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# Inclusion and Geometric Properties of Mixed Morrey Double-Sequence Spaces

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## Abstract

In the continuous setting, Morrey spaces have been studied extensively, especially since the late 1960s. Meanwhile, Morrey sequence spaces, which are also known as discrete Morrey spaces, have only been developed by Gunawan et al. since 2018. In this article, we extend some known results on their inclusion properties and their (lack of) uniform nonsquareness to mixed Morrey double-sequence spaces, i.e. Morrey double-sequence spaces equipped with a mixed norm. As in the calculation of three geometric constants of Morrey spaces by Gunawan et al. in 2019, we also compute three geometric constants, namely Von Neumann-Jordan constant, James constant, and Dunkl-Williams constant for mixed Morrey double-sequence spaces. These constants measure uniformly nonsquareness of any Banach space. Through the values of the three constants, we reveal that mixed Morrey double-sequence spaces are not uniformly nonsquare. A relation between mixed Morrey double-sequence spaces and mixed Morrey spaces is also discussed.

**Keywords** Morrey double-sequence spaces · Discrete Morrey spaces · Mixed norms

**Mathematics Subject Classification** 42B35 · 46B45 · 46E30

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## 1 Introduction: Morrey Sequence Spaces

The 'continuous' Morrey spaces $M_q^p = M_q^p(\mathbb{R}^n)$ were introduced by C. B. Morrey [19]. A function $f$ on $\mathbb{R}^n$ belongs to $M_q^p$ $(1 \le p \le q < \infty)$ if and only if

$$\|f\|_{M_q^p} := \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p dy\right)^{1/p} < \infty.$$

Here, $B(a,r)$ is the ball centered at $a \in \mathbb{R}^n$ and of radius $r > 0$. While continuous Morrey spaces have been developed since the 1930 s (see [21, 22] and references therein), discrete Morrey spaces or Morrey sequence spaces $\ell_q^p = \ell_q^p(\mathbb{Z})$ $(1 \le p \le q < \infty)$ were first studied by Gunawan et al. [9] and have attracted many researchers since then (see, for examples, [3, 11, 12]).

A sequence $\{x_j\} := \{x_j\}_{j\in\mathbb{Z}}$ belongs to $\ell_q^p$ if and only if

$$\|\{x_j\}\|_{\ell_q^p} := \sup_{m\in\mathbb{Z}, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j\in S_{m,N}} |x_j|^p\right)^{1/p} < \infty.$$

Here, $S_{m,N} = \{m-N, m-N+1, \dots, m-1, m, m+1, \dots, m+N-1, m+N\}$. Note that for $p = q$, we have $\ell_p^p = \ell^p$.

One property that Morrey spaces have is the inclusion property [8, 20]. For Morrey sequence spaces $\ell_q^p$, the following theorem is found in [12]:

**Theorem 1.1** [12] *Let* $1 \le p_1 \le q_1 < \infty$ *and* $1 \le p_2 \le q_2 < \infty$. *Then* $\ell_{q_2}^{p_2} \subseteq \ell_{q_1}^{p_1}$ *if and only if* $q_2 \le q_1$ *and* $\dfrac{p_1}{q_1} \le \dfrac{p_2}{q_2}$.

***Remark 1.2*** Note that, the inclusion results in [12] are more general than Theorem 1.1, namely, including the results on sequences defined in higher dimension $\mathbb{Z}^n$, the quasi-Banach case $(0 < p_1, q_1, p_2, q_2 < 1)$, and compactness of embedding. An extension of this inclusion result to weak type Morrey sequence space can be found in [7]. In this paper, we shall concentrate on the case of Morrey sequence spaces as Banach spaces and also the sequence with index in $\mathbb{Z}$ only.

We note from the above theorem that for a fixed $q \in [1, \infty)$, the largest Morrey sequence space is $\ell_q^1$.

Besides the inclusion property, geometric properties of Morrey spaces, which are seen through some geometric constants, are quite interesting.

