let $0\leq b_1, b_2<\frac{1}{2}$, $s\in\mathbb{R}$, then

$$
\|\phi(t)\int_0^t e^{-\delta(t-t')}F(t')\|_{Y^{s,b_2}}\lesssim\|F\|_{Y^{s,-b_1}}.
$$

Proof According to the introduction of the space $Y^{s,b}$, we obtain the estimate as follows:

$$
\begin{aligned}
&\|\phi(t)\int_0^t e^{-\delta(t-t')}F(t')\|_{Y^{s,b_2}}\\
&= \|\langle\xi\rangle^s\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)\int_0^t e^{-\delta(t-t')}\mathscr{F}_x F(t',\xi)dt')(\tau)\|_{L^2_\tau L^2_\xi}\\
&\lesssim \|\langle\xi\rangle^s\langle\tau-\delta\rangle^{b_2}\mathscr{F}_t(\phi(t)\int_{\mathbb{R}}\frac{e^{it\tau_1}-e^{-t\delta}}{i\tau_1+\delta}\widehat{F}(\tau_1,\xi)d\tau_1)(\tau)\|_{L^2_\tau L^2_\xi}\\
&\lesssim \|\langle\xi\rangle^s\|\langle\tau-\delta\rangle^{b_2}\widehat{k_\xi}(\tau)\|_{L^2_\tau}\|_{L^2_\xi}\\
&\lesssim \|\langle\xi\rangle^s\|\frac{\widehat{F}(\tau,\xi)}{\langle\tau-\delta\rangle^{b_1}}\|_{L^2_\tau}\|_{L^2_\xi}.
\end{aligned}
$$

The proposition is proved, where the last inequality can be obtained by Proposition 2.3. □

Finally, we get the lemmas [16] below which we will use throughout the paper.

Lemma 2 If $b_1 \geq b_2 \geq 0$ and $b_1 + b_2 > 1$, then

$$\int \frac{1}{\langle\lambda-a\rangle^{b_1}\langle\lambda-b\rangle^{b_2}}d\lambda \lesssim \langle a-b\rangle^{-b_2}\varphi_{b_1}(a-b),$$

where

$$\varphi_{b_1}(a) = \begin{cases} 1,\ b_1 > 1,\\ log(1+\langle a\rangle),\ b_1 = 1,\\ \langle a\rangle^{1-b_1},\ b_1 < 1. \end{cases}$$

Lemma 3 If $b_0, b_1, b_2 \geq 0$ and $b_0 + b_1 + b_2 > 1$. Let $l := \max(1, b_0, b_1, b_2)$. Then

$$\int \frac{1}{\langle\lambda-a\rangle^{b_0}\langle\lambda\rangle^{b_1}\langle\lambda+a\rangle^{b_2}}d\lambda \lesssim \langle a\rangle^{-b_0-b_1-b_2+l}\langle a\rangle^{0+},$$

the term $\langle a\rangle^{0+}$ can be discarded unless $\max(b_0, b_1, b_2) = 1$.

Lemma 4 For fixed $\beta \in (\frac{1}{2}, 1)$, we have

$$\int \frac{1}{\langle\lambda\rangle^{\beta}\sqrt{|\lambda-a|}}dx \lesssim \frac{1}{\langle a\rangle^{\beta-\frac{1}{2}}}.$$

3 A Priori Estimates

Our destination of this section is to establish a priori estimates of the system (1) which allows us to prove the local results. We start with the linear estimates on two equations separately. Then we prove the crucial bilinear estimates in detail.

3.1 Linear Estimates

The corresponding linear equations are posed first:

$$\begin{cases} iu_t + u_{xx} = 0, \ (x, t) \in \mathbb{R}^+ = (0, \infty), \\ n_t + \delta n = 0, \ (x, t) \in \mathbb{R}^+ = (0, \infty), \\ u(x, 0) = u_0(x), \ n(x, 0) = n_0(x), \\ u(0, t) = h(t), \ n(0, t) = f(t). \end{cases} \tag{9}$$

For this system, we have the following lemmas.

Lemma 5 For $s \geq 0$. For any $u_0 \in H^s(\mathbb{R})$, it holds that $\phi(t)W_{\mathbb{R}}u_0 \in C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R} \times \mathbb{R})$, and we have

$$\|\phi W_{\mathbb{R}} u_0\|_{L_x^\infty H_t^{\frac{2s+1}{4}}} \lesssim \|u_0\|_{H^s(\mathbb{R})}.$$

Proof Because the argument is well-known, we shall omit the details here. (see [16]) □

Lemma 6 For $s \in \mathbb{R}$. For any $n_0 \in H^s(\mathbb{R})$, it holds that $\phi(t)e^{-\delta t}n_0 \in C_x^0 H_t^s(\mathbb{R} \times \mathbb{R})$, and we have

$$\|\phi e^{-\delta t} n_0\|_{L_x^\infty H_t^s} \lesssim \|n_0\|_{H^s(\mathbb{R})}.$$

Proof It is not hard to achieve the lemma since the operator is continuous. □

Lemma 7 For $s \geq 0$. Suppose h with $\chi_{(0,\infty)}h \in H^{\frac{2s+1}{4}}(\mathbb{R})$, it holds that $W_0^t(0, h) \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$, and $\phi(t)W_0^t(0, h) \in C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R} \times \mathbb{R})$.

Proposition 7 For $b \leq \frac{1}{2}$ and $s \geq 0$. Suppose h with $\chi_{(0,\infty)}h \in H^{\frac{2s+1}{4}}(\mathbb{R})$, it holds that

$$\|\phi(t)W_0^t(0, h)\|_{X^{s,b}} \lesssim \|\chi_{(0,\infty)}h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}.$$

Proof It can be found in [15]. □

Lemma 8 For $s \in \mathbb{R}$. Suppose f with $\chi_{(0,\infty)}f \in H^s(\mathbb{R})$, it holds that $V_0^t(0, f) \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$, and $\phi(t)V_0^t(0, f) \in C_x^0 H_t^s(\mathbb{R} \times \mathbb{R})$.

Proposition 8 For $b \in \mathbb{R}$ and $s > -\frac{1}{2}$. Suppose f with $\chi_{(0,\infty)} f \in H^s(\mathbb{R})$, it holds that

$$\|\phi(t) V_0^t(0, f)\|_{Y^{s,b}} \lesssim \|\chi_{(0,\infty)} f\|_{H_t^s(\mathbb{R})}.$$

Proof The proof is not particularly difficult but will not be reproduced here. □

3.2 Nonlinear Estimates

Let us estimate the forced IBVP:

$$\begin{cases} iu_t + u_{xx} = F, (x,t) \in \mathbb{R}^+ = (0,\infty), \\ n_t + \delta n = G, (x,t) \in \mathbb{R}^+ = (0,\infty), \\ u(x,0) = n(x,0) = 0, \\ u(0,t) = n(0,t) = 0. \end{cases} \tag{10}$$

Proposition 9 For any $b < \frac{1}{2}$, it holds that

$$\begin{aligned} &\|\phi \int_0^t W_{\mathbb{R}}(t-\tau) F d\tau\|_{C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R}\times\mathbb{R})} \\ &\lesssim \begin{cases} \|F\|_{X^{s,-b}}, \; 0 \le s \le \dfrac{1}{2}, \\ \|F\|_{X^{s,-b}} + \|\displaystyle\int_{\mathbb{R}} \langle \lambda + \xi^2 \rangle^{\frac{2s-3}{4}} |\widehat{F}(\xi,\lambda)| d\xi\|_{L_\lambda^2}, \; \dfrac{1}{2} < s. \end{cases} \end{aligned}$$

Proof The proof can be found in [15]. □

For the second equation, we have a similar proposition:

Proposition 10 For any $b' < \frac{1}{2}$, it holds that

$$\begin{aligned} &\|\phi \int_0^t e^{-\delta(t-\tau)} G d\tau\|_{C_x^0 H_t^s(\mathbb{R}\times\mathbb{R})} \\ &\lesssim \begin{cases} \|G\|_{Y^{s,-b'}} + \|\langle\lambda\rangle^s \displaystyle\int_{\mathbb{R}} \frac{1}{\langle \lambda - \delta \rangle} |\widehat{G}(\xi,\lambda)| d\xi\|_{L_\lambda^2}, \; -\dfrac{1}{2} < s < \dfrac{1}{2}, \\ \|G\|_{Y^{s,-b'}} + \|\displaystyle\int_{\mathbb{R}} \langle \lambda - \delta \rangle^{s-1} |\widehat{G}(\xi,\lambda)| d\xi\|_{L_\lambda^2}, \; \dfrac{1}{2} < s. \end{cases} \end{aligned}$$

Proof Now that $Y^{s,b}$ norm is independent of space translation, it turns to estimate the bound above for $\phi D_0(\int_0^t e^{-\delta(t-\tau)} G d\tau)$. At $x = 0$, we get

$$D_0\Big(\int_0^t e^{-\delta(t-\tau)} G d\tau\Big) = \int_{\mathbb{R}} \int_0^t e^{-\delta(t-\tau)} G(\widehat{\xi}, \tau\Big) d\tau d\xi.$$

Using

$$G(\widehat{\xi,\ \tau})=\int_{\mathbb{R}}e^{i\tau\lambda}\widehat{G}(\xi,\ \lambda)d\lambda$$

and

$$\int_0^t e^{i\tau\lambda+\tau\delta}d\tau=\frac{e^{t(i\lambda+\delta)}-1}{i\lambda+\delta},$$

we obtain

$$\begin{aligned}D_0\bigg(\int_0^t e^{-\delta(t-\tau)}Gd\tau\bigg)&=\int_{\mathbb{R}^2}\frac{e^{it\lambda}-e^{-t\delta}}{i\lambda+\delta}\widehat{G}(\xi,\ \lambda)d\xi d\lambda\\&\sim\int_{\mathbb{R}^2}\frac{e^{it\lambda}-e^{-t\delta}}{\lambda-\delta}\widehat{G}(\xi,\ \lambda)d\xi d\lambda.\end{aligned}$$

