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# Characterizations of Pluriharmonic Bloch Functions and Composition Operators in Bounded Symmetric Domains

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## Abstract

Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple $X$. First, we extend the definition for pluriharmonic Bloch functions to $\mathbb{B}_X$ by using the infinitesimal Kobayashi metric. Next, we develop some methods to investigate Bloch functions, and composition operators of pluriharmonic Bloch spaces on bounded symmetric domains. The obtained results provide the improvements and extensions of the corresponding known results.

**Keywords** Bloch space · Bounded symmetric domains · Composition operator · JB*-triple · The Kobayashi metric · Pluriharmonic function

**Mathematics Subject Classification** Primary: 31C10 · 32A18 · 32M15 · 47B33 · Secondary: 30H30

## 1 Preliminaries

For Banach spaces $X$ and $Y$ with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, let $L(X, Y)$ be the space of all continuous linear operators from $X$ into $Y$ with the standard operator norm

$$\|A\| = \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X},$$

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where $A \in L(X, Y)$. $L(X, Y)$ is a Banach space with respect to this norm. Denote by $X^*$ the dual space of the real or complex Banach space $X$. For $x \in X\backslash\{0\}$, let

$$T(x) = \{\ell_x \in X^* : \ \ell_x(x) = \|x\|_X \text{ and } \|\ell_x\|_{X^*} = 1\}.$$

Then the well known Hahn-Banach theorem implies that $T(x) \neq \emptyset$.

### Holomorphic Functions in Complex Banach Spaces

Let $\psi$ be a mapping of a domain $\Omega \subset X$ into a Banach space $Y$, where $X$ is a complex Banach space. We say that $\psi$ is differentiable at $z \in \Omega$ if there exists a bounded real linear operator $D\psi(z) : X \to Y$ such that

$$\lim_{\|\tau\|_X \to 0^+} \frac{\|\psi(z+\tau) - \psi(z) - D\psi(z)\tau\|_Y}{\|\tau\|_X} = 0.$$

Here $D\psi(z)$ is called the Fréchet derivative of $\psi$ at $z$. If $Y$ is a complex Banach space and $D\psi(z)$ is bounded complex linear for each $z \in \Omega$, then $\psi$ is said to be holomorphic in $\Omega$. Given domains $\Omega_1$ and $\Omega_2$ in complex Banach spaces $X$ and $Y$, respectively, we denote by $H(\Omega_1, \Omega_2)$ the set of holomorphic mappings from $\Omega_1$ into $\Omega_2$. The set $H(\Omega_1, \Omega_1)$ of self-mappings will be abbreviated to $H(\Omega_1)$.

### Pluriharmonic Functions in Complex Banach Spaces

Let $\mathbb{C}$ be the complex plane and let $\Omega$ be a domain in a complex Banach space $X$. A $C^2$ mapping $f$ of $\Omega$ into $\mathbb{C}$ is said to be *pluriharmonic* if the restriction of $f$ to every holomorphic curve in $\Omega$ is harmonic (cf. [3–5, 12, 16, 19, 20, 22, 26]). In particular, if $X = \mathbb{C}^n$ and $\Omega$ is a simply connected domain of $\mathbb{C}^n$, then a function $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if $f$ has a decomposition $f = f_1 + \overline{f_2}$, where $f_1, f_2 \in H(\Omega, \mathbb{C})$ (see [26]). This decomposition is unique up to an additive constant. Furthermore, if $n = 1$, then the pluriharmonic functions are equivalent to complex-valued harmonic functions (or harmonic mappings) (see [11]). Let $\mathscr{PH}(\Omega)$ denote the set of all pluriharmonic functions of $\Omega$ into $\mathbb{C}$ in the form $f = f_1 + \overline{f_2}$, where $f_1, f_2 \in H(\Omega, \mathbb{C})$. Note that if $X$ is finite dimensional and $\Omega$ is simply connected, then $\mathscr{PH}(\Omega)$ coincides with the set of all pluriharmonic functions of $\Omega$ into $\mathbb{C}$. In the following, if we write $f = f_1 + \overline{f_2}$ for $f \in \mathscr{PH}(\Omega)$, we always assume that $f_1, f_2 \in H(\Omega, \mathbb{C})$ with $f_2(0) = 0$, where $\Omega$ is a domain in a complex Banach space $X$ with $0 \in \Omega$.