The *Von Neumann-Jordan constant* $\mathrm{C_{NJ}}(X)$ (see [15]), the *James constant* $\mathrm{C}_J(X)$ (see [13]) and the *Dunkl-Williams constant* $\mathrm{C_{DW}}(X)$ (see [4]) for a Banach space $X$ are given by

$$\mathrm{C_{NJ}}(X) := \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2(\|x\|_X^2 + \|y\|_X^2)} : x, y \in X \setminus \{0\}\right\},$$

$$\mathrm{C_J}(X) := \sup\{\min\{\|x+y\|_X, \|x-y\|_X\} : x, y \in X,\ \|x\|_X = \|y\|_X = 1\},$$

and

$$\mathrm{C_{DW}}(X) := \sup\left\{ \frac{\|x\|_X + \|y\|_X}{\|x-y\|_X} \left\| \frac{x}{\|x\|_X} - \frac{y}{\|y\|_X} \right\|_X : x, y \in X,\ x \neq 0, y \neq 0, x \neq y \right\},$$

respectively. As a consequence of the definition of $C_{\mathrm{NJ}}(X)$ and the triangle inequality, it is well known that $1 \leq C_{\mathrm{NJ}}(X) \leq 2$ for every Banach space $X$ (see [15, Theorem II]), and that $C_{\mathrm{NJ}}(X) = 1$ if and only if $X$ is a Hilbert space. Meanwhile, $\sqrt{2} \leq C_{\mathrm{J}}(X) \leq 2$ holds for every Banach space $X$, and $C_{\mathrm{J}}(X) = \sqrt{2}$ if (but not only if) $X$ is a Hilbert space (see [1, 5]). As for the Dunkl-Williams constant, we have $2 \leq \mathrm{C_{DW}}(X) \leq 4$ and $\mathrm{C_{DW}}(X) = 2$ if and only if $X$ is a Hilbert space (see [4]). Note that the larger the constant, the lesser round the unit ball in the space. Von Neumann-Jordan constant, James constant, and Dunkl-Williams constant are used to quantify convexity properties such as strict convexity, uniform convexity, and uniform squareness (see definitions in Subsection 2.2.3). In particular, $C_{NJ}(X) < 2$ is a necessary condition for uniform convexity of $X$ (see [16]). Moreover, $C_{NJ}(X) < 2$ is equivalent to uniform squareness of $X$ (see [23]). In addition, the condition $C_{NJ}(X) < 2$ imply the fixed point property for nonexpansive mappings on $X$ (see [6]).

For Lebesgue spaces $L^p = L^p(\mathbb{R}^n)$ where $1 \leq p \leq \infty$, it is known that $C_{\mathrm{NJ}}(L^p) = \max\{2^{2/p-1}, 2^{1-2/p}\}$ (see [2]) and $C_{\mathrm{J}}(L^p) = \max\{2^{1/p}, 2^{1-1/p}\}$ (see [17]). Meanwhile, for the Dunkl-Williams constant, we know that $C_{\mathrm{DW}}(L^1) = C_{\mathrm{DW}}(L^\infty) = 4$ (see [14]).

For continuous Morrey spaces $M^p_q$ and discrete Morrey spaces $\ell^p_q$, the following results are obtained in [10]:

**Theorem 1.3** [10] *If* $1 \leq p < q < \infty$, *then*

(i) $C_{\mathrm{NJ}}(M^p_q) = C_{\mathrm{J}}(M^p_q) = 2$ *and* $\mathrm{C_{DW}}(M^p_q) = 4$.
(ii) $C_{\mathrm{NJ}}(\ell^p_q) = C_{\mathrm{J}}(\ell^p_q) = 2$ *and* $\mathrm{C_{DW}}(\ell^p_q) = 4$.

Note that the three constants take the largest possible values, which mean that both continuous and discrete Morrey spaces are lacking the nonsquareness property (see Definition 2.7 for the definition of uniformly nonsquare Banach spaces).

In the following section, we present our results on the inclusion properties and geometric properties of mixed Morrey double-sequence spaces, which we shall define below. We shall also discuss the relation between mixed Morrey double-sequence spaces with mixed 'continuous' Morrey spaces (see [24] for the inclusion results of mixed Morrey spaces).