Let us write

$$\phi(t)D_0\bigg(\int_0^t e^{-\delta(t-\tau)}Gd\tau\bigg)=S_1+S_2+S_3,$$

where

$$\begin{aligned}S_1&=\phi(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}-e^{-t\delta}}{\lambda-\delta}\psi(\lambda-\delta)\widehat{G}(\xi,\ \lambda)d\xi d\lambda,\\S_2&=\phi(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}}{\lambda-\delta}\psi^c(\lambda-\delta)\widehat{G}(\xi,\ \lambda)d\xi d\lambda,\\S_3&=\phi(t)\int_{\mathbb{R}^2}\frac{e^{-t\delta}}{\lambda-\delta}\psi^c(\lambda-\delta)\widehat{G}(\xi,\ \lambda)d\xi d\lambda\end{aligned}$$

and ψ is defined as a smooth cut-off function in $[-1,\ 1]$, and $\psi^c=1-\psi$. According to Taylor expansion, we obtain

$$\frac{e^{it\lambda}-e^{-t\delta}}{\lambda-\delta}=e^{it\lambda}\sum_{k=1}^{\infty}\frac{(t)^k}{k!}(\lambda-\delta)^{k-1}.$$

Using an inequality that $\|uv\|_{H^s} \leq \|u\|_{H^1}\|v\|_{H^s}$, we have

$$\begin{aligned}\|S_1\|_{H^s} &\lesssim \sum_{k=1}^{\infty} \frac{\|\phi(t)t^k\|_{H^1}}{k!} \|\int_{\mathbb{R}^2} e^{it\lambda}(\lambda-\delta)^{k-1}\psi(\lambda-\delta)\widehat{G}(\xi,\lambda)d\xi d\lambda\|_{H^s} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \|\langle\lambda\rangle^s \int_{\mathbb{R}} (\lambda-\delta)^{k-1}\psi(\lambda-\delta)\widehat{G}(\xi,\lambda)d\xi\|_{L^2_\lambda} \\ &\lesssim \|\langle\lambda\rangle^s \int_{\mathbb{R}} \frac{1}{\langle\lambda-\delta\rangle}|\widehat{G}(\xi,\lambda)|d\xi\|_{L^2_\lambda},\end{aligned}$$

where we use that $\|\phi t^k\|_{H^1} \leq k$ and $\sum_{k=1}^{\infty}\frac{1}{(k-1)!}$ is convergent. This completes the proof for $s \in \mathbb{R}$.

For $s > \frac{1}{2}$, by Cauchy–Schwartz inequality in ξ, it is easy to find that

$$\|S_1\|_{H^s} \lesssim \sup_{\lambda}(\langle\lambda\rangle^{2s} \int_{|\lambda-\delta|\leq 1} \langle\xi\rangle^{-2s}d\xi)^{\frac{1}{2}} \cdot \|G\|_{Y^{s,-b'}}.$$

We can bound it by $\|G\|_{Y^{s,-b'}}$ since $\int\langle\xi\rangle^{-2s}d\xi$ is convergent.

Next we prove the second term S_2,

$$\begin{aligned}\|S_2\|_{H^s(\mathbb{R})} &\lesssim \|\phi\|\|_{H^1}\|\langle\lambda\rangle^s \int_{\mathbb{R}} \frac{1}{\lambda-\delta}\psi^c(\lambda-\delta)\widehat{G}(\xi,\lambda)d\xi\|_{L^2_\lambda} \\ &\lesssim \|\langle\lambda\rangle^s \int_{\mathbb{R}} \frac{1}{\langle\lambda-\delta\rangle}|\widehat{G}(\xi,\lambda)|d\xi\|_{L^2_\lambda}.\end{aligned}$$

For $-\frac{1}{2} < s < \frac{1}{2}$, the proof is completed.

For $s > \frac{1}{2}$, we see that

$$\|S_2\|_{H^s(\mathbb{R})} \lesssim \|\int_{\mathbb{R}} \langle\lambda-\delta\rangle^{s-1}|\widehat{G}(\xi,\lambda)|d\xi\|_{L^2_\lambda} + \|\int_{\mathbb{R}} \frac{\delta^s}{\langle\lambda-\delta\rangle}|\widehat{G}(\xi,\lambda)|d\xi\|_{L^2_\lambda}$$

where we used the inequality $\langle\lambda\rangle \lesssim \langle\lambda-\delta\rangle + \delta$. Applying Cauchy–Schwarz inquality, the second summand is bounded as follows

$$\begin{aligned}&\left(\int_{\mathbb{R}} \langle\delta\rangle^{2s}\left(\int \frac{1}{\langle\lambda-\delta\rangle^{2-2b'}\langle\xi\rangle^{2s}}d\xi\right)\left(\int \frac{\langle\xi\rangle^{2s}}{\langle\lambda-\delta\rangle^{2b'}}|\widehat{G}(\xi,\lambda)|^2 d\xi\right)d\lambda\right)^{\frac{1}{2}} \\ &\lesssim \sup_{\lambda}\left(\int \frac{\langle\delta\rangle^{2s}}{\langle\lambda-\delta\rangle^{2-2b'}\langle\xi\rangle^{2s}}d\xi\right)^{\frac{1}{2}} \cdot \|G\|_{Y^{s,-b'}} \\ &\lesssim \|G\|_{Y^{s,-b'}}.\end{aligned}$$

To compute $\|S_3\|_{H^s(\mathbb{R})}$, it can be completed by the method analogous to that used above. □

3.3 Bilinear Estimate

In this section, our goal is to give bilinear estimates which is of great use to obtain local well-posedness. Due to the presence of the boundary conditions, we shall deal with more bilinear estimates than the Cauchy problem of LS type equations.

Proposition 11 For any admissible s_0, s_1 and for any $a_0 < \min(\frac{1}{2}, s_1 + \frac{1}{2}, s_1 - s_0 + 1)$, for $\frac{1}{2} - \epsilon < b, b_1 < \frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\|nu\|_{X^{s_0+a_0,-b}} \lesssim \|n\|_{Y^{s_1,b_1}} \|u\|_{X^{s_0,b}}.$$

Proof According to the relation of the Fourier transform and its convolution, it is obvious to get

$$\widehat{nu}(\xi, \tau) = \int \widehat{n}(\xi_1, \tau_1)\widehat{u}(\xi - \xi_1, \tau - \tau_1)d\xi_1 d\tau_1,$$

Therefore,

$$\|nu\|^2_{X^{s_0+a_0,-b}} = \|\int \frac{\langle\xi\rangle^{s_0+a_0}\widehat{n}(\xi_1, \tau_1)\widehat{u}(\xi - \xi_1, \tau - \tau_1)}{\langle\tau + \xi^2\rangle^b} d\xi_1 d\tau_1\|^2_{L^2_\xi L^2_\tau}.$$

We define

$$\begin{aligned} f(\xi, \tau) &= |\widehat{n}(\xi, \tau)| \langle\xi\rangle^{s_1} \langle\tau - \delta\rangle^{b_1}, \\ g(\xi, \tau) &= |\widehat{u}(\xi, \tau)| \langle\xi\rangle^{s_0} \langle\tau + \xi^2\rangle^{b}, \end{aligned}$$

and

$$M(\xi, \xi_1, \tau, \tau_1) = \frac{\langle\xi\rangle^{s_0+a_0} \langle\xi_1\rangle^{-s_1} \langle\xi - \xi_1\rangle^{-s_0}}{\langle\tau + \xi^2\rangle^b \langle\tau_1 - \delta\rangle^{b_1} \langle\tau - \tau_1 + (\xi - \xi_1)^2\rangle^b}.$$

Then it turns to prove that

$$\begin{aligned} &\|\int M(\xi, \xi_1, \tau, \tau_1) f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1\|^2_{L^2_\xi L^2_\tau} \\ &\lesssim \|f\|^2_{L^2} \|g\|^2_{L^2} \\ &= \|n\|^2_{Y^{s_1,b_1}} \|u\|^2_{X^{s_0,b}}. \end{aligned}$$

Applying Cauchy–Schwarz in the $\xi_1\tau_1$ integral and Hölder's inequality, the norm above can be estimated by

$$\begin{aligned}
&\|\Big(\int M^2 d\xi_1 d\tau_1\Big)^{1/2}\Big(\int f^2(\xi_1,\tau_1)g^2(\xi-\xi_1,\tau-\tau_1)d\xi_1 d\tau_1\Big)^{1/2}\|^2_{L^2_\xi L^2_\tau}\\
&\le \sup_{\xi,\tau}\Big(\int M^2 d\xi_1 d\tau_1\Big)\|\int f^2(\xi_1,\tau_1)g^2(\xi-\xi_1,\tau-\tau_1)\|_{L^1_\xi L^1_\tau}\\
&= \sup_{\xi,\tau}\Big(\int M^2 d\xi_1 d\tau_1\Big)\|f\|^2_{L^2}\|g\|^2_{L^2}.
\end{aligned}$$

Thus it turns to prove the proposition in the case that the supremum above is finite. Applying the Lemma 2.5, it yields the bound

$$\sup_{\xi,\tau}\int \frac{\langle\xi\rangle^{2s_0+2a_0}\langle\xi_1\rangle^{-2s_1}\langle\xi-\xi_1\rangle^{-2s_0}}{\langle\tau+\xi^2\rangle^{2b}\langle\tau-\delta+(\xi-\xi_1)^2\rangle^{2b}}d\xi_1,$$

With the help of the inequality $\langle a-b\rangle\lesssim\langle\tau-a\rangle\langle\tau-b\rangle$, the supremum is reduced to

$$\begin{aligned}
&\sup_{\xi}\int \frac{\langle\xi\rangle^{2s_0+2a_0}\langle\xi_1\rangle^{-2s_1}\langle\xi-\xi_1\rangle^{-2s_0}}{\langle\xi^2+\delta-(\xi-\xi_1)^2\rangle^{1-}}d\xi_1\\
&\quad=\sup_{\xi}\int \frac{\langle\xi\rangle^{2s_0+2a_0}\langle\xi_1\rangle^{-2s_1}\langle\xi-\xi_1\rangle^{-2s_0}}{\langle\delta-\xi_1^2+2\xi\xi_1\rangle^{1-}}d\xi_1.
\end{aligned}$$

Observe that when $|\xi|\lesssim 1$, it is not hard to obtain the bound

$$\int \frac{\langle\xi_1\rangle^{-2s_1-2s_0}}{\langle\delta-\xi_1^2\rangle^{1-}}d\xi_1\lesssim 1$$

provided $s_1+s_0>-\frac{1}{2}$.