### JB*-Triples

A complex Banach space $X$ is called a JB*-triple if it admits a continuous Jordan triple product $\{\cdot, \cdot, \cdot\} : X^3 \longrightarrow X$ which is symmetric and linear in the outer variables, but conjugate linear in the middle variable, and satisfies

(i) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\}$;

(ii) $a\Box a$ is a hermitian operator on $X$ and has non-negative spectrum;
(iii) $\|a\Box a\|_X = \|a\|_X^2$

for $a, b, c, x, y \in X$, where the box operator $a\Box b : X \to X$ is defined by $a\Box b(\cdot) = \{a, b, \cdot\}$ and satisfies $\|a\Box b\|_X \leq \|a\|_X \|b\|_X$.

#### The Möbius Transformations in Bounded Symmetric Domains

Let $\Omega$ be a domain in a complex Banach space $X$. Denote by $\mathrm{Aut}(\Omega)$ the set of biholomorphic automorphisms of $\Omega$. A domain $\Omega \subset X$ is said to be homogeneous if for each $x, y \in \Omega$, there exists $f \in \mathrm{Aut}(\Omega)$ such that $f(x) = y$. Every bounded symmetric domain in a complex Banach space $X$ is homogeneous. Conversely, the open unit ball $\mathbb{B}$ of $X$ admits a symmetry $s(z) = -z$ at 0 and if $\mathbb{B}$ is homogeneous, then $\mathbb{B}$ is a symmetric domain. The Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, the polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$ and the classical Cartan domains are bounded symmetric domains in $\mathbb{C}^n$. Banach spaces with homogeneous open unit ball are precisely the JB*-triples. In fact, every bounded symmetric domain in a complex Banach space is biholomorphic to the open unit ball of a JB*-triple [7]. We refer to [6, 7] for more details of JB*-triples and bounded symmetric domains.

Throughout this paper, we use $\mathbb{B}_X$ to denote the bounded symmetric domain realized as the open unit ball of a JB*-triple $X$. For every $x, y \in X$, the Bergman operator $B_X(x, y) \in L(X)$ is defined by

$$B_X(x, y)z = z - 2(x\Box y)(z) + \{x, \{y, z, y\}, x\} \qquad (z \in X).$$

For $\|x\|_X < 1$, the operator $B_X(x, x)$ has non-negative spectrum (see [6, Lemma 2.5.21]) and hence the square roots $B_X(x, x)^{\pm 1/2}$ exist. For each element $a$ in the open unit ball $\mathbb{B}_X$ of $X$, the Möbius transformation $g_a \in \mathrm{Aut}(\mathbb{B}_X)$, induced by $a$, is given by

$$g_a(x) = a + B_X(a, a)^{1/2}(I_X + x\Box a)^{-1}(x), \tag{1.1}$$

with $g_a^{-1} = g_{-a}$, $g_a(-a) = 0$, $g_a(0) = a$ and $Dg_a(0) = B(a, a)^{1/2}$.

Throughout this paper, we use the symbol $C$ to denote the various positive constants, whose value may change from one occurrence to another.