## 2 Mixed Morrey Double-Sequence Spaces

Let $1 \leq p \leq q < \infty$ and $1 \leq r \leq s < \infty$. The Morrey double-sequence spaces with mixed norm $\ell^p_q(\ell^r_s) = \ell^p_q(\ell^r_s)(\mathbb{Z}^2)$ is defined to be the set all double-sequences $\{x_{ij}\} = \{x_{ij}\}_{i,j\in\mathbb{Z}}$ for which

$$\|\{x_{ij}\}\|_{\ell^p_q(\ell^r_s)} := \|\{\|\{x_{ij}\}\|_{\ell^r_s,i}\}\|_{\ell^p_q,j} < \infty.$$

The notation $\|\{\cdot\}\|_{\ell^r_s,i}$ means the norm is calculated for a sequence with index $i$. From now on, we shall abbreviate the term Morrey double-sequence spaces with mixed norm by **mixed Morrey double-sequence spaces**.

### 2.1 A Key Lemma

We observe that an example of a member of a mixed Morrey double-sequence space can be obtained by taking the product of two sequences in associated Morrey sequence spaces. This fact is given in the following lemma.

**Lemma 2.1** *Let $1 \le p \le q < \infty$ and $1 \le r \le s < \infty$. Suppose that $\{y_i\} \in \ell^r_s$ and $\{z_j\} \in \ell^p_q$. If $x_{ij} := y_i z_j$, then $\{x_{ij}\} \in \ell^p_q(\ell^r_s)$ with*

$$\|\{x_{ij}\}\|_{\ell^p_q(\ell^r_s)} = \|\{y_i\}\|_{\ell^r_s}\|\{z_j\}\|_{\ell^p_q}. \tag{1}$$

***Proof*** The identity (1) follows directly from the definition of mixed Morrey double-sequence spaces. In fact,

$$\begin{aligned}\|\{x_{ij}\}\|_{\ell^p_q(\ell^r_s)} &= \|\{\|\{y_i z_j\}\|_{\ell^r_{s,i}}\}\|_{\ell^p_{q,j}} \\ &= \|\{\|\{y_i\}\|_{\ell^r_s,i} \cdot z_j\}\|_{\ell^p_q,j} = \|\{y_i\}\|_{\ell^r_s,i}\|\{z_j\}\|_{\ell^p_q,j} < \infty.\end{aligned}$$

Thus, $\{x_{ij}\} \in \ell^p_q(\ell^r_s)$ and the identity (1) holds. □

We shall use this lemma to prove the inclusion property and convexity properties of mixed Morrey double-sequence spaces.

### 2.2 Main Results

#### 2.2.1 Inclusion Properties

Our first result is the following theorem on the inclusion property of mixed Morrey double-sequence spaces.

**Theorem 2.2** *Let $1 \le p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $1 \le r_1 \le s_1 < \infty$, and $1 \le r_2 \le s_2 < \infty$. If $q_2 \le q_1$, $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$, $s_2 \le s_1$, and $\frac{r_1}{s_1} \le \frac{r_2}{s_2}$, then $\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2}) \subseteq \ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})$.*

***Proof*** Let $x = \{x_{ij}\} \in \ell^{p_2}_{q_2}(\ell^{r_2}_{s_2})$. Since $\ell^{r_2}_{s_2} \subseteq \ell^{r_1}_{s_1}$ with $\|\cdot\|_{\ell^{r_1}_{s_1}} \le \|\cdot\|_{\ell^{r_2}_{s_2}}$, we see that

$$\|\{x_{ij}\}\|_{\ell^{r_1}_{s_1},i} \le \|\{x_{ij}\}\|_{\ell^{r_2}_{s_2},i}$$

for every $j \in \mathbb{Z}$. The inclusion $\ell^{p_2}_{q_2} \subseteq \ell^{p_1}_{q_1}$ implies

$$\begin{aligned}\|x\|_{\ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})} &\le \|\{\|x_{ij}\|_{\ell^{r_2}_{s_2},i}\}\|_{\ell^{p_1}_{q_1},j}\\ &\le \|\{\|x_{ij}\|_{\ell^{r_2}_{s_2},i}\}\|_{\ell^{p_2}_{q_2},j} = \|x\|_{\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2})} < \infty.\end{aligned}$$