When $|\xi|\gg 1$, we take the following cases into consideration: $i)$ $|\xi_1|<1$, $ii)$ $|\xi_1-2\xi|<1$ and $iii)$ $|\xi_1|>1$ and $|\xi_1-2\xi|>1$.

In the first case, the transformation $\eta=\xi_1^2-2\xi\xi_1$, $d\eta=2(\xi_1-\xi)d\xi_1$ contributes to the following:

$$\sup_{\xi}\int_{|\eta|\lesssim|\xi|}\frac{|\xi|^{2a_0-1}}{\langle\delta-\eta\rangle^{1-}}d\eta\lesssim 1,$$

provided that $a_0<\frac{1}{2}$.

In the second case, after the same variables substitution, the integral is estimated by

$$\sup_{\xi}\int_{|\eta|\lesssim|\xi|}\frac{|\xi|^{2a_0-2s_1-1}}{\langle\delta-\eta\rangle^{1-}}d\eta\lesssim 1.$$

provided that $a_0 < s_1 + \frac{1}{2}$.

In the third case, which implies that $\langle \delta - \xi_1^2 + 2\xi\xi_1 \rangle \sim \langle \xi_1^2 - 2\xi\xi_1 \rangle$, it is not difficult to estimate the integral (after the variable substitution $\xi_1 \to \xi_1 + \xi$)

$$\sup_{\xi} \int \frac{\langle \xi \rangle^{2s_0+2a_0}}{\langle \xi_1 - \xi \rangle^{1-} \langle \xi_1 \rangle^{2s_0} \langle \xi_1 + \xi \rangle^{2s_1+1-}} d\xi_1.$$

Combining Lemma 2.5, we bound this by

$$\begin{aligned} &\langle \xi \rangle^{2s_0+2a_0} \langle \xi_1 \rangle^{-2s_1-2s_0-2+\max(1,2s_1+1,2s_0)+} \\ &= \langle \xi \rangle^{2a_0-2s_1-2+\max(1,2s_1+1,2s_0)+}, \end{aligned}$$

which is bounded for

$$a_0 < s_1 + 1 - \frac{1}{2}\max(1, 2s_1 + 1, 2s_0) = \min(s_1 + \frac{1}{2}, s_1 - s_0 + 1).$$

□

Proposition 12 For any admissible s_0, s_1 and for any $\frac{1}{2} - s_0 < a_0 < \min(\frac{1}{2}, s_1 + \frac{1}{2}, s_1 - s_0 + 1)$, for $\frac{1}{2} - \epsilon < b, b_1 < \frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\| \int_{\mathbb{R}} \langle \lambda + \xi^2 \rangle^{\frac{2(s_0+a_0)-3}{4}} |\widehat{nu}(\xi, \lambda)| d\xi \|_{L^2_\lambda} \lesssim \|n\|_{Y^{s_1,b_1}} \|u\|_{X^{s_0,b}}.$$

Proof This proposition can be shown in a similar way as before. First, we have

$$\begin{aligned} &\| \int_{\mathbb{R}^3} \langle \lambda + \xi^2 \rangle^{\frac{2(s_0+a_0)-3}{4}} |\widehat{n}(\xi_1, \lambda_1)| |\widehat{u}(\xi - \xi_1, \lambda - \lambda_1)| d\xi_1 d\lambda_1 d\xi \|_{L^2_\lambda} \\ &= \| \int_{\mathbb{R}^3} \frac{\langle \lambda + \xi^2 \rangle^{\frac{2(s_0+a_0)-3}{4}} f(\xi_1, \lambda_1) g(\xi - \xi_1, \lambda - \lambda_1)}{\langle \xi_1 \rangle^{s_1} \langle \xi - \xi_1 \rangle^{s_0} \langle \lambda_1 - \delta \rangle^{b_1} \langle \lambda - \lambda_1 + (\xi - \xi_1)^2 \rangle^{b}} d\xi_1 d\lambda_1 d\xi \|_{L^2_\lambda}. \end{aligned}$$

Using Cauchy–Schwarz inequality in ξ_1, λ_1, ξ variables, it turns to verify that

$$\sup_{\lambda} \int_{\mathbb{R}^3} \frac{\langle \lambda + \xi^2 \rangle^{s_0+a_0-\frac{3}{2}}}{\langle \xi_1 \rangle^{2s_1} \langle \xi - \xi_1 \rangle^{2s_0} \langle \lambda_1 - \delta \rangle^{2b_1} \langle \lambda - \lambda_1 + (\xi - \xi_1)^2 \rangle^{2b}} d\xi_1 d\lambda_1 d\xi < \infty.$$

Using Lemma 2.5 in the λ_1 integral, we obtain the bound

$$\sup_{\lambda} \int_{\mathbb{R}^2} \frac{\langle \lambda + \xi^2 \rangle^{s_0+a_0-\frac{3}{2}}}{\langle \xi_1 \rangle^{2s_1} \langle \xi - \xi_1 \rangle^{2s_0} \langle \lambda + (\xi - \xi_1)^2 - \delta \rangle^{1-}} d\xi_1 d\xi. \tag{11}$$

First we consider the case $\frac{3}{2} \le s_0 + a_0$.

Using $\langle \lambda + \xi^2 \rangle \lesssim \langle \lambda + (\xi - \xi_1)^2 - \delta \rangle \langle \xi_1 \rangle \langle \xi_1 - 2\xi \rangle$, we bound the integral above by

$$\int_{\mathbb{R}^2} \frac{\langle \xi_1 - 2\xi \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi_1 \rangle^{2s_1 + \frac{3}{2} - s_0 - a_0} \langle \xi - \xi_1 \rangle^{2s_0}} d\xi_1 d\xi.$$

For $|\xi| \leq |\xi_1|$, we bound this by

$$\int_{\mathbb{R}^2} \frac{1}{\langle \xi_1 \rangle^{2s_1 + 3 - 2s_0 - 2a_0} \langle \xi - \xi_1 \rangle^{2s_0}} d\xi_1 d\xi \lesssim 1$$

since $2s_0 > 1$ and $2s_1 + 3 - 2s_0 - 2a_0 > 1$.

For $|\xi| \gg |\xi_1|$, we bound it by

$$\int \frac{1}{\langle \xi_1 \rangle^{2s_1 + \frac{3}{2} - s_0 - a_0} \langle \xi \rangle^{s_0 - a_0 + \frac{3}{2}}} d\xi d\xi_1 \lesssim \int \frac{1}{\langle \xi_1 \rangle^{2s_1 + 3 - 2a_0}} d\xi_1 \lesssim 1$$

since $a_0 < s_1 + \frac{1}{2}$.

Now we consider the case $\frac{1}{2} < s_0 + a_0 < \frac{3}{2}$.

Applying the inequality $\langle a - b \rangle \lesssim \langle \tau - a \rangle \langle \tau - b \rangle$, we bound (11) by

$$\int_{\mathbb{R}^2} \frac{1}{\langle \xi_1 \rangle^{2s_1} \langle \xi - \xi_1 \rangle^{2s_0} \langle \xi_1^2 - 2\xi_1 \xi - \delta \rangle^{\frac{3}{2} - s_0 - a_0}} d\xi_1 d\xi.$$

For $|\xi_1| \lesssim 1$, the integral is estimated after the variables substitution $\eta = \xi_1^2 - 2\xi\xi_1$, $d\eta = 2(\xi_1 - \xi) d\xi_1$:

$$\int_{\mathbb{R}^2} \frac{1}{\langle \xi \rangle^{2s_0 + 1} \langle \eta - \delta \rangle^{\frac{3}{2} - s_0 - a_0 + \frac{1}{2}}} d\eta d\xi \lesssim 1$$

since $s_0 > 0$ and $s_0 + a_0 < 1$. For $|\xi - 2\xi_1| \lesssim 1$, the integral is estimated after the change of variable as before $\eta = \xi_1^2 - 2\xi\xi_1$, $d\eta = 2(\xi_1 - \xi) d\xi_1$:

$$\int_{\mathbb{R}^2} \frac{1}{\langle \xi \rangle^{2s_1 + 2s_0 + 1} \langle \eta - \delta \rangle^{\frac{3}{2} - s_0 - a_0 + \frac{1}{2}}} d\eta d\xi \lesssim 1$$

since $s_1 + s_0 > 0$ and $s_0 + a_0 < 1$. For $|\xi_1| \gtrsim 1$ and $|\xi - 2\xi_1| \gtrsim 1$, which implies that $\langle \delta - \xi_1^2 + 2\xi\xi_1 \rangle \sim \langle \xi_1^2 - 2\xi\xi_1 \rangle$, we bound the integral by

$$\int_{\mathbb{R}^2} \frac{1}{\langle \xi \rangle^{2s_1 + \frac{3}{2} - s_0 - a_0} \langle \xi - \xi_1 \rangle^{2s_0} \langle \xi_1 - 2\xi \rangle^{\frac{3}{2} - s_0 - a_0}} d\xi_1 d\xi.$$

Using Lemma 2.6 in the ξ_1 integral, it is not difficult to get the bound

$$\int_{\mathbb{R}} \frac{1}{\langle\xi\rangle^{2s_1+3-2a_0-\max(1,2s_1+\frac{3}{2}-s_0-a_0,2s_0\frac{3}{2}-s_0-a_0)+}}d\xi.$$

Under the assumption of the proposition, the integral is finite. □

Proposition 13 For any admissible s_0, s_1 and for any $a_0 < \min(2s_0 + \frac{1}{2}, 1)$, for $\frac{1}{2} - \epsilon < b, b_1 < \frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\||u|^2u\|_{X^{s_0+a_0,-b}} \lesssim \|u\|^3_{X^{s_0,b}}.$$

Proof The proof can be found in [16], so we omit it here. □

Proposition 14 For any admissible s_0, s_1 and for any $\frac{1}{2} - s_0 < a_0 < min(1, s_0 + \frac{1}{2}, 4s_0)$, for $\frac{1}{2} - \epsilon < b, b_1 < \frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\|\int_{\mathbb{R}} \langle\lambda+\xi^2\rangle^{\frac{2(s_0+a_0)-3}{4}}|\widehat{|u|^2u}(\xi,\lambda)|d\xi\|_{L^2_\lambda} \lesssim \|u\|^3_{X^{s_0,b}}.$$