## 2 Introduction and Main Results

#### The Kobayashi Metric and Pluriharmonic Bloch Functions

For a JB*-triple $X$, let

$$\kappa_X(z, w) = \inf\{\eta > 0 : \exists \phi \in H(\mathbb{D}, \mathbb{B}_X), \phi(0) = z, D\phi(0)\eta = w\}$$

be the infinitesimal Kobayashi metric on $\mathbb{B}_X$, where $\mathbb{D} := \mathbb{B}^1$ is the unit disk in $\mathbb{C}$. Then $\kappa_X(0, w) = \|w\|_X$ for all $w \in X$, and

$$\kappa_X(z, w) = \|Dg_{-z}(z)w\|_X = \|B_X(z, z)^{-1/2}w\|_X, \quad z \in \mathbb{B}_X, w \in X, \tag{2.1}$$

where $g_{-z} \in \mathrm{Aut}(\mathbb{B}_X)$ is the Möbius transformation induced by $-z$ given by (1.1) and $B_X(\cdot, \cdot)$ is the Bergman operator. Furthermore, for $z \in \mathbb{B}_X$ and $w \in X$, we have

$$\kappa_X(z, w) \leq \frac{\|w\|_X}{1 - \|z\|_X^2}. \tag{2.2}$$

The Kobayashi metric $\rho$ on $\mathbb{B}_X$, which is the integral form of the infinitesimal Kobayashi metric $\kappa_X$ and generalizes the Poincaré metric $\rho_{\mathbb{D}}$ on $\mathbb{D}$, can be described by a Möbius transformation: $\rho(a, b) = \tanh^{-1} \|g_{-a}(b)\|_X$ for $a, b \in \mathbb{B}_X$, where $g_{-a} \in \mathrm{Aut}(\mathbb{B}_X)$ is the Möbius transformation induced by $-a$ given by (1.1) (cf. [7, 8]). In particular, $\rho(a, 0) = \tanh^{-1} \|a\|_X$. Moreover, for $z,\ w \in \mathbb{B}_X$, we have

$$\rho(z, w) = \frac{1}{2} \log \frac{1 + \|\varphi_z(w)\|_X}{1 - \|\varphi_z(w)\|_X},$$

where $\varphi_z \in \mathrm{Aut}(\mathbb{B}_X)$ with $\varphi_z(z) = 0$ (see [7, Theorem 3.5.9]).

In [25], Bergman metric plays an essential role in the definition and equivalent conditions for (holomorphic) Bloch functions in finite dimensional case. On bounded symmetric domains $\mathbb{B}_X$ realized as the open unit balls of JB*-triples $X$, the Bergman metric is not available in general. So, Chu et al. [8] used the infinitesimal Kobayashi metric instead to circumvent this difficulty. Similarly, we generalize the definition of Bloch functions for holomorphic functions on $\mathbb{B}_X$ to pluriharmonic functions $\mathbb{B}_X$ as follows.

**Definition 1** Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple $X$. A function $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$ is called a *pluriharmonic Bloch function* if

$$\sup\{Q_f(z) : z \in \mathbb{B}_X\} < \infty,$$

where

$$Q_f(z) = \sup\left\{\frac{|Df_1(z)x + \overline{Df_2(z)x}|}{\kappa_X(z, x)} : x \in X \setminus \{0\}\right\}.$$

The class of all pluriharmonic Bloch functions will be denoted by $\mathscr{B}(\mathbb{B}_X)$.

For each $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$, let

$$\Lambda_f(z) = \sup\{|Df_1(z)x| + |Df_2(z)x| : \|x\|_X = 1\}, \quad z \in \mathbb{B}_X$$

and let

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s} = \sup\left\{\Lambda_{f\circ\Phi}(0) : \Phi \in \operatorname{Aut}(\mathbb{B}_X)\right\}.$$

By using an argument similar to that in the proof of [8, Lemma 3.2], we have

**Theorem 1** *Suppose that X is a JB*-triple. Let* $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$. *Then*

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s} = \sup\{Q_f(z) : z \in \mathbb{B}_X\}.$$

