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Divisorial Ideals in the Power Series Ring $A + XB[\![X]\!]$

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Abstract

Let $A \subseteq B$ be an extension of integral domains, $B[\![X]\!]$ be the power series ring over B, and $R = A + XB[\![X]\!]$ be a subring of $B[\![X]\!]$. In this paper, we give a complete description of v-invertible v-ideals (with nonzero trace in A) of R. We show that if B is a completely integrally closed domain and I is a fractional divisorial v-invertible ideal of R with nonzero trace over A, then $I = u(J_1 + XJ_2[\![X]\!])$ for some $u \in qf(R)$, J_2 an integral divisorial v-invertible ideal of B and $J_1 \subseteq J_2$ a nonzero ideal of A.

Keywords t-ideal · t-invertible ideal · Class group

Mathematics Subject Classification 13A15 · 13C20 · 13G05

1 Introduction

Let A be an integral domain with quotient field K. Let $\mathcal{F}(A)$ be the set of nonzero fractional ideals of D. For an $I \in \mathcal{F}(A)$, set $I^{-1} = \{x \in K \mid xI \subseteq A\}$. The mapping on $\mathcal{F}(A)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$ is called the v-operation on A. A nonzero fractional ideal I is said to be a *v-ideal* or *divisorial* if $I = I_v$, and I is said to be *v-invertible* if $(II^{-1})_v = A$. For properties of the v-operation the reader is referred to [8, Section 34]. However, we will be mostly interested in the t-operation defined on $\mathcal{F}(A)$ by $I \mapsto I_t = \cup\{J_v,\ J$ is a nonzero finitely generated fractional subideal of $I\}$. (For properties of the t-operation the reader may consult [1]). A fractional ideal I is called a *t-ideal* if $I = I_t$. A t-ideal (respectively, v-ideal) I has t- (respectively, v-) *finite type* if $I = J_t$ (respectively, $I = J_v$) for some finitely generated fractional ideal J of A. The set of v-ideals may be a proper subset of the set of t-ideals. A fractional ideal I is said to be *t-invertible* if $(II^{-1})_t = A$. The set $T(A)$ of t-invertible fractional t-ideals of A is a group under the t-multiplication $I \star J := (IJ)_t$, and the set $P(A)$ of nonzero

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principal fractional ideals of A is a subgroup of $T(A)$. Following [7], we define the *t-class group* of A, denoted $\mathrm{Cl}_t(A)$, to be the t-group of t-invertible fractional t-ideals of A under t-multiplication modulo its subgroup of principal fractional ideals that is, $\mathrm{Cl}_t(A) = T(A)/P(A)$.The t-class group of an integral domain was studied by many authors ([7–11]).

Let $A \subseteq B$ be an extension of integral domains. In [2], the authors study when the natural mapping $\varphi : Cl_t(A) \to Cl_t(A + XB[X])$; $[I] \mapsto [I(A + XB[X])]$ is an isomorphism. They showed that if B is an integrally closed domain and $\mathrm{qf}(A) \subseteq B$, then $Cl_t(A) \cong Cl_t(A + XB[X])$ ([2, Theorem 4.7]). Also, the authors study the form of v-invertible (respectively, t-invertible) ideals of the polynomial ring of the form $A + XB[X]$. Let $A \subseteq B$ be an extension of integral domains such that B is an integrally closed domain and $A + XB[X]$. The authors proved that if I is a fractional divisorial v-invertible ideal of R, then $I = u(J_1 + XJ_2[X])$ for some $u \in qf(A + XB[X])$, J_2 an integral divisorial v-invertible ideal of B and $J_1 \subseteq J_2$ a nonzero ideal of A ([2, Theorem 2.3]). In this paper, we extend these results to the ring of formal power series of the form $A + XB[\![X]\!]$. In particular, we give a relationship between v-invertible v-ideals of an integral domain and those of its power series ring of the form $A + XB[\![X]\!]$.