Proof Similarly to Proposition 3.9. It is straightforward to show that

$$\begin{aligned}
&\|\int_{\mathbb{R}} \langle\lambda+\xi^2\rangle^{\frac{2(s_0+a_0)-3}{4}}|\widehat{|u|^2u}(\xi,\lambda)|d\xi\|_{L^2_\lambda}\\
&= \|\int_{\mathbb{R}^5} \langle\lambda+\xi^2\rangle^{\frac{2(s_0+a_0)-3}{4}}\widehat{u}(\xi_1,\lambda_1)\overline{\widehat{u}(\xi_2,\lambda_2)}\\
&\quad\cdot \widehat{u}(\xi-\xi_1-\xi_2,\lambda-\lambda_1-\lambda_2)d\xi_1 d\xi_2 d\lambda_1 d\lambda_2 d\xi\|_{L^2_\lambda}\\
&= \|\int_{\mathbb{R}^5} \frac{\langle\lambda+\xi^2\rangle^{\frac{2(s_0+a_0)-3}{4}} f(\xi_1,\lambda_1)\overline{f}(\xi_2,\lambda_2)f(\xi-\xi_1-\xi_2,\lambda-\lambda_1-\lambda_2)}{\langle\xi_1\rangle^{s_0}\langle\xi_2\rangle^{s_0}\langle\xi-\xi_1-\xi_2\rangle^{s_0}\langle\lambda_1+\xi_1^2\rangle^b\langle\lambda_2-\xi_2^2\rangle^b}\\
&\quad\cdot \frac{1}{\langle\lambda-\lambda_1-\lambda_2+(\xi-\xi_1-\xi_2)^2\rangle^b}d\xi_1 d\xi_2 d\lambda_1 d\lambda_2 d\xi\|_{L^2_\lambda}.
\end{aligned}$$

With aid of Cauchy–Schwarz inequality in the ξ_1, ξ_2, λ_1, λ_2, ξ variables (after the change of variable $\xi_2 \to -\xi_2$, $\lambda_2 \to -\lambda_2$), it turns to prove that

$$\begin{aligned}
&\sup_\lambda \int_{\mathbb{R}^5} \frac{\langle\lambda+\xi^2\rangle^{s_0+a_0-\frac{3}{2}}}{\langle\xi_1\rangle^{2s_0}\langle\xi_2\rangle^{2s_0}\langle\xi-\xi_1+\xi_2\rangle^{2s_0}\langle\lambda_1+\xi_1^2\rangle^{2b}\langle\lambda_2+\xi_2^2\rangle^{2b}}\\
&\quad\cdot \frac{1}{\langle\lambda-\lambda_1+\lambda_2-(\xi-\xi_1+\xi_2)^2\rangle^{2b}}d\xi_1 d\xi_2 d\lambda_1 d\lambda_2 d\xi < \infty.
\end{aligned}$$

Using Lemma 2.5 in the λ_1 and λ_2 integral, the supremum is bounded by

$$\sup_{\lambda} \int_{\mathbb{R}^3} \frac{\langle \lambda + \xi^2 \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi_1 \rangle^{2s_0} \langle \xi_2 \rangle^{2s_0} \langle \xi - \xi_1 + \xi_2 \rangle^{2s_0} \langle \lambda + \xi_1^2 - \xi_2^2 + (\xi - \xi_1 + \xi_2)^2 \rangle^{1-}} d\xi_1 d\xi_2 d\xi. \tag{12}$$

First we consider the case $\frac{3}{2} \leq s_0 + a_0$.

Note that $\langle \lambda + \xi^2 \rangle \lesssim \langle \lambda + \xi_1^2 - \xi_2^2 + (\xi - \xi_1 + \xi_2)^2 \rangle \langle \xi - \xi_1 \rangle \langle \xi_1 - \xi_2 \rangle$, we bound (12) by

$$\int_{\mathbb{R}^3} \frac{\langle \xi_1 - \xi_2 \rangle^{s_0 + a_0 - \frac{3}{2}} \langle \xi_1 - \xi \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi_1 \rangle^{2s_0} \langle \xi_2 \rangle^{2s_0} \langle \xi - \xi_1 + \xi_2 \rangle^{2s_0}} d\xi_1 d\xi_2 d\xi.$$

After the variables substitution $\xi_2 \to \xi_1 + \xi_2$, $\xi_1 \to \xi + \xi_1$, we bound it by

$$\int_{\mathbb{R}^3} \frac{\langle \xi_1 \rangle^{s_0 + a_0 - \frac{3}{2}} \langle \xi_2 \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi + \xi_1 \rangle^{2s_0} \langle \xi + \xi_2 \rangle^{2s_0} \langle \xi + \xi_1 + \xi_2 \rangle^{2s_0}} d\xi_1 d\xi_2 d\xi.$$

Considering the subcases $|\xi + \xi_1 + \xi_2| \gtrsim |\xi|$ and $|\xi + \xi_1| \gtrsim |\xi|$, we can bound it by

$$\int \langle \xi \rangle^{-2s_0} \left(\int \frac{\langle \xi_1 \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi + \xi_1 \rangle^{2s_0}} d\xi_1 \right)^2 d\xi + \int \frac{\langle \xi_1 \rangle^{s_0 + a_0 - \frac{3}{2}} \langle \xi_2 \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi \rangle^{2s_0} \langle \xi + \xi_1 + \xi_2 \rangle^{2s_0} \langle \xi + \xi_2 \rangle^{2s_0}} d\xi_1 d\xi_2 d\xi$$
$$=: S_1 + S_2.$$

Notice that $\langle \xi + \xi_1 \rangle \langle \xi \rangle \gtrsim \langle \xi_1 \rangle$, we have

$$\begin{aligned} S_1 &\lesssim \int \langle \xi \rangle^{-2s_0} \left(\int \frac{\langle \xi \rangle^{s_0 + a_0 - \frac{3}{2}}}{\langle \xi + \xi_1 \rangle^{s_0 - a_0 + \frac{3}{2}}} d\xi_1 \right)^2 d\xi \\ &= \int \langle \xi \rangle^{2a_0 - 3} \left(\int \frac{1}{\langle \xi + \xi_1 \rangle^{s_0 - a_0 + \frac{3}{2}}} d\xi_1 \right)^2 d\xi \\ &\lesssim 1 \end{aligned}$$

provided $3 - 2a_0 > 1$ and $s_0 - a_0 + \frac{3}{2} > 1$. Using Lemma 2.5 and $\langle \xi + \xi_2 \rangle \langle \xi + \xi_1 + \xi_2 \rangle \gtrsim \langle \xi_1 \rangle$ and $\langle \xi + \xi_2 \rangle \langle \xi \rangle \gtrsim \langle \xi_2 \rangle$, we get

$$\begin{aligned} S_2 &\lesssim \int \frac{1}{\langle \xi + \xi_2 \rangle^{3 - 2a_0} \langle \xi + \xi_1 + \xi_2 \rangle^{s_0 - a_0 + \frac{3}{2}} \langle \xi \rangle^{s_0 - a_0 + \frac{3}{2}}} d\xi_1 d\xi_2 d\xi \\ &\lesssim \int \frac{1}{\langle \xi_1 \rangle^{3 - 2a_0} \langle \xi \rangle^{s_0 - a_0 + \frac{3}{2}}} d\xi_1 d\xi \\ &\lesssim 1 \end{aligned}$$

provided $3-2a_0>1$ and $s_0-a_0+\frac{3}{2}>1$.

Now we consider the case $\frac{1}{2}<s_0+a_0<\frac{3}{2}$. For $s_0\geq\frac{1}{2}$, applying the relation $\langle a-b\rangle\lesssim\langle\tau-a\rangle\langle\tau-b\rangle$, we bound (12) by

$$\begin{aligned}
&\int_{\mathbb{R}^3}\frac{1}{\langle\xi_1\rangle^{2s_0}\langle\xi_2\rangle^{2s_0}\langle\xi-\xi_1+\xi_2\rangle^{2s_0}\langle\xi_1-\xi\rangle^{\frac{3}{2}-s_0-a_0}\langle\xi_1-\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1d\xi_2d\xi\\
&\lesssim\int_{\mathbb{R}^2}\frac{1}{\langle\xi_1\rangle^{2s_0}\langle\xi_2\rangle^{s_0-a_0+\frac{3}{2}}\langle\xi_1-\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1d\xi_2\\
&\lesssim\int_{\mathbb{R}}\frac{1}{\langle\xi_1\rangle^{2s_0}\langle\xi_1\rangle^{-s_0-a_0+\frac{3}{2}}}d\xi_1\\
&\lesssim 1
\end{aligned}$$

provided $s_0-a_0+\frac{3}{2}>1$.

For $s>\frac{1}{2}$, we take the following cases into consideration: $i)$ $|\xi_1-\xi|<1$ or $|\xi_1-\xi_2|<1$, $ii)$ $|\xi_1-\xi|\geq 1$ and $|\xi_1-\xi_2|\geq 1$.