$\|f\|_{\mathscr{B}(\mathbb{B}_X),s}$ will be called the *Bloch semi-norm* of $f$. We equip $\mathscr{B}(\mathbb{B}_X)$ with a norm, called the *Bloch norm*, defined by

$$\|f\|_{\mathscr{B}(\mathbb{B}_X)} = |f(0)| + \|f\|_{\mathscr{B}(\mathbb{B}_X),s} \qquad (f \in \mathscr{B}(\mathbb{B}_X))$$

and call $\mathscr{B}(\mathbb{B}_X)$ the *pluriharmonic Bloch space* on $\mathbb{B}_X$. Since $\mathscr{B}(\mathbb{B}_X) \cap H(\mathbb{B}_X, \mathbb{C})$ is a complex Banach space ([8, Proposition 3.6]), $\mathscr{B}(\mathbb{B}_X)$ is also a complex Banach space. We compare $\sup\{Q_f(z) : z \in \mathbb{B}_X\}$ and

$$\sup_{z\in\mathbb{B}_X}\left\{(1-\|z\|_X^2)\Lambda_f(z)\right\}$$

as follows.

**Remark 1** (i) If $\mathbb{B}_X = \mathbb{D}$, then $\kappa_{\mathbb{C}}(z, w) = |w|/(1-|z|^2)$ for $z \in \mathbb{D}$ and $w \in \mathbb{C}$. So, we have

$$\sup\{Q_f(z), z \in \mathbb{D}\} = \sup_{z\in\mathbb{D}}\left\{(1-|z|^2)\Lambda_f(z)\right\}$$

and Definition 1 coincides with the usual definition of harmonic Bloch functions (cf. [10]).

(ii) If $f = f_1 + \overline{f}_2 \in \mathscr{PH}(\mathbb{B}_X)$, then, by using (2.2), we have

$$\Lambda_f(z) \le \frac{Q_f(z)}{1-\|z\|_X^2}.$$

This implies that if $f = f_1 + \overline{f}_2 \in \mathscr{B}(\mathbb{B}_X)$, then

$$\sup_{z\in\mathbb{B}_X}\left\{(1-\|z\|_X^2)\Lambda_f(z)\right\} < \infty.$$

(iii) Let $\mathbb{B}_X = \mathbb{B}_H$ be the unit ball of a complex Hilbert space $H$ and let $f = f_1 + \overline{f}_2 \in \mathscr{PH}(\mathbb{B}_H)$. Then we can show that

$$\sup_{z\in\mathbb{B}_H}\left\{(1-\|z\|_H^2)\Lambda_f(z)\right\} < \infty$$

implies that $f = f_1 + \overline{f}_2 \in \mathscr{B}(\mathbb{B}_H)$ as in [2, Theorem 3.8].

(iv) In [9, Example 2.10] and [17, Proposition 2.5], it is shown independently that there exists $f \in H(\mathbb{D}^2, \mathbb{C})$ such that

$$\sup_{z\in\mathbb{D}^2} \left\{(1 - \|z\|_\infty^2)\|Df(z)\|\right\} < \infty$$

and

$$\sup_{z\in\mathbb{D}^2} Q_f(z) = \infty,$$

where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{C}^2$.

Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. Given a holomorphic mapping $\phi \in H(\mathbb{B}_X, \mathbb{B}_Y)$, define the Kobayashi constant of $\phi$ by

$$K_\phi := \sup_{z\in\mathbb{B}_X} \sup_{x\in X\setminus\{0\}} \frac{\kappa_Y(\phi(z), D\phi(z)x)}{\kappa_X(z, x)}.$$

In contrast to the Bergman constant $B_\phi$ defined in [1], we always have $K_\phi \le 1$ by the contractive property of $\kappa$ (cf. [1, p.682; Open question (3), p.687]).

**Proposition 1** *Let $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_Y)$ and $\phi \in H(\mathbb{B}_X, \mathbb{B}_Y)$, where $\mathbb{B}_X$ and $\mathbb{B}_Y$ are bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. Then $Q_{f\circ\phi}(z) \le K_\phi Q_f(\phi(z))$ for each $z \in \mathbb{B}_X$.*

For $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$ and $z_0 \in \mathbb{B}_X$, we define a family $F_f(z_0)$ by

$$F_f(z_0) = \{f \circ \Phi - (f \circ \Phi)(z_0) : \Phi \in \mathrm{Aut}(\mathbb{B}_X)\}.$$

Analogy with the holomorphic case in [25, Theorem 3.4] and [8, Theorem 3.8], we give several characterizations of pluriharmonic Bloch functions in infinite dimensional bounded symmetric domains as follows by using Theorem 1 and Proposition 1. Let $\mathbb{B}_X(0, r) = \{z \in X : \|z\| < r\}$.