Let $A \subseteq B$ be an extension of integral domains, $B[\![X]\!]$ be the power series ring over B, and $R = A+XB[\![X]\!]$. In the first part of this paper, we study when the natural mapping

$$\begin{array}{rcl} \varphi : Cl_t(A) & \to & Cl_t(R) \\ [I] & \mapsto & [(IR)_t] \end{array}$$

is an injective homomorphism. We show that if B is a flat A-module, then the mapping φ is an injective homomorphism. Also, we prove that the mapping φ is not surjective in general (Remark 2.7). In the second part of this paper, we give a complete description of v-invertible v-ideals (with nonzero trace in A) of $A+XB[\![X]\!]$. First, we show that if $A \subseteq B$ is an extension of integral domains such that B is completely integrally closed, then for each divisorial ideal I of $R = A+XB[\![X]\!]$ such that $I \cap A \neq (0)$, there exist a divisorial ideal J of B and a nonzero ideal $H \subseteq J$ of A such that $I = H + XJ[\![X]\!]$ (Proposition 3.2). Based on the above result, we prove that if I is a fractional divisorial v-invertible ideal of R such that $I \cap A \neq (0)$, then $I = u(J_1 + XJ_2[\![X]\!])$ for some $u \in qf(R)$, J_2 an integral divisorial v-invertible ideal of B and $J_1 \subseteq J_2$ a nonzero ideal of A, where B satisfies $\circledast$.

2 The *t*-Class Group of A+XB[\![X]\!]

Let A be an integral domain. A fractional ideal I of A is said to be *v-invertible* (respectively, *t-invertible*, *invertible*) if $(II^{-1})_v = A$ (respectively, $(II^{-1})_t = A$, $II^{-1} = A$). Following [7], we define the *t-class group* of A, denoted by $Cl_t(A)$, to be the group $T(A)$ of t-invertible fractional t-ideals of A under t-multiplication (i.e., $I \star J := (IJ)_t$) modulo its subgroup $P(A)$ of principal fractional ideals, that is, $Cl_t(A) = T(A)/P(A)$. When A is a Krull domain, then the t-class group and the divisor class group coincide. We denote by $[I]$ the equivalence class of a t-invertible t-ideal I of A. Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[\![X]\!]$. In this section we show that the natural mapping $\varphi : Cl_t(A) \to Cl_t(R)$; $[I] \mapsto [(IR)_t]$ is an injective homomorphism. To prove it, we need the following lemmas.

Lemma 2.1 *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[\![X]\!]$. Let F_1 (respectively, F_2) be a fractional ideal of A (respectively, B) such that $F_1 \subseteq F_2$. Then $F_1 + XF_2[\![X]\!]$ is a fractional ideal of R and*

$$(F_1 + XF_2[\![X]\!])^{-1} = F_1^{-1} \cap F_2^{-1} + XF_2^{-1}[\![X]\!].$$

Proof Let $I = F_1 + XF_2[\![X]\!]$. Since $F_1 \subseteq I$, we obtain $I^{-1} = R : I \subseteq R : F_1$, where $R : F_1 = \{g \in qf(R), gF_1 \subseteq R\}$. This implies that $I^{-1} \subseteq K[\![X]\!]$, where $K = qf(B)$. Indeed, let $u \in I^{-1}$ and $\alpha \in F_1 \backslash (0)$. Since $uF_1 \subseteq R$, $u = \dfrac{\alpha u}{\alpha} \in \dfrac{1}{\alpha} R \subseteq K[\![X]\!]$.

Now we show that $u \in I^{-1}$ if and only if $u(0)F_1 \subseteq A$ and $uF_2[\![X]\!] \subseteq B[\![X]\!]$.
($\Rightarrow$) Let $u \in I^{-1}$. Since $uI \subseteq R$, we get $uF_1 + uXF_2[\![X]\!] \subseteq A + XB[\![X]\!]$. Chose $X = 0$, we obtain $u(0)F_1 \subseteq A$. Moreover, $uF_2[\![X]\!] \subseteq B[\![X]\!]$.

($\Leftarrow$) Assume that $u(0)F_1 \subseteq A$ and $uF_2[\![X]\!] \subseteq B[\![X]\!]$. We prove that $u \in I^{-1}$. As $u \in K[\![X]\!]$, we can write $u = \sum_{i=0}^{\infty} a_i X^i$, where $a_i \in K$. It is clearly that

$$uI = uF_1 + XuF_2[\![X]\!] \subseteq u(0)F_1 + (\sum_{i=1}^{\infty} a_i X^i)F_1 + XuF_2[\![X]\!].$$