Hence, $x \in \ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})$. Thus, $\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2}) \subseteq \ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})$ with $\|\cdot\|_{\ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})} \le \|\cdot\|_{\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2})}$. □

***Remark 2.3*** Assume that $s_1 = s_2$, $r_1 < r_2$, $q_1 = q_2$, and $p_1 < p_2$. By using a similar argument as in the proof of Theorem 2.2 in [7], we can construct a sequence $x = \{x_i\} \in \ell^{r_1}_{s_1} \setminus \ell^{r_2}_{s_2}$. Similarly, one can construct $y = \{y_j\} \in \ell^{p_1}_{q_1} \setminus \ell^{p_2}_{q_2}$. If $z_{ij} = x_i y_j$, then $z = \{z_{ij}\} \in \ell^{p_1}_{q_1}(\ell^{r_1}_{s_1}) \setminus \ell^{p_2}_{q_2}(\ell^{r_2}_{s_2})$. Thus, the inclusion in Theorem 2.2 is proper under this assumption.

As the converse of Theorem 2.2, we have the following theorem.

**Theorem 2.4** *Let* $1 \le p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $1 \le r_1 \le s_1 < \infty$, *and* $1 \le r_2 \le s_2 < \infty$. *If* $\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2}) \subseteq \ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})$, *then* $\frac{1}{s_1} + \frac{1}{q_1} \le \frac{1}{s_2} + \frac{1}{q_2}$.

***Proof*** Let $K \in \mathbb{N}$. Define $y_i := \begin{cases} 1, & |i| \le K, \\ 0, & |i| > K \end{cases}$ and $z_j := \begin{cases} 1, & |j| \le K, \\ 0, & |j| > K \end{cases}$. Let $x_{ij} := y_i z_j$. Observe that, for $k = 1, 2$, we have $\|\{y_i\}\|_{\ell^{r_k}_{s_k},i} = (2K+1)^{\frac{1}{s_k}}$ and $\|\{z_j\}\|_{\ell^{p_k}_{q_k},j} = (2K+1)^{\frac{1}{q_k}}$. Hence, by Lemma 2.1, we obtain

$$\|\{x_{ij}\}\|_{\ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})} = (2K+1)^{\frac{1}{s_1}+\frac{1}{q_1}} \quad \text{and} \quad \|\{x_{ij}\}\|_{\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2})} = (2K+1)^{\frac{1}{s_2}+\frac{1}{q_2}}.$$

Since $\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2}) \subseteq \ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})$, there must exist a constant $C > 0$ (see [9]) such that

$$\|\{x_{ij}\}\|_{\ell^{p_1}_{q_1}(\ell^{r_1}_{s_1})} \le C\|\{x_{ij}\}\|_{\ell^{p_2}_{q_2}(\ell^{r_2}_{s_2})},$$

whence

$$(2K+1)^{\frac{1}{s_1}+\frac{1}{q_1}-(\frac{1}{s_2}+\frac{1}{q_2})} \le C.$$

As this is true for every $K \in \mathbb{N}$, we conclude that $\frac{1}{s_1} + \frac{1}{q_1} \le \frac{1}{s_2} + \frac{1}{q_2}$. □

### 2.2.2 Geometric Constants

Now we move to the geometric properties of mixed Morrey double-sequence spaces. As we have mentioned before, we shall study them through three geometric constants. Our result is the following.

**Theorem 2.5** *If* $1 \le p < q < \infty$ *and* $1 \le r < s < \infty$, *then* $C_{\mathrm{NJ}}(\ell^p_q(\ell^r_s)) = C_J(\ell^p_q(\ell^r_s)) = 2$ *and* $C_{\mathrm{DW}}(\ell^p_q(\ell^r_s)) = 4$.