**Theorem 2** *Let $X$ be a JB*-triple and let $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$. The following conditions are equivalent:*

*(1) $f \in \mathscr{B}(\mathbb{B}_X)$.*
*(2) $f$ is uniformly continuous as a function from the metric space $(\mathbb{B}_X, \rho)$ to the metric space ($\mathbb{C}$, Euclidean distance).*
*(3) The family $F_f(z_0)$ is bounded on $\mathbb{B}_X(0, r)$ for $0 < r < 1$ and $z_0 \in \mathbb{B}_X$.*
(4) $\|f\|_{\mathscr{B}(\mathbb{B}_X),s} < \infty$.
(5) *The family $\{f \circ \psi : \psi \in H(\mathbb{D}, \mathbb{B}_X)\}$ consists of harmonic Bloch functions on $\mathbb{D}$ with uniformly bounded Bloch semi-norm.*

(6) *The family* $\{f \circ \psi - (f \circ \psi)(0) : \psi \in H(\mathbb{D}, \mathbb{B}_X)\}$ *is locally uniformly bounded on* $\mathbb{D}$.

For a JB*-triple $X$ and $f \in \mathscr{PH}(\mathbb{B}_X)$, the Lipschitz number of $f$ is defined by

$$\mathscr{L}_f = \sup_{z,w \in \mathbb{B}_X, z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

Actually, the Lipschitz number in this case is given by the Bloch seminorm of $f$, as follows (see [1, Theorem 3.1] and [9, Proposition 2.3] in the holomorphic case).

**Theorem 3** *Let X be a JB*-triple and let* $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$. *Then* $f \in \mathscr{B}(\mathbb{B}_X)$ *if and only if* $\mathscr{L}_f < \infty$. *Moreover,*

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s} = \mathscr{L}_f.$$

For harmonic mappings $f$ of $\mathbb{D}$ into itself, Colonna [10] proved the following result.

**Theorem A** *([10, Theorems 3 and 4]) Let* $f$ *be a harmonic mapping of* $\mathbb{D}$ *into itself. Then* $\mathscr{L}_f \leq 4/\pi$. *The constant* $4/\pi$ *in this inequality can not be improved.*

In the following, we extend Theorem A to bounded symmetric domains by using Theorem 3.

**Theorem 4** *For a JB*-triple X, let* $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$ *with* $\|f\|_\infty = \sup_{z \in \mathbb{B}_X} |f(z)| < \infty$. *Then* $\mathscr{L}_f \leq 4\|f\|_\infty/\pi$. *The constant* $4/\pi$ *in this inequality is sharp.*

We obtain the following corollary from Theorems 3 and 4.

**Corollary 1** *Let X be a JB*-triple and let* $f = f_1 + \overline{f_2} \in \mathscr{PH}(\mathbb{B}_X)$. *If* $f$ *is bounded on* $\mathbb{B}_X$, *then* $f \in \mathscr{B}(\mathbb{B}_X)$.

Holland-Walsh [15] gave the following characterization of Bloch functions on $\mathbb{D}$.

