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On *k*-Wise *L*-Intersecting Families for Simplicial Complexes

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Abstract

A family Δ of subsets of $\{1, 2, \ldots, n\}$ is a simplicial complex if all subsets of F are in Δ for any $F \in \Delta$, and the element of Δ is called the face of Δ. Let $V(\Delta) = \bigcup_{F \in \Delta} F$. A simplicial complex Δ is a near-cone with respect to an apex vertex $v \in V(\Delta)$ if for every face $F \in \Delta$, the set $(F \backslash \{w\}) \cup \{v\}$ is also a face of Δ for every $w \in F$. Denote by $f_i(\Delta) = |\{A \in \Delta : |A| = i+1\}|$ and $h_i(\Delta) = |\{A \in \Delta : |A| = i+1, n \notin A\}|$ for every i, and let $\text{link}_\Delta(v) = \{E : E \cup \{v\} \in \Delta, v \notin E\}$ for every $v \in V(\Delta)$. Assume that p is a prime and $k \geqslant 2$ is an integer. In this paper, some extremal problems on k-wise L-intersecting families for simplicial complexes are considered. (i) Let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of s nonnegative integers. If $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of the simplicial complex Δ such that $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \in L$ for any collection of k distinct sets from $\mathscr{F}$, then $m \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\Delta)$. In addition, if the size of every member of $\mathscr{F}$ belongs to the set $K := \{k_1, k_2, \ldots, k_r\}$ with $\min K > s - r$, then $m \leqslant (k-1) \sum_{i=s-r}^{s-1} f_i(\Delta)$. (ii) Let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$ such that $\min K > s - 2r + 1$. Assume that Δ is a simplicial complex with $n \in V(\Delta)$ and $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of Δ such that $|F_j| \pmod p \in K$ for every j and $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod p \in L$ for any collection of k distinct sets from $\mathscr{F}$. Then $m \leqslant (k-1) \sum_{i=s-2r}^{s-1} h_i(\Delta)$. (iii) Let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$. Assume that Δ is a near-cone

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with apex vertex v and $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of Δ such that $|F_j| \pmod{p} \notin L$ for every j and $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod{p} \in L$ for any collection of k distinct sets from $\mathscr{F}$. Then $m \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\mathrm{link}_\Delta(v))$.

Keywords Simplicial complex · Erdős–Ko–Rado theorem · k-wise L-intersecting families · Multilinear polynomials

Mathematics Subject Classification 05D05 · 05C65

1 Introduction

We begin with introducing some background information that will lead to our main results.

1.1 Background

Throughout our paper, let $[n] = \{1, 2, \ldots, n\}$ and let $2^{[n]}$ be the family of all subsets of $[n]$. A family $\mathscr{F}$ of subsets of $[n]$ is called *intersecting* if every pair of distinct subsets $E, F \in \mathscr{F}$ have a nonempty intersection. Let $L = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. A family $\mathscr{F}$ of subsets of $[n]$ is called *k-wise L-intersecting* if $|F_1 \cap F_2 \cap \cdots \cap F_k| \in L$ for any collection of k ($\geqslant 2$) distinct subsets from $\mathscr{F}$. When $k = 2$, such a family $\mathscr{F}$ is called *L-intersecting*. A family $\mathscr{F}$ is *k-uniform* if it is a collection of k-element subsets of $[n]$. Thus, a k-uniform intersecting family is L-intersecting for $L = \{1, 2, \ldots, k-1\}$. Two families $\mathscr{A}$ and $\mathscr{B}$ of $[n]$ are called *cross L-intersecting* if $|A \cap B| \in L$ for each member A from $\mathscr{A}$ and B from $\mathscr{B}$.

A family Δ of subsets of $[n]$ is said to be a *simplicial complex* (a *hereditary family*, a *downset* or an *ideal*) if all subsets of any set in Δ are in Δ. Let $V(\Delta) = \bigcup_{F \in \Delta} F$. Then $V(\Delta) \subseteq [n]$. The elements of Δ are called the faces of Δ. For $S \in \Delta$, the dimension of S is $|S|-1$. The *dimension* of Δ is defined as $\dim(\Delta) = \max\{|A|-1 : A \in \Delta\}$. For a simplicial complex Δ with dimension $d-1$, let $f_{i-1}(\Delta) = |\{A \in \Delta : |A| = i\}|$ for $i = 0, 1, \ldots, d$. Clearly, $f_{-1} = 1$. The 0-dimensional faces are called the vertices of Δ, and $F \in \Delta$ is called a *facet* of Δ if there does not exist $F' \in \Delta$ such that $F \subset F'$. For a vertex v of Δ, denote the *link* of v in Δ to be

$$\mathrm{link}_\Delta(v) = \{E : E \cup \{v\} \in \Delta, \quad v \notin E\},$$

i.e., it is the star at v, with v itself removed from each set thereof. Obviously, $\mathrm{link}_\Delta(v)$ is also a simplicial complex. A simplicial complex Δ is called a *near-cone* with respect to an *apex vertex* v if for every face $F \in \Delta$, the set $(F \backslash \{w\}) \cup \{v\}$ is also a face of Δ for each vertex $w \in F$.

Problems and results concerning the maximum cardinality of set systems with certain restrictions on the intersections of its members are at the heart of extremal set theory. In 1961, Erdős et al. [11] obtained a classical result, which is one of the most celebrated theorems in extremal set theory.

Theorem 1.1 (Erdős–Ko–Rado theorem, [11]) *Let $n \geqslant 2k$ and let $\mathscr{F}$ be a k-uniform intersecting family of subsets of $[n]$. Then*

$$|\mathscr{F}| \leqslant \binom{n-1}{k-1}$$

with equality if and only if $\mathscr{F}$ consists of all the k-subsets containing a common element provided $n > 2k$.

Since then, a vast amount of beautiful results concerning intersecting families have appeared; see [1, 6, 10, 13, 15, 18, 28, 33, 39]. In particular, notice that a k-uniform intersecting family is also L-intersecting for $L = \{1, 2, \ldots, k-1\}$. In 1975, Ray-Chaudhuri and Wilson [32] derived an upper bound for k-uniform L-intersecting family, where L is an arbitrary set of s nonnegative integers. This is the well-known Ray-Chaudhuri–Wilson theorem. In 1981, Frankl and Wilson [14] used a method of higher incidence matrices to obtain a remarkable theorem, which extends the Ray-Chaudhuri–Wilson theorem by allowing different subset sizes.

Theorem 1.2 ([14]) *Let L be a set of s nonnegative integers. If $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is an L-intersecting family of subsets of $[n]$, then*

$$m \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

In terms of the parameters n and s, this inequality is best possible, as shown by the set of all subsets of size at most s of $[n]$ with $L = \{0, 1, \ldots, s-1\}$. In 1991, Alon et al. [2] used a very elegant algebraic method to derive the following result which is a common generalization to the Ray-Chaudhuri–Wilson theorem and Theorem 1.2.

Theorem 1.3 ([2]) *Let L be a set of s nonnegative integers and let K be a set of r positive integers satisfying* $\min K > s - r$. *If $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is an L-intersecting family of subsets of $[n]$ such that $|F_i| \in K$ for every $1 \leqslant i \leqslant m$, then*

$$m \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

This upper bound is best possible as shown by the set of all subsets of $[n]$ of sizes at least $s - r + 1$ and at most s. In 2002, Grolmusz and Sudakov [17] obtained the following result which extends Theorems 1.2 and 1.3 to the k-wise L-intersecting families for $k \geqslant 2$.

Theorem 1.4 ([17]) *Let $k \geqslant 2$ and let L be a set of s nonnegative integers. If $\mathscr{F}$ is a k-wise L-intersecting family of subsets of $[n]$, then*

$$m \leqslant (k-1)\left[\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}\right]. \tag{1.1}$$

If in addition the size of every member of $\mathscr{F}$ *belongs to the set* $\{k_1, k_2, \ldots, k_r\}$ *and* $k_i > s - r$ *for every* i, *then*

$$m \leqslant (k-1)\left[\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}\right]. \tag{1.2}$$

In the same paper [17], Grolmusz and Sudakov also showed that the same bounds in (1.1) and (1.2) remain true if L is a set of residues modulo a prime p, and the cardinality of k-wise intersections of members of $\mathscr{F}$ modulo p is in L, but the size of every member of $\mathscr{F}$ modulo p is not in L.

Note that the set L in the above theorems may contain 0. When L does not contain 0, Snevily [36] proved the next result, which implies Theorem 1.2.