***Proof*** We shall begin with Von Neumann-Jordan constant. Note that, $C_{\mathrm{NJ}}(\ell_q^p(\ell_s^r)) \le 2$. Therefore, we only need to prove that $C_{\mathrm{NJ}}(\ell_q^p(\ell_s^r)) \ge 2$. Since $q > p$ and $s > r$, there exists a positive, even integer $n$ such that $n > 2^{\frac{q}{q-p}} - 1$ and $n > 2^{\frac{s}{s-r}} - 1$. Define $\{x_i\}$ and $\{y_i\}$ by

$$x_i := \begin{cases} 1, & i \in \{0, n\} \\ 0, & i \notin \{0, n\} \end{cases} \quad \text{and} \quad y_i := \begin{cases} 1, & i = 0 \\ -1, & i = n \\ 0, & i \notin \{0, n\}. \end{cases}$$

Observe that

$$\begin{aligned} \|\{x_i\}\|_{\ell_q^p} &= \max\left\{ |S_{0,0}|^{\frac{1}{q}-\frac{1}{p}}|x_0|, |S_{n,0}|^{\frac{1}{q}-\frac{1}{p}}|x_n|, |S_{n/2,n/2}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{i\in S_{\frac{n}{2},\frac{n}{2}}} |x_i|^p \right)^{\frac{1}{p}} \right\} \\ &= \max\left\{1, (n+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}}\right\}. \end{aligned}$$

Since $n > 2^{\frac{q}{q-p}} - 1$, we have $n + 1 > 2^{\frac{q}{q-p}}$, so $(n+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}} < (2^{\frac{q}{q-p}})^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}} = 1$. Consequently, $\|\{x_i\}\|_{\ell_q^p} = 1$. By a similar argument, we obtain $\|\{x_i\}\|_{\ell_s^r} = 1$, $\|\{y_i\}\|_{\ell_q^p} = 1$, and $\|\{y_i\}\|_{\ell_s^r} = 1$.

Now let us define the double-sequences $a = \{a_{ij}\}$ and $b = \{b_{ij}\}$ by

$$a_{ij} = x_i x_j \quad \text{and} \quad b_{ij} = y_i y_j, \tag{2}$$

respectively. By virtue of Lemma 2.1, we have

$$\|a\|_{\ell_q^p(\ell_q^r)} = \|\{x_i\}\|_{\ell_s^r,i}\|\{x_j\}\|_{\ell_q^p,j} = 1 \quad \text{and} \quad \|b\|_{\ell_q^p(\ell_q^r)} = \|\{y_i\}\|_{\ell_s^r,i}\|\{y_j\}\|_{\ell_q^p,j} = 1. \tag{3}$$

According to the definition of $\{x_i\}$ and $\{y_i\}$, we see that

$$x_i + y_i = \begin{cases} 2, & i = 0 \\ 0, & i = n \\ 0, & i \notin \{0, n\} \end{cases} \quad \text{and} \quad x_i - y_i = \begin{cases} 0, & i = 0 \\ 2, & i = n \\ 0, & i \notin \{0, n\} \end{cases}.$$

Hence we have $\|\{x_i + y_i\}\|_{\ell_s^r} = 2$ and $\|\{x_i - y_i\}\|_{\ell_s^r} = 2$.

We now calculate $\|a+b\|_{\ell_q^p(\ell_s^r)}$ as follows. For every $i \in \mathbb{Z}$ and fixed $j \notin \{0, n\}$, we have $a_{ij}+b_{ij} = x_i \cdot 0 + y_i \cdot 0 = 0$. Therefore, $\|\{a_{ij}+b_{ij}\}\|_{\ell_s^r,i} = 0$ for every $j \notin \{0, n\}$. Meanwhile, for $j = 0$, we have $\|\{a_{ij} + b_{ij}\}\|_{\ell_s^r,i} = \|\{x_i + y_i\}\|_{\ell_s^r} = 2$. Similarly, for $j = n$, we obtain $\|\{a_{ij} + b_{ij}\}\|_{\ell_s^r,i} = \|\{x_i - y_i\}\|_{\ell_s^r} = 2$. These calculations can be summarized as

$$\|\{a_{ij}+b_{ij}\}\|_{\ell_s^r,i} = \begin{cases} 2, & j \in \{0,n\} \\ 0, & j \notin \{0,n\} \end{cases} = 2x_j.$$