**Theorem B** ([15, Theorem 3]) *Let* $f$ *be a holomorphic function on* $\mathbb{D}$. *Then* $f$ *is a Bloch function if and only if*

$$S_{\mathbb{D}}(f) = \sup_{z,w \in \mathbb{D}, z \neq w} \left\{ (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} \right\} < \infty.$$

Ren and Tu [21] extended Theorem B to holomorphic functions on $\mathbb{B}^n$ of $\mathbb{C}^n$, and Chu et al. [9] extended it to holomorphic functions on the complex Hilbert balls. It is also remarked that it can not be extended to holomorphic functions on bounded symmetric domains (see [9]). In order to extend Theorem B to bounded symmetric domains, we modify $S_{\mathbb{D}}(f)$ into the following form:

$$S_{\mathbb{D}}(f) = \sup_{z,w \in \mathbb{D}, z \neq w} \left\{ (1 - |\phi_z(w)|^2)^{1/2} \frac{|f(z) - f(w)|}{|\phi_z(w)|} \right\},$$

where $\phi_a(\zeta)$, $a \in \mathbb{D}$, is the Möbius transformation given by

$$\phi_a(\zeta) = \frac{a - \zeta}{1 - \zeta\overline{a}}, \quad \zeta \in \mathbb{D}.$$

For a JB*-triple $X$ and $f \in \mathscr{PH}(\mathbb{B}_X)$, we define

$$S(f) = \sup_{z,w\in\mathbb{B}_X, z\neq w} \left\{(1 - \|\Phi_z(w)\|_X^2)^{1/2} \frac{|f(z) - f(w)|}{\|\Phi_z(w)\|_X}\right\},$$

where $\Phi_a(x) = g_a(-x) \in \mathrm{Aut}(\mathbb{B}_X)$ and $g_a$ is the Möbius transformation, induced by $a$, given by (1.1). Then, by using $S(f)$ and Theorem 3, we extend Theorem B to pluriharmonic functions on bounded symmetric domains $\mathbb{B}_X$ as follows.

**Theorem 5** *Let $X$ be a JB*-triple, and let $f \in \mathscr{PH}(\mathbb{B}_X)$. Then $f \in \mathscr{B}(\mathbb{B}_X)$ if and only if $S(f) < \infty$.*

### Composition Operators in Bounded Symmetric Domains

Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. For a given $\phi \in H(\mathbb{B}_X, \mathbb{B}_Y)$, the composition operator $C_\phi : \mathscr{PH}(\mathbb{B}_Y) \to \mathscr{PH}(\mathbb{B}_X)$ is defined by

$$C_\phi(f) = f \circ \phi,$$

where $f \in \mathscr{PH}(\mathbb{B}_Y)$.

In 1987, Shapiro [23] gave a complete characterization of compact composition operators on $\mathscr{H}^2(\mathbb{D})$, with a number of interesting consequences for peak sets, essential norm of composition operators, and so on. Recently, the study of composition operators from one holomorphic Bloch type space to another has attracted much attention of many mathematicians (see e.g. [1, 8, 9, 18, 24, 27–29]). In particular, Chu et al. [8, 9], Zhou and Shi [28] investigated the composition operators of holomorphic Bloch spaces in bounded symmetric domains. It is motivated by these articles that we first establish the boundedness of a composition operator $C_\phi$ between the pluriharmonic Bloch spaces on infinite dimensional bounded symmetric domains. Since the operator norm of $C_\phi$ depends on the norm of the underlying Banach space, it should be pointed out that in the literature for finite dimensional domains $\mathbb{B}_X$, the operator $C_\phi$ on the Bloch spaces is considered by the Bergman metric, whereas we consider $C_\phi$ on the Bloch spaces by the infinitesimal Kobayashi metric. By using Theorem 3 and Proposition 1, we extend [9, Theorem 3.2], [1, Theorem 3.2 and Corollary 3.1] and [27, Theorem 2 and Corollary 1] to the following form.

**Theorem 6** *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. Let $\phi \in H(\mathbb{B}_X, \mathbb{B}_Y)$. Then $C_\phi : \mathscr{B}(\mathbb{B}_Y) \to \mathscr{B}(\mathbb{B}_X)$ is bounded and*

$$\max\{1, \rho_Y(\phi(0), 0)\} \le \|C_\phi\| \le \max\left\{1, \rho_Y(\phi(0), 0) + K_\phi\right\}.$$

*If* $\phi(0) = 0$, *then* $\|C_\phi\| = 1$.