Moreover, $(\sum_{i=1}^{\infty} a_i X^i)F_1 = (u - u(0))F_1 \subseteq uF_1 + u(0)F_1$. Then $uF_1 + u(0)F_1 \subseteq B[\![X]\!]$, because $uF_1 \subseteq uF_2 \subseteq B[\![X]\!]$ and $u(0)F_1 \subseteq A \subseteq B[\![X]\!]$. This implies that $(\sum_{i=1}^{\infty} a_i X^i)F_1 \subseteq B[\![X]\!]$. Now let P be an element of $(\sum_{i=1}^{\infty} a_i X^i)F_1$. Then there exists an element α of F_1 such that $P = X(\sum_{i=1}^{\infty} a_i X^{i-1})\alpha$. Since $(\sum_{i=1}^{\infty} a_i X^i)F_1 \subseteq B[\![X]\!]$, $(\sum_{i=1}^{\infty} a_i X^{i-1})\alpha \in B[\![X]\!]$. Thus $P \in XB[\![X]\!]$, and so $(\sum_{i=1}^{\infty} a_i X^i)F_1 \subseteq XB[\![X]\!]$. This shows that

$$\begin{aligned} uI &\subseteq u(0)F_1 + XuF_2[\![X]\!] + (\sum_{i=1}^{\infty} a_i X^i)F_1 \\ &\subseteq A + XB[\![X]\!] \\ &= R. \end{aligned}$$

Hence $u \in I^{-1}$.

Now $u \in I^{-1}$ if and only if $u(0)F_1 \subseteq A$ and $uF_2[\![X]\!] \subseteq B[\![X]\!]$ which equivalent to $u(0) \in F_1^{-1}$ and $u \in (F_2[\![X]\!])^{-1}$. But $(F_2[\![X]\!])^{-1} = F_2^{-1}[\![X]\!]$. Hence $u \in I^{-1}$ if and only if $u \in F_1^{-1} \cap F_2^{-1} + XF_2^{-1}[\![X]\!]$. □

Example 2.2 Let $A = \mathbb{Z}$, $B = \mathbb{Z}[i]$ and $R = \mathbb{Z} + X\mathbb{Z}[i][\![X]\!]$. Let $I = 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][\![X]\!]$. We show that I is a divisorial ideal of R, i.e., $I_v = I$.

It is clear that I is an ideal of R. Now by Lemma 2.1,

$$\begin{aligned} I^{-1} &= \tfrac{1}{2}\mathbb{Z} \bigcap ((1+i)\mathbb{Z}[i])^{-1} + X((1+i)\mathbb{Z}[i])^{-1}[\![X]\!] \\ &= \tfrac{1}{2}\mathbb{Z} \bigcap (1+i)^{-1}\mathbb{Z}[i] + (1+i)^{-1}X\mathbb{Z}[i][\![X]\!]. \end{aligned}.$$

But if $x \in \frac{1}{2}\mathbb{Z} \bigcap \frac{1}{1+i}\mathbb{Z}[i]$, then $x = \frac{1}{2}r = \frac{1}{1+i}u$, with $r \in \mathbb{Z}$ and $u \in \mathbb{Z}[i]$. This implies that $(1+i)r = 2u$. Write $u = \alpha + i\beta$. Then $2\alpha = r$ and $2\beta = r$ thus 2 divided r, and so $x = \alpha \in \mathbb{Z}$. Hence $I^{-1} = \mathbb{Z} + X\frac{1-i}{2}\mathbb{Z}[i][[\![X]\!]$. Again by Lemma 2.1,

$$\begin{aligned} I_v &= (I^{-1})^{-1} \\ &= \mathbb{Z} \bigcap ((1+i)^{-1}\mathbb{Z}[i])^{-1} + X((1+i)^{-1}\mathbb{Z}[i])^{-1}[\![X]\!] \\ &= \mathbb{Z} \bigcap (1+i)\mathbb{Z}[i] + (1+i)X\mathbb{Z}[i][\![X]\!] \\ &= 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][\![X]\!] \\ &= I. \end{aligned}.$$

This shows that I is a divisorial ideal of R.

Let $A \subseteq B$ be an extension of integral domains. Following [3], we say that B is *t-linked over A*, if for each finitely generated fractional ideal I of A with $I^{-1} = A$, we have $(IB)^{-1} = B$.

Lemma 2.3 *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[\![X]\!]$. If B is t-linked over A, then the extension $A \subseteq R$ is t-linked.*

Proof Let I be a finitely generated fractional ideal of A such that $I^{-1} = A$. Since $IR \subseteq I + (IB)[\![X]\!]$, then by Lemma 2.1,

$$I^{-1} \cap (IB)^{-1} + X(IB)^{-1}[\![X]\!] = (I + (IB)[\![X]\!])^{-1} \subseteq (IR)^{-1}.$$

But B is t-linked over A, then $R = A + XB[\![X]\!] = I^{-1} \cap (IB)^{-1} + X(IB)^{-1}[\![X]\!] \subseteq (IR)^{-1}$, and hence $R \subseteq (IR)^{-1}$.

Now we will show that $(IR)^{-1} \subseteq R$. Let u be an element of $(IR)^{-1}$. It is easy to prove that $u \in L + XK[\![X]\!]$, where $L = qf(A)$ and $K = qf(B)$. Put $u = \sum_{i=0}^{\infty} a_i X^i \in L + XK[\![X]\!]$, and let $\alpha \in I$. Since $\alpha u = \sum_{i=0}^{\infty}(\alpha a_i)X^i \in R$, $\alpha a_0 \in A$, and hence $a_0 \in I^{-1}$. Moreover, if $r \in IB$, then $urX \in u(IR) \subseteq R$. This implies that for each $i \geq 1$, $ra_i \in B$. Therefore for each $i \geq 1$, $a_i \in (IB)^{-1}$. Hence $u \in I^{-1} + X(IB)^{-1}[\![X]\!] = A + XB[\![X]\!] = R$ since B is t-linked over A. Hence $(IR)^{-1} = R$. □

Proposition 2.4 *Let $A \subseteq B$ be an extension of integral domains such that B is t-linked over A. Then the mapping*

$$\begin{aligned} \varphi : Cl_t(A) &\rightarrow Cl_t(R) \\ [I] &\mapsto [(IR)_t] \end{aligned}$$

is an homomorphism.

Proof Follows from Lemma 2.3 and [3, Theorem 2.2]. □

Let $A \subseteq B$ be an extension of integral domains and I a finitely generated ideal of A. It well known that $I.A[\![X]\!] = (IA)[\![X]\!] = I[\![X]\!]$. Using the same proof we can prove that $I.B[\![X]\!] = (IB)[\![X]\!]$.

Lemma 2.5 *Let $A \subseteq B$ be an extension of integral domains such that B is a flat A-module, I an ideal of A and $R = A + XB[\![X]\!]$. We assume that I and I^{-1} are v-ideals of finite type. Then $(IR)_v = I + X(IB)[\![X]\!]$.*

Proof Since I and I^{-1} are v-ideals of finite type, $I = J_v$ and $I^{-1} = L_v$ for some finitely generated ideals J and L of A. Since $JR = J + X(JB)[\![X]\!]$, by Lemma 2.1,

$$\begin{aligned} (JR)^{-1} &= (J + X(JB)[\![X]\!])^{-1} \\ &= J^{-1} \cap (JB)^{-1} + X(JB)^{-1}[\![X]\!] \\ &= J^{-1} \cap J^{-1}B + X(J^{-1}B)[\![X]\!] \\ &= J^{-1}, \end{aligned}$$

where the third equality follows from the fact that B is a flat A-module.
Again apply Lemma 2.1, $(JR)_v = J_v \cap (J^{-1}B)^{-1} + X(J^{-1}B)^{-1}[\![X]\!]$.
Since $L_v = I^{-1} = J^{-1}$,

$$(J^{-1}B)^{-1} = (L_vB)^{-1} = (LB)^{-1} = L^{-1}B = J_vB,$$

where the second equality follow from the proof of [5, Proposition 2.2]. So

$$\begin{aligned} (JR)_v &= J_v \cap (J_vB) + X(J_vB)[\![X]\!] \\ &= J_v + X(J_vB)[\![X]\!] \\ &= I + X(IB)[\![X]\!]. \end{aligned}$$

This implies that $I + X(IB)[\![X]\!] \subseteq (IR)_v$. Now, using Lemma 2.1, we can prove that

$$(I + X(IB)[\![X]\!])_v = I + X(IB)[\![X]\!].$$

This shows that $(IR)_v \subseteq I + X(IB)[\![X]\!]$, and hence $(IR)_v = I + X(IB)[\![X]\!]$. □

We are now ready to prove the main result of this section.

Theorem 2.6 *Let $A \subseteq B$ be an extension of integral domains such that B is a flat A-module. Then the mapping*

$$\begin{aligned} \varphi : Cl_t(A) &\rightarrow Cl_t(R) \\ [I] &\mapsto [(IR)_t] \end{aligned}$$

is an injective homomorphism.

Proof Since B is a flat A-module, B is t-linked over A. So by Proposition 2.4, the mapping φ is an homomorphism. We show that φ is injective. Let I be a t-invertible t-ideal of A such that $(IR)_t$ is a principal ideal of R. We will prove that I is principal. Since $(IR)_t$ is principal, $(IR)_t = fR$ for some $f \in (IR)_t$.
Case 1: I is an integral ideal of A.
As $(IR)_t = fR$, then $(IR)_v = fR$. By Lemma 2.5, $(IR)_v = I + X(IB)[\![X]\!]$; so $I = f(0)A$ is a principal ideal of A.
Case 2: I is a fractional ideal of A.
Let $d \in A\backslash(0)$ such that $dI \subseteq A$. Put $I' = dI$. Then I' is an integral t-invertible t-ideal of A. Moreover, $(I'R)_t = dfR$ is a principal ideal of R. By case 1, I' is a principal ideal of A. So I is a principal ideal of A, and hence φ is injective. □

Remark 2.7 Let $A \subseteq B$ be an extension of integral domains and let $\varphi : Cl_t(A) \to Cl_t(R)$ be the natural mapping. Note that φ is not surjective in general. Indeed, let $A = \mathbb{Z}$, $B = \mathbb{Z}[i]$ and $R = \mathbb{Z} + X\mathbb{Z}[i][\![X]\!]$. Assume that φ is surjective.
By [6, Chapter 1, Proposition 2], $\mathbb{Z}[i] = \mathbb{Z} \oplus i\mathbb{Z}$ is a flat $\mathbb{Z}$-module; so by Theorem 2.6, φ is an injective homomorphism, and hence φ is an isomorphism. This implies that

$$Cl_t(\mathbb{Z}) \cong Cl_t(\mathbb{Z} + X\mathbb{Z}[i][\![X]\!]).$$

Since $\mathbb{Z}$ is a PID (principal ideal domain), $Cl_t(\mathbb{Z}) = 0$ which implies that $Cl_t(\mathbb{Z} + X\mathbb{Z}[i][\![X]\!]) = 0$. Now we prove that $Cl_t(\mathbb{Z} + X\mathbb{Z}[i][\![X]\!]) \neq 0$, and hence we obtain a contradiction. Let $I = 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][\![X]\!]$.
Claim 1: I and I^{-1} are ideals of R of v-finite type.
It is clear that $(2, (1+i)X) \subseteq I$. Conversely, let $f \in I$. Then $f = 2r + X(1+i)Q$, for some $r \in \mathbb{Z}$ and $Q \in \mathbb{Z}[i][\![X]\!] = \mathbb{Z} + i\mathbb{Z} + X\mathbb{Z}[i][\![X]\!]$. So there exist $s, t \in \mathbb{Z}$ and $h \in \mathbb{Z}[i][\![X]\!]$ such that $f = 2r + X(1+i)(s+it+Xh) = 2(r-tX) + (1+i)X(s+t+Xh) \in (2, (1+i)X)$. Hence $I = (2, (1+i)X)$. Now, by Example 2.2, $I^{-1} = \mathbb{Z} + X\frac{1-i}{2}\mathbb{Z}[i][\![X]\!]$. In the same way, we can show that $I^{-1} = (1, \frac{1-i}{2}X)$.
Claim 2: I is a v-invertible ideal of R.

Note that

$$II^{-1} = (1, \frac{1-i}{2}X)(2, (1+i)X) = (2, (1+i)X, (1-i)X, X^2).$$

Let $u \in \mathrm{qf}(R)$ such that $(2, (1+i)X, (1-i)X, X^2) \subseteq uR$. Since $2 \in (2, (1+i)X, (1-i)X, X^2) \subseteq uR$, then $u = \frac{2}{f}$, with $f \in R$ and $X^2 \in (2, (1+i)X, (1-i)X, X^2) \subseteq uR = \frac{2}{f}R$. Thus $X^2 f = 2g$, for some $g = a_0 + a_1X + \cdots + a_nX^n \in R$. This implies that $a_0 = a_1 = 0$, and so $g = X^2h$, where $h = (a_2 + \cdots + a_nX^{n-2}) \in \mathbb{Z}[i][\![X]\!]$. Then $f(0) = 2h(0) \in \mathbb{Z}$. But $\mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$, then $h(0) = s + it \in \mathbb{Z} + i\mathbb{Z}$. Since $2h(0) \in \mathbb{Z}$, then $h(0) \in \mathbb{Z}$, and so $1 = uh \in uR$. Thus

$$(II^{-1})_v = (2, (1+i)X, (1-i)X, X^2)_v = R.$$

Using claim 1 and 2, it is easy to prove that I is a t-invertible t-ideal of R. This implies that $[I] \in Cl_t(R)$. Now we show that $[I] \neq 0$ which equivalent to I is not a principal ideal of R. Assume the contrary that I is principal. Then $I = PR$ for some $P \in R$. Since $2 \in I$, $P(0) \neq 0$. In fact $P(0) \in \{\pm 1, \pm 2\}$. Moreover, as $(1+i)X \in I$, we obtain $P(0) \in \{\pm 1\}$ which implies that $P(0)$ is a unit in $\mathbb{Z}$. A routine calculation (by induction) shows that P is a unit in R. This implies that $I = PR = R$, a contradiction. Then $[I] \neq 0$, and hence $Cl_t(\mathbb{Z} + X\mathbb{Z}[i][\![X]\!]) \neq 0$.

3 *v*-Invertible *v*-Ideals of A+XB[[X]]

In this section, we investigate a relationship between v-invertible v-ideals of an integral domain and those of its power series ring of the form $A + XB[\![X]\!]$, where $A \subseteq B$ is an extension of integral domain. We begin this section by the following proposition.

Proposition 3.1 *Let $A \subseteq B$ be an extension of integral domain, J an ideal of A and $R = A + XB[\![X]\!]$.*

(1) If $(JR)_v = R$, then $J_v = A$.
(2) If $(JR)_t = R$, then $J_t = A$.

Proof(1). Assume that $(JR)_v = R$ and let $u \in \mathrm{qf}(A)$ such that $J \subseteq uA$. Then $JR \subseteq uAR \subseteq uR$ which implies that $R = (JR)_v \subseteq (uR)_v = uR$. Thus

$$A \subseteq \bigcap_{u \in \mathrm{qf}(A),\, J \subseteq uA} Au = J_v.$$

This shows that $A \subseteq J_v \subseteq A$, and hence $J_v = A$.

(2). Suppose that $(JR)_t = R$. Then

$$R = \bigcup \{(FR)_v,\ F \subseteq J \text{ of finite type of } A\}.$$

Thus there exists a finitely generated ideal F_0 of A such that $F_0 \subseteq J$ and $1 \in (F_0R)_v$. This implies that $R = (F_0R)_v$. Now, by (1), $(F_0)_v = A$; so

$$A \subseteq \bigcup \{F_v,\ F \subseteq J \text{ of finite type of } A\} = J_t \subseteq A.$$

Hence $A = J_t$.

□

Let A be an integral domain. According to [12, Theorem 2.11], A is completely integrally closed if and only if for each $f, g \in A[\![X]\!]$, $(A_f A_g)_v = (A_{fg})_v$. Using this result we prove a complete description of v-invertible v-ideals (with nonzero trace in A) of R. First we need to prove the following proposition.

Proposition 3.2 *Let $A \subseteq B$ be an extension of integral domains such that B completely integrally closed and $R = A + XB[\![X]\!]$. Then for each divisorial ideal I of R such that $I \cap A \neq (0)$, there exist a divisorial ideal J of B and a nonzero ideal $H \subseteq J$ of A such that $I = H + XJ[\![X]\!]$.*

Proof Let $H = I \bigcap A$ and J the ideal of B generated the coefficients of all elements of I.

It is clear that $H \subseteq J$ and $H \subseteq I$. We show that $XJ_v[\![X]\!] \subseteq I$. Let $f, g \in R,\ g \neq 0$ such that $I \subseteq \frac{f}{g}R$. Let $0 \neq a \in H$. Since $a \in H \subseteq I \subseteq \frac{f}{g}R$, then there exists an $r \in R\backslash(0)$ such that $\frac{a}{r} = \frac{f}{g}$. Let $0 \neq h \in I \subseteq \frac{f}{g}R = \frac{a}{r}R$. Then $rh \in aR$ which implies that $rh \in aB[\![X]\!]$. So $(A_{rh})_v \subseteq aB$. By hypothesis B is a completely integrally closed domain, then $A_r A_h \subseteq (A_r A_h)_v \subseteq aB$. This implies that $rA_h[\![X]\!] \subseteq aB[\![X]\!]$. Now we show that $rJ[\![X]\!] \subseteq aB[\![X]\!]$. Indeed, if $f \in rJ[\![X]\!]$, then $f = rf_1$ for some $f_1 = \sum_{i=0}^{\infty} a_i X^i \in J[\![X]\!]$. Put $r = \sum_{i=0}^{\infty} \beta_i X^i$. Then $f = \sum_{n=0}^{\infty}(\sum_{i=0}^{n} a_i \beta_{n-i})X^n$. But $a_i = \sum_{k=0}^{m_i} \alpha_{i,k} t_{i,k}$ with $t_{i,k} \in B$, $\alpha_{i,k} \in A_{f_{i,k}}$, then

$$a_i \beta_{n-i} = \sum_{k=0}^{n_i} \alpha_{i,k} t_{i,k} \beta_{n-i} \in A_r A_{f_{i,k}} \subseteq aB.$$

Which implies that $rJ[\![X]\!] \subseteq aB[\![X]\!]$. So

$$r(J[\![X]\!])_v = (rJ[\![X]\!])_v \subseteq (aB[\![X]\!])_v = a(B[\![X]\!])_v = aB[\![X]\!].$$

Since $(J[\![X]\!])_v = J_v[\![X]\!]$, $rJ_v[\![X]\!] \subseteq aB[\![X]\!]$. This implies that $\frac{aX}{r}B[\![X]\!] \subseteq \frac{a}{r}R$; so $\frac{rXJ_v[\![X]\!]}{r} \subseteq \frac{aX}{r}B[\![X]\!] \subseteq \frac{a}{r}R = \frac{f}{g}R$ which implies that $XJ_v[\![X]\!] \subseteq \frac{f}{g}R$. Thus

$$XJ_v[\![X]\!] \subseteq \cap_{f,g \in R, I \subseteq \frac{f}{g}} \frac{f}{g}R = I_v = I,$$

and hence $H + XJ[\![X]\!] \subseteq H + XJ_v[\![X]\!] \subseteq I$. Now we will show that $I \subseteq H + XJ[\![X]\!]$. Let $f \in I$. Then $f = a_0 + \sum_{i=1}^{\infty} a_i X^i$, where $a_0 \in A$ and $a_i \in B$ for each $i \geq 1$. As $J = < A_f,\ f \in I >$, then for each $i \geq 1$, $a_i \in J$; so

$$\sum_{i=1}^{\infty} a_i X^i = X \sum_{i=1}^{\infty} a_i X^{i-1} \in XJ[\![X]\!] \subseteq XJ_v[\![X]\!].$$

Since $XJ_v[\![X]\!] \subseteq I$, $\sum_{i=1}^{\infty} a_i X^i \in I$. This implies that $a_0 = f - \sum_{i=1}^{\infty} a_i X^i \in I$. Thus $a_0 \in A \cap I = H$, and hence $f \in H + XJ[\![X]\!]$. Now we have

$$H + XJ[\![X]\!] \subseteq H + XJ_v[\![X]\!] \subseteq I \subseteq H + XJ[\![X]\!].$$

Hence $I = H + XJ[\![X]\!]$ and $J_v = J$. □

Our next result give a complete description of v-invertible v-ideals of $A+XB[\![X]\!]$ with nonzero trace in A.

Theorem 3.3 *Let $A \subseteq B$ be an extension of integral domains such that B is completely integrally closed and $R = A + XB[\![X]\!]$. Let I be a fractional divisorial v-invertible ideal of R such that $I \cap A \neq (0)$. Then $I = u(J_1 + XJ_2[\![X]\!])$ for some $u \in qf(R)$, J_2 an integral divisorial v-invertible ideal of B and $J_1 \subseteq J_2$ a nonzero ideal of A.*

Proof Since I is a divisorial ideal of R and B is completely integrally closed, by Proposition 3.2, $I = H + XJ[\![X]\!]$ for some divisorial ideal J of B and a nonzero ideal $H \subseteq J$ of A. We show that there exists nonzero $c \in K$ such that $cH \subseteq A$ and $cJ \subseteq B$.
Let $a \in H$ be a nonzero element. We have aI^{-1} is a divisorial ideal of R. Using Lemma 2.1, it is easy to prove that $aI^{-1} \cap A \neq (0)$. Then by Proposition 3.2, $aI^{-1} = H' + XJ'[\![X]\!]$ for some divisorial ideal J' of B and a nonzero ideal $H' \subseteq J'$ of A.

$$\begin{aligned} aR &= a(II^{-1})_v \\ &= (a(II^{-1}))_v \\ &= (I(aI^{-1}))_v \\ &= ((H + XJ[\![X]\!])(H' + XJ'[\![X]\!]))_v. \end{aligned}$$

So $(H + XJ[\![X]\!])(H' + XJ'[\![X]\!]) \subseteq aR = aA + aXB[\![X]\!]$. Then $HH' \subseteq aA$ and $JJ' \subseteq aB$. This implies that $\frac{1}{a}HH' \subseteq A$ and $\frac{1}{a}JJ' \subseteq B$.

Let $c \in \frac{1}{a}H'$ be a nonzero element. Then $J_1 = cH \subseteq \frac{1}{a}HH' \subseteq A$ and $J_2 = cJ \subseteq B$. We have $J_1 \neq (0)$ and J_2 is a divisorial ideal of B.
Since $I = H + XJ[\![X]\!]$, then

$$I = \frac{1}{c}(cH + XcJ[\![X]\!]) = \frac{1}{c}(J_1 + XJ_2[\![X]\!]) = u(J_1 + XJ_2[\![X]\!]),$$

where $u = \frac{1}{c} \in qf(R)$. Now we will show that J_2 is v-invertible. By Lemma 2.1, we have

$$I^{-1} = \frac{1}{u}(J_1^{-1} \cap J_2^{-1} + XJ_2^{-1}[\![X]\!]).$$

Thus

$$\begin{aligned} II^{-1} &\subseteq J_1(J_1^{-1} \cap J_2^{-1}) + XJ_2(J_2^{-1}[\![X]\!]) \\ &\subseteq J_1J_1^{-1} + X(J_2J_2^{-1})[\![X]\!] \\ &\subseteq A + XB[\![X]\!] \\ &= R. \end{aligned}$$

Since I is v-invertible, we get

$$R = (J_1(J_1^{-1} \cap J_2^{-1}) + X(J_2J_2^{-1})[\![X]\!])^{-1}.$$

Again by Lemma 2.1, $R = (J_1(J_1^{-1} \cap J_2^{-1}))^{-1} \cap (J_2J_2^{-1})^{-1} + X(J_2J_2^{-1})^{-1}[\![X]\!]$. Then $B[\![X]\!] = (J_2J_2^{-1})^{-1}[\![X]\!]$, and this implies that $B = (J_2J_2^{-1})^{-1}$. Hence J_2 is v-invertible. ◻

Clearly that every Krull domain is completely integrally closed. Using Theorem 3.3, we obtain a new characterization of divisorial v-invertible ideals of the power series ring of the form $A + XB[\![X]\!]$.

Corollary 3.4 *Let I be a fractional divisorial v-invertible ideal of $R = A + XB[\![X]\!]$ such that $I \cap A \neq (0)$. Assume that B is a Krull domain. Then $I = u(J_1 + XJ_2[\![X]\!])$ for some $u \in qf(R)$, J_2 an integral divisorial v-invertible ideal of B and $J_1 \subseteq J_2$ a nonzero ideal of A.*

Recall from [4] that an integral domain A is called *formally integrally closed* if $(A_{fg})_t = (A_f A_g)_t$ for all $f, g \in A[\![X]\!]\backslash(0)$. It was shown in [4] that if A is formally integrally closed, then A is completely integrally closed, but the converse is false in general ([4, Example 3.2]).

Proposition 3.5 *[4, Proposition 3.6] Let A be a formally integrally closed domain. If I is a finite type v-ideal of $A[\![X]\!]$ with $J \cap A \neq 0$, then $I = J[\![X]\!]$ for some v-ideal J of A.*

Note that in [4] Anderson and Kang characterized the v-ideals of finite type of the power series ring $A[\![X]\!]$ with nonzero trace in A in the case when A is a formally integrally closed domain. Now, using Proposition 3.2, in the particular case when $A = B$, we obtain a new approach to characterize the divisorial ideals of the ring $A[\![X]\!]$ with nonzero trace in A.

Proposition 3.6 *Let A be a completely integrally closed domain and I a fractional divisorial ideal of $A[\![X]\!]$ such that $I \cap A \neq (0)$. Then $I = J_1 + XJ_2[\![X]\!]$ for some nonzero ideal J_1 of A and some divisorial ideal J_2 of A such that $J_1 \subseteq J_2$.*

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Declaration

Conflict of interest The author states that there is no Conflict of interest.

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