Consequently,

$$\|a+b\|_{\ell_q^p(\ell_s^r)} = \|\|\{a_{ij}+b_{ij}\}\|_{\ell_s^r,i}\|_{\ell_q^p,j} = \|\{2x_j\}\|_{\ell_q^p,j} = 2. \tag{4}$$

By a similar argument, we also have

$$\|a-b\|_{\ell_q^p(\ell_s^r)} = 2. \tag{5}$$

We combine (3)–(5) to obtain

$$C_{N_J}(\ell_q^p(\ell_s^r)) \geq \frac{\|a+b\|^2_{\ell_q^p(\ell_s^r)} + \|a-b\|^2_{\ell_q^p(\ell_s^r)}}{2(\|a\|^2_{\ell_q^p(\ell_s^r)} + \|b\|^2_{\ell_q^p(\ell_s^r)})} = \frac{2^2+2^2}{2(1+1)} = 2,$$

as desired.

As for James constant, we let the sequences $a$ and $b$ be defined by (2). As a consequence of (4) and (5), we have

$$\begin{aligned} &C_J(\ell_q^p(\ell_s^r)) \\ &\quad = \sup\{\min\{\|a+b\|_{\ell_q^p(\ell_s^r)}, \|a-b\|_{\ell_q^p(\ell_s^r)}\} : \|a\|_{\ell_q^p(\ell_s^r)} = \|b\|_{\ell_q^p(\ell_s^r)} = 1\} = 2. \end{aligned}$$

Finally, we move to Dunkl-Williams constant. Let $t > 0$ and let $a$ and $b$ be the sequences be defined by (2). Define

$$u = a+b \quad , \quad v = a-b, \quad \text{and} \quad w = (1+t)u + (1-t)v. \tag{6}$$

Note that, $\|u+v\|_{\ell_q^p(\ell_s^r)} = \|v-u\|_{\ell_q^p(\ell_s^r)} = 2$. We now prove that

$$\|w\|_{\ell_q^p(\ell_s^r)} = 2+2t. \tag{7}$$

Observe that

$$w = (1+t)u + (1-t)v = (1+t)(a+b) + (1-t)(a-b) = 2a + 2tb.$$

Therefore,

$$w_{ij} = 2a_{ij} + 2tb_{ij} = 2x_i x_j + 2ty_i y_j.$$

Using the definition of $\{x_i\}$ and $\{y_i\}$, we have

$$w_{i0} = 2x_i + 2ty_i = \begin{cases} 2+2t & i=0 \\ 2-2t, & i=n \\ 0, & i \notin \{0,n\} \end{cases}$$

and

$$w_{in} = 2x_i - 2ty_i = \begin{cases} 2-2t & i=0 \\ 2+2t, & i=n \\ 0, & i \notin \{0,n\}. \end{cases}$$

Consequently, $\|w_{i0}\|_{\ell_s^r} = \|w_{in}\|_{\ell_s^r} = 2+2t$. Using this fact, we obtain

$$\| \|w_{ij}\|_{\ell_s^r,i}\|_{\ell_q^p,j} = \|(2+2t)\{x_j\}\|_{\ell_q^p,j} = (2+2t)\|\{x_j\}\|_{\ell_q^p,j} = 2+2t,$$

as desired. Thus, it follows from (7) that

$$\begin{aligned} \frac{\|u+v\|+\|w\|}{t\|v-u\|}\left\|\frac{u+v}{\|u+v\|}-\frac{w}{\|w\|}\right\| &= \frac{4+2t}{2t}\left\|\frac{u+v}{2}-\frac{w}{2+2t}\right\| \\ &= \frac{2+2t}{t}\left\|\frac{(1+t)(u+v)-w}{2+2t}\right\| \\ &= \frac{2+t}{2t+2t^2}\|2tv\| = \frac{4t+2t^2}{2t+2t^2}\|v\| = \frac{4+2t}{1+t}. \end{aligned}$$

Taking $t \to 0^+$, we find that $C_{\mathrm{DW}}(\ell_q^p(\ell_s^r)) = 4$. □

### 2.2.3 Convexity Properties

Besides the geometric constants, we also obtain the following results about the convexity properties of mixed Morrey double-sequence spaces.