Theorem 1.5 ([36]) *Let* L *be a set of* s *positive integers. If* $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ *is an* L*-intersecting family of subsets of* $[n]$, *then*

$$m \leqslant \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

The upper bound in Theorem 1.5 is also best possible as shown by the set of all subsets of $[n]$ which contain a common element and have at most $s + 1$ elements. Snevily [35] also proved that the above bound remains true if L is a set of residues modulo a prime p, and the cardinality of pairwise intersections of members of $\mathscr{F}$ modulo p is in L, but the size of every member of $\mathscr{F}$ modulo p is not in L. For more advances on L-interesting family, please refer to [16, 20, 24, 25, 31] and the references therein. In studying k-wise L-intersecting families, it often involves cross L-intersecting families, see for example [17]. For more advances on cross L-intersecting families, one may be referred to [21–23, 26, 37].

Note that the family of all independent sets of a graph forms a simplicial complex. As another generalization of Erdős–Ko–Rado theorem, Holroyd and Talbot [19] first defined the Erdős–Ko–Rado property of graphs in terms of the independent sets of graphs. Fakhari [12] studied the Erdős–Ko–Rado type theorems for simplicial complexes associated to the independent sets of graphs. Borg [3] proved a conjecture of Holroyd and Talbot by giving multi-level solution of simplicial complexes. Olarte et al. [30] showed that the family of facets of a pure simplicial complex of dimension up to three satisfies the Erdős–Ko–Rado property whenever it is flag and has no boundary ridges. Woodroofe [40] generalized Erdős–Ko–Rado property to near-cone by using algebraic shift method. Recently, Wang [38] extended Theorems 1.2 and 1.3 to simplicial complex and also extended Theorem 1.5 to near-cone by using linear algebra method.

Bear in mind that one of the outstanding open problems in extremal set theory is the following Chvátal's conjecture. It concerns the largest intersecting family of simplicial complex.

Conjecture 1.6 ([7]) *Let* $\mathscr{F}$ *be any family of subsets of* $[n]$ *such that* $S \in \mathscr{F}, T \subset S$ *implies* $T \in \mathscr{F}$, *then some largest intersecting subfamily of* $\mathscr{F}$ *has the form* $\{A \in \mathscr{F} : x \in A\}$ *for some* $x \in [n]$.

This conjecture can also be described in the following form.

Conjecture 1.7 *Let Δ be a simplicial complex and let $\mathscr{F}$ be an interesting family of faces of Δ. Then*

$$|\mathscr{F}| \leqslant \max_{v \in V(\Delta)} \left(\sum_r f_r(link_\Delta(v)) \right).$$

Recall that $2^{[n]}$ is a simplicial complex, so Conjecture 1.7 follows directly from Theorem 1.1 when $\Delta = 2^{[n]}$. Chvátal [8, 9] verified this conjecture for the case when Δ is compressed (that is, if $j \in F \in \Delta$, then for any $i \in V(\Delta) \backslash F$ with $i < j$, we have $(F \backslash \{j\}) \cup \{i\} \in \Delta$). Snevily [34] confirmed this conjecture when Δ is a near-cone with respect to an apex vertex, which is the best result so far on this conjecture. Many other results have been inspired by this conjecture, we refer the readers to [4, 27, 29] for more details.

Motivated by Theorem 1.4, Conjecture 1.7 and the main results in [38, 40], it is natural and interesting for us to consider the maximum cardinality of k-wise L-intersecting families on simplicial complex in the current paper.

1.2 Main Results

Our first result determines the maximum cardinality of k-wise L-intersecting families on simplicial complex. On the one hand our result extends [17, Theorem 1] to simplicial complex, on the other hand it also extends [38, Theorems 2.1 and 2.2] to the k-wise L-intersecting families for $k \geqslant 2$.

Theorem 1.8 *Let $k \geqslant 2$ and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of s nonnegative integers. If $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of simplicial complex Δ such that $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \in L$ for any collection of k distinct sets from $\mathscr{F}$, then*

$$m \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\Delta).$$

In addition, if the size of every member of $\mathscr{F}$ belongs to the set $K = \{k_1, k_2, \ldots, k_r\}$ with $\min K > s - r$, *then*

$$m \leqslant (k-1) \sum_{i=s-r}^{s-1} f_i(\Delta).$$

Our next result is a modular version of Theorem 1.8, which strengthens the Grolmusz–Sudakov theorem, i.e., [17, Theorem 2].

Theorem 1.9 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Assume that $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of simplicial* *complex* Δ *such that* $|F_i| \pmod{p} \notin L$ *for every* $1 \leqslant i \leqslant m$ *and* $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod{p} \in L$ *for any collection of* $k \geqslant 2$ *distinct sets from* $\mathscr{F}$. *Then*

$$m \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\Delta).$$

In addition, let $K = \{k_1, k_2, \ldots, k_r\}$ *be a subset of* $\{0, 1, \ldots, p-1\}$ *with* $\min K > s - r$. *If* $|F_i| \pmod{p} \in K$ *for every* $1 \leqslant i \leqslant m$, *then*

$$m \leqslant (k-1) \sum_{i=s-r}^{s-1} f_i(\Delta).$$

The following result improves Theorem 1.8 when p is larger than n.

Theorem 1.10 *Let* p *be a prime and let* $L = \{l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$ *be two disjoint subsets of* $\{0, 1, \ldots, p-1\}$ *satisfying* $\min K > s - 2r + 1$. *Assume that* Δ *is a simplicial complex with* $n \in V(\Delta)$ *and* $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ *is a family of faces of* Δ *such that* $|F_i| \pmod{p} \in K$ *for every* $1 \leqslant i \leqslant m$ *and* $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod{p} \in L$ *for any collection of* $k \geqslant 2$ *distinct sets from* $\mathscr{F}$. *Then*

$$m \leqslant (k-1) \sum_{i=s-2r}^{s-1} h_i(\Delta),$$

where $h_i(\Delta)$ *is the number of the* i*-dimensional faces which don't contain* n *in* Δ.

If $\Delta = 2^{[n]}$, then $h_i(\Delta) = \binom{n-1}{i+1}$ for $s - 2r \leqslant i \leqslant s - 1$. By Theorem 1.10, the next result holds directly.

Corollary 1.11 *Let* p *be a prime and let* $L = \{l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$ *be two disjoint subsets of* $\{0, 1, \ldots, p-1\}$ *satisfying* $\min K > s - 2r + 1$. *Assume that* $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ *is a family of subsets of* $[n]$ *with* $|F_i| \pmod{p} \in K$ *for any* $1 \leqslant i \leqslant m$ *and* $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod{p} \in L$ *for any collection of* $k \geqslant 2$ *distinct sets from* $\mathscr{F}$. *Then*

$$m \leqslant (k-1)\left[\binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}\right].$$

Note that from the condition $\min K > \max L$ or $\min K > s - r$, one has $\min K > s - 2r + 1$. Thus, Corollary 1.11 implies the following two results, which were obtained by Chen and Liu [5] and by Liu and Yang [23], respectively.

Corollary 1.12 ([5]) *Let* p *be a prime and let* $L = \{l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$ *be two disjoint subsets of* $\{0, 1, \ldots, p-1\}$ *satisfying* $\min K > \max L$. *Assume that*

$\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of $[n]$ satisfying $|F_i| \pmod p \in K$ for every $1 \leqslant i \leqslant m$ and $|F_i \cap F_j| \pmod p \in L$ for every pair $i \neq j$. Then

$$m \leqslant \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

Corollary 1.13 ([23]) *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$ such that $\min K > s - r$. Assume that $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for every $1 \leqslant i \leqslant m$ and $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod p \in L$ for any collection of $k \geqslant 2$ distinct sets from $\mathscr{F}$. Then*

$$m \leqslant (k-1)\left[\binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}\right].$$

Our last main result improves Theorems 1.8 and 1.9.

Theorem 1.14 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Assume that Δ is a near-cone with apex vertex v and $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of Δ satisfying $|F_i| \pmod p \notin L$ for every $1 \leqslant i \leqslant m$, and $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod p \in L$ for any collection of $k \geqslant 2$ distinct sets from $\mathscr{F}$. Then*

$$m \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(link_\Delta(v)).$$

Note that the family $2^{[n]}$ is a near-cone and any vertex in $[n]$ is an apex vertex of $2^{[n]}$. By setting $k = 2$ and $\Delta = 2^{[n]}$ and choosing any vertex $v \in [n]$ in Theorem 1.14, the next result, which was obtained by Snevily in 1994 [35], follows immediately.