In the former case we find that $\langle\xi_1\rangle\langle\xi-\xi_1+\xi_2\rangle\sim\langle\xi_2\rangle\langle\xi\rangle$, which yields the bound

$$\int\frac{\langle\xi\rangle^{-2s_0}\langle\xi_2\rangle^{-4s_0}}{\langle\xi_1-\xi\rangle^{\frac{3}{2}-s_0-a_0}\langle\xi_1-\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1d\xi_2d\xi.$$

Subsitituing $\eta=(\xi_1-\xi)(\xi_1-\xi_2)$ in the ξ_1 integral, we obtain

$$\begin{aligned}
&\int\frac{\langle\xi\rangle^{-2s_0}\langle\xi_2\rangle^{-4s_0}}{\langle\eta\rangle^{\frac{3}{2}-s_0-a_0}\sqrt{|4\eta+(\xi-\xi_2)^2|}}d\eta d\xi_2d\xi\\
&\lesssim\int\frac{\langle\xi\rangle^{-2s_0}\langle\xi_2\rangle^{-4s_0}}{\langle\xi-\xi_2\rangle^{2-2s_0-a_0}}d\xi_2d\xi\\
&\lesssim\begin{cases}\int\frac{\langle\xi\rangle^{-2s_0}}{\langle\xi\rangle^{2-2s_0-a_0}}d\xi,\ s_0>\frac{1}{4},\\[2mm] \int\frac{\langle\xi\rangle^{-2s_0}}{\langle\xi\rangle^{2-2s_0-a_0+4s_0-1}}d\xi,\ s_0\leq\frac{1}{4},\end{cases}
\end{aligned}$$

provided $a_0<\min(1,4s_0)$. In the latter case, after the variables substitution $\xi_2\to\xi_1+\xi_2$, $\xi_1\to\xi+\xi_1$, the integral is bounded by

$$\begin{aligned}
&\int_{\mathbb{R}^3}\frac{1}{\langle\xi_1\rangle^{2s_0}\langle\xi_2\rangle^{2s_0}\langle\xi-\xi_1+\xi_2\rangle^{2s_0}\langle\xi_1-\xi\rangle^{\frac{3}{2}-s_0-a_0}\langle\xi_1-\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1d\xi_2d\xi\\
&=\int_{\mathbb{R}^3}\frac{\langle\xi+\xi_1\rangle^{-2s_0}\langle\xi+\xi_2\rangle^{-2s_0}\langle\xi+\xi_1+\xi_2\rangle^{-2s_0}}{\langle\xi_1\rangle^{\frac{3}{2}-s_0-a_0}\langle\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1d\xi_2d\xi.
\end{aligned}$$

By symmetry, we obtain two inequalities: $|\xi+\xi_1+\xi_2|\gtrsim|\xi|$ and $|\xi+\xi_1|\gtrsim|\xi|$, leading to the bound

$$\begin{aligned}
&\int\langle\xi\rangle^{-2s_0}\left(\int\frac{\langle\xi+\xi_1\rangle^{-2s_0}}{\langle\xi_1\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1\right)^2d\xi\\
&\quad+\int\frac{\langle\xi\rangle^{-2s_0}\langle\xi+\xi_2\rangle^{-2s_0}\langle\xi+\xi_1+\xi_2\rangle^{-2s_0}}{\langle\xi_1\rangle^{\frac{3}{2}-s_0-a_0}\langle\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_1d\xi_2d\xi\\
&\lesssim\int\langle\xi\rangle^{-2s_0}(\langle\xi\rangle^{s_0+a_0-\frac{3}{2}})^2d\xi+\int\frac{\langle\xi\rangle^{-2s_0}\langle\xi+\xi_2\rangle^{-2s_0}}{\langle\xi+\xi_2\rangle^{\frac{3}{2}-s_0-a_0}\langle\xi_2\rangle^{\frac{3}{2}-s_0-a_0}}d\xi_2d\xi\\
&\lesssim\int\langle\xi\rangle^{-2s_0}(\langle\xi\rangle^{s_0+a_0-\frac{3}{2}})^2d\xi+\int\langle\xi\rangle^{-2s_0}\langle\xi\rangle^{s_0+a_0-\frac{3}{2}}d\xi\\
&\lesssim 1
\end{aligned}$$

provided $a_0<\min(1,s_0+\frac{1}{2})$. □

Proposition 15 For any admissible s_0, s_1 and for any $a_1<\min(s_0-s_1,2s_0-s_1-\frac{1}{2})$, for $\frac{1}{2}-\epsilon<b,b_1<\frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\|\partial_x|u|^2\|_{Y^{s_1+a_1,-b_1}}\lesssim\|u\|^2_{X^{s_0,b}}.$$

Proof Similarly to Proposition 3.9. Notice that

$$\widehat{\partial_x(|u|^2)}(\xi,\tau)=c\xi(\widehat{u}*\widehat{\overline{u}})(\xi,\tau),$$

Therefore,

$$\|\partial_x(|u|^2)\|^2_{Y^{s_1+a_1,-b_1'}}=\|c\int\frac{\langle\xi\rangle^{s_1+a_1}\widehat{u}(\xi_1,\tau_1)\widehat{\overline{u}}(\xi-\xi_1,\tau-\tau_1)|\xi|}{\langle\tau-\delta\rangle^{b_1}}d\xi_1d\tau_1\|^2_{L^2_\xi L^2_\tau}.$$

Let

$$f(\xi,\tau)=|\widehat{u}(\xi,\tau)|\langle\xi\rangle^{s_0}\langle\tau+\xi^2\rangle^b$$

and

$$M(\xi,\xi_1,\tau,\tau_1)=\frac{\langle\xi\rangle^{s_1+a_1}|\xi|\langle\xi\rangle^{-s_0}\langle\xi-\xi_1\rangle^{-s_0}}{\langle\tau-\delta\rangle^{b_1}\langle\tau_1+\xi_1^2\rangle^b\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^b}.$$

Then we estimate the supremum by

$$\sup_{\xi}\int \frac{\langle\xi\rangle^{2s_1+2a_1}|\xi|^2\langle\xi_1\rangle^{-2s_0}\langle\xi-\xi_1\rangle^{-2s_0}}{\langle\xi^2-2\xi\xi_1-\delta\rangle^{2b+2b_1-1}}d\xi_1$$
$$\lesssim \sup_{\xi}\int \frac{\langle\xi\rangle^{2s_1+2a_1}|\xi|^2\langle\xi_1\rangle^{-2s_0}\langle\xi-\xi_1\rangle^{-2s_0}}{\langle\xi^2-2\xi\xi_1-\delta\rangle^{1-}}d\xi_1.$$

When $|\xi|\lesssim 1$, we bound the supermum by

$$\sup_{|\xi|\leq 1}\int \frac{|\xi|^2}{\langle\xi\xi_1\rangle^{1-}\langle\xi_1\rangle^{4s_0}}d\xi_1 \lesssim 1.$$

When $|\xi|\gg 1$, we estimate the supremum by

$$\sup_{\xi}\int \frac{\langle\xi\rangle^{2s_1+2a_1}|\xi|^2\langle\xi_1\rangle^{-2s_0}\langle\xi-\xi_1\rangle^{-2s_0}}{\langle\xi^2-2\xi\xi_1\rangle^{1-}}d\xi_1.$$

Then we consider the following two cases: when $|\xi-2\xi_1|<1$, we bound the integral by

$$\int_{|\xi-2\xi_1|<1}\frac{\langle\xi\rangle^{2s_1+2a_1+2-4s_0}}{\langle\xi^2-2\xi\xi_1\rangle^{1-}}d\xi_1=\int_{|\rho|<1}\frac{\langle\xi\rangle^{2s_1+2a_1+2-4s_0}}{\langle\xi\rho\rangle^{1-}}d\rho\lesssim\langle\xi\rangle^{2s_1+2a_1+1-4s_0},$$

which is bounded by $a_1<2s_0-s_1-\frac{1}{2}$.

when $|\xi-2\xi_1|>1$, we bound the integral by

$$\int \frac{\langle\xi\rangle^{2s_1+2a_1+1+}}{\langle\xi_1\rangle^{2s_0}\langle\xi-\xi_1\rangle^{2s_0}\langle\xi-2\xi_1\rangle^{1-}}d\xi_1\lesssim\langle\xi\rangle^{2s_1+2a_1-4s_0+\max(2s_0,1)+},$$

this holds since $a_1<\min(s_0-s_1,2s_0-s_1-\frac{1}{2})$. □

Proposition 16 For any admissible s_0, s_1 and for any $\frac{1}{2}-s_1<a_1<1-s_1$, for $\frac{1}{2}-\epsilon<b$, $b_1<\frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\|\int_{\mathbb{R}}\langle\lambda-\delta\rangle^{s_1+a_1-1}|\widehat{\partial_x|u|^2}|d\xi\|_{L^2_\lambda}\lesssim\|u\|^2_{X^{s_0,b}}.$$

Proof Similarly to Proposition 3.9, we continue to show that

$$\sup_{\lambda}\int_{\mathbb{R}^3}\frac{\langle\lambda-\delta\rangle^{2(s_1+a_1-1)}|\xi|^2}{\langle\xi_1\rangle^{2s_0}\langle\xi-\xi_1\rangle^{2s_0}\langle\lambda_1-\xi_1^2\rangle^{2b}\langle\lambda-\lambda_1+(\xi-\xi_1)^2\rangle^{2b}}d\xi_1 d\lambda_1 d\xi<\infty.$$

With the help of Lemma 2.5 in the λ_1 integral and the variable substitution $\eta = \xi(\xi - 2\xi_1)$ in the integral, the supremun is bounded as

$$\sup_\lambda \int_{\mathbb{R}^2} \frac{\langle \lambda - \delta \rangle^{2(s_1+a_1-1)} |\xi|}{\langle \xi - \frac{\eta}{\xi} \rangle^{2s_0} \langle \xi + \frac{\eta}{\xi} \rangle^{2s_0} \langle \lambda + \eta \rangle^{1-}} d\eta d\xi.$$

Without loss of generality $|\xi - \frac{\eta}{\xi}| \gtrsim |\xi|$, we get

$$\sup_\lambda \int_{\mathbb{R}^2} \frac{\langle \lambda - \delta \rangle^{2(s_1+a_1-1)} |\xi|}{\langle \xi \rangle^{2s_0} \langle \xi + \frac{\eta}{\xi} \rangle^{2s_0} \langle \lambda + \eta \rangle^{1-}} d\eta d\xi.$$

Using $s_0 > \frac{1}{2}$, then integrating in η, we have the bound in a similar way as before,

$$\begin{aligned} &\sup_\lambda \int_{\mathbb{R}^2} \frac{\langle \lambda - \delta \rangle^{2(s_1+a_1-1)} |\xi|^2}{\langle \xi \rangle^{2s_0} \langle \xi^2 + \eta \rangle^{2s_0} \langle \lambda + \eta \rangle^{1-}} d\eta d\xi \\ &\lesssim \sup_\lambda \int_{\mathbb{R}} \frac{\langle \lambda - \delta \rangle^{2(s_1+a_1-1)} |\xi|^2 \langle \xi \rangle^{-2s_0}}{\langle \lambda - \xi^2 \rangle^{1-}} d\xi, \end{aligned}$$

When $|\xi| \lesssim 1$, we have the bound

$$\sup_\lambda \langle \lambda - \delta \rangle^{2(s_1+a_1)-3} \int \frac{1}{|\xi|^{2s_0-2}} d\xi \lesssim 1,$$

provided $a_1 < \frac{3}{2} - s_1$.

When $|\xi| \gg 1$, we have the bound

$$\sup_\lambda \langle \lambda - \delta \rangle^{2(s_1+a_1)-2} \int \frac{|\xi|^2 \langle \xi \rangle^{-2s_0}}{\langle \xi \rangle^{1-}} d\xi \lesssim 1,$$

provided $a_1 < 1 - s_1$. □

Proposition 17 For any admissible s_0, s_1 and for any $-\frac{1}{2} - s_1 < a_1 < \min(s_0 - s_1 + 1, 1 - s_1)$, for $\frac{1}{2} - \epsilon < b, b_1 < \frac{1}{2}$ where ϵ is sufficient small, it holds that

$$\| \langle \lambda \rangle^{s_1+a_1} \int_{\mathbb{R}} | \frac{\widehat{\partial_x |u|^2|}}{\langle \lambda - \delta \rangle} d\xi \|_{L^2_\lambda} \lesssim \|u\|^2_{X^{s_0,b}}.$$

Proof Similarly to Proposition 3.9, the goal is to prove that

$$\sup_\lambda \int_{\mathbb{R}^3} \frac{\langle \lambda \rangle^{2s_1+2a_1} \langle \lambda - \delta \rangle^{-2} |\xi|^2}{\langle \xi_1 \rangle^{2s_0} \langle \xi - \xi_1 \rangle^{2s_0} \langle \lambda_1 - \xi_1^2 \rangle^{2b} \langle \lambda - \lambda_1 + (\xi - \xi_1)^2 \rangle^{2b}} d\xi_1 d\lambda_1 d\xi < \infty.$$

Using Lemma 2.5 in the λ_1 integral and the variable substitution $\eta = \xi(\xi - 2\xi_1)$, the integral is estimated as

$$\sup_{\lambda} \int_{\mathbb{R}^2} \frac{\langle\lambda\rangle^{2(s_1+a_1)}\langle\lambda-\delta\rangle^{-2}|\xi|}{\langle\xi - \frac{\eta}{\xi}\rangle^{2s_0}\langle\xi + \frac{\eta}{\xi}\rangle^{2s_0}\langle\lambda+\eta\rangle^{1-}} d\eta d\xi.$$

Without loss of generality $|\xi - \frac{\eta}{\xi}| \gtrsim |\xi|$, it can be inferred that

$$\begin{aligned}
&\sup_{\lambda} \int_{\mathbb{R}^2} \frac{\langle\lambda\rangle^{2(s_1+a_1)}\langle\lambda-\delta\rangle^{-2}|\xi|}{\langle\xi - \frac{\eta}{\xi}\rangle^{2s_0}\langle\xi + \frac{\eta}{\xi}\rangle^{2s_0}\langle\lambda+\eta\rangle^{1-}} d\eta d\xi \\
&\lesssim \sup_{\lambda} \int_{\mathbb{R}} \frac{\langle\lambda\rangle^{2(s_1+a_1)}\langle\lambda-\delta\rangle^{-2}|\xi|^{1+\min(2s_0,1)}\langle\xi\rangle^{-2s_0}}{\langle\lambda-\xi^2\rangle^{\min(2s_0,1)-}} d\xi.
\end{aligned}$$

When $|\xi| \lesssim 1$, we have the bound

$$\sup_{\lambda} \langle\lambda\rangle^{2(s_1+a_1)-2-\min(2s_0,1)} \int \frac{1}{\langle\xi\rangle^{2s_0-1-\min(2s_0,1)}} d\xi \lesssim 1,$$

provided that $a_1 < \min(s_0 - s_1 + 1, \frac{3}{2} - s_1)$.

When $|\xi| \gg 1$, we have the bound

$$\sup_{\lambda} \int \frac{\langle\lambda\rangle^{2(s_1+a_1)}\langle\lambda-\delta\rangle^{-2}|\xi|^{1+\min(2s_0,1)}\langle\xi\rangle^{-2s_0}}{\langle\xi^2\rangle^{\min(2s_0,1)-}} d\xi \lesssim 1,$$

provided that $a_1 < 1 - s_1$. □

3.4 Counter-Examples

We are devoted to give two counter-examples, which shows that the restrictions on s_0 and s_1 are of necessity.

Theorem 18 For any $b, b_1 \in \mathbb{R}$, the estimate $\|nu\|_{X^{s_0+a_0,-b}} \lesssim \|n\|_{Y^{s_1,b_1}} \|u\|_{X^{s_0,b}}$ holds only if $s_0 - s_1 < 1$.

Proof Choose $N \in Z^+$ a large integer. Let

$$\begin{aligned}
A_1 =& \{(\xi,\tau) \in \mathbb{R}^2; 0 \le \xi \le \frac{1}{N} \; and \; |\tau + \xi^2| \le 1\}, \\
B_1 =& \{(\xi,\tau) \in \mathbb{R}^2; N \le \xi \le N + \frac{1}{N^2} \; and \; |\tau - \delta| \le 1\}.
\end{aligned}$$

Put $\widehat{f} := \chi_{A_1}$ and $\widehat{g} := \chi_{B_1}$. A straightforward computation gives that

$$\|fg\|_{X^{s_0,-\frac{1}{2}}} \sim \left(\frac{1}{N^2}\left(\frac{N^{s_0}}{N}\right)^2\right)^{\frac{1}{2}} \sim N^{s_0-2},$$

$$\|f\|_{X^{s_0,b}} \sim N^{-\frac{1}{2}} \text{ and } \|g\|_{Y^{s_1,b_1}} \sim N^{s_1-\frac{1}{2}}.$$

Thus, $\|fg\|_{X^{s_0,-\frac{1}{2}}} \lesssim \|f\|_{X^{s_0,b}}\|g\|_{Y^{s_1,b_1}}$ implies that $s_0 - s_1 < 1$, which completes the proof. □

Theorem 19 For any $b, b_1 \in \mathbb{R}$, the estimate $\|\partial_x |u|^2\|_{Y^{s_1+a_1,-b_1}} \lesssim \|u\|^2_{X^{s_0,b}}$ holds only if $2s_0 - s_1 \geq \frac{1}{2}$.

Proof Choose $N \in Z^+$ a large integer. Let

$$A_2 = \{(\xi, \tau) \in \mathbb{R}^2; N \leq \xi \leq N + \frac{1}{N} \, and \, |\tau + \xi^2| \leq 1\},$$

Put $\widehat{f} := \chi_{A_2}$. A straightforward computation gives that

$$\|\partial_x f^2\|_{Y^{s_1,-\frac{1}{2}}} \sim (\frac{1}{N}(N\frac{N^{s_1}}{N})^2)^{\frac{1}{2}} \sim N^{s_1-\frac{1}{2}},$$
$$\|f\|_{X^{s_0,b}} \sim N^{s_0-\frac{1}{2}}.$$

Thus, $\|\partial_x f^2\|_{Y^{s_1,-\frac{1}{2}}} \lesssim \|f\|^2_{X^{s_0,b}}$ implies that $2s_0 - s_1 \geq \frac{1}{2}$. The proof is completed. □

4 Local Theory

Now, we consider the IBVP:

$$\begin{cases} iu_t + u_{xx} = nu - |u|^2 u, \ (x, t) \in \mathbb{R}^+ = (0, \infty), \\ n_t + \delta n + |u|^2_x = 0, \ (x, t) \in \mathbb{R}^+ = (0, \infty), \\ u(x, 0) = u_0(x), \ n(x, 0) = n_0(x), \\ u(0, t) = h(t), \ n(0, t) = f(t). \end{cases} \tag{13}$$

We write the IBVP(13) in its integral equation form:

$$\Gamma_1(u, n)(t) = \phi(t) W_0^t(\tilde{u}_0, h) - i\phi(t)\int_0^t W_{\mathbb{R}}(t-\tau)F(u, n)d\tau + i\phi(t)W_0^t(0, q),$$
$$\Gamma_2(u, n)(t) = \phi(t)e^{-\delta t}\tilde{n}_0(x) + \phi(t)\int_0^t e^{-\delta(t-\tau)}G(u, n)d\tau + \phi(t)V_0^t(0, z).$$

Let

$$B_r = \{(u, n) \in X^{s_0,b} \times Y^{s_1,b_1} : \|u\|_{X^{s_0,b}} + \|n\|_{Y^{s_1,b_1}} \leq r\}.$$

A proof will be given that $\Gamma = (\Gamma_1(u,n)(t), \Gamma_2(u,n)(t))$ is a contraction map(see [5]) from B_r to B_r for appropriate r and T. Applying (6), (7), Proposition 3.9 and Proposition 3.11, it yields that

$$\begin{aligned}\|\phi(t)\int_0^t W_{\mathbb{R}}(t-\tau)F(u,n)d\tau\|_{X^{s_0,b}} &\lesssim \|F(u,n)\|_{X^{s_0,-\frac{1}{2}+}} \\ &\lesssim T^{\frac{1}{2}-b-}\|nu-|u|^2u\|_{X^{s_0,-b}} \\ &\lesssim T^{\frac{1}{2}-b-}(\|u\|_{X^{s_0,b}}\|n\|_{Y^{s_1,b_1}}+\|u\|^3_{X^{s_0,b}}).\end{aligned}$$

Notice that

$$\phi(t)W_0^t(\tilde{u}_0,h)+i\phi(t)W_0^t(0,q)=\phi(t)W_{\mathbb{R}}\tilde{u}_0+\phi(t)W_0^t(0,h-p+iq),$$

By (5), we get

$$\|\phi(t)W_{\mathbb{R}}\tilde{u}_0\|_{X^{s_0,b}}\lesssim \|\tilde{u}_0\|_{H^{s_0}}\lesssim \|u_0\||_{H^{s_0}(\mathbb{R}^+)}.$$