Recall that a normed space $X$ is *strictly convex* if for every $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ we have $\|x+y\| < 2$, and is *uniformly convex* if for every $\epsilon \in (0, 2]$ there exists a $\delta \in (0, 1)$ such that for every $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \epsilon$ we have $\|x+y\| \leq 2(1-\delta)$. Note that, by definition, uniform convexity is stronger than strict convexity.

For mixed Morrey double-sequence spaces, we have the following theorem.

**Theorem 2.6** *If $1 \leq p < q < \infty$ and $1 \leq r < s < \infty$, then $\ell_q^p(\ell_s^r)$ is not strictly convex. Consequently, $\ell_q^p(\ell_s^r)$ is not uniformly convex.*

***Proof*** Let the sequences $a = \{a_{ij}\}$ and $b = \{b_{ij}\}$ be defined as in the proof of Theorem 2.5. Note that, $a \neq b$, $\|a\|_{\ell_q^p(\ell_s^r)} = 1$, $\|b\|_{\ell_q^p(\ell_s^r)} = 1$, and $\|a+b\|_{\ell_q^p(\ell_s^r)} = 2$. Thus, $\ell_q^p(\ell_s^r)$ is not strictly convex, and accordingly is not uniformly convex. □

We now discuss the lack uniformly nonsquarenes of mixed Morrey double-sequence spaces. Let us recall the following definition.

**Definition 2.7** [13] A Banach space $X$ is called *uniformly nonsquare* if there exists a $\delta > 0$ such that for every $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq 2(1-\delta)$ we have $\|x + y\| \leq 2(1-\delta)$.

A necessary and sufficient condition for uniform nonsquareness of a Banach space is given by the following theorem.

**Theorem 2.8** [23] *A Banach space $X$ is uniformly nonsquare if and only if $C_{NJ}(X) < 2$.*

Based on Theorems 2.5 and 2.8, we obtain the following corollary.

**Corollary 2.9** *If $1 \leq p < q < \infty$ and $1 \leq r < s < \infty$, then $\ell_q^p(\ell_s^r)$ is not uniformly nonsquare.*

### 2.3 An Additional Result: Relation with Continuous Morrey Spaces

As shown in [18], there is a relation between Morrey sequence spaces and Morrey spaces. From now on, $M_q^p$ denotes Morrey spaces over $\mathbb{R}$.

**Theorem 2.10** [18] *Let $1 \leq p \leq q < \infty$. For every $a = \{a_j\} \in \ell_q^p$, define*

$$\overline{a}(t) = \left( \sum_{j \in \mathbb{Z}} |a_j|^p \chi_{[j,j+1)}(t) \right)^{1/p}.$$

*Then, there exist positive constants $C_1$ and $C_2$, independent of $a$, such that*

$$C_1 \|a\|_{\ell_q^p} \leq \|\overline{a}\|_{M_q^p} \leq C_2 \|a\|_{\ell_q^p}.$$

***Remark 2.11*** A relation between Morrey spaces and Morrey sequence spaces is also given in [12, Remark 2.4].

A relation between mixed Morrey double-sequence spaces and Morrey spaces with mixed norm is given in the following proposition. Let us recall the definition of mixed Morrey spaces. The mixed Morrey space $M_q^p(M_s^r)(\mathbb{R}^2)$ is the set of all measurable functions $f$ on $\mathbb{R}^2$ for which

$$\|f\|_{M_q^p(M_s^r)} = \| \|f(x, y)\|_{M_s^r} \|_{M_q^p}$$

is finite. In the next proposition and its proof, the notation $\|\overline{x}\|_{M_q^p} \lesssim \|x\|_{\ell_q^p}$ means that there exists a constant $C > 0$ such that $\|\overline{x}\|_{M_q^p} \leq C\|x\|_{\ell_q^p}$.

**Proposition 2.12** *Let* $1 \le p \le q < \infty$ *and* $1 \le r \le s < \infty$. *For* $x = \{x_{ij}\} \in \ell_q^p(\ell_s^r)$, *define*

$$\overline{x}(t_1, t_2) = \left( \sum_{j\in\mathbb{Z}} \left( \sum_{i\in\mathbb{Z}} |x_{ij}|^r \chi_{[i,i+1)}(t_1) \right)^{p/r} \chi_{[j,j+1)}(t_2) \right)^{1/p}, \quad t_1, t_2 \in \mathbb{R}.$$

*Then* $\overline{x} \in M_q^p(M_s^r)(\mathbb{R}^2)$ *with* $\|\overline{x}\|_{M_q^p(M_s^r)} \lesssim \|x\|_{\ell_q^p(\ell_s^r)}$.

***Proof*** Observe that, for every $t_2 \in \mathbb{R}$, we have

$$\begin{aligned} \|\overline{x}(\cdot, t_2)\|_{M_s^r} &= \left\| \left\| \left\{ \left( \sum_{i\in\mathbb{Z}} |x_{ij}|^r \chi_{[i,i+1)}(\cdot) \right)^{1/r} \chi_{[j,j+1)}(t_2) \right\}_j \right\|_{\ell^p} \right\|_{M_s^r} \\ &\le \left\| \left\{ \left\| \left( \sum_{i\in\mathbb{Z}} |x_{ij}|^r \chi_{[i,i+1)}(\cdot) \right)^{1/r} \right\|_{M_s^r} \chi_{[j,j+1)}(t_2) \right\}_j \right\|_{\ell^p}. \end{aligned}$$

Applying Theorem 2.10 to $a_j := \{x_{ij}\}_i$ for each $j$, we have

$$\|\overline{x}(\cdot, t_2)\|_{M_s^r} \lesssim \left\| \left\{ \left\| \{x_{ij}\} \right\|_{\ell_s^r, i} \chi_{[j,j+1)}(t_2) \right\}_j \right\|_{\ell^p} = \left( \sum_{j\in\mathbb{Z}} \|x_{ij}\|_{\ell_s^r, i}^p \chi_{[j,j+1)}(t_2) \right)^{1/p}.$$

Applying Theorem 2.10 once again to $b := \{b_j\}$ and

$$\overline{b}(t_2) := \left( \sum_{j\in\mathbb{Z}} |b_j|^p \chi_{[j,j+1)}(t_2) \right)^{1/p}$$

with $b_j := \|\{x_{ij}\}\|_{\ell_s^r, i}$, we obtain

$$\|\overline{x}\|_{M_q^p(M_s^r)} \lesssim \left\| \left\| \left\{ \left\| \{x_{ij}\} \right\|_{\ell_s^r, i} \chi_{[j,j+1)}(\cdot) \right\}_j \right\|_{\ell^p} \right\|_{M_q^p} \lesssim \| \|\{x_{ij}\|_{\ell_s^r, i} \|_{\ell_q^p, j} = \|x\|_{\ell_q^p(\ell_s^r)},$$

as desired. □

***Remark 2.13*** At this time we do not know whether or not we have the inequality $\|x\|_{\ell_q^p(\ell_s^r)} \lesssim \|\overline{x}\|_{M_q^p(M_s^r)}$ for $1 \le p \le q < \infty$ and $1 \le r \le s < \infty$.

**Future works** It is interesting to investigate a generalization of our results to mixed Morrey double-sequence spaces defined on $\mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}$ and the case of quasi-Banach spaces. In addition, the generalization of our results to other extension of Morrey spaces such as weak Morrey spaces and generalized Morrey spaces is worth investigating.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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