Corollary 1.15 ([35]) *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Suppose that $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of $[n]$ such that $|F_i| \pmod p \notin L$ for every $1 \leqslant i \leqslant m$ and $|F_i \cap F_j| \pmod p \in L$ for every pair $i \neq j$. Then*

$$m \leqslant \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

Our paper is organized as follows. In the remainder of this section, we introduce some notations which will be used in the subsequent sections. In Sect. 2, we give the proofs of Theorems 1.8 and 1.9. In Sect. 3, we give the proof of Theorem 1.10. In Sect. 4, we give the proof of Theorem 1.14. In the last section, we conclude our paper with further research problems.

Let p be a prime and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector with n variables. Let $\mathbb{F}[x_1, x_2, \ldots, x_n]$ denote the polynomial ring, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{F}_p$. A polynomial $f(\mathbf{x})$ in variables x_i, $i = 1, 2, \ldots, n$, is called *multilinear* if the power of each variable

x_i in each term is at most one. Obviously, if each variable x_i takes only the value 0 or 1, then any polynomial in variables $x_i, i = 1, 2, \ldots, n$, is multilinear since any positive power of a variable x_i may be replaced by one. In this paper, all multilinear polynomials considered are from $\mathbb{F}[x_1, x_2, \ldots, x_n]$.

For any subset $A \subseteq [n]$, we define the *characteristic vector* of A as the vector $\mathbf{v}_A = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ with $v_i = 1$ if $i \in A$ and $v_i = 0$ otherwise. For two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n), \mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$, let $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$ denote their standard *inner product*. Moreover, we index the monic multilinear monomials by the set of their variables:

$$\mathbf{x}_A := \prod_{i \in A} x_i.$$

In particular, if $A = \emptyset$, define $\mathbf{x}_\emptyset := 1$.

In the following sections, we use $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to denote a vector of n variables with each variable x_j taking values 0 or 1.

2 Proofs of Theorems 1.8 and 1.9

In this section, we give the proofs of Theorems 1.8 and 1.9. In order to complete our proof, we need some preliminaries.

Lemma 2.1 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Choose $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ to be two families of faces of simplicial complex Δ such that (1) $|A_i \cap B_i| \pmod p \notin L$ for every $1 \leqslant i \leqslant m$; (2) $|A_j \cap B_i| \pmod p \in L$ for every pair $j > i$. Then*

$$m \leqslant \sum_{i=-1}^{s-1} f_i(\Delta).$$

Proof For $1 \leqslant i \leqslant m$, define

$$\phi_i(\mathbf{x}) = \prod_{j=1}^{s} \left(\mathbf{v}_{B_i} \cdot \mathbf{x} - l_j\right). \tag{2.1}$$

Then each $\phi_i(\mathbf{x})$ is a polynomial of degree at most s. By condition (1), $\phi_i(\mathbf{v}_{A_i}) = \prod_{j=1}^{s}(|A_i \cap B_i| - l_j) \not\equiv 0 \pmod p$ for every $1 \leqslant i \leqslant m$. By condition (2), $\phi_i(\mathbf{v}_{A_j}) = \prod_{t=1}^{s}(|A_j \cap B_i| - l_t) = 0 \pmod p$ for every pair $i < j$.

Now we show that the polynomials $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_m(\mathbf{x})$ are linearly independent over the finite field $\mathbb{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero, i.e.,

$$\sum_{i=1}^{m} \alpha_i \phi_i(\mathbf{x}) = 0 \tag{2.2}$$

with all coefficients α_i being in $\mathbb{F}_p$. We show that all coefficients must be zero modulo p as follows.

Claim 1 $\alpha_i = 0 \pmod{p}$ *for each* $1 \leqslant i \leqslant m$.

Proof of Claim 1 Suppose, to the contrary, that i_0 is the largest subscript such that $\alpha_{i_0} \neq 0 \pmod{p}$. As pointed out above, $\phi_i(\mathbf{v}_{A_{i_0}}) = 0 \pmod{p}$ for $i < i_0$. By evaluating $\mathbf{x} = \mathbf{v}_{A_{i_0}}$ in Eq. (2.2), we have $\alpha_{i_0}\phi_{i_0}(\mathbf{v}_{A_{i_0}}) = 0 \pmod{p}$. Combined with $\phi_{i_0}(\mathbf{v}_{A_{i_0}}) \neq 0 \pmod{p}$, it follows that $\alpha_{i_0} = 0 \pmod{p}$, a contradiction. So Claim 1 holds. ◻

By Claim 1, the polynomials $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_m(\mathbf{x})$ are linearly independent over the field $\mathbb{F}_p$. On the other hand, by the definitions of simplicial complex Δ and $\phi_i(\mathbf{x})$, we see that the monomial $x_{j_1}x_{j_2}\ldots x_{j_t}$ appears in $\phi_i(\mathbf{x})$ only if $\{j_1, j_2, \ldots, j_t\}$ is a face of Δ. This means that

$$m \leqslant \sum_{i=-1}^{s-1} f_i(\Delta).$$

This completes the proof. ◻

Lemma 2.2 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two subsets of* $\{0, 1, \ldots, p-1\}$ *with* $\min K > s - r$. *Choose $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ to be two families of faces of simplicial complex Δ such that (1) $|A_i \cap B_i| \pmod{p} \notin L$ for every $1 \leqslant i \leqslant m$; (2) $|A_j \cap B_i| \pmod{p} \in L$ for every pair $j > i$; (3) $|A_i| \pmod{p} \in K$ for every $1 \leqslant i \leqslant m$. Then*

$$m \leqslant \sum_{i=s-r}^{s-1} f_i(\Delta).$$

Proof Let $\mathscr{W}$ be the family of the faces of Δ with dimensions at most $s - r - 1$. Then $|\mathscr{W}| = \sum_{i=-1}^{s-r-1} f_i(\Delta)$. For each $I \in \mathscr{W}$, define

$$h_I(\mathbf{x}) = P(\mathbf{x}) \prod_{j \in I} x_j, \tag{2.3}$$

where $P(\mathbf{x}) = \prod_{k_j \in K}(\sum_{i=1}^{n} x_i - k_j)$. Obviously, $h_I(\mathbf{x})$ is a polynomial of degree at most s.

Let $A_I(\mathbf{x})$ be the sum of all the monomials $x_{j_1}x_{j_2}\ldots x_{j_t}$ in the expansion of $h_I(\mathbf{x})$ such that $\{j_1, j_2, \ldots, j_t\}$ are the faces of Δ. Then each $A_I(\mathbf{x})$ is a polynomial of degree at most s. For any given $A_i \in \mathscr{A} \subseteq \Delta$ and $I \in \mathscr{W}$, if $I \nsubseteq A_i$, then $A_I(\mathbf{v}_{A_i}) = 0 \pmod{p}$. While if $I \subseteq A_i$, then $h_I(\mathbf{v}_{A_i}) = P(\mathbf{v}_{A_i})$. Note that $A_i \in \Delta$ and Δ is a simplicial complex, we obtain that $A_I(\mathbf{v}_{A_i}) = P(\mathbf{v}_{A_i})$. Together with the fact that $|A_i| \pmod{p} \in K$, it follows that $A_I(\mathbf{v}_{A_i}) = 0 \pmod{p}$.

Recall that the polynomials $\phi_i(\mathbf{x})$, $i = 1, 2, \ldots, m$, are defined in (2.1). Now, we prove that the polynomials in

$$\{\phi_i(\mathbf{x}) : 1 \leqslant i \leqslant m\} \cup \{A_I(\mathbf{x}) : I \in \mathscr{W}\} \tag{2.4}$$

are linearly independent over the field $\mathbb{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero, i.e.,

$$\sum_{i=1}^{m} \alpha_i \phi_i(\mathbf{x}) + \sum_{I \in \mathscr{W}} \mu_I A_I(\mathbf{x}) = 0 \tag{2.5}$$

with all coefficients α_i and μ_I being in $\mathbb{F}_p$. In order to complete our proof, we show the following two claims.

Claim 2 $\alpha_i = 0 \pmod{p}$ *for each* $1 \leqslant i \leqslant m$.

Proof of Claim 2 Suppose, to the contrary, that i_0 is the largest subscript such that $\alpha_{i_0} \neq 0 \pmod{p}$. Substituting $\mathbf{x} = \mathbf{v}_{A_{i_0}}$ in (2.5), we have $A_I(\mathbf{v}_{A_{i_0}}) = 0 \pmod{p}$ for each $I \in \mathscr{W}$. Hence, all terms in the second sum of (2.5) vanish. In this case, by the same discussion as in the proof of Claim 1, we deduce that all $\alpha_i = 0 \pmod{p}$. So Claim 2 holds. □

Claim 3 $\mu_I = 0 \pmod{p}$ *for each* $I \in \mathscr{W}$.

Proof of Claim 3 Suppose, to the contrary, that I' is a subset of the smallest size in $\mathscr{W}$ such that $\mu_{I'} \neq 0 \pmod{p}$. It is routine to check that $A_I(\mathbf{v}_{I'}) \equiv 0 \pmod{p}$ for each $|I| \geqslant |I'|$ and $I \neq I'$. Evaluating $\mathbf{x} = \mathbf{v}_{I'}$ in Eq. (2.5) gives us $\mu_{I'} A_{I'}(\mathbf{v}_{I'}) = 0 \pmod{p}$. By a direct calculation, we have $h_{I'}(\mathbf{v}_{I'}) = P(\mathbf{v}_{I'}) = \prod_{k_j \in K}(\sum_{i=1}^{n} x_i - k_j)$. Combined with the condition $\min K > s - r$ and $I' \in \Delta$, we have $A_{I'}(\mathbf{v}_{I'}) = P(\mathbf{v}_{I'}) \neq 0 \pmod{p}$. Thus, $\mu_{I'} = 0 \pmod{p}$, a contradiction. So Claim 3 holds. □

By Claims 2 and 3, we obtain that the polynomials in (2.4) are linearly independent over the field $\mathbb{F}_p$. Notice that $\phi_i(\mathbf{x}), i = 1, 2, \ldots, m$ and $A_I(\mathbf{x}), I \in \mathscr{W}$ are the polynomials of degree at most s, and the monomial $x_{j_1} x_{j_2} \ldots x_{j_t}$ appears in these polynomials only if $\{j_1, j_2, \ldots, j_t\}$ is a face of Δ. Thus,

$$m + \sum_{i=-1}^{s-r-1} f_i(\Delta) \leqslant \sum_{i=-1}^{s-1} f_i(\Delta),$$

which is equivalent to $m \leqslant \sum_{i=s-r}^{s-1} f_i(\Delta)$.

This completes the proof. □

Lemma 2.3 *Let* $L = \{l_1, l_2, \ldots, l_s\}$ *be a subset of* s *nonnegative integers. Assume that* $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ *and* $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ *are two families of faces of simplicial complex* Δ *such that (1)* $|A_i \cap B_i| = |B_i|$ *for each* $1 \leqslant i \leqslant m$; *(2)* $|A_j \cap B_i| \in L$ *and* $|A_j \cap B_i| < |B_i|$ *for every pair* $j > i$. *Then*

$$m \leqslant \sum_{i=-1}^{s-1} f_i(\Delta).$$

Proof For $1 \leqslant i \leqslant m$, define

$$\phi_i(\mathbf{x}) = \prod_{l_j<|B_i|} \left(\mathbf{v}_{B_i} \cdot \mathbf{x} - l_j\right). \tag{2.6}$$

Then each $\phi_i(\mathbf{x})$ is a polynomial of degree at most s. By condition (1), $\phi_i(\mathbf{v}_{A_i}) = \prod_{l_j<|B_i|}(|A_i \cap B_i| - l_j) \neq 0$ for every $1 \leqslant i \leqslant m$. By condition (2), $\phi_i(\mathbf{v}_{A_j}) = \prod_{l_t<|B_i|}(|A_j \cap B_i| - l_t) = 0$ for every pair $i < j$.

By a similar discussion as in the proof of Lemma 2.1, we can also get that the polynomials $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_m(\mathbf{x})$ are linearly independent, whose procedure is omitted here. On the other hand, by the definitions of simplicial complex Δ and $\phi_i(\mathbf{x})$, we see that the monomial $x_{j_1}x_{j_2}\ldots x_{j_t}$ appears in $\phi_i(\mathbf{x})$ only if $\{j_1, j_2, \ldots, j_t\}$ is a face of Δ. Thus,

$$m \leqslant \sum_{i=-1}^{s-1} f_i(\Delta).$$

This completes the proof. □

Lemma 2.4 *Let* $L = \{l_1, l_2, \ldots, l_s\}$ *and* $K = \{k_1, k_2, \ldots, k_r\}$ *be two subsets of non-negative integers such that* $\min K > s - r$. *Assume that* $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ *and* $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ *are two families of faces of simplicial complex* Δ *such that (1)* $|A_i \cap B_i| = |B_i|$ *for every* $1 \leqslant i \leqslant m$; *(2)* $|A_j \cap B_i| \in L$ *and* $|A_j \cap B_i| < |B_i|$ *for every pair* $j > i$; *(3)* $|A_i| \in K$ *for every* $1 \leqslant i \leqslant m$. *Then*

$$m \leqslant \sum_{i=s-r}^{s-1} f_i(\Delta).$$

Proof Let the polynomial $A_I(\mathbf{x})$ be defined the same as that in the proof of Lemma 2.2 and let the polynomial $\phi_i(\mathbf{x})$ be defined as that in (2.6). By a similar discussion as in the proof of Lemma 2.2, we can deduce that the polynomials in $\{\phi_i(\mathbf{x}) : 1 \leqslant i \leqslant m\} \cup \{A_I(\mathbf{x}) : I \in \mathscr{W}\}$ are linearly independent, whose procedure is omitted here. Notice that $\phi_i(\mathbf{x}), i = 1, 2, \ldots, m$ and $A_I(\mathbf{x}), I \in \mathscr{W}$ are the polynomials of degree at most s, and the monomial $x_{j_1}x_{j_2}\ldots x_{j_t}$ appears in these polynomials only if $\{j_1, j_2, \ldots, j_t\}$ is a face of Δ. Thus,

$$m + \sum_{i=-1}^{s-r-1} f_i(\Delta) \leqslant \sum_{i=-1}^{s-1} f_i(\Delta),$$

which is equivalent to

$$m \leqslant \sum_{i=s-r}^{s-1} f_i(\Delta).$$

This completes the proof. □

Now we are ready to give the proof of Theorem 1.9, and then we modify it to show Theorem 1.8.

Proof of Theorem 1.9 Let $\mathscr{F}$ be a family satisfying assertion of Theorem 1.9. We repeat the following procedure until $\mathscr{F}$ is empty. At round i if $\mathscr{F} \neq \emptyset$, we choose a maximal collection $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$ such that $|\cap_{i=1}^{d'} F_i| \pmod{p} \notin L$ for all $1 \leqslant d' \leqslant d$, but for any additional set $F' \in \mathscr{F}$ we obtain that $|(\cap_{i=1}^{d} F_i) \cap F'| \pmod{p} \in L$. Clearly, by definition such family always exists and $1 \leqslant d \leqslant k-1$. Denote $A_i = F_1$ and $B_i = \cap_{j=1}^{d} F_j$ and remove all sets $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$. As the result of this process, we obtain at least $m' \geqslant |\mathscr{F}|/(k-1)$ pairs of sets A_i, B_i. By definition, we get that $|A_i \cap B_i| = |B_i| \pmod{p} \notin L$ and $|A_j \cap B_i| \pmod{p} \in L$ for any $j > i$.

In addition, note that F_i is a face of Δ, and so A_i is a face of Δ. Note that the intersection of F_i and F_j is still a face of Δ, we obtain that B_i is also a face of Δ. Thus, we derive that $\mathscr{A} = \{A_1, A_2, \ldots, A_{m'}\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_{m'}\}$ are two families of faces of Δ. So, by Lemma 2.1, we have

$$|\mathscr{F}| \leqslant (k-1)m' \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\Delta).$$

We now extend the idea above to prove the second part of the theorem. Let $K = \{k_1, k_2, \ldots, k_r\}$ be a subset of $\{0, 1, \ldots, p-1\}$ with $\min K > s - r$. Since $|F_i| \pmod{p} \in K$, we get that $|A_i| \pmod{p} \in K$ for each i. According to the above construction process, we see that these two families $\mathscr{A}$ and $\mathscr{B}$ satisfy the conditions of Lemma 2.2. Thus, by Lemma 2.2, we have

$$|\mathscr{F}| \leqslant (k-1)m' \leqslant (k-1) \sum_{i=s-r}^{s-1} f_i(\Delta).$$

This completes the proof. □

Theorem 1.8 can be proved by a simple modification of the above proof.

Proof of Theorem 1.8 Let $\mathscr{F}$ be a family satisfying assertion of Theorem 1.8. We repeat the following procedure. At step i, if $|F| \in L$ for all $F \in \mathscr{F}$, then let F_1 be the largest set remaining in $\mathscr{F}$. Denote $A_i = B_i = F_1$ and remove F_1 from $\mathscr{F}$. Otherwise, there exists at least $F \in \mathscr{F}$ such that $|F| \notin L$. We choose a maximal collection $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$ such that $|\cap_{i=1}^{d'} F_i| \notin L$ for all $1 \leqslant d' \leqslant d$, but for any additional set $F' \in \mathscr{F}$ we obtain that $|(\cap_{i=1}^{d} F_i) \cap F'| \in L$. Denote $A_i = F_1$ and $B_i = \cap_{j=1}^{d} F_j$ and remove all sets $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$. As the result of this process, we obtain at least $m' \geqslant |\mathscr{F}|/(k-1)$ pairs of sets A_i, B_i. By definition, we get that $|A_i \cap B_i| = |B_i|$, $|A_j \cap B_i| \in L$ and $|A_j \cap B_i| < |B_i|$ for any $j > i$.

Since F_i is a face of Δ, we get that A_i is also a face of Δ. Clearly, the intersection of F_i and F_j is still a face of Δ, we obtain that B_i is also a face of Δ. Thus, we derive that $\mathscr{A} = \{A_1, A_2, \ldots, A_{m'}\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_{m'}\}$ are two families of faces of Δ. So, by Lemma 2.3, we have

$$|\mathscr{F}| \leqslant (k-1)m' \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\Delta).$$

The proof of the second part of this theorem is identical with that of Theorem 1.9 (based on Lemma 2.4) and we omit it here.

This completes the proof. □

3 Proof of Theorem 1.10

In this section, we give the proof of Theorem 1.10. In order to do so, we need the following lemma.

Lemma 3.1 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of* $\{0, 1, \ldots, p-1\}$ *satisfying* $\min K > s - 2r + 1$. *Assume that Δ is a simplicial complex with $n \in V(\Delta)$ and choose $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ to be two families of faces of Δ such that* (1) $|A_i \cap B_i| \pmod p \notin L$ *for every* $1 \leqslant i \leqslant m$; (2) $|A_j \cap B_i| \pmod p \in L$ *for every pair* $j > i$; (3) *there exists an integer t satisfying $n \in A_i$ for $i > t$ and $n \notin A_i \cup B_i$ for $i \leqslant t$*; (4) $|A_i| \pmod p \in K$ *for every* $1 \leqslant i \leqslant m$. *Then*

$$m \leqslant \sum_{i=s-2r}^{s-1} h_i(\Delta),$$

where $h_i(\Delta)$ is the number of the i-dimensional faces which don't contain n in Δ.

Proof For $1 \leqslant i \leqslant m$, define

$$\phi_i(\mathbf{x}) = \prod_{j=1}^{s} \left(\mathbf{v}_{B_i} \cdot \mathbf{x} - l_j\right).$$

Then each $\phi_i(\mathbf{x})$ is a polynomial of degree at most s. By condition (1), $\phi_i(\mathbf{v}_{A_i}) = \prod_{j=1}^{s}(|A_i \cap B_i| - l_j) \neq 0 \pmod p$ for every $1 \leqslant i \leqslant m$. By condition (2), $\phi_i(\mathbf{v}_{A_j}) = \prod_{q=1}^{s}(|A_j \cap B_i| - l_q) = 0 \pmod p$ for every pair $i < j$.

Let $\mathscr{Q}$ be the family of the faces of Δ with dimensions at most $s-1$ which contain the vertex n. For each $L \in \mathscr{Q}$, define

$$q_L(\mathbf{x}) = (1 - x_n) \prod_{j \in L \setminus \{n\}} x_j.$$

Let $\mathscr{W}$ be the family of the faces of Δ with dimensions at most $s - 2r - 1$ which don't contain the vertex n. Let $H = \{(k_i - 1) \pmod p : k_i \in K\} \cup K$. Then $|H| \leqslant 2r$. For each $I \in \mathscr{W}$, define

$$T_I(\mathbf{x}) = P(\mathbf{x}) \prod_{j \in I} x_j,$$

where $P(\mathbf{x}) = \prod_{h \in H} (\sum_{i=1}^{n-1} x_i - h)$. Let $A_I(\mathbf{x})$ be the sum of all the monomials $x_{j_1} x_{j_2} \dots x_{j_q}$ in the expansion of $T_I(\mathbf{x})$ such that $\{j_1, j_2, \dots, j_q\}$ are the faces of Δ. Then each $A_I(\mathbf{x})$ is a polynomial of degree at most s. For any given $A_i \in \mathscr{A} \subseteq \Delta$ and $I \in \mathscr{W}$, if $I \nsubseteq A_i$, then $A_I(\mathbf{v}_{A_i}) = 0 \pmod p$, whereas if $I \subseteq A_i$, then $T_I(\mathbf{v}_{A_i}) = P(\mathbf{v}_{A_i})$. Note that $A_i \in \Delta$, we obtain that $A_I(\mathbf{v}_{A_i}) = P(\mathbf{v}_{A_i})$. Together with the fact that $|A_i| \pmod p \in K$, it follows that $A_I(\mathbf{v}_{A_i}) = 0 \pmod p$.

Now, we prove that the polynomials in

$$\{\phi_i(\mathbf{x}) : 1 \leqslant i \leqslant m\} \cup \{q_L(\mathbf{x}) : L \in \mathscr{Q}\} \cup \{A_I(\mathbf{x}) : I \in \mathscr{W}\} \tag{3.1}$$

are linearly independent over the field $\mathbb{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero, i.e.,

$$\sum_{i=1}^{m} \alpha_i \phi_i(\mathbf{x}) + \sum_{L \in \mathscr{Q}} \beta_L q_L(\mathbf{x}) + \sum_{I \in \mathscr{W}} \gamma_I A_I(\mathbf{x}) = 0. \tag{3.2}$$

with all coefficients α_i, β_L and γ_I being in $\mathbb{F}_p$. In what follows we show that all coefficients must be zero modulo p.

Claim 4 $\alpha_i = 0 \pmod p$ *for each* $i > t$.

Proof of Claim 4 Suppose, to the contrary, that i_0 is the largest subscript such that $\alpha_{i_0} \neq 0 \pmod p$ and $i_0 > t$. By condition (3), we have $n \in A_{i_0}$. Then $q_L(\mathbf{v}_{A_{i_0}}) = 0$ for each $L \in \mathscr{Q}$. Note that $A_I(\mathbf{v}_{A_{i_0}}) = 0$ for each $I \in \mathscr{W}$ and $\phi_i(\mathbf{v}_{A_{i_0}}) = 0 \pmod p$ for $i < i_0$. By evaluating Eq. (3.2) with $\mathbf{x} = \mathbf{v}_{A_{i_0}}$, we have $\alpha_{i_0} \phi_{i_0}(\mathbf{v}_{A_{i_0}}) = 0 \pmod p$. Combined with $\phi_{i_0}(\mathbf{v}_{A_{i_0}}) \neq 0 \pmod p$, it follows that $\alpha_{i_0} = 0 \pmod p$, a contradiction. So Claim 4 holds. □

Claim 5 $\alpha_i = 0 \pmod p$ *for each* $i \leqslant t$.

Proof of Claim 5 Suppose, to the contrary, that i_0 is the largest subscript such that $\alpha_{i_0} \neq 0 \pmod p$ and $i_0 \leqslant t$. Then $n \notin A_{i_0}$. Let $A'_{i_0} = A_{i_0} \cup \{n\}$. Then $x_n = 1$ in $\mathbf{v}_{A'_{i_0}}$. This means that $q_L(\mathbf{v}_{A'_{i_0}}) = 0$ for each $L \in \mathscr{Q}$.

From the condition $|A_{i_0}| \pmod p \in K$, we have $P(\mathbf{v}_{A'_{i_0}}) = P(\mathbf{v}_{A_{i_0}}) = 0 \pmod p$. Then for any $I \in \mathscr{W}$, if $I \nsubseteq A'_{i_0}$, then $A_I(\mathbf{v}_{A'_{i_0}}) = 0 \pmod p$. While if $I \subseteq A'_{i_0}$, then $T_I(\mathbf{v}_{A'_{i_0}}) = P(\mathbf{v}_{A'_{i_0}})$, i.e., $T_I(\mathbf{v}_{A'_{i_0}}) = 0 \pmod p$. Since $A_{i_0} \in \Delta$, we have $A_I(\mathbf{v}_{A'_{i_0}}) = T_I(\mathbf{v}_{A'_{i_0}}) = 0 \pmod p$.

Note that $n \notin B_i$ for each $i \leqslant t$, we have $\phi_i(\mathbf{v}_{A'_{i_0}}) = \phi_i(\mathbf{v}_{A_{i_0}}) = 0$ for $i < i_0$. Then by evaluating Eq. (3.2) with $\mathbf{x} = \mathbf{v}_{A'_{i_0}}$, we have $\alpha_{i_0} \phi_{i_0}(\mathbf{v}_{A'_{i_0}}) = 0 \pmod p$. Combined with $\phi_{i_0}(\mathbf{v}_{A'_{i_0}}) \neq 0 \pmod p$, it follows that $\alpha_{i_0} = 0 \pmod p$, a contradiction. So Claim 5 holds. □

Combining Claims 4 and 5 reduces (3.2) to

$$\sum_{L\in\mathscr{Q}} \beta_L q_L(\mathbf{x}) + \sum_{I\in\mathscr{W}} \gamma_I A_I(\mathbf{x}) = 0. \tag{3.3}$$

Rewrite (3.3) as

$$\left(\sum_{L\in\mathscr{Q}} \beta_L q_L'(\mathbf{x}) + \sum_{I\in\mathscr{W}} \gamma_I A_I(\mathbf{x})\right) - x_n \left(\sum_{L\in\mathscr{Q}} \beta_L q_L'(\mathbf{x})\right) = 0, \tag{3.4}$$

where $q_L'(\mathbf{x}) = \prod_{j\in L\setminus\{n\}} x_j$. Notice that x_n doesn't appear in the first parentheses of (3.4). Setting $x_n = 0$ in (3.4) gives us

$$\sum_{L\in\mathscr{Q}} \beta_L q_L'(\mathbf{x}) + \sum_{I\in\mathscr{W}} \gamma_I A_I(\mathbf{x}) = 0$$

and

$$x_n \left(\sum_{L\in\mathscr{Q}} \beta_L q_L'(\mathbf{x})\right) = 0.$$

By setting $x_n = 1$, we have

$$\sum_{L\in\mathscr{Q}} \beta_L q_L'(\mathbf{x}) = 0. \tag{3.5}$$

As the set of all multilinear monomials in variables x_i, $1 \leqslant i \leqslant n$, of degree at most $s-1$ are linearly independent over the field $\mathbb{F}_p$. Thus, by (3.5), we obtain the following claim directly.

Claim 6 $\beta_L = 0 \pmod p$ *for each* $L \in \mathscr{Q}$.

In order to complete the proof, in view of Claims 4–6, we just need to show that $A_I(\mathbf{x})$, $I \in \mathscr{W}$, are linearly independent over the field $\mathbb{F}_p$.

Claim 7 $\gamma_I = 0 \pmod p$ *for each* $I \in \mathscr{W}$.

Proof of Claim 7 Suppose, to the contrary, that I_0 is a subset of the smallest size in $\mathscr{W}$ such that $\gamma_{I_0} \neq 0 \pmod p$. From the condition $\min K > s - 2r + 1$, we have $P(\mathbf{v}_I) \neq 0 \pmod p$ for each $I \in \mathscr{W}$. For each $I \in \mathscr{W}\setminus\{I_0\}$ with $|I| \geqslant |I_0|$, it is routine to check that $A_I(\mathbf{v}_{I_0}) = 0 \pmod p$. By evaluating Eq. (3.2) with $\mathbf{x} = \mathbf{v}_{I_0}$, we have $\gamma_{I_0} A_{I_0}(\mathbf{v}_{I_0}) = 0 \pmod p$. Combined with $I_0 \in \Delta$, we obtain that $A_{I_0}(\mathbf{v}_{I_0}) = P(\mathbf{v}_{I_0}) \neq 0 \pmod p$. Then $\gamma_{I_0} = 0 \pmod p$, a contradiction. So Claim 7 holds. □

By Claims 4–7, we see that the polynomials in (3.1) are linearly independent over the field $\mathbb{F}_p$. By the definition of simplicial complex and the definitions of the polynomials $\phi_i(\mathbf{x})$, $q_L(\mathbf{x})$ and $A_I(\mathbf{x})$, we get that each monomial $x_{j_1}x_{j_2}\ldots x_{j_k}$ appearing in $\phi_i(\mathbf{x})$ [resp. $q_L(\mathbf{x})$ and $A_I(\mathbf{x})$] satisfies that $\{j_1, j_2, \ldots, j_k\}$ is a face of Δ. Thus,

$$m + |\mathscr{Q}| + |\mathscr{W}| \leqslant \sum_{i=-1}^{s-1} f_i(\Delta).$$

By the definitions of $\mathscr{Q}$ and $\mathscr{W}$, we have $|\mathscr{Q}| = \sum_{i=-1}^{s-1} f_i'(\Delta)$ and $|\mathscr{W}| = \sum_{i=-1}^{s-1} h_i(\Delta)$, where $f_i'(\Delta)$ and $h_i(\Delta)$ denote the number of the i-dimensional faces which contain n and the number of the i-dimensional faces which do not contain n in Δ, respectively. Hence,

$$m \leqslant \sum_{i=-1}^{s-1} f_i(\Delta) - \sum_{i=-1}^{s-1} f_i'(\Delta) - \sum_{i=-1}^{s-2r-1} h_i(\Delta) = \sum_{i=s-2r}^{s-1} h_i(\Delta).$$

This completes the proof. □

Now we are ready to show Theorem 1.10.

Proof of Theorem 1.10 Let $\mathscr{F}$ be a family satisfying assertion of Theorem 1.10. We repeat the following procedure until $\mathscr{F}$ is empty. At round i if $\mathscr{F} \neq \emptyset$, we choose a maximal collection $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$ such that $n \notin F_1$ whenever there exists $F \in \mathscr{F}$ with $n \notin F$, and $|\cap_{i=1}^{d'} F_i| \pmod{p} \notin L$ for all $1 \leqslant d' \leqslant d$, but for any additional set $F' \in \mathscr{F}$ we obtain that $|(\cap_{i=1}^{d} F_i) \cap F'| \pmod{p} \in L$. Clearly, by definition such family always exists and $1 \leqslant d \leqslant k-1$. Denote $A_i = F_1$ and $B_i = \cap_{j=1}^{d} F_j$ and remove all sets $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$. As the result of this process, we obtain at least $m' \geqslant |\mathscr{F}|/(k-1)$ pairs of sets A_i, B_i. By definition, we get that $|A_i| \pmod{p} \in K$ and $|A_i \cap B_i| = |B_i| \pmod{p} \notin L$ for every $1 \leqslant i \leqslant m$, and $|A_j \cap B_i| \pmod{p} \in L$ for any $j > i$. Moreover, we also obtain that there must exist an integer t such that $n \in A_i$ for $i > t$ and $n \notin A_i \cup B_i$ for $i \leqslant t$.

In addition, since F_i is a face of Δ, we get that A_i is a face of Δ. Note that the intersection of F_i and F_j is still a face of Δ, we obtain that B_i is also a face of Δ. Thus, we derive that $\mathscr{A} = \{A_1, A_2, \ldots, A_{m'}\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_{m'}\}$ are two families of faces of Δ. So, by Lemma 3.1, we have

$$|\mathscr{F}| \leqslant (k-1)m' \leqslant (k-1) \sum_{i=s-2r}^{s-1} h_i(\Delta).$$

This completes the proof. □

4 Proof of Theorem 1.14

In this section, we give the proof of Theorem 1.14. In order to do so, we need the following lemma.