Applying Lemma 2.1 and Proposition 3.4, it can be inferred that

$$\|\phi(t)W_0^t(0,h-p+iq)(t)\|_{X^{s_0,b}}\lesssim \|h\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R}^+)}+\|p\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R})}+\|q\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R})}.$$

By Lemma 3.1, we can obtain

$$\|p\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R})}\lesssim \|u_0\|_{H^{s_0}(\mathbb{R}^+)}.$$

According to (7), Propositions 3.7, 3.9, 3.10, 3.11 and 3.12, we see that

$$\|q\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R})}\lesssim T^{\frac{1}{2}-b-}(\|u\|_{X^{s_0,b}}\|n\|_{Y^{s_1,b_1}}+\|u\|^3_{X^{s_0,b}}).$$

In view of these estimates, it shows that

$$\begin{aligned}\|\Gamma_1(u,n)\|_{X^{s_0,b}}\lesssim\ &\|u_0\|_{H^{s_0}(\mathbb{R}^+)}+\|h\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R}^+)} \\ &+T^{\frac{1}{2}-b-}(\|u\|_{X^{s_0,b}}\|n\|_{Y^{s_1,b_1}}+\|u\|^3_{X^{s_0,b}}).\end{aligned}$$

Then, applying Propositions 2.4, 3.8 and 3.13, it yields that

$$\|\phi(t)\int_0^t e^{-\delta(t-\tau)}G(u,n)d\tau\|_{Y^{s_1,b_1}}\lesssim T^{\frac{1}{2}-b-}(\|u\|^2_{X^{s_0,b}}).$$

Notice that

$$\phi(t)V_0^t(\tilde{n}_0,f)+\phi(t)V_0^t(0,z)=\phi(t)e^{-\delta t}\tilde{n}_0+\phi(t)V_0^t(0,f-r-z).$$

Using Lemma 2.1, Propositions 2.2 and 3.5, it follows that

$$\|\phi(t)V_0^t(0,\ f-r-z)\|_{Y^{s_1,b_1}}\lesssim \|f\|_{H_t^{s_1+1}(\mathbb{R}^+)}+\|r\|_{H_t^{s_1+1}(\mathbb{R})}+\|z\|_{H_t^{s_1+1}(\mathbb{R})}.$$

By Lemma 3.2, we see that

$$\|r\|_{H_t^{s_1}(\mathbb{R})}\lesssim \|n_0\|_{H^{s_1}(\mathbb{R}^+)}.$$

Finally, according to (7), Propositions 3.8, 3.14 and 3.15, it is obvious to find

$$\|z\|_{H_t^{s_1}(\mathbb{R})}\lesssim T^{\frac{1}{2}-b-}(\|u\|^2_{X^{s_0,b}}).$$

With aid of these estimates, it yields that

$$\|\Gamma_2(u,n)\|_{Y^{s_1,b_1}}\lesssim \|n_0\|_{H^{s_1}(\mathbb{R}^+)}+\|f\|_{H_t^{s_1}(\mathbb{R}^+)}+T^{\frac{1}{2}-b-}(\|u\|^2_{X^{s_0,b}}).$$

Hence, the main point of this problem is the following inequality. For any $u,\ n\in B_r$,

$$\begin{aligned}\|\Gamma_1(u,n)\|_{X^{s_0,b}}\lesssim&\|u_0\|_{H^{s_0}(\mathbb{R}^+)}+\|h\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R}^+)}\\&+T^{\frac{1}{2}-b-}(\|u\|_{X^{s_0,b}}\|n\|_{Y^{s_1,b_1}}+\|u\|^3_{X^{s_0,b}}),\\ \|\Gamma_2(u,n)\|_{Y^{s_1,b_1}}\lesssim&\|n_0\|_{H^{s_1}(\mathbb{R}^+)}+\|f\|_{H_t^{s_1+1}(\mathbb{R}^+)}\\&+T^{\frac{1}{2}-b-}(\|u\|^2_{X^{s_0,b}}),\end{aligned}$$

Choosing $r>0$ so that

$$\begin{cases} r=4C(\|u_0\|_{H^{s_0}(\mathbb{R}^+)}+\|h\|_{H_t^{\frac{2s_0+1}{4}}(\mathbb{R}^+)}+\|n_0\|_{H^{s_1}(\mathbb{R}^+)}+\|f\|_{H_t^{s_1+1}(\mathbb{R}^+)}),\\ T^{\frac{1}{2}-b-}(r+r^2)\le\dfrac{1}{4},\\ C(2r^2+4r)\le\dfrac{1}{2}.\end{cases}\tag{14}$$

Then

$$\|\Gamma_1(u,n)\|_{X^{s_0,b}}+\|\Gamma_2(u,n)\|_{Y^{s_1,b_1}}\le r.$$

Therefore, with such a choice of r, $(\Gamma_1(u,n),\ \Gamma_2(u,n))$ maps B_r into B_r. The same inequalities permit one to infer that for r as in (5),

$$\begin{aligned}&\|\Gamma_1(u_1,n_1)-\Gamma_1(u_2,n_2)\|_{X^{s_0,b}}+\|\Gamma_2(u_1,n_1)-\Gamma_2(u_2,n_2)\|_{Y^{s_1,b_1}}\\&\le\frac{1}{2}(\|u_1-u_2\|_{X^{s_0,b}}+\|n_1-n_2\|_{Y^{s_1,b_1}}),\end{aligned}$$

for any $(u_1, n_1), (u_2, n_2) \in B_r$. In other words, the map $(\Gamma_1(u, n), \Gamma_2(u, n))$ is a contraction mapping of B_r.

Hence, we get the local well-posedness in $X^{s_0,b} \times Y^{s_1,b_1}$. Now we check that $u \in C_t^0 H_x^{s_0}([0, T] \times \mathbb{R})$. Applying Lemma 3.1, we observe that the Schrödinger group operator $W_{\mathbb{R}}\tilde{u}_0$ and the boundary term $W_0^t(0, f - p + iq)$ is continuous, which means that the first and the third part of Γ_1 is continuous in H^{s_0}. We also obtain the same result for the second term owing to the embedding $X^{s,b} \subset C_t^0 H_x^s$ for $b > 1/2$ and (6) together with Propositions 3.9 and 3.11. The result that $u \in C_x^0 H_t^{\frac{2s_0+1}{4}}(\mathbb{R} \times [0, T])$ is shown from Lemmas 3.1, 3.3 and Proposition 3.7. In addition, the statements for n are proved in a similar way. The well-posedness depending on the initial and boundary conditions follows from the a priori estimates and the above contraction mapping argument.

5 Ill-Posedness

Here, it is natural to be concerned about the ill-posedness of the LS type equation for the completeness of the research. Therefore, we shall give a proof of Theorem 1.3.

Proof Suppose for the contradiction that the LS type equations (1) are locally well-posed on $[0, T]$ for $T \in (0, 1)$, and the solution map $(u_0, n_0) \mapsto (u, n)$ is C^2 from $H^{s_0}(\mathbb{R}) \times H^{s_1}(\mathbb{R})$ to $C_t^0([0, T]; H^{s_0}(\mathbb{R}) \times H^{s_1}(\mathbb{R}))$. Then. by Picard iterative scheme, so is the operator $A = (A_1, A_2) : H^{s_0}(\mathbb{R}) \times H^{s_1}(\mathbb{R}) \to C_t^0([0, T]; H^{s_0}(\mathbb{R}) \times H^{s_1}(\mathbb{R}))$ defined as

$$A_1(u_0, n_0) = -i \int_0^t W(t-\tau)[W(\tau)u_0 \cdot V(\tau)n_0 + |W(\tau)u_0|^2 W(\tau)u_0]d\tau;$$
$$A_2(u_0, n_0) = \int_0^t V(t-\tau)[\partial_x(|W(\tau)u_0|^2)(\tau)]d\tau.$$

Suppose N is large enough, we consider the initial data (u_0, n_0) such that

$$\widehat{u_0}(\xi) = \epsilon_0 N^{-s_0-\frac{1}{4}}\chi_{[-10,10]}(\xi); \widehat{n_0}(\xi) = \epsilon_0 N^{-\frac{n}{2}}\chi_{[-10,10]}(\xi),$$

then $\|u_0\|_{H^{s_0}}, \|n_0\|_{H^{s_1}} \sim \epsilon_0$. We may set T=1 by choosing ϵ_0 small enough. Further, $\|A_2\|_{C_t^0([0,1];H^{s_1}}$ is equal to

$$\sup_{0\le t\le 1} \|\int_0^t V(t-\tau)[\partial_x(|W(\tau)u_0|^2)(\tau)]d\tau\|_{H^{s_1}}$$
$$= \sup_{0\le t\le 1} \|\xi\langle\xi\rangle^{s_1} \int_0^t \int \exp\{i(t-\tau)\delta\}[\exp\{-i\tau(\xi-\xi_2)^2\}$$

$$\cdot \exp\{i\tau\xi_2^2\}\widehat{u_0}(\xi-\xi_2)\widehat{u_0}(\xi_2)]d\xi_2 d\tau\|_{L^2_\xi}$$

$$\geq \sup_{0\leq t\leq 1}\|\xi\langle\xi\rangle^{s_1}\int_0^t\int \exp\{it\delta\}\exp\{-i\tau(\xi^2-2\xi\xi_2-\delta)\}\widehat{u_0}(\xi-\xi_2)\widehat{u_0}(\xi_2)d\xi_2 d\tau\|_{L^2_\xi}.$$

This term has a lower bound of

$$N^{s_1+1}\sup_{0\leq t\leq 1}\|\int_0^t\int \exp\{it\delta\}exp\{-i\tau(\xi^2-2\xi\xi_2-\delta)\}\widehat{u_0}(\xi-\xi_2)\widehat{u_0}(\xi_2)d\xi_2 d\tau\|_{L^2_\xi}. \tag{15}$$

Note that N, set $t := \frac{1}{100N^4}$ and localize to the region where $-1 \leq \xi \leq 1$, one can confirm that

$$\Re(\exp\{it\delta\}\exp\{-i\tau(\xi^2-2\xi\xi_2-\delta)\}) > \frac{1}{2}$$

where $0 \leq \tau \leq t$.

Thus, we obtain

$$(15) \gtrsim \epsilon_0^2 N^{s_1+1}N^{-2s_0-\frac{1}{2}} \sim \epsilon_0^2 N^{s_1-2s_0+\frac{1}{2}}.$$

Therefore, by choosing N large enough, we have

$$\|A_2\|_{C_t^0([0,1];H^{s_1}(\mathbb{R})}\gtrsim \epsilon_0^2 N^{s_1-2s_0+\frac{1}{2}}.$$

Since A_2 is C^2-differentiable, we must have

$$\|A_2\|_{C_t^0([0,1];H^{s_1}(\mathbb{R})}\lesssim \|u_0\|^2_{H^{s_0}}+\|n_0\|^2_{H^{s_1}},$$

but it fails to hold when $s_1 - 2s_0 + \frac{1}{2} > 0$. This completes the proof. □

6 Global Well-Posedness

We shall investigate global results in energy space $H^1 \times L^2$. To begin with, we recall the equations which need to be solved:

$$\begin{cases} iu_t + u_{xx} = nu - |u|^2u,\ x \in \mathbb{R}^+ = (0,\infty),\ t \in \mathbb{R}^+, \\ n_t + \delta n = -|u|_x^2,\ x \in \mathbb{R}^+ = (0,\infty),\ t \in \mathbb{R}^+, \\ u(x,0) = u_0(x),\ n(x,0) = n_0(x), \\ u(0,t) = h(t),\ n(0,t) = f(t). \end{cases} \tag{16}$$

To obtain the global solution, we establish several a priori estimates and give the following identities. A few computational techniques are needed for the desired results.

Multiplying (16) by u, it is simple to yield

$$\frac{d}{dt}\int_0^\infty |u|^2 dx - 2\Im u_x(0,t)\overline{h}(t) = 0. \tag{17}$$

Multiplying (16) by $2u_t + u$, integrating and taking the real part, it holds that

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^\infty u_x^2 dx + \int_0^\infty n|u|^2 dx - \frac{1}{2}\int_0^\infty |u|^4 dx\right) + \int_0^\infty u_x^2 dx \\ &+ \int_0^\infty n|u|^2 dx - \int_0^\infty |u|^4 dx - \int_0^\infty |u|^2 n_t dx - \Re(iu_t, u) \\ &- \Re u_x(0,t)\overline{h}(t) + 2\Re u_x(0,t)\overline{h}'(t) = 0.\end{aligned} \tag{18}$$

From (16) we infer that

$$\begin{aligned}\int_0^\infty |u|^2 n_t dx &= \int_0^\infty |u|^2(-\delta n - |u|_x^2)dx \\ &= -\delta\int_0^\infty |u|^2 n dx - \int_0^\infty |u|^2|u|_x^2 dx \\ &= -\delta\int_0^\infty |u|^2 n dx + \frac{1}{2}h^4(t).\end{aligned} \tag{19}$$

By substituting (19) into (18), we see that

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^\infty u_x^2 dx + \int_0^\infty n|u|^2 dx - \frac{1}{2}\int_0^\infty |u|^4 dx\right) + \int_0^\infty u_x^2 dx \\ &+ (\delta+1)\int_0^\infty n|u|^2 dx - \int_0^\infty |u|^4 dx - \frac{1}{2}h^4(t) - \Re(iu_t, u) \\ &- \Re u_x(0,t)\overline{h}(t) + 2\Re u_x(0,t)\overline{h}'(t) = 0.\end{aligned} \tag{20}$$

Multiplying (16) by u_x, integrating and taking the real part, it holds that

$$-\frac{d}{dt}\Im\int_0^\infty u\overline{u}_x dx - 2\Re\int_0^\infty \overline{u}_x(un)dx + |u_x(0,t)|^2 - \Im h(t)\overline{h}'(t) - \frac{1}{2}h^4(t) = 0. \tag{21}$$

From (16) we infer that

$$\begin{aligned}2\Re\int_0^\infty \overline{u}_x(un)dx &= \int_0^\infty n|u|_x^2 dx \\ &= \int_0^\infty n(-n_t - \delta n)dx.\end{aligned} \tag{22}$$

By substituting (22) into (21), we see that

$$\frac{d}{dt}\Im\int_0^\infty u\overline{u}_x dx - \frac{1}{2}\frac{d}{dt}\int_0^\infty n^2 dx - \delta\int_0^\infty n^2 dx + |u_x(0,t)|^2 + \Im h(t)\overline{h}'(t) + \frac{1}{2}h^4(t) = 0. \tag{23}$$

Multipying (16) by n, we obtain

$$\frac{d}{dt}\int_0^\infty n^2 dx + 2\delta\int_0^\infty n^2 dx + 2\int_0^\infty n|u|_x^2 dx = 0. \tag{24}$$

From (16) we find

$$\begin{aligned}\int_0^\infty n|u|_x^2 dx &= 2\Re\int_0^\infty \overline{u}_x(un)dx\\ &= 2\Re\int_0^\infty \overline{u}_x(iu_t + u_{xx} + |u|^2u)dx\\ &= -\Im\frac{d}{dt}\int_0^\infty u\overline{u}_x dx - \Im h(t)\overline{h}'(t) - |u_x(0,t)|^2 - \frac{1}{2}h^4(t).\end{aligned} \tag{25}$$

Inserting (25) into (24), we get

$$\frac{d}{dt}\int_0^\infty n^2 dx - 2\frac{d}{dt}\Im\int_0^\infty u\overline{u}_x dx + 2\delta\int_0^\infty n^2 dx - 2\Im h(t)\overline{h}'(t) - 2|u_x(0,t)|^2 - h^4(t) = 0. \tag{26}$$

By (20)+(26), we derive that

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^\infty u_x^2 dx + \int_0^\infty n|u|^2 dx - \frac{1}{2}\int_0^\infty |u|^4 dx - 2\Im\int_0^\infty u\overline{u}_x dx + \int_0^\infty n^2 dx\right)\\ &+\int_0^\infty u_x^2 dx + (\delta+1)\int_0^\infty n|u|^2 dx - \int_0^\infty |u|^4 dx + 2\delta\int_0^\infty n^2 dx - \Re(iu_t, u)\\ &-\Re u_x(0,t)\overline{h}(t) + 2\Re u_x(0,t)\overline{h}'(t) - 2\Im h(t)\overline{h}'(t) - 2|u_x(0,t)|^2 - \frac{3}{2}h^4(t) = 0.\end{aligned} \tag{27}$$

Let

$$\begin{aligned}E(t) &:= \int_0^\infty u_x^2 dx + \int_0^\infty n|u|^2 dx - \frac{1}{2}\int_0^\infty |u|^4 dx - 2\Im\int_0^\infty u\overline{u}_x dx + \int_0^\infty n^2 dx,\\ K(t) &:= (2\delta-1)\int_0^\infty u_x^2 dx + (\delta-1)\int_0^\infty n|u|^2 dx + (1-\delta)\int_0^\infty |u|^4 dx\end{aligned}$$

$$-4\delta\int_0^\infty uu_x dx + \Re(iu_t, u) + \Re u_x(0, t)\overline{h}(t) - 2\Re u_x(0, t)\overline{h}'(t)$$
$$+2\Im h(t)\overline{h}'(t) + 2|u_x(0, t)|^2 + \frac{3}{2}h^4(t),$$

Then we have

$$\frac{d}{dt}E(t) + 2\delta E(t) = K(t).$$

Let

$$A = \int_0^t |u_x(0, s)|^2 ds.$$

By integrating (17) with respect to t, it infers that

$$\|u\|_{L^2}^2 - \|u_0\|_{L^2}^2 = \Im\int_0^t u_x(0, s)\overline{h}(s)ds.$$

Therefore, we obtain

$$\|u\|_{L^2}^2 \le C(\sqrt{A} + 1).$$

Using Gagliardo-Nirenberg inequality, we see that

$$\left(\int_0^\infty |u|^4 dx\right)^{\frac{1}{4}} \lesssim \|u\|_{L^2}^{\frac{3}{4}}\|u_x\|_{L^2}^{\frac{1}{4}}.$$

In addition, note that

$$|\int_0^\infty n|u|^2 dx| \lesssim \rho\|n\|_{L^2}^2 + C(\rho)\|u_x\|_{L^2}^2 + C(A),$$
$$|\int_0^\infty u\overline{u}_x dx| \lesssim \|u_x\|_{L^2}^2 + C(A),$$
$$|\Re(iu_t, u)| \lesssim \rho\|n\|_{L^2}^2 + C(\rho)\|u_x\|_{L^2}^2 + C(A),$$

where $C(A)$ represents a constant related to A. Then, we conclude that

$$E(t) \gtrsim \frac{1}{2}(\|u_x\|^2 + \|n\|^2),$$
$$K(t) \le (2\delta - 1)\|u_x\|^2 + C(\rho)(\delta - 1)\|u_x\|^2 + \rho(\delta - 1)\|n\|^2 + (1-\delta)\|u\|^3\|u_x\|$$
$$+4\delta\|u_x\|^2 + C(\rho)\|u_x\|^2 + \rho\|n\|^2 + C(h, f, A)$$
$$\le (2\delta - 1 + \delta C(\rho) - 4\delta + \epsilon^{-\frac{1}{7}}(1-\delta))\|u_x\|^2 + \delta\rho\|n\|^2 + C(h, f, A)$$
$$\le \delta(\|u_x\|^2 + \|n\|^2) + C(h, f, A),$$

where $\rho \leq 1$, $2\delta - 1 + \delta C(\rho) - 4\delta + \epsilon^{-\frac{1}{7}}(1-\delta) \leq \delta$. Therefore, we infer that

$$\frac{d}{dt}E(t) + \delta E(t) \leq C(h,\ f,\ A). \tag{28}$$

Using Gagliardo-Nirenberg inequality, we also see that

$$E(t) \lesssim \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|n\|_{L^2}^2,$$
$$\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|n\|_{L^2}^2 \lesssim E(t),$$

Then combining (23), we get that $A \lesssim E(t) + 1$. Substituting this result into (28) and using Gronwall's inequality, we deduce $\|u\|_{H^1} + \|n\|_{L^2}$ remains bounded.

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Declarations

Conflict of interest Authors have no conflict of interest to declare.

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