Bull. Malays. Math. Sci. Soc. (2024) 47:151
https://doi.org/10.1007/s40840-024-01727-y

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

The Harmonious Song of a Moufang Quartet

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Received: 4 January 2024 / Revised: 30 May 2024 / Accepted: 4 June 2024 /
Published online: 5 August 2024

Abstract

The variety of Moufang loops is axiomatized by any one of four well known (equivalent) identities. We prove that this axiomatic harmony holds in a broader setting by obtaining two alternate, generalized versions of the (traditional) definition of a Moufang loop using four "local" identities, each derived from one of the four "global" Moufang identities, one for loops, the other for magmas with the right or left inverse property.

Keywords Magma · Loop · Moufang

Mathematics Subject Classification 20N05

1 Prelude

A *magma* is a set, Q, together with a single binary operation, $\cdot$, and a two-sided neutral element. We denote this (unique) neutral element as e for an arbitrary magma, and 0 in our examples. A *loop* is a magma such that in $x \cdot y = z$, knowledge of any two of x, y and z specifies the third uniquely. For an overview of the theory of loops, see [1, 4, 7, 13]. Loops, per se, are so general that they resist mathematical analysis; one needs

Communicated by V. Ravichandran.

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more structure. For example, loops that satisfy the associative law have elegant—even tight—structure: they are groups.

In the 1930's, Ruth Moufang [11] initiated the study of loops that satisfy a weakening of the associative law: $x(y \cdot xz) = (xy \cdot x)z$. Note that we have adopted the simplifying notational convention that the product of juxtaposed elements has priority over the displayed binary operation $\cdot$ in terms to be multiplied, i.e., the identity we have just written is shorthand for $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z$. In addition to much else, Moufang showed that, in the variety of loops, this identity is equivalent to its mirror image: $(zx \cdot y)x = z(x \cdot yx)$. She also initiated a study of loops that satisfy another "nearly associative" identity: $x(yz \cdot x) = xy \cdot zx$, which has, of course, a "nearly associative" mirror: $(x \cdot zy)x = xz \cdot yx$.

Bol [2] showed that in the variety of loops, these four identities are equivalent, and so these identities came to be known as the *Moufang identities*. Not surprisingly, the variety of loops axiomatized by any one of them is called the variety of *Moufang loops*. Later, Bruck [3] provided a shorter proof of the equivalence of the Moufang identities, and in [4], described more properties of these loops. The smallest Moufang loop that is not a group has order 12 [6]; it is the only nonassociative Moufang loop of order 12. The smallest Moufang loop of odd order that is not a group has order 81 [12] (in fact, there are five nonassociative Moufang loops of order 81 [5, 12]). In short, Moufang loops are very close to groups; they share many deep structural properties and equational laws. For example, both satisfy the *flexible law* ($xy \cdot x = x \cdot yx$), *right alternative law* ($yx \cdot x = y \cdot xx$) and *left alternative law* ($x \cdot xy = xx \cdot y$). Moreover, each element, x, has a unique 2-sided inverse, denoted by x^{-1}, and satisfy both the *right* and *left inverse properties*: $yx \cdot x^{-1} = y$ and $x^{-1} \cdot xy = y$ respectively, and hence Moufang loops are *inverse property loops*. Moufang loops also satisfy Hall's theorem, as well as Lagrange's Theorem [7, 8]. An arbitrary loop need not satisfy any of these laws.

The theory of Moufang loops is sophisticated and quite deep, and includes a large collection of highly specialized and technical results [6, 16]. The starting point in this theory is the basic theorem mentioned above, i.e., that the four Moufang identities are equivalent in the variety of loops. In this paper, we offer a proof that applies not only to loops, but also in the more general setting of magmas, thus allowing us to strengthen the theorem.

In Sect. 2, we introduce the notion of local identities and elements, and we prove the equivalence of a pair among the four localized Moufang identities in a more general magma setting, as well as the equivalence of all four of these identities for magmas with the inverse property—a harmonious song. Along the way, we establish a number of useful properties of magmas with either right, left or both inverse properties. In Sect. 3, the harmony is pared down to its simplest elegance, as we give distinguishing examples, showing that various assumptions from the previous section are not only sufficient, but necessary.

Our investigations were aided by the automated reasoning tool Prover9 [9]; most of our examples were constructed using the finite model builder Mace 4.0 [10].

2 Scherzo—Local Laws

A *local* version of an identity is what results when one of the variables in the identity is held constant. So for example, the familiar *commutant* of a loop L, which is given by $C(L) = \{c \in L : cx = xc, \forall x \in L\}$, is the local version of the (*global*) commutative law (in fact, it is the only local version of the commutative law).

The more complicated associative law can be localized in three different ways, resulting in the familiar *left, middle* and *right nuclei*: $\mathrm{N}_\lambda(L) = \{a \in L : a \cdot xy = ax \cdot y, \forall x, y \in L\}$, $\mathrm{N}_\mu(L) = \{a \in L : x \cdot ay = xa \cdot y, \forall x, y \in L\}$, and $\mathrm{N}_\rho(L) = \{a \in L : x \cdot ya = xy \cdot a, \forall x, y \in L\}$ respectively.

The Moufang identities are more complicated still. Each of the four Moufang identities has three variables; there are thus 12 possible ways to "localize" the following four Moufang identities:

$$(A) : z(xy \cdot z) = zx \cdot yz \qquad (C) : z(x \cdot zy) = (zx \cdot z)y$$
$$(B) : (z \cdot xy)z = zx \cdot yz \qquad (D) : (xz \cdot y)z = x(z \cdot yz)$$

For example, the identity (A) is localized in three ways, thusly:

$$a(xy \cdot a) = ax \cdot ya, \;\; z(ay \cdot z) = za \cdot yz, \;\; z(xa \cdot z) = zx \cdot az$$

Let S be a set, together with a binary operation, $\cdot$. We introduce the following notation: $A_\lambda(S) = \{a \in S : x(ay \cdot x) = xa \cdot yx, \forall x, y \in S\}$, $A_\mu(S) = \{a \in S : a(xy \cdot a) = ax \cdot ya, \forall x, y \in S\}$, and $A_\rho(S) = \{a \in S : x(ya \cdot x) = xy \cdot ax, \forall x, y \in S\}$ where a is the local element. We shall write $A_\lambda(S)$ simply as A_λ if there is no confusion about the set. We focus on the following four identities (for both historical and algebraic reasons summarized in [15]). We say that *$a \in S$ has the property* (A_μ), (B_μ), (C_μ) or (D_μ) respectively, if for all $x, y \in S$, it satisfies any of the identities:

$$(A_\mu) : a(xy \cdot a) = ax \cdot ya \qquad (C_\mu) : a(x \cdot ay) = (ax \cdot a)y$$
$$(B_\mu) : (a \cdot xy)a = ax \cdot ya \qquad (D_\mu) : (xa \cdot y)a = x(a \cdot ya)$$

We will call the element $a \in A_\mu(S)$ an A_μ *element*. The analogous definitions for B_μ, C_μ and D_μ are now clear. For any magma, none of A_μ, B_μ, C_μ or D_μ are empty sets since $e \in A_\mu \cap B_\mu \cap C_\mu \cap D_\mu$. Also $A_\mu = B_\mu$; other than this trivial equivalence, no one of the twelve local Moufang identities implies any other in the variety of magmas [15], and none of the three "Moufang subsets" must be a submagma [14]. We define a *Moufang magma* as a magma which satisfies all four Moufang identities (A), (B), (C) and (D). Though these identities are equivalent in the variety of loops, only (A) and (B) are equivalent in the variety of magmas (Corollary 2.2, and Example 3.1 and its dual).

Any identity can be localized, of course. We define three local elements, a, that appear later in the paper:

$$\textit{right alternative element} : xa \cdot a = x \cdot aa$$

$$left\ alternative\ element: a \cdot ax = aa \cdot x$$
$$flexible\ element: ax \cdot a = a \cdot xa$$

Many of the results in this section pertain to local versions of the right and left inverse properties: a is a *right* (alternatively, *left*) *inverse element* if $xa \cdot a^{-1} = xa^{-1} \cdot a = x$ (alternatively, $a^{-1} \cdot ax = a \cdot a^{-1}x = x$) for all $x \in Q$.

We prove the following three results for a general magma before proceeding to prove results that hold for magmas with the right, or left, or both inverse properties.

Lemma 2.1 *For any magma, every element in $A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$ is flexible.*

Proof Let $a \in A_\mu$. In $a(xy \cdot a) = ax \cdot ya$, set $y = e$ to get that a is a flexible element.
Let $a \in B_\mu$. In $(a \cdot xy)a = ax \cdot ya$, set $x = e$ to get that a is a flexible element.
Let $a \in C_\mu$. In $a(x \cdot ay) = (ax \cdot a)y$, set $y = e$ to get that a is a flexible element.
Let $a \in D_\mu$. In $(xa \cdot y)a = x(a \cdot ya)$, set $x = e$ to get that a is a flexible element. □

The following is a simple corollary of the lemma above:

Corollary 2.2 $A_\mu = B_\mu$ *for every magma.*

Proof By Lemma 2.1, for every x, y in a magma Q, $a(xy \cdot a) = (a \cdot xy)a$. Hence $A_\mu = B_\mu$ for every magma Q by using the identities (A_μ) and (B_μ). □

Lemma 2.3 *For any magma, every element in C_μ is a left alternative element, whereas every element in D_μ is a right alternative element.*

Proof Let $a \in C_\mu$. In $a(x \cdot ay) = (ax \cdot a)y$, set $x = e$ to get that a is a left alternative element.
Let $a \in D_\mu$. In $(xa \cdot y)a = x(a \cdot ya)$, set $y = e$ to get that a is a right alternative element. □

Let Q be a magma. It is clear that e, the two-sided neutral element, is also the unique right, and the unique left, neutral element of Q. As we shall see in the next lemma, inverses in magmas have familiar properties.

Lemma 2.4 *Let Q be a magma.*
Suppose for every $x \in Q$ there exists some $x' \in Q$ such that $yx \cdot x' = y\ \forall y \in Q$. Then $x'' = x$, and hence, $yx' \cdot x = y$.
Alternatively, suppose $x' \cdot xy = y\ \forall y \in Q$. Then $x'' = x$, and hence, $x \cdot x'y = y$.
In both cases above, x' is the unique inverse of x; more specifically, x' is the unique right, and the unique left, inverse of x.

Proof Assume that Q satisfies $yx \cdot x' = y$. Now $xx' = ex \cdot x' = e$; so x' is a right inverse of x. Thus $x'' = ex'' = xx' \cdot x'' = x$. Then $yx' \cdot x = yx' \cdot x'' = y$. Also $x'x = x'x'' = e$. This proves that x' is also a left inverse of x. To prove that x' is the unique left inverse of x, let $u \in Q$ be another left inverse of x: $ux = e$. Then $u = ux \cdot x' = ex' = x'$. Likewise, if $xv = e$ for some $v \in Q$, then $v = v'' = (ev')' = (xv \cdot v')' = x'$. This completes the proof of the case where $yx \cdot x' = y\ \forall x, y \in Q$.

The proof for the case where Q satisfies $x' \cdot xy = y$ is the mirror of the proof given here. □

Lemma 2.4 allows us to use the standard notation, x^{-1} for the inverse of x, instead of $x^{'}$, whenever the magma satisfies $yx \cdot x^{'} = y$ or $x^{'} \cdot xy = y$.

In much of the following results and proofs, we will be using the notations “mirror of an identity", “mirror of a property (or law)” and “mirror of a proof (or argument)”. The *mirror of an equation* is simply an equation being written from right to left instead of left to right. So (A_{μ}) and (B_{μ}) are mirror identities. The same is true for the identities (C_{μ}) and (D_{μ}). Likewise, the right inverse property and left inverse property, and the right cancelation law and left cancelation law are two examples of pairs of properties that are mirrors of each other. Similarly, the mirror of a proof would require each step of the proof being written from left to right instead of right to left (and vice-versa) using the relevant mirror identities, laws or properties given in the original proof.

Lemma 2.5 *Let Q be a magma.*

(i) Suppose Q has the right inverse property. Then the mirror of the properties satisfied by elements in $A_{\mu} \cup B_{\mu}$ also hold for elements in this set if Q has the left inverse property.
Alternatively, suppose Q has the left inverse property. Then the mirror of the properties satisfied by elements in $A_{\mu} \cup B_{\mu}$ also hold for elements in this set if Q has the right inverse property.

(ii) Suppose Q has the right inverse property. Then the mirror of the properties satisfied by elements in C_{μ} also hold for elements in D_{μ} if Q has the left inverse property, and vice-versa.
Alternatively, suppose Q has the left inverse property. Then the mirror of the properties satisfied by elements in C_{μ} also hold for elements in D_{μ} if Q has the right inverse property, and vice-versa.

Proof Since the right and left inverse properties are mirrors of each other, the proof of (i) is due to Corollary 2.2, and the identities (A_{μ}) and (B_{μ}) being mirrors of each other, whereas the proof of (ii) is due to the identities (C_{μ}) and (D_{μ}) being mirrors of each other. □

The next nine results give general properties of magmas with either the right or left inverse property.

Lemma 2.6 *Let Q be a magma with the right inverse property. Then the right cancelation law applies to every element in Q.*

Alternatively, let Q be a magma with the left inverse property. Then the left cancelation law applies to every element in Q.

Proof Suppose Q has the right inverse property.

If $yx = zx$ for some $x, y, z \in Q$, then $y = yx \cdot x^{-1} = zx \cdot x^{-1} = z$. Thus, the right cancelation law applies to every element in Q.

The proof for the case where Q has the left inverse property is the mirror of the proof above. □

Lemma 2.7 *Let Q be a magma with the right inverse property and $x \in Q$. Then the following three properties hold for all $a \in A_{\mu} \cup B_{\mu} \cup C_{\mu} \cup D_{\mu}$.*

(i) The left cancelation law applies to a,
(ii) $a^{-1} \cdot ax = x$*, and*
(iii) $a \cdot a^{-1}x = x$.

Moreover, the following three properties hold for all $a \in A_\mu \cup B_\mu \cup C_\mu$.

(iv) $(ax)^{-1} = x^{-1}a^{-1}$,
(v) $(xa)^{-1} = a^{-1}x^{-1}$*, and*
(vi) $x^{-1} \cdot xa = a$.

Proof Let $a \in C_\mu$.

By Lemma 2.1 and the right inverse property of a, we get $(a \cdot a^{-1}a^{-1})a = a(a^{-1}a^{-1} \cdot a) = aa^{-1} = e$. Then for any $u \in Q$, $a(a^{-1}a^{-1} \cdot au) = ((a \cdot a^{-1}a^{-1})a)u = u$. Now suppose $ax = ay$ for some $x, y \in Q$. Then $x = a(a^{-1}a^{-1} \cdot ax) = a(a^{-1}a^{-1} \cdot ay) = y$. Hence, the left cancelation law applies to any C_μ element. This proves (i).

Since $a(a^{-1} \cdot ax) = (aa^{-1} \cdot a)x = ax$, use (i) to obtain (ii).

In the proof of (i), we showed that $x = az$ if $z = (a^{-1}a^{-1} \cdot ax) \in Q$. From (ii), we get $a(a^{-1} \cdot az) = az$. Replacing az with x proves (iii).

By the flexibilty of a and since $a \in C_\mu$, we get $a(x^{-1}a^{-1} \cdot ax) = a(x^{-1}a^{-1} \cdot a) \cdot x = ax^{-1} \cdot x = ae$. By (i), we get $x^{-1}a^{-1} \cdot ax = e$. By Lemma 2.4, $x^{-1}a^{-1}$ is the (unique) inverse of ax, thereby proving (iv).

By (iv) and the right inverse property, we get $(a(xa)^{-1})^{-1} = xa \cdot a^{-1} = x$. Taking inverses on both sides, we get $a(xa)^{-1} = x^{-1}$. Then, by (ii), $(xa)^{-1} = a^{-1} \cdot a(xa)^{-1} = a^{-1}x^{-1}$, which proves (v).

To prove (vi), we use (iii), (v) and the right inverse property: $x^{-1} \cdot xa = (a \cdot a^{-1}x^{-1}) \cdot xa = a(xa)^{-1} \cdot xa = a$. This completes the proof for $a \in C_\mu$.

Now, let $a \in A_\mu$. Then, $xa = aa^{-1} \cdot xa = (a \cdot a^{-1}x)a$ by Corollary 2.2. By Lemma 2.6, we have proven (iii).

Replacing x in (iii) with x^{-1} and multiplying by xa on the right, we get $x^{-1} \cdot xa = (a \cdot a^{-1}x^{-1}) \cdot xa = a(a^{-1}x^{-1} \cdot x) \cdot a = aa^{-1} \cdot a = a$, which proves (vi).

Replacing x in (vi) with $x^{-1}a^{-1}$ we get $(x^{-1}a^{-1})^{-1}(x^{-1}a^{-1} \cdot a) = a$. Applying the right inverse property gives $(x^{-1}a^{-1})^{-1}x^{-1} = ax \cdot x^{-1}$. Then (iv) follows by using the right cancelation law on x^{-1}, then taking the inverses of both sides.

Next, suppose $ax = ay$ for some $x, y \in Q$. Then, using the right inverse property and (iv), we get $x^{-1} = x^{-1}a^{-1} \cdot a = (ax)^{-1}a = (ay)^{-1}a = y^{-1}a^{-1} \cdot a = y^{-1}$. Taking inverses on both sides gives us the left cancelation law for a, thereby proving (i).

Next, replacing x with ax in (iii), we get $a(a^{-1} \cdot ax) = ax$. Then use (i) to obtain (ii).

From (iii), the right inverse property and (vi), we have $a \cdot a^{-1}x^{-1} = x^{-1} = (x^{-1} \cdot xa)(xa)^{-1} = a(xa)^{-1}$. Then use (i) to obtain (v). This completes the proof when $a \in A_\mu \cup B_\mu$.

Now, suppose $a \in D_\mu$. Then, by the right inverse property, (D_μ) property of a and right inverse property again, $a^{-1} \cdot ax = a^{-1} \cdot a(xa^{-1} \cdot a) = (a^{-1}a \cdot xa^{-1})a = xa^{-1} \cdot a = x$, which proves (ii).

By (ii) and identity (D_μ), we get $e = a^{-1}a = [a^{-1}(a \cdot x^{-1}a) \cdot (a \cdot x^{-1}a)^{-1}]a = (x^{-1}a \cdot (a \cdot x^{-1}a)^{-1})a = x^{-1}(a \cdot (a \cdot x^{-1}a)^{-1}a)$. So, by Lemma 2.4, $x = a \cdot (a \cdot$

$x^{-1}a)^{-1}a$. Then by (ii), $a \cdot a^{-1}x = a(a^{-1}(a \cdot (a \cdot x^{-1}a)^{-1}a) = aa \cdot (a \cdot x^{-1}a)^{-1}a = x$; this proves (iii).

Finally, if $ax = ay$ for some $x, y \in Q$, multiply both sides with a^{-1} on the left and use (ii) to prove (i). □

Note that the properties (iv), (v) and (vi) are not generally true for $a \in D_\mu$. This makes it possible for $A_\mu = B_\mu = C_\mu$ to be a proper subset of D_μ as we shall see in Lemma 2.12 and Example 3.4.

By Lemma 2.5, we obtain the following lemma:

Lemma 2.8 *Let Q be a magma with the left inverse property and $x \in Q$. Then the following three properties hold for all $a \in A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$.*

(i) The right cancelation law applies to a,
(ii) $xa \cdot a^{-1} = x$, and
(iii) $xa^{-1} \cdot a = x$.

Moreover, the following three properties hold for all $a \in A_\mu \cup B_\mu \cup D_\mu$.

(iv) $(xa)^{-1} = a^{-1}x^{-1}$,
(v) $(ax)^{-1} = x^{-1}a^{-1}$, and
(vi) $ax \cdot x^{-1} = a$.

Lemma 2.9 *Let Q be a magma with the right or left inverse property. Then every element in $A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$ is both a right and left inverse element.*

Proof If Q has the right inverse property, then $xa \cdot a^{-1} = xa^{-1} \cdot a = x$, and $a^{-1} \cdot ax = a \cdot a^{-1}x = x$ by Lemmas 2.7(ii) and (iii). The proof when Q has the left inverse property uses Lemmas 2.8(ii) and (iii). □

Lemma 2.10 *Let Q be a magma with the right or left inverse property. Then every element in $A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$ is both a left and right alternative element.*

Proof We prove this lemma for magmas with the right inverse property using Lemma 2.7; the proof for magmas with the left inverse property is a mirror of this proof by using Lemma 2.8.

Let $a \in A_\mu$. Then, by Lemma 2.1, $aa \cdot x = aa \cdot (xa^{-1} \cdot a) = a \cdot (a \cdot xa^{-1})a = a \cdot a(xa^{-1} \cdot a) = a \cdot ax$; hence a is a left alternative element. The proof that a is a right alternative element uses the mirror of the proof above, and Lemma 2.7(iii) and Corollary 2.2. The proof for the case $a \in B_\mu$ follows from Corollary 2.2.

Suppose $a \in C_\mu$. Then, by using identity (C_μ) and Lemma 2.1, $a(x \cdot aa) = (ax \cdot a)a = (a \cdot xa)a = a(xa \cdot a)$. Use Lemma 2.7(i) for left cancelation of a to prove that a is right alternative. a is left alternative by Lemma 2.3.

The proof that every element in D_μ is left alternative is a mirror of the proof that every element in C_μ is right alternative by using identity (D_μ) and Lemma 2.1, and Lemma 2.6 in the place of Lemma 2.8(i). $a \in D_\mu$ is right alternative by Lemma 2.3. □

Lemma 2.11 *Let Q be a magma with the right or left inverse property. Then $a \in A_\mu$ if and only if $a^{-1} \in A_\mu$. Likewise, $a \in B_\mu$ if and only if $a^{-1} \in B_\mu$, $a \in C_\mu$ if and only if $a^{-1} \in C_\mu$, and $a \in D_\mu$ if and only if $a^{-1} \in D_\mu$. Moreover, $a \cdot xa^{-1} = ax \cdot a^{-1}$ and $a^{-1} \cdot xa = a^{-1}x \cdot a$ for all $a \in A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$ and $x \in Q$.*

Proof We begin by proving the third statement of this lemma for a magma Q with the right inverse property. Let $a \in B_\mu$ and $x, y \in Q$. By Lemma 2.1, the (B_μ) identity and Lemma 2.9, $a \cdot (a^{-1}x \cdot ya^{-1})a = a(a^{-1}x \cdot ya^{-1}) \cdot a = (a \cdot a^{-1}x)(ya^{-1} \cdot a) = xy$. Multiplying both sides by a^{-1} on the left, and then by a^{-1} on the right, Lemma 2.9 gives $a^{-1}x \cdot ya^{-1} = (a^{-1} \cdot xy)a^{-1}$. So $a^{-1} \in B_\mu$. Hence, if $a^{-1} \in B_\mu$, then $(a^{-1})^{-1} = a \in B_\mu$. The proof for the case $a \in A_\mu$ follows from Corollary 2.2.

Now, let $a \in C_\mu$ and $x, y \in Q$; and write $u = a^{-1}x \cdot a^{-1}$ and $v = a^{-1}y$. Then, by Lemmas 2.1 and 2.9, $au \cdot a = a \cdot ua = a \cdot (a^{-1}x \cdot a^{-1})a = a \cdot a^{-1}x = x$ and $av = a \cdot a^{-1}y = y$. Replacing these in $a(u \cdot av) = (au \cdot a)v$, we get $a \cdot (a^{-1}x \cdot a^{-1})y = x \cdot a^{-1}y$. Multiplying both sides by a^{-1} on the left and using Lemma 2.9 gives $(a^{-1}x \cdot a^{-1})y = a^{-1}(x \cdot a^{-1}y)$. Thus, $a^{-1} \in C_\mu$. The proof of this case is complete since $(a^{-1})^{-1} = a$.

The proof for the case $a \in D_\mu$ is the mirror of the proof for the C_μ case.

Corollary 2.2 completes the proof of the second and third statements of this lemma when Q has the right inverse property.

The proof of the same for the case where Q has the left inverse property is the mirror of the proofs above.

The proof of the first equation in the last statement of this lemma uses Lemmas 2.9, 2.1 and 2.9 again; the proof of the second equation is the mirror of this proof:
$a \cdot xa^{-1} = (a \cdot xa^{-1})a \cdot a^{-1} = a(xa^{-1} \cdot a) \cdot a^{-1} = ax \cdot a^{-1}$. □

Lemma 2.12 *Let Q be a magma with the right inverse property. Then $A_\mu = B_\mu = C_\mu \subseteq D_\mu$.*

Proof Let $a \in B_\mu$ and $x, y \in Q$. If $u = x \cdot ay$ and $v = (ay)^{-1} = y^{-1}a^{-1}$ by Lemma 2.7(iv), then $au = a(x \cdot ay)$, and by the right inverse property, $va = y^{-1}a^{-1} \cdot a = y^{-1}$ and $uv = (x \cdot ay)(ay)^{-1} = x$. Replace these in $au \cdot va = (a \cdot uv)a$ to get $a(x \cdot ay) \cdot y^{-1} = ax \cdot a$. Multiply by y on the right and use the right inverse property to get $a \in C_\mu$.

If $a \in C_\mu$ and $x, y \in Q$, by Lemma 2.7(v), $(ya)^{-1} = a^{-1}y^{-1}$. Now, by Lemma 2.9, $(a \cdot xy)a = ((a \cdot xy)a \cdot (ya)^{-1}) \cdot ya = a(xy \cdot (a \cdot a^{-1}y^{-1})) \cdot ya = a(xy \cdot y^{-1}) \cdot ya = ax \cdot ya$, and we have shown that $a \in B_\mu$.

This, with Corollary 2.2 complete the proof that $A_\mu = B_\mu = C_\mu$.

Next, by Lemmas 2.9 and 2.1, if $a \in B_\mu = C_\mu$, compute $x(a \cdot ya) = (a \cdot a^{-1}x)(ay \cdot a) = a(a^{-1}x \cdot ay) \cdot a = ((a \cdot a^{-1}x)a \cdot y)a = (xa \cdot y)a$ and we have shown that $a \in D_\mu$ which proves the subset containment. □

Lemma 2.12 harmonizes with its mirror, Lemma 2.13:

Lemma 2.13 *Let Q be a magma with the left inverse property. Then $A_\mu = B_\mu = D_\mu \subseteq C_\mu$.*

The following lemma proves that magmas with both inverse properties are, in fact, loops:

Lemma 2.14 *Let S be a set, together with a binary operation, $\cdot$.*

Suppose $\forall x \in S$, there exists some (not necessarily unique) $x' \in S$ such that $yx \cdot x' = y$ $\forall y \in S$. Then every element of S appears once and only once in each column of the Cayley table.

Alternatively, suppose $\forall x \in S$, there exists some (not necessarily unique) $x' \in S$ such that $x' \cdot xy = y$ $\forall y \in S$. Then every element of S appears once and only once in each row of the Cayley table.

Proof Suppose S satisfies the condition $yx \cdot x' = y$. Let $u, v \in S$. Then there exists $w = vu' \in S$. Now $v = vu' \cdot u'' = wu'' = (wu \cdot u')u'' = wu$. So, every element of S appears at least once in every column of the Cayley table. Now, if $v = zu$ for some $z \in S$, $zu = wu$ implies that $z = zu \cdot u' = wu \cdot u' = w$. So, each element of S appears only once in each column of the Cayley table. The proof when $x' \cdot xy = y$ $\forall x \in S$ is a mirror of the proof above. □

The converse of this lemma is not true as shown in Example 3.7. Moreover, in Example 3.8 we see that the lemma above holds even if $x \in S$ does not have a unique $x \in S$ such that $yx \cdot x' = y$ for every $y \in S$.

By combining Lemmas 2.12, 2.13 and 2.14, the quartet emerges in full harmony:

Corollary 2.15 *Let Q be a magma with the inverse property. Then $A_\mu = B_\mu = C_\mu = D_\mu$, and Q is an (inverse property) loop.*

Now we proceed to obtain results that will culminate in two forms of generalization of the definition of Moufang loops.

Corollary 2.16 *Let Q be a magma with the right or left inverse property. Suppose every element of Q belongs to at least one of A_μ, B_μ, C_μ or D_μ. Then Q is a loop which satisfies each of the Moufang identities (A), (B), (C) and (D), (and hence is a Moufang loop).*

Proof Since $Q = A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$, by Lemma 2.9, it has both right and left inverse properties. Hence, by Corollary 2.15, $Q = A_\mu = B_\mu = C_\mu = D_\mu$, and hence is a Moufang loop. □

Lemma 2.17 *Let Q be a magma with the right or left inverse property. In its Cayley table, every element of Q appears once and only once in each row and column whose leading element is from $A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$.*

Proof Suppose Q has the right inverse property. By Lemma 2.14, every element of Q (including any element $a \in A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$) appears once and only once in each column of the Cayley table.

Now, for any $x \in Q$, there exists $u = a^{-1}x \in Q$. Then, by Lemma 2.9, $au = a \cdot a^{-1}x = x$. So every element of Q appears at least once in the row where a is the leading element. The proof of the uniqueness of u in the equation $au = x$ follows from Lemma 2.7(i).

The proof when Q has the left inverse property is a mirror of the proof above by using Lemma 2.8(i). □

We note that the "right or left inverse property" condition in Lemma 2.17 is nontrivial (i.e., necessary), as we shall see in Example 3.5.

Lemma 2.18 *Let L be a loop such that each $x \in L$ has an inverse, $x^{-1} \in L$. Then*

(i) $x^{-1} \cdot xy = y$ for each $y \in L$ if and only if $x \cdot x^{-1}y = y$ for each $y \in L$, and
(ii) $yx \cdot x^{-1} = y$ for each $y \in L$ if and only if $yx^{-1} \cdot x = y$ for each $y \in L$.

Proof Since L is a loop, left and right cancelation laws apply here. Suppose $x^{-1} \cdot xy = y$ for each $y \in L$. Substitute y with $x^{-1}y$ and use the left cancelation law on x^{-1} to obtain $x \cdot x^{-1}y = y$ for each $y \in L$. To prove that the second equation in (i) implies the first, substitute y with xy in the second equation and use the left cancelation law on x. This proves (i).

To prove (ii), use the mirror of the proof above and the right cancelation law. □

Lemma 2.19 *Let L be a loop. Suppose every element of L has at least one of the properties (A_μ), (B_μ), (C_μ) or (D_μ). Then L has the inverse property.*

Proof Since a loop is a variety of magmas, we are free to use the lemmas in this section that apply to general magmas (i.e., those that don't impose the right or left inverse properties).

Let x be an arbitrary element of L. Since L is a loop, there exist unique elements $u, v \in L$ such that $ux = xv = e$. We show that $u = v$.

By the premise of the lemma, $x \in A_\mu \cup B_\mu \cup C_\mu \cup D_\mu = L$. So by Lemma 2.1, $x \cdot vx = xv \cdot x = ex = x = xe$. Left cancelation of x (since L is a loop) gives $vx = e = ux$. Right cancelation of x produces the desired $v = u$, i.e., x has a unique two-sided inverse, x^{-1}.

To complete the proof of this lemma, by Lemma 2.18, we only need to prove that either one of the two equations in each of Lemma 2.18(i) and (ii) hold.

Now, let $y \in L$.

Case 1. Suppose $x \in B_\mu$, i.e., $x \in A_\mu$ by Corollary 2.2. Then $yx = xx^{-1} \cdot yx = (x \cdot x^{-1}y)x$ by the (B_μ) property of x. Right cancelation of x gives $y = x \cdot x^{-1}y$. Since identities (A_μ) and (B_μ) are mirrors of each other, the mirror of the proof above proves that $y = yx^{-1} \cdot x$.

Case 2. Suppose $x \in C_\mu$. By the (C_μ) property of x we have $x(x^{-1} \cdot xy) = (xx^{-1} \cdot x)y = xy$. Left cancelation of x gives $x^{-1} \cdot xy = y$. Now $x \cdot x^{-1}y = x(x^{-1}y \cdot xx^{-1}) = (x \cdot x^{-1}y)x \cdot x^{-1}$ by the (C_μ) property of x. Then $y = yx \cdot x^{-1}$ since $x \cdot x^{-1}y = y$ by Lemma 2.18(i).

Case 3. Suppose $x \in D_\mu$. The proof is a mirror of the proof in the case above. □

Our scherzo has now reached its crescendo in full harmony:

Theorem 2.20 *A loop is a Moufang loop if and only if every one of its element has at least one of the properties (A_μ), (B_μ), (C_μ) or (D_μ).*

Proof If a loop is a Moufang loop, then all its elements has all of the properties (A_μ), (B_μ), (C_μ) and (D_μ) by definition.

Now suppose every element of a loop, L, has at least one of the properties (A_μ), (B_μ), (C_μ) or (D_μ). So $L = A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$. By Lemma 2.19, L

has the inverse property. Then, since a loop is also a magma, by Corollary 2.15, $A_\mu = B_\mu = C_\mu = D_\mu$. So $L = A_\mu \cap B_\mu \cap C_\mu \cap D_\mu$.

So every element of L satisfies all four Moufang identities (A), (B), (C) and (D), and hence, L is a Moufang loop by definition.

This completes the proof of this theorem. □

Corollary 2.16 and Theorem 2.20 provide an alternate, and somewhat generalized, definition of Moufang loops: A *Moufang loop* is either a magma with the right or left inverse property, or a loop, whose every element has at least one of the local properties (A_μ), (B_μ), (C_μ) or (D_μ).

We summarize (as well as generalize) some of the main results obtained in this section (specifically, Lemmas 2.4 and 2.9 - 2.13):

Theorem 2.21 *Let Q be a magma such that for each $x \in Q$ there exists some $x^{'} \in Q$ satisfying the stated condition.*

(1) Suppose Q satisfies at least one of the identities $yx \cdot x' = y$ or $x' \cdot xy = y$ for all $x, y \in Q$. Then

(a) $x^{'}$ is the (unique) inverse of x,
(b) every element in $A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$ is both a right and left alternative element,
(c) every element in $A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$ is both a right and left inverse element, and
(d) $a \in A_\mu$ (respectively B_μ, C_μ, D_μ) if and only if $a' \in A_\mu$ (respectively B_μ, C_μ, D_μ).

(2) (a) Suppose $yx \cdot x' = y$ for all $x, y \in Q$. Then $A_\mu = B_\mu = C_\mu \subseteq D_\mu$.
(b) Suppose $x' \cdot xy = y$ for all $x, y \in Q$. Then $A_\mu = B_\mu = D_\mu \subseteq C_\mu$.