Lemma 4.1 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Assume that Δ is a near-cone with apex vertex v and $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_m\}$ are two families of faces of Δ such that (1) $v \notin B_i$ and $B_i \cup \{v\} \in \Delta$ for every $1 \leqslant i \leqslant t$; (2) $v \notin B_i$ and $B_i \cup \{v\} \notin \Delta$ for every $t+1 \leqslant i \leqslant r$; (3) $v \in B_i$ for every $r+1 \leqslant i \leqslant m$; (4) $B_i \subseteq A_i$ and $|B_i| \pmod{p} \notin L$ for every $1 \leqslant i \leqslant m$; (5) $|A_j \cap B_i| \pmod{p} \in L$ for every pair $j > i$. Then*

$$m \leqslant \sum_{i=-1}^{s-1} f_i(link_\Delta(v)).$$

Proof According to the conditions (1)–(3), we denote $\mathscr{B}_1 = \{B_1, B_2, \ldots, B_t\}$, $\mathscr{B}_2 = \{B_{t+1}, B_{t+2}, \ldots, B_r\}$ and $\mathscr{B}_3 = \{B_{r+1}, B_{l+2}, \ldots, B_m\}$ for short. For $B_i \in \mathscr{B}_1$, define

$$\phi_i(\mathbf{x}) = \prod_{j=1}^{s} \left(\mathbf{v}_{B_i} \cdot \mathbf{x} - l_j\right).$$

For $B_i \in \mathscr{B}_2$ or $B_i \in \mathscr{B}_3$, define

$$\phi_i(\mathbf{x}) = \prod_{l_j < |B_i|} \left(\mathbf{v}_{B_i} \cdot \mathbf{x} - l_j\right).$$

Then each $\phi_i(\mathbf{x})$ is a polynomial of degree at most s. By the definition of $\phi_i(\mathbf{x})$, we first claim that $\phi_i(\mathbf{v}_{A_i}) \neq 0 \pmod{p}$ for every $1 \leqslant i \leqslant m$ and $\phi_i(\mathbf{v}_{A_j}) = 0 \pmod{p}$ for every pair $i < j$. On the one hand, by condition (4), we obtain that $|A_i \cap B_i| = |B_i| \pmod{p} \notin L$ for every $1 \leqslant i \leqslant m$. This means that $\phi_i(\mathbf{v}_{A_i}) \neq 0 \pmod{p}$ for every $1 \leqslant i \leqslant m$. On the other hand, for any given pair $i < j$, by condition (5), we obtain that $\phi_i(\mathbf{v}_{A_j}) = \prod_{q=1}^{s}(|A_j \cap B_i| - l_q) = 0 \pmod{p}$ for $i \leqslant t$. Furthermore, combining conditions (4) and (5), we derive that $|A_j \cap B_i| < |B_i|$ for $i < j$. Then $\phi_i(\mathbf{v}_{A_j}) = \prod_{l_q < |B_i|}(|A_j \cap B_i| - l_q) = 0 \pmod{p}$ for $i \geqslant t+1$. Thus, we get that $\phi_i(\mathbf{v}_{A_j}) = 0 \pmod{p}$ for every pair $i < j$.

Let $\mathscr{Q}$ be the family of the faces of Δ with dimensions at most $s-1$ which contain the apex vertex v. For each $L \in \mathscr{Q}$, define

$$q_L(\mathbf{x}) = (x_v - 1) \prod_{j \in L \setminus \{v\}} x_j.$$

Let $\mathscr{H}$ be the family of the faces of Δ with dimensions at most $s-1$ satisfying that for each $R \in \mathscr{H}$, we obtain that $v \notin R$ and $R \cup \{v\} \notin \Delta$. For each $R \in \mathscr{H}$, define

$$h_R(\mathbf{x}) = \prod_{j \in R} x_j.$$

Now, we prove that the polynomials in

$$\{\phi_i(\mathbf{x}) : 1 \leqslant i \leqslant m\} \cup \{q_L(\mathbf{x}) : L \in \mathscr{Q}\} \cup \{h_R(\mathbf{x}) : R \in \mathscr{H}\} \tag{4.1}$$

are linearly independent over the field $\mathbb{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero, i.e.,

$$\sum_{i=1}^{m} \alpha_i \phi_i(\mathbf{x}) + \sum_{L \in \mathscr{Q}} \beta_L q_L(\mathbf{x}) + \sum_{R \in \mathscr{H}} \gamma_R h_R(\mathbf{x}) = 0. \tag{4.2}$$

with all coefficients α_i, β_L and γ_R being in $\mathbb{F}_p$. We show that all coefficients must be zero modulo p in what follows.

Claim 8 $\gamma_R = 0 \pmod{p}$ *for each* $R \in \mathscr{H}$.

Proof of Claim 8 Suppose, to the contrary, that R_0 is a face in $\mathscr{H}$ such that $\gamma_{R_0} \neq 0 \pmod{p}$. We consider the coefficient of the monomial $\prod_{j \in R_0} x_j$ in (4.2). Since Δ is a near-cone with apex vertex v and $R_0 \cup \{v\}$ is not a face of Δ, we claim that R_0 is a facet of Δ. Otherwise, there exists $F \in \Delta$ such that $R_0 \subset F$. If $v \in F$, then we find that $R_0 \cup \{v\} \in \Delta$, a contradiction. If $v \notin F$, since Δ is a near-cone with apex vertex v, we obtain that for any subset S of F, $S \cup \{v\}$ is a face of Δ. Thus, $R_0 \cup \{v\} \in \Delta$, a contradiction. So, R_0 is a facet of Δ. By the definitions of $\phi_i(\mathbf{x})$ and $q_L(\mathbf{x})$, it is straightforward to check that $\prod_{j \in R_0} x_j$ doesn't appear in $\phi_i(\mathbf{x})$ and $q_L(\mathbf{x})$. It follows that the monomial $\prod_{j \in R_0} x_j$ only appears in $h_{R_0}(\mathbf{x})$. Thus, the coefficient of $\prod_{j \in R_0} x_j$ in (4.2) is γ_{R_0}. As the set of all multilinear monomials in variables x_i, $1 \leqslant i \leqslant n$, of degree at most s are linearly independent over the field $\mathbb{F}_p$. Thus, $\gamma_{R_0} = 0 \pmod{p}$, a contradiction. So Claim 8 holds. □

Claim 9 $\alpha_i = 0 \pmod{p}$ *for each* $r+1 \leqslant i \leqslant m$.

Proof of Claim 9 Suppose, to the contrary, that i_0 is the largest subscript such that $\alpha_{i_0} \neq 0 \pmod{p}$ and $r+1 \leqslant i_0 \leqslant m$. Then $v \in B_{i_0}$. In view of condition (4), we have $B_i \subseteq A_i$ for every $1 \leqslant i \leqslant m$. This means that $v \in A_{i_0}$. Then $x_v = 1$ in $\mathbf{v}_{A_{i_0}}$. Thus, $q_L(\mathbf{v}_{A_{i_0}}) = 0$ for each $L \in \mathscr{Q}$. Note that $\phi_i(\mathbf{v}_{A_{i_0}}) = 0 \pmod{p}$ for any $i < i_0$. By evaluating Eq. (4.2) with $\mathbf{x} = \mathbf{v}_{A_{i_0}}$, we have $\alpha_{i_0}\phi_{i_0}(\mathbf{v}_{A_{i_0}}) = 0 \pmod{p}$. Together with the fact that $\phi_{i_0}(\mathbf{v}_{A_{i_0}}) \neq 0 \pmod{p}$, it follows that $\alpha_{i_0} = 0 \pmod{p}$, a contradiction. So Claim 9 holds. □

Combining Claims 8 and 9 reduces (4.2) to

$$\sum_{i=1}^{r} \alpha_i \phi_i(\mathbf{x}) + \sum_{L \in \mathscr{Q}} \beta_L q_L(\mathbf{x}) = 0. \tag{4.3}$$

which is equivalent to

$$\left(\sum_{i=1}^{r}\alpha_i\phi_i(\mathbf{x})-\sum_{L\in\mathscr{Q}}\beta_L q'_L(\mathbf{x})\right)+x_v\left(\sum_{L\in\mathscr{Q}}\beta_L q'_L(\mathbf{x})\right)=0, \tag{4.4}$$

where $q'_L(\mathbf{x})=\prod_{j\in L\setminus\{v\}}x_j$. Notice that x_v doesn't appear in the first parentheses of Eq. (4.4). Setting $x_v=0$ in Eq. (4.4) gives us

$$\sum_{i=1}^{r}\alpha_i\phi_i(\mathbf{x})-\sum_{L\in\mathscr{Q}}\beta_L q'_L(\mathbf{x})=0$$

and

$$x_v\left(\sum_{L\in\mathscr{Q}}\beta_L q'_L(\mathbf{x})\right)=0.$$

By setting $x_v=1$, we have

$$\sum_{L\in\mathscr{Q}}\beta_L q'_L(\mathbf{x})=0. \tag{4.5}$$

As the set of all multilinear monomials in variables x_i, $1\leqslant i\leqslant n$, of degree at most $s-1$ are linearly independent over the field $\mathbb{F}_p$. By (4.5), we obtain the following claim directly.

Claim 10 $\beta_L=0 \pmod p$ *for each* $L\in\mathscr{Q}$.

In order to complete the proof, we need to show the following claim. That is, we need only to show that $\phi_i(\mathbf{x})$, $1\leqslant i\leqslant r$, are linearly independent over the field $\mathbb{F}_p$.

Claim 11 $\alpha_i=0 \pmod p$ *for each* $i=1,\ldots,r$.

Proof of Claim 11 Suppose, to the contrary, that i_0 is the largest subscript such that $\alpha_{i_0}\neq 0 \pmod p$ and $1\leqslant i_0\leqslant r$. Note that $\phi_i(\mathbf{v}_{A_{i_0}})=0 \pmod p$ for any $i<i_0$. By evaluating Eq. (4.3) with $\mathbf{x}=\mathbf{v}_{A_{i_0}}$, we have $\alpha_{i_0}\phi_{i_0}(\mathbf{v}_{A_{i_0}})=0 \pmod p$. Combined with $\phi_{i_0}(\mathbf{v}_{A_{i_0}})\neq 0 \pmod p$, it follows that $\alpha_{i_0}=0 \pmod p$, a contradiction. So Claim 11 holds. □

By Claims 8–11, we obtain that the polynomials in (4.1) are linearly independent over the field $\mathbb{F}_p$. By the definition of simplicial complex Δ and the definitions of the polynomials $\phi_i(\mathbf{x})$, $q_L(\mathbf{x})$ and $h_R(\mathbf{x})$, we obtain that each monomial $x_{j_1}x_{j_2}\ldots x_{j_k}$ appearing in $\phi_i(\mathbf{x})$ (resp. $q_L(\mathbf{x})$ and $h_R(\mathbf{x})$) satisfies that $\{j_1,j_2,\ldots,j_k\}$ is a face of Δ. Thus we have

$$m+|\mathscr{Q}|+|\mathscr{H}|\leqslant\sum_{i=-1}^{s-1}f_i.$$

By the definitions of $\mathscr{Q}$ and $\mathscr{H}$, we have $|\mathscr{Q}| = \sum_{i=-1}^{s-1} f_i'$ and $|\mathscr{H}| = \sum_{i=-1}^{s-1} f_i''$, where f_i' and f_i'' denote the number of the i-dimensional faces which contain v and the number of the i-dimensional faces which do not contain v and unite v is not a face, respectively. Hence, we get

$$m \leqslant \sum_{i=-1}^{s-1} f_i - \sum_{i=-1}^{s-1} f_i' - \sum_{i=-1}^{s-1} f_i'' = \sum_{i=-1}^{s-1} f_i(\mathrm{link}_\Delta(v)).$$

This completes the proof. □

Now we give the proof of Theorem 1.14.

Proof of Theorem 1.14 Let $\mathscr{F}$ be a family satisfying assertion of Theorem 1.14. Based on $\mathscr{F}$, we define three families as follows:

$$\mathscr{F}_1 = \{F_i \in \mathscr{F} : v \notin F_i \quad \text{and} \quad F_i \cup \{v\} \in \Delta\},$$
$$\mathscr{F}_2 = \{F_i \in \mathscr{F} : v \notin F_i \quad \text{and} \quad F_i \cup \{v\} \notin \Delta\},$$
$$\mathscr{F}_3 = \{F_i \in \mathscr{F} : v \in F_i\}.$$

Clearly, $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ is a partition of $\mathscr{F}$. We repeat the following procedure until $\mathscr{F}$ is empty. At round i if $\mathscr{F} \neq \emptyset$, we choose a maximal collection $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$ such that $|\cap_{i=1}^{d'} F_i| \pmod p \notin L$ for all $1 \leqslant d' \leqslant d$, but for any additional set $F' \in \mathscr{F}$ we obtain that $|(\cap_{i=1}^{d} F_i) \cap F'| \pmod p \in L$. In addition, the following conditions should be satisfied when selecting F_i from $\mathscr{F}$: we select the set F_i from $\mathscr{F}_1$ if $\mathscr{F}_1 \neq \emptyset$, and select the set F_i from $\mathscr{F}_2$ if $\mathscr{F}_1 = \emptyset$ and $\mathscr{F}_2 \neq \emptyset$, and select the set F_i from $\mathscr{F}_3$ if $\mathscr{F}_1 = \mathscr{F}_2 = \emptyset$ and $\mathscr{F}_3 \neq \emptyset$.

Since $\mathscr{F}$ is a k-wise L-intersecting family and $|F_i| \pmod p \notin L$ for each i, we obtain that such family always exists and $1 \leqslant d \leqslant k-1$. Denote $A_i = F_1$ and $B_i = \cap_{j=1}^{d} F_j$ and remove all sets $F_1, F_2, \ldots, F_d$ from $\mathscr{F}$. As the result of this process, we obtain at least $m' \geqslant |\mathscr{F}|/(k-1)$ pairs of sets A_i, B_i. By definition, we get that $B_i \subseteq A_i$ and $|B_i| = |\cap_{j=1}^{d} F_j| \pmod p \notin L$ and $|A_j \cap B_i| \pmod p \in L$ for any $j > i$. Furthermore, we obtain that there must exist two integers t, r satisfying (1) $v \notin B_i$ and $B_i \cup \{v\} \in \Delta$ for every $i \leqslant t$; (2) $v \notin B_i$ and $B_i \cup \{v\} \notin \Delta$ for every $t+1 \leqslant i \leqslant r$; (3) $v \in B_i$ for every $r+1 \leqslant i \leqslant m$. Since $F_1 \in \mathscr{F}$ is a face of Δ, we obtain that A_i is a face of Δ. Since the intersection of F_i and F_j is still a face of Δ, we obtain that B_i is also a face of Δ. Therefore, we derive that these two families $\mathscr{A} = \{A_1, A_2, \ldots, A_{m'}\}$ and $\mathscr{B} = \{B_1, B_2, \ldots, B_{m'}\}$ satisfy the conditions of Lemma 4.1. So, by Lemma 4.1, we have

$$|\mathscr{F}| \leqslant (k-1)m' \leqslant (k-1) \sum_{i=-1}^{s-1} f_i(\mathrm{link}_\Delta(v)).$$

This completes the proof. □

5 Concluding Remarks

In this paper, we present some upper bounds for the k-wise L-intersecting families of faces of simplicial complex Δ. In particular, in Theorem 1.14, we derive that if Δ is a near-cone with apex vertex v, and $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of Δ satisfying $|F_i| \pmod{p} \notin L$ for every $1 \leqslant i \leqslant m$, and $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod{p} \in L$ for any collection of $k \geqslant 2$ distinct sets from $\mathscr{F}$, then

$$m \leqslant (k-1)\sum_{i=-1}^{s-1} f_i(\mathrm{link}_\Delta(v)).$$

Thus, it is, of course, interesting to consider the following problem.

Problem 5.1 *Let p be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Assume that Δ is a near-cone with apex vertex v and $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of faces of Δ satisfying $|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| \pmod{p} \in L$ for any collection of $k \geqslant 2$ distinct sets from $\mathscr{F}$. Then*

$$m \leqslant (k-1)\sum_{i=-1}^{s-1} f_i(link_\Delta(v)).$$

Notice that Problem 5.1 follows directly from Theorem 1.14 under the restriction $|F_i| \pmod{p} \notin L$ for every $1 \leqslant i \leqslant m$. Hence, the rest case is $|F_i| \pmod{p} \in L$ for some $1 \leqslant i \leqslant m$.

In addition, motivated by Chvátal's conjecture (i.e., Conjecture 1.7) and Theorem 1.14, we conclude this paper by proposing the following problem.

Problem 5.2 *Let $k \geqslant 2$ and let $L = \{l_1, l_2, \ldots, l_s\}$ be a subset of s nonnegative integers. If $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ is a k-wise L-interesting family of faces of simplicial complex Δ, then*

$$m \leqslant \max_{v \in V(\Delta)} \left[(k-1)\sum_{i=-1}^{s-1} f_i(link_\Delta(v))\right].$$

It is routine to check that Problem 5.2 implies Chvátal's conjecture when $k = 2$ and $L = \{1, 2, \ldots, \dim(\Delta)\}$, where $\dim(\Delta)$ stands for the dimension of Δ.

We intend to do exactly the above challenging problems in the near future.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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