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Neighborhoods of Harmonic and Stable Harmonic Mappings

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Abstract

Based on the well known notion of the neighborhoods of univalent analytic functions and other related developments, in this article, at first, we obtain an interesting result on neighborhoods of sense-preserving harmonic mappings. Thereafter, we discuss about neighborhoods of stable harmonic univalent mappings and some of its subclasses.

Keywords Neighborhoods of univalent functions · Convolution · Stable harmonic mapping · Fully starlike function · Fully convex function

Mathematics Subject Classification 30C45 · 30C55 · 31A05

1 Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D}$ be the open unit disc in $\mathbb{C}$, i.e. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We denote $\mathcal{A}$ by the class of all functions f analytic in $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$ and, let $\mathcal{S}$ be the class of all univalent (one to one) functions in $\mathcal{A}$. The starlike and convex subclasses of $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. It is well known that $\mathcal{K} \subsetneq \mathcal{S}^*$. We refer [12, Chap. 2] for more details about these classes of functions. In 1981, St. Ruscheweyh (c.f. [15]) introduced the notion of neighborhoods of analytic functions as follows: for $\delta \geq 0$, the δ-neighborhood of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ is defined as

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$$N_\delta(f) = \left\{ F(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} n|a_n - c_n| \le \delta \right\}.$$

In [11], A. W. Goodman proved that, for the identity function $e(z) = z$, $N_1(e) \subset \mathcal{S}^*$. R. Fournier also studied the δ-neighborhood of analytic functions and derived some interesting results on the δ-neighborhood of various subclasses of $\mathcal{A}$ in [8, 9]. Later in 1985, J. E. Brown generalized (compare [4]) some results of Ruscheweyh and Fournier on δ-neighborhood.

In this article, we aim to investigate neighborhoods of harmonic mappings which are motivated by the research work done in the analytic case as mentioned above. These results are important in their own right, because, though we consider similar problems like in the analytic case, but, the techniques that are used to obtain these results are quite general in the harmonic case. We first start with some basic definitions and results on harmonic functions. Let Ω be a domain in $\mathbb{C}$. A complex valued harmonic function $f : \Omega \to \mathbb{C}$ is of the form $f(z) = u(x, y) + iv(x, y)$, $z = x + iy \in \Omega$ where, u and v are real valued harmonic functions on Ω. On each simply connected domain Ω, f can be expressed as $f = h + \overline{g}$, where h and g are analytic functions on Ω and they are known as the analytic and the co-analytic parts of f, respectively. It is well known from a result of H. Lewy that a harmonic function $f = h + \overline{g}$ is locally univalent in Ω if and only if its Jacobian $J_f = |h'|^2 - |g'|^2$ is non-vanishing on Ω. We call f is sense-preserving if $J_f > 0$.

Let $\mathcal{H}$ denote the class of all harmonic functions $f = h + \overline{g}$ in $\mathbb{D}$ with the normalizations $h(0) = 0 = g(0)$, $f_z(0) = 1$, and $\mathcal{H}_0$ be the class of functions $f \in \mathcal{H}$ with $f_{\bar{z}}(0) = 0$. The subclass of $\mathcal{H}$ containing all sense-preserving univalent (one-to-one) harmonic functions is denoted by $\mathcal{S}_\mathcal{H}$, and the corresponding subclass of $\mathcal{H}_0$ is denoted by $\mathcal{S}_\mathcal{H}^0$. We clarify here that, a complex valued one-to-one harmonic function defined on a domain Ω is called *harmonic mapping*. Each function $f \in \mathcal{S}_\mathcal{H}^0$ has the following Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n}, \quad z \in \mathbb{D}. \tag{1.1}$$

For more information about the class $\mathcal{S}_\mathcal{H}^0$, we refer to [6, 7]. Now, in order to describe our results we need to recall some subclasses of harmonic functions.

A harmonic function $f \in \mathcal{H}$ is said to be fully convex (fully starlike) if it maps every circle $|z| = r$ in a one-to-one manner onto a convex (starlike) curve (see [5]). We denote $\mathcal{FS}_\mathcal{H}^{0*}$ and $\mathcal{FK}_\mathcal{H}^0$ by the fully starlike and fully convex subclasses of $\mathcal{S}_\mathcal{H}^0$ respectively. Let f and F be two functions in $\mathcal{H}$ having the following expansions in $\mathbb{D}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \tag{1.2}$$

and

$$F(z) = z + \sum_{n=2}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d_n z^n}. \tag{1.3}$$

The convolution of these two harmonic functions is defined by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n + \sum_{n=1}^{\infty} \overline{b_n d_n z^n}.$$

In [13], Hernandez and Martin introduced the notion of stable harmonic mappings. They defined a (sense-preserving) harmonic function $f = h + \overline{g}$ to be a stable harmonic univalent mapping (respectively, stable harmonic starlike, stable harmonic convex and stable harmonic close-to-convex) if all the functions $f_\lambda = h + \lambda\overline{g}$ with $|\lambda| = 1$ are univalent (respectively, starlike, convex and close-to-convex). In the same article, they proved that, a sense-preserving harmonic function $f = h + \overline{g}$ is stable harmonic univalent (respectively, stable harmonic starlike, stable harmonic convex and stable harmonic close-to-convex) in $\mathbb{D}$ if and only if the analytic functions $F_\lambda = h + \lambda g$ are univalent (respectively, starlike, convex and close-to-convex) in $\mathbb{D}$ for each $|\lambda| = 1$. Motivated by the work in [13], we also study the class of stable harmonic mappings in [3].

We now move on to define formally the neighborhoods of harmonic functions. For $\delta \geq 0$, the δ-neighborhood of a function $f \in \mathcal{H}$ having expansion of the form (1.2) is defined and denoted by

$$N_\delta^H(f) := \left\{ F(z) \in \mathcal{H} : \sum_{n=2}^{\infty} n|a_n - c_n| + \sum_{n=1}^{\infty} n|b_n - d_n| \leq \delta \right\},$$

where F is of the form (1.3). In particular, if the function $f \in \mathcal{H}_0$ has expansion of the form (1.1), then the δ-neighborhood of f will be defined by

$$N_\delta^H(f) = \left\{ F(z) = z + \sum_{n=2}^{\infty} c_n z^n + \sum_{n=2}^{\infty} \overline{d_n z^n} \in \mathcal{H}_0 : \sum_{n=2}^{\infty} n(|a_n - c_n| + |b_n - d_n|) \leq \delta \right\}.$$

We refer to the article [17] by Yasar and Yalcin who also worked on the δ-neighborhood of harmonic functions with varying arguments. It follows directly from a result of Bharanedhar and Ponnusamy (see [2, Lemma 2]) that, for $\delta = 1$, $N_1^H(e) \subset \mathcal{FS}_{\mathcal{H}}^{0*}$, where, $e(z) = z$ is the identity function. This result can be thought as a generalization of the result of Goodman ([11]) mentioned in the first paragraph of this section. In [15], Ruscheweyh has shown that $N_\delta(f) \subset \mathcal{S}$ at most for $\delta = \inf_{z \in \mathbb{D}} |f'(z)|$ where $f \in \mathcal{A}$. As a generalization of this result for harmonic functions, we prove that if $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving, then $N_\delta^H(f) \subset \mathcal{S}_{\mathcal{H}}$ at most for $\delta = \inf_{z \in \mathbb{D}}\{|h'(z)| + |g'(z)|\}$, which is the content of the Theorem 1 in the next section. It was proved in [15] that, if $f \in \mathcal{K}$, then $N_{1/4}(f) \subset \mathcal{S}^*$. After the initiation of the work of Ruscheweyh, there has been a considerable interest on the topic of neighborhoods of univalent analytic functions (cf. [14]). Next, in Theorem 2, we prove that, if $f \in \mathcal{S}_{\mathcal{H}}^0$ is a stable harmonic convex mapping, then $N_{1/4}^H(f)$ is contained in the stable harmonic starlike subclass of $\mathcal{S}_{\mathcal{H}}^0$. In [15, Theorem 1], Ruscheweyh proved the following result

Theorem A *If $f \in \mathcal{A}$ and $(f(z)+\varepsilon z)/(1+\varepsilon) \in M$ for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$ $(\delta > 0)$, where M is any of the classes $\mathcal{S}$ or $\mathcal{S}^*$, then $N_\delta(f) \subset M$.*

Motivated by the above theorem, we have shown that, for $f \in \mathcal{H}_0$ if $f(z) + \varepsilon z$ is stable harmonic univalent for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$, then $N_\delta^H(f)$ is contained in the subclass of $\mathcal{S}_{\mathcal{H}}^0$ that contains all stable harmonic univalent mappings. This is the content of Theorem 3 in the upcoming section. In our penultimate result, i.e. in Theorem 4, we prove, as an application of Theorem 3, that if $f \in \mathcal{H}_0$ and $(f(z)+\varepsilon z)/(1+\varepsilon) \in \mathcal{FS}_{\mathcal{H}}^{0*}$ such that $f(z) + \varepsilon z$ is stable harmonic univalent, for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$, then $N_\delta^H(f) \subset \mathcal{FS}_{\mathcal{H}}^{0*}$. Sheil-Small and Silvia (see for instance [16]) introduced more general notion of neighborhood of an analytic function known as the T_δ-neighborhood. For $\delta \geq 0$ and $T = \{T_n\}_{n=1}^\infty$, a sequence of nonnegative reals, the T_δ-neighborhood of $f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{A}$ is defined by

$$TN_\delta(f) = \left\{ F(z) = z + \sum_{n=2}^\infty c_n z^n \in \mathcal{A} : \sum_{n=2}^\infty T_n |a_n - c_n| \leq \delta \right\}.$$

In particular, for $T = \{n\}_{n=1}^\infty$, the notion of T_δ-neighborhood coincides with the well known δ-neighborhood introduced by Ruscheweyh. Analogously, for $\delta \geq 0$ and $T = \{T_n\}_{n=1}^\infty$, the T_δ-neighborhood of a function $f \in \mathcal{H}_0$ with expansion (1.1) is defined as follows

$$TN_\delta^H(f) = \left\{ F \in \mathcal{H}_0 : \sum_{n=2}^\infty T_n |a_n - c_n| + \sum_{n=2}^\infty T_n |b_n - d_n| \leq \delta \right\},$$

where $F(z) = z + \sum_{n=2}^\infty c_n z^n + \sum_{n=2}^\infty \overline{d_n z^n}$. Theorem A cannot be extended for functions in the class $\mathcal{K}$. However, it can be easily checked that, if $f \in \mathcal{A}$ and $(f(z) + \varepsilon z)/(1 + \varepsilon) \in \mathcal{K}$ for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$, then $TN_\delta(f) \subset \mathcal{K}$ for $T = \{n^2\}_{n=1}^\infty$. Analogously, as an application of Theorem 3, we prove that, if $f \in \mathcal{H}_0$ and $(f(z) + \varepsilon z)/(1+\varepsilon) \in \mathcal{FK}_{\mathcal{H}}^0$ such that $f(z)+\varepsilon z$ is stable harmonic univalent, with $|\varepsilon| < \delta$ and $\delta > 0$, then $TN_\delta^H(f) \subset \mathcal{FK}_{\mathcal{H}}^0$ for $T = \{n^2\}_{n=1}^\infty$. This is the content of Theorem 5.

2 Main Results

We start with our first result.

Theorem 1 *If $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving, then $N_\delta^H(f) \subset \mathcal{S}_{\mathcal{H}}$ at most for $\delta = \inf_{z\in\mathbb{D}}\{|h'(z)| + |g'(z)|\}$.*

Proof We claim that, for any $\delta' > \delta$, we can choose $\eta \in \mathbb{D}$ such that

$$\frac{|h'(\eta)| + |g'(\eta)|}{|\eta|} < \delta'.$$

Since f is sense-preserving, $h'(z) \neq 0$. If $g'(z) = 0$ for all $z \in \mathbb{D}$, then by the normalization of f, we have $g(z) = 0$. In that case $f \in \mathcal{A}$ and our claim follows from the proof of [15, Theorem 1]. Now suppose $g'(z) \neq 0$. Since f is sense-preserving, the analytic function $\psi(z) = h'(z) + cg'(z) \neq 0$ in $\mathbb{D}$ for all $c \in \mathbb{D}$. Let $\sigma = \inf_{z\in\mathbb{D}}\{|\psi(z)|\}$. Therefore $\sigma \leq \delta$. Consequently, it follows from the minimum modulus principle that there exists a sequence $\{z_n\}_{n\geq 1}$ with $|z_n| \to 1$ as $n \to \infty$ such that

$$\left|\frac{\psi(z_n)}{z_n}\right| \to \sigma \leq \delta < \delta', \text{ as } n \to \infty.$$

Hence, there exists some $z_k \in \{z_n\}_{n\geq 1}$ such that $|\psi(z_k)/z_k| < \delta'$ for every choice of complex constant $c \in \mathbb{D}$. This implies $(|h'(z_k)| + |g'(z_k)|)/|z_k| < \delta'$. Hence, our claim is established (taking $z_k = \eta$). Now let us consider the function

$$F(z) = f(z) - \frac{h'(\eta)}{2\eta}z^2 - \overline{\frac{g'(\eta)}{2\eta}z^2}, \ z \in \mathbb{D},$$

where η is chosen as before. Then clearly $F \in N^H_{\delta'}(f)$, but, the Jacobian $J_F(\eta) = 0$. Therefore $F \notin \mathcal{S}_{\mathcal{H}}$. Hence, proof of the theorem is complete.

The next theorem is a generalization of a well known result of St. Ruscheweyh ([15, Corollary 1]) for $n = 1$ to the stable harmonic convex mappings.

Theorem 2 *Let $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}$ be a stable harmonic convex mapping. Then $N^H_{1/4}(f)$ is contained in the stable harmonic starlike subclass of $\mathcal{S}^0_{\mathcal{H}}$. The number $1/4$ is best possible.*

Proof Since $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}$ is stable harmonic convex, then by [13, Corollary 3.2], the mappings $F_\lambda = h + \lambda g$ are convex for all $\lambda \in \overline{\mathbb{D}}$, where, $\overline{\mathbb{D}}$ denotes the closed unit disc. Let f have an expansion of the form (1.1) in $\mathbb{D}$. We first show that if

$$\rho(z) = \varphi(z) + \overline{\psi(z)} = z + \sum_{n=2}^{\infty} c_n z^n + \sum_{n=2}^{\infty} \overline{d_n z^n} \in N^H_\delta(f),$$

for $\delta \geq 0$, then $P_\lambda = \varphi + \lambda\psi \in N_\delta(F_\lambda)$. Now if $\rho \in N^H_\delta(f)$, then by the definition we have

$$\sum_{n=2}^{\infty} n(|a_n - c_n| + |b_n - d_n|) \leq \delta.$$

Now

$$\begin{aligned}\sum_{n=2}^{\infty} n|(a_n+\lambda b_n)-(c_n+\lambda d_n)| &= \sum_{n=2}^{\infty} n|(a_n-c_n)+\lambda(b_n-d_n)|\\ &\le \sum_{n=2}^{\infty} n(|a_n-c_n|+|b_n-d_n|)\\ &\le \delta.\end{aligned}$$

Therefore $P_\lambda=\varphi+\lambda\psi\in N_\delta(F_\lambda)$. Since $F_\lambda\in\mathcal{K}$, it follows from [15, Corollary 1], that $N_{1/4}(F_\lambda)\subset\mathcal{S}^*$. So in particular for $\delta=1/4$, we can say that if $\rho=\varphi+\overline{\psi}\in N_{1/4}^H(f)$, then $P_\lambda=\varphi+\lambda\psi\in\mathcal{S}^*$ for all $\lambda\in\overline{\mathbb{D}}$. Evidently P_λ is locally univalent, which implies $\varphi'(z)+\lambda\psi'(z)\neq 0$ for all $z\in\mathbb{D}$ and for all $\lambda\in\overline{\mathbb{D}}$. This implies $|\varphi'(z)|\neq|\lambda\psi'(z)|$ for all $z\in\mathbb{D}$ and for all $\lambda\in\overline{\mathbb{D}}$ and hence $|\psi'(z)|<|\varphi'(z)|$ for all $z\in\mathbb{D}$. Because if $|\psi'(z_0)|\ge|\varphi'(z_0)|$ for some $z_0\in\mathbb{D}$, then it will imply $|\varphi'(z_0)|=|\gamma\psi'(z_0)|$ for some $\gamma\in\overline{\mathbb{D}}$, which is a contradiction. Therefore $|\psi'(z)|<|\varphi'(z)|$ for all $z\in\mathbb{D}$ and hence $\rho=\varphi+\overline{\psi}$ is sense-preserving in $\mathbb{D}$. Therefore, by [13, Theorem 4.2], $\rho=\varphi+\overline{\psi}$ is a stable harmonic starlike mapping. Hence, the result follows.

Finally, we prove that the value $\delta=1/4$ is best possible. To see this, suppose $N_\delta^H(f)$ is contained in the stable harmonic starlike subclass of $\mathcal{S}_{\mathcal{H}}^0$ for some $\delta>1/4$. If $P_\lambda=\varphi+\lambda\psi\in N_\delta(F_\lambda)$ for all $\lambda\in\overline{\mathbb{D}}$, then

$$\sum_{n=2}^{\infty} n|(a_n+\lambda b_n)-(c_n+\lambda d_n)| = \sum_{n=2}^{\infty} n|(a_n-c_n)+\lambda(b_n-d_n)|\le\delta.$$

Thus, it follows that for some suitable λ with $|\lambda|=1$,

$$\sum_{n=2}^{\infty} n(|a_n-c_n|+|b_n-d_n|) = \sum_{n=2}^{\infty} n|(a_n-c_n)+\lambda(b_n-d_n)|\le\delta.$$

This implies $\rho(z)=\varphi(z)+\overline{\psi(z)}\in N_\delta^H(f)$. Therefore, ρ belongs to the stable harmonic starlike subclass of $\mathcal{S}_{\mathcal{H}}^0$. We thus get $P_\lambda=\varphi+\lambda\psi\in\mathcal{S}^*$ for all $\lambda\in\overline{\mathbb{D}}$ (see, [13, Corollary 4.3]). Hence $N_\delta(F_\lambda)\subset\mathcal{S}^*$ for some $\delta>1/4$, which contradicts sharpness of the [15, Corollary 1] for $n=1$.

Our next theorem is an extension of Theorem A to stable harmonic univalent mappings.

Theorem 3 *Let* $f=h+\overline{g}\in\mathcal{H}_0$ *and* $\delta>0$. *Assume that for all* $\varepsilon\in\mathbb{C}$ *such that* $|\varepsilon|<\delta$, $f(z)+\varepsilon z$ *is stable harmonic univalent. Then* $N_\delta^H(f)$ *is contained in the stable harmonic univalent subclass of* $\mathcal{S}_{\mathcal{H}}^0$.

Proof Since $f(z)+\varepsilon z$ is stable harmonic univalent, therefore by [13, Proposition 2.1], the functions

$$\frac{F_\lambda(z)+\varepsilon z}{1+\varepsilon} \in \mathcal{S} \text{ for all } \lambda \in \partial\mathbb{D},$$

where $F_\lambda = h+\lambda g$ and $\partial\mathbb{D}$ denotes the unit circle. Therefore by Theorem A, we have $N_\delta(F_\lambda) \subset \mathcal{S}$. Now let us assume that $\rho = \varphi + \overline{\psi} \in N_\delta^H(f)$. Then by a similar argument as in the proof of Theorem 2, we have $P_\lambda = \varphi + \lambda\psi \in N_\delta(F_\lambda)$. This implies $P_\lambda = \varphi+\lambda\psi \in \mathcal{S}$ for all $\lambda \in \partial\mathbb{D}$. Again following a similar argument as in Theorem 2, we can say that $\rho = \varphi + \overline{\psi}$ is sense-preserving in $\mathbb{D}$. Hence by [13, Proposition 2.1], $\rho = \varphi + \overline{\psi}$ is stable harmonic univalent. Hence the proof of the theorem is complete.

As an application of the Theorem 3, we prove our next result.

Theorem 4 *Let $f \in \mathcal{H}_0$ and $\delta > 0$ such that for all $|\varepsilon| < \delta$, the function $(f(z)+\varepsilon z)/(1+\varepsilon) \in \mathcal{FS}_{\mathcal{H}}^{0*}$, $z \in \mathbb{D}$. If $f(z)+\varepsilon z$ is stable harmonic univalent then $N_\delta^H(f) \subset \mathcal{FS}_{\mathcal{H}}^{0*}$.*

Proof Define for $z \in \mathbb{D}$,

$$Q' = \left\{ \frac{z + \frac{(\zeta-1)}{2}z^2}{(1-z)^2} - \frac{\zeta\bar{z} - \frac{(\zeta-1)}{2}\bar{z}^2}{(1-\bar{z})^2} \; : \; |\zeta| = 1 \right\}. \tag{2.1}$$

Then for each function $f \in \mathcal{S}_{\mathcal{H}}^0$, $f \in \mathcal{FS}_{\mathcal{H}}^{0*}$ if and only if $(f*\varphi)(z) \neq 0, 0 < |z| < 1$, for all $\varphi \in Q'$ (see [2, Lemma 1]). Now if

$$\varphi(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n + \sum_{n=1}^{\infty} \overline{\beta_n z^n} \in Q',$$

then it follows from (2.1) that

$$\alpha_n = \frac{n+1}{2} + \frac{(n-1)\zeta}{2}, \; \beta_n = -\left(\frac{n-1}{2} + \frac{(n+1)\bar{\zeta}}{2}\right), \; |\zeta| = 1.$$

Thus, $|\alpha_n| \leq n$ and $|\beta_n| \leq n$, for all $n = 1, 2, 3, \ldots$. Now, let $f \in \mathcal{H}_0$ having expansion of the form (1.1) be such that for all $\varepsilon \in \mathbb{C}$, with $|\varepsilon| < \delta$, $(f(z)+\varepsilon z)/(1+\varepsilon) \in \mathcal{FS}_{\mathcal{H}}^{0*}$, $z \in \mathbb{D}$. Then we have for $\varphi \in Q'$,

$$\frac{(f*\varphi)(z)+\varepsilon z}{1+\varepsilon} \neq 0, \; 0 < |z| < 1, \; |\varepsilon| < \delta,$$

which implies $(f*\varphi)/z \neq -\varepsilon$, $z \in \mathbb{D}$. Therefore it follows that $|(f*\varphi)/z| \geq \delta$, for $z \in \mathbb{D}$. Assume $F \in N_\delta^H(f)$, where F has the following expansion

$$F(z) = z + \sum_{n=2}^{\infty} c_n z^n + \sum_{n=2}^{\infty} \overline{d_n z^n}, \, z \in \mathbb{D}. \tag{2.2}$$

Since $f(z)+\varepsilon z$ is stable harmonic univalent, it is clear from Theorem 3 that $F \in \mathcal{S}_{\mathcal{H}}^0$. Therefore, to prove $F \in \mathcal{FS}_{\mathcal{H}}^{0*}$, it is sufficient to show that $F * \varphi \neq 0$ for all $\varphi \in Q'$ with $0 < |z| < 1$. Now for $\varphi \in Q'$, we have

$$\left|\frac{F*\varphi}{z}\right| = \left|\frac{f*\varphi}{z} + \frac{(F-f)*\varphi}{z}\right| \geq \delta - \left|\frac{(F-f)*\varphi}{z}\right|.$$

But, for $z \in \mathbb{D}$,

$$\begin{aligned}\left|\frac{(F-f)*\varphi}{z}\right| &= \left|\sum_{n=2}^{\infty}(c_n-a_n)\alpha_n z^{n-1} + \sum_{n=2}^{\infty}\frac{1}{z}\overline{(d_n-b_n)\beta_n z^n}\right|\\ &\leq \sum_{n=2}^{\infty}|c_n-a_n||\alpha_n||z^{n-1}| + \sum_{n=2}^{\infty}|d_n-b_n||\beta_n||z^{n-1}|\\ &< \sum_{n=2}^{\infty}n|c_n-a_n| + \sum_{n=2}^{\infty}n|d_n-b_n|\\ &\leq \delta.\end{aligned}$$

Therefore $|(F*\varphi)/z| > 0$, $z \in \mathbb{D}$, which implies $(F*\varphi)(z) \neq 0$ for $0 < |z| < 1$ for all $\varphi \in Q'$. Hence $F \in \mathcal{FS}_{\mathcal{H}}^{0*}$.

Remark 1. In particular, if we take $f(z) = e(z) = z$ and $\delta = 1$, then it follows from Theorem 4 that $N_1^H(e) \subset \mathcal{FS}_{\mathcal{H}}^{0*}$.

2. In a recent article ([10]), Fournier has shown that, in particular, if we take $M = \mathcal{S}^*$ in Theorem A, then this result is a consequence of the Kobori-Noshiro result, which states that if a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the condition $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then f is univalent and starlike in $\mathbb{D}$. For harmonic functions, it is known from [2, Lemma 2] that, if $f \in \mathcal{H}_0$ with expansion of the form (1.1) satisfies the condition

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=2}^{\infty} n|b_n| \leq 1,$$

then $f \in \mathcal{FS}_{\mathcal{H}}^{0*}$. We shall see that Theorem 4 can be considered as a consequence of [2, Lemma 2]. Now let us assume the function $f \in \mathcal{H}_0$ with expansion of the form (1.1) satisfies the hypothesis of Theorem 4 and let $F \in N_\delta^H(f)$ having expansion of the form (2.2). Therefore $F \in \mathcal{S}_{\mathcal{H}}^0$ and

$$\frac{1}{\delta}\left(\sum_{n=2}^{\infty} n|c_n-a_n| + \sum_{n=2}^{\infty} n|d_n-b_n|\right) \leq 1,$$

then it follows from [2, Lemma 2] that for each θ,

$$\mu(z) = z + \frac{F(z) - f(z)}{\delta e^{i\theta}} \in \mathcal{FS}_{\mathcal{H}}^{0*}.$$

Hence $\mu(z) * \varphi(z) \neq 0$, $0 < |z| < 1$, for all $\varphi \in Q'$, which gives

$$0 \neq \frac{\mu(z) * \varphi(z)}{z} = 1 + \frac{F(z) * \varphi(z) - f(z) * \varphi(z)}{\delta e^{i\theta} z}, \ z \in \mathbb{D}.$$

Therefore, we have for $z \in \mathbb{D}$,

$$\left| \frac{F(z) * \varphi(z)}{z} - \frac{f(z) * \varphi(z)}{z} \right| < \delta.$$

Again, from hypothesis of Theorem 4, it follows that $|(f * \varphi)/z| \geq \delta$, $z \in \mathbb{D}$. Therefore $(F * \varphi)/z \neq 0$, for all $\varphi \in Q'$, which implies $F \in \mathcal{FS}_{\mathcal{H}}^{0*}$. Hence $N_\delta^H(f) \subset \mathcal{FS}_{\mathcal{H}}^{0*}$. Therefore from the above discussion we conclude that Theorem 4 is a consequence of [2, Lemma 2].

To prove our next result, we need [1, Theorem 2.8]. It is known that a function $f = h + \overline{g}$ is convex in $|z| < r$ for each $r < 1$ if and only if

$$\frac{\partial}{\partial \theta}\left[\arg\left(\frac{\partial}{\partial \theta} f(re^{i\theta})\right)\right] = \operatorname{Re} \frac{z(zh'(z))' + \overline{z(zg'(z))'}}{zh'(z) - \overline{zg'(z)}} > 0,$$

for $0 < |z| < 1$. In the proof of [1, Theorem 2.8], one can easily see that the limit

$$\lim_{z \to 0} \frac{z(zh'(z))' + \overline{z(zg'(z))'}}{zh'(z) - \overline{zg'(z)}}$$

does not exist if $g'(0) \neq 0$. The above limit exists and is equal to 1 only when $g'(0) = 0$. Therefore, in the statement of [1, Theorem 2.8], f must belong to the class $\mathcal{S}_{\mathcal{H}}^0$ instead of $\mathcal{S}_{\mathcal{H}}$. With this slight modification, we recall [1, Theorem 2.8] in the following form:

Lemma 1 *Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$. Then $f \in \mathcal{FK}_{\mathcal{H}}^0$ if and only if*

$$h(z) * \frac{z + \zeta z^2}{(1 - z)^3} + \overline{g(z)} * \frac{\zeta \bar{z} + \bar{z}^2}{(1 - \bar{z})^3} \neq 0, \ |\zeta| = 1, \ 0 < |z| < 1.$$

We now state and prove our final result.

Theorem 5 *Let $f \in \mathcal{H}_0$ and $\delta > 0$ such that for all $|\varepsilon| < \delta$, the function $(f(z) + \varepsilon z)/(1+\varepsilon) \in \mathcal{FK}_{\mathcal{H}}^0$, $z \in \mathbb{D}$. If $f(z) + \varepsilon z$ is stable harmonic univalent then $TN_\delta^H(f) \subset \mathcal{FK}_{\mathcal{H}}^0$ for $T = \{n^2\}_{n=1}^{\infty}$.*

Proof For $z \in \mathbb{D}$, define

$$Q'_c = \left\{ \frac{z + \zeta z^2}{(1-z)^3} + \frac{\zeta \bar{z} + \bar{z}^2}{(1-\bar{z})^3} : \ |\zeta| = 1 \right\}. \tag{2.3}$$

Then it follows from the Lemma 1 that, for $f \in \mathcal{S}^0_{\mathcal{H}}$, $f \in \mathcal{FK}^0_{\mathcal{H}}$ if and only if $(f * \varphi)(z) \neq 0$, $0 < |z| < 1$, for all $\varphi \in Q'_c$. Again, we see that if

$$\varphi(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n + \sum_{n=1}^{\infty} \overline{\beta_n z^n} \in Q'_c,$$

then, from (2.3) it follows that

$$\alpha_n = \frac{n(n+1)}{2} + \frac{n(n-1)}{2}\zeta, \ \beta_n = \frac{n(n-1)}{2} + \frac{n(n+1)}{2}\bar{\zeta}, \ |\zeta| = 1.$$

This implies $|\alpha_n| \leq n^2$ and $|\beta_n| \leq n^2$, for all $n = 1, 2, 3, \ldots$. Let $f \in \mathcal{H}_0$ having expansion of the form (1.1) be such that, for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$, $(f(z) + \varepsilon z)/(1 + \varepsilon) \in \mathcal{FK}^0_{\mathcal{H}}$. Then for $\varphi \in Q'_c$, we have

$$\frac{(f * \varphi)(z) + \varepsilon z}{1 + \varepsilon} \neq 0, \ 0 < |z| < 1, \ |\varepsilon| < \delta,$$

which implies $(f * \varphi)/z \neq -\varepsilon, z \in \mathbb{D}$. Therefore it follows that $|(f * \varphi)/z| \geq \delta, z \in \mathbb{D}$. Let $F \in TN^H_\delta(f)$ having expansion of the form (2.2), where $T = \{n^2\}_{n=1}^{\infty}$. Since $f(z) + \varepsilon z$ is stable harmonic univalent, it follows from the Theorem 3 that $F \in \mathcal{S}^0_{\mathcal{H}}$. Therefore to prove the result it is sufficient to show that $F * \varphi \neq 0$ for all $\varphi \in Q'_c$ with $0 < |z| < 1$. Now for $\varphi \in Q'_c$, we have

$$\begin{aligned}
\left| \frac{(F - f) * \varphi}{z} \right| &= \left| \sum_{n=2}^{\infty} (c_n - a_n)\alpha_n z^{n-1} + \sum_{n=2}^{\infty} \frac{1}{z} \overline{(d_n - b_n)\beta_n z^n} \right| \\
&\leq \sum_{n=2}^{\infty} |c_n - a_n||\alpha_n||z^{n-1}| + \sum_{n=2}^{\infty} |d_n - b_n||\beta_n||z^{n-1}| \\
&< \sum_{n=2}^{\infty} n^2 |c_n - a_n| + \sum_{n=2}^{\infty} n^2 |d_n - b_n| \\
&\leq \delta.
\end{aligned}$$

Therefore

$$\left| \frac{F * \varphi}{z} \right| = \left| \frac{f * \varphi}{z} + \frac{(F - f) * \varphi}{z} \right| \geq \delta - \left| \frac{(F - f) * \varphi}{z} \right| > \delta - \delta = 0.$$

This implies $F * \varphi \neq 0$ for all $\varphi \in Q'_c$ with $0 < |z| < 1$. Hence $F \in \mathcal{FK}^0_{\mathcal{H}}$ and the proof is complete.

Remark In [13, Corollary 4.1], Hernandez and Martin proved that, if a sense-preserving harmonic mapping is stable harmonic convex, then it is fully convex. Thus, in the hypothesis of Theorem 5, if we consider $f \in \mathcal{H}_0$ such that, for all $|\varepsilon| < \delta$, $(\delta > 0)$ the functions $f(z) + \varepsilon z$ are sense-preserving and stable harmonic convex, then the conclusion of Theorem 5 follows as well.

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