3 Intermezzo—Examples

The following example shows that inverse properties are necessary in Lemmas 2.12 and 2.13. By Lemma 2.14, this magma, Q, has neither the right nor left inverse property. Moreover, in this magma, $C_\mu = D_\mu = \{0\} \subset A_\mu = B_\mu = Q$. So Q satisfies the (global) Moufang identities (A) and (B), but not (C) or (D); hence Q is not a Moufang magma. Also, without the right inverse property, it is possible for D_μ to be a proper subset of B_μ and $B_\mu \neq C_\mu$, and without the left inverse property, it is possible for C_μ to be a proper subset of B_μ, and $B_\mu \neq D_\mu$.

Example 3.1

0	1	2
1	0	0
2	0	0

By Lemma 2.14, we know that the magma, Q, in the next example has neither the right nor left inverse property. In this example, $A_\mu = B_\mu = C_\mu = \{0, 4\} \subset D_\mu = Q$

which shows that the converse of Lemma 2.12 is not true. The dual (i.e., the transpose) of this example is a magma with neither the right nor left inverse property such that $A_\mu = B_\mu = D_\mu = \{0, 4\} \subset C_\mu = Q$ which shows that the converse of Lemma 2.13 is not true either. Neither Q nor its dual satisfy the Moufang identities (A) or (B); hence, both of them are not Moufang magmas. However Q satisfies (D) (but not (C)), whereas its dual satisfies (C) (but not (D)).

Example 3.2

0	1	2	3	4
1	1	2	3	4
2	1	2	4	4
3	1	4	3	4
4	4	4	4	4

Examples 3.1 and 3.2 (and its dual), with Corollary 2.2, show that, unlike in the variety of loops, for magmas in general (specifically, for those with neither the right nor left inverse property), only two of the global versions of the four Moufang identities, i.e., (A) and (B), are equivalent. However, imposing on Q the condition "satisfies the right or left inverse property" renders all four *global* identities equivalent (i.e., a special case of Corollary 2.16). But with the addition of such a condition, the magma becomes a loop if it satisfies any one of the four global identities, as per Corollary 2.16. Regardless, Example 3.3 below shows that it is possible for a non-loop magma to satisfy all four global Moufang identities. However, by Corollary 2.16, we know that such an example of a magma, Q, exists because it has neither the right nor left inverse property. Here $Q = A_\mu = B_\mu = C_\mu = D_\mu$; this is an example of a Moufang magma which is not a Moufang loop. This example also shows that the converse of the first part of Corollary 2.15 is not true, i.e., it is possible that $A_\mu = B_\mu = C_\mu = D_\mu$ for a magma Q, yet Q needs not have the inverse property.

Example 3.3

0	1	2
1	1	1
2	1	1

Example 3.3 forecloses on the possibility of generalizing Corollary 2.16 to "a magma whose every element is an A_μ, B_μ, C_μ and D_μ element is a Moufang loop." Hence, the "right or left inverse property" condition is not trivial but necessary in Corollary 2.16 for the magma to be a Moufang loop. The next example shows that Lemma 2.12 is not trivial (i.e., that the subset containment can be proper). This magma, Q, has the right, but not the left, inverse property. In this magma, $A_\mu = B_\mu = C_\mu = \{0\} \subset D_\mu = \{0, 1\}$.

Example 3.4

0	1	2	3
1	0	3	2
2	2	0	1
3	3	1	0

We now compare Example 3.4 with Lemma 2.7. We have shown that a D_μ element in a magma with the right inverse property must satisfy the first three properties of Lemma 2.7. But a D_μ element in a magma with the right inverse property need not satisfy the final three properties in Lemma 2.7, as Example 3.4 shows. Explicitly, the D_μ element 1 does not satisfy (iv) from Lemma 2.7, since $(1\cdot 2)^{-1} = 3^{-1} = 3 \neq 2 = 2\cdot 1 = 2^{-1}\cdot 1^{-1}$. Nor does it satisfy (v) from Lemma 2.7, since $(2\cdot 1)^{-1} = 2^{-1} = 2 \neq 3 = 1\cdot 2 = 1^{-1}\cdot 2^{-1}$. And finally, the D_μ element 1 does not satisfy (vi) from Lemma 2.7, since $2^{-1}\cdot(2\cdot 1) = 2^{-1}\cdot 2 = 2\cdot 2 = 0 \neq 1$. We note that in Lemma 2.12, we use the properties (iv) and (v) from Lemma 2.7 that are not satisfied by a D_μ element in general to prove that $B_\mu = C_\mu$ in right inverse property magmas.

Note, of course, that the dual of this example is a magma that has the left, but not the right, inverse property and in which $A_\mu = B_\mu = D_\mu = \{0\} \subset C_\mu = \{0, 1\}$. Example 3.4 and its dual show that the condition of the magma satisfying both inverse properties in Corollary 2.15 is necessary and not trivial to ensure the equality $A_\mu = B_\mu = C_\mu = D_\mu$.

In the following example, Q is a magma with the right (but not the left) inverse property, and $A_\mu = B_\mu = C_\mu = D_\mu = \{0, 1\} \neq Q$. Hence Q is not a Moufang loop, and as per Lemma 2.17, each element appears exactly once in the rows with these (and only these) two elements (0 and 1) as leading entries.

Example 3.5

0	1	2	3	4	5
1	0	3	2	5	4
2	3	0	1	2	3
3	2	1	0	3	2
4	5	4	5	0	1
5	4	5	4	1	0

The next example shows that the “right or left inverse property" condition in Corollary 2.16 is not trivial. In this example, Q is a magma without the right or left inverse property, and for which $Q = A_\mu \cup B_\mu \cup C_\mu \cup D_\mu$, i.e., every element of Q has at least one of the properties (A_μ), (B_μ), (C_μ) or (D_μ). Specifically, $A_\mu = B_\mu = D_\mu = \{0, 2, 3\}$ and $C_\mu = \{0, 1\}$. However, Q is not a loop, and it is certainly not Moufang since $A_\mu \cap B_\mu \cap C_\mu \cap D_\mu = \{0\} \neq Q$.

This example also proves that Lemma 2.17 is not trivial since $2 \in D_\mu$ but there is a repetition of elements of Q in the row where 2 is the leading element.

Example 3.6

0	1	2	3
1	0	2	3
2	3	2	2
3	2	2	2

The example below shows that the converse of Lemma 2.14 is not true even for a magma. Though every element of $Q = \{0, 1, 2\}$ appears once and only once in each column of the Cayley table, taking $y = 1, x = 1 \in Q$ in the equation $yx \cdot x^{'} = y$ gives $(1 \cdot 1) \cdot x^{'} = 2 \cdot x^{'} = 1$. However, $2 \cdot z \neq 1, \forall z \in Q$, which shows that none of the elements of Q qualify as $x^{'}$.

Example 3.7

0	1	2
1	2	1
2	0	0

In the following example, which is not a magma, the condition of the existence of some $x^{'} \in S$ for each $x \in S$ such that $yx \cdot x^{'} = y$ for each $y \in S$ is satisfied. But note that $(1 \cdot 1) \cdot 1 = (1 \cdot 1) \cdot 2 = 1$ and $(2 \cdot 1) \cdot 1 = (2 \cdot 1) \cdot 2 = 2$. So, both 1 and 2 satisfy the criteria for $1^{'}$. Likewise, $(1 \cdot 2) \cdot 1 = (1 \cdot 2) \cdot 2 = 1$ and $(2 \cdot 2) \cdot 1 = (2 \cdot 2) \cdot 2 = 2$. So, both 1 and 2 satisfy the criteria for $2^{'}$. Thus, neither $1^{'}$ nor $2^{'}$ is a unique element of S. Yet, in this example, every element appears once and only once in each column of the Cayley table, as per Lemma 2.14. The repetition of elements in a row of the Cayley table is displayed in the proof of Lemma 2.14, where $v = wu^{''} = wu$ with the possibility of $u^{''} \neq u$. Hence the element v may be repeated in the row where w is the leading element, i.e., in both the columns where $u^{''}$ and u are the leading elements. Moreover, in this example, $A_\mu = B_\mu = S$. This shows that the converse about the equality $A_\mu = B_\mu$ for magmas, in Corollary 2.2, is not true, i.e., this equality can be true in a variety that is more general than magmas.

Example 3.8

1	1
2	2

The following example shows that in Corollary 2.2, the condition of Q being a magma is not trivial, i.e., it is possible that $A_\mu \neq B_\mu$ for a general set Q. In this example, Q is not a magma, $A_\mu = \{3, 4\}$ and $B_\mu = \{1, 3, 4\}$. For completeness, we note that $D_\mu = A_\mu \neq B_\mu = C_\mu$, and Q is both right and left alternative.

Example 3.9

3	4	3	3
1	2	3	2
3	3	3	3
1	4	3	4

4 Presto—Conclusion

Since all four Moufang identities are equivalent in the variety of loops [2, 3], to show that a given loop is Moufang, one must show that each triple of elements from the loop satisfies any one of the four Moufang identities. Our harmony reveals an alternate, and in some sense sharper, approach, i.e., it suffices, via Theorem 2.20, to show that each element in a loop satisfies only one, *any* one, of the four local identities (A_μ), (B_μ), (C_μ) and (D_μ) (instead of one of the four “global" Moufang identities) in order for the loop to be Moufang, which in turn, of course, allows us to conclude that every element of the loop satisfies all four of the global identities. In fact, Corollary 2.16 establishes this in the more general setting of magmas with either the right or left inverse property. Hence, Corollary 2.16 and Theorem 2.20 are two alternate, generalized versions of the theorem on the equivalence of the four Moufang identities for loops.

References

1. Белоусов, В.Д.: *ОсновыТеорииКвазигруппиЛуп* Издательство <<Наука>> Москва (1967)
2. Bol, G.: Gewebe und gruppen. Math. Ann. **114**, 414–437 (1937)
3. Bruck, R.H.: Contributions to the theory of loops. Trans. Am. Math. Soc. **60**, 245–354 (1946)
4. Bruck, R.H.: A Survey of Binary Systems. Springer, Cham (1971)
5. Chee, W.L., Rajah, A.: Product rules and classification of Moufang loops of order 81. Int. J. Algebra Comput. **30**(4), 711–730 (2020)
6. Chein, O.: Moufang loops of small order. Mem. Am. Math. Soc. **13197**, 31–51 (1978)
7. Glauberman, G.: On loops of odd order. J. Algebra **1**, 374–396 (1964)
8. Grishkov, A., Zavarnitsine, A.: Lagrange’s theorem for Moufang loops. Math. Proc. Cambridge Philos. Soc. **139**(1), 41–57 (2005)
9. McCune, W.W.: Prover9 3.3 Reference Manual and Guide, Argonne National Laboratory Technical Memorandum ANL/MCS-TM-263 (2003). http://www.mcs.anl.gov/AR/Prover9/
10. McCune, W.W.: Mace 4.0 Reference Manual and Guide, Argonne National Laboratory Technical Memorandum ANL/MCS-TM-264 (2003). http://www.mcs.anl.gov/AR/mace4/
11. Moufang, R.: Zur struktur von alternativkörpern. Math. Ann. **110**, 416–430 (1935)
12. Nagy, G.P., Vojtěchovský, P.: The Moufang loops of order 64 and 81. J. Symb. Comput. **42**, 871–883 (2007)
13. Pflugfelder, H.O.: Quasigroups and Loops, Introduction, Sigma Series 7 in Pure Mathematics. Helderman-Verlag, Berlin (1990)
14. Phillips, J.D.: Moufang and Commutant Elements in Magmas, in Nonassociative Mathematics and its Applications, 213–226, Contemporary Mathematics 721. American Mathematical Society, Providence, RI (2019)
15. Phillips, J.D.: The Moufang laws, global and local. J. Algebra Appl. **8**(4), 477–492 (2009)
16. Rajah, A.: Moufang loops of odd order pq^3. J. Algebra **235**(1), 66–93 (2001)