In [9, Proposition 3.3], Chu et al. obtained a sufficient condition for $C_\phi$ to be an isometry on $\mathscr{B}(\mathbb{B}_X) \cap H(\mathbb{B}_X, \mathbb{C})$ (cf. [1, Theorem 5.1]). We extend it to the pluriharmonic case as follows by using Theorem 3.

**Theorem 7** *Let $X$ be a JB*-triple, and let $\phi \in H(\mathbb{B}_X)$ with $\phi(0) = 0$. If there is a sequence $\{S_j\}$ in $\mathrm{Aut}(\mathbb{B}_X)$ such that $\{\phi \circ S_j\}$ converges locally uniformly to the identity mapping on $\mathbb{B}_X$, then $C_\phi$ is an isometry on $\mathscr{B}(\mathbb{B}_X)$.*

For compactness of $C_\phi$, we obtain the following lemma easily.

**Lemma 1** *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. Let $\phi \in H(\mathbb{B}_X, \mathbb{B}_Y)$. Then $C_\phi : \mathscr{B}(\mathbb{B}_Y) \to \mathscr{B}(\mathbb{B}_X)$ is compact if and only if $C_\phi : \mathscr{B}(\mathbb{B}_Y) \cap H(\mathbb{B}_Y, \mathbb{C}) \to \mathscr{B}(\mathbb{B}_X) \cap H(\mathbb{B}_X, \mathbb{C})$ is compact.*

By using Lemma 1, many results in [8] related to the compactness of $C_\phi$ can be generalized to the pluriharmonic Bloch type space. We omit the details.

The proofs of Theorems 1–5 and Proposition 1 will be presented in Sect. 3, and the proofs of Theorems 6 and 7 will be given in Sect. 4.

## 3 The Kobayashi Metric and Pluriharmonic Bloch Functions

### Proof of Theorem 1

Let $z \in \mathbb{B}_X \setminus \{0\}$ be fixed, and let $g_z \in \mathrm{Aut}(\mathbb{B}_X)$ be the Möbius transformation, induced by $z$, given by (1.1). Then, by (2.1), we have

$$|Df_1(z)x + \overline{Df_2(z)x}| \leq \Lambda_{f \circ g_z}(0)\|Dg_{-z}(z)x\|_X \leq \kappa_X(z, x)\|f\|_{\mathscr{B}(\mathbb{B}_X),s}.$$

Also, we have

$$|Df_1(0)x + \overline{Df_2(0)x}| \leq \|x\|_X \|f\|_{\mathscr{B}(\mathbb{B}_X),s} = \kappa_X(0, x)\|f\|_{\mathscr{B}(\mathbb{B}_X),s}.$$

This gives $\sup\{Q_f(z) : z \in \mathbb{B}_X\} \leq \|f\|_{\mathscr{B}(\mathbb{B}_X),s}$.

On the other hand, for $x \in X \setminus \{0\}$ and $\Phi \in \mathrm{Aut}(\mathbb{B}_X)$, let

$$\Psi(x) = \left| D(f_1 \circ \Phi)(0)\left(\frac{x}{\|x\|_X}\right) + \overline{D(f_2 \circ \Phi)(0)\left(\frac{x}{\|x\|_X}\right)} \right|.$$

Then we have

$$\Psi(x) = \frac{|Df_1(\Phi(0))D\Phi(0)x + \overline{Df_2(\Phi(0))D\Phi(0)x}|}{\kappa_X(\Phi(0), D\Phi(0)x)} \leq Q_f(\Phi(0)). \tag{3.1}$$

Note that,

$$\begin{aligned}
&\sup\left\{\Psi(x):\ x\in\ X\backslash\{0\}\right\}\\
&\quad=\sup\left\{|D(f_1\circ\Phi)(0)e^{i\theta}\varsigma+\overline{D(f_2\circ\Phi)(0)e^{i\theta}\varsigma}|:\theta\in\mathbb{R},\ \|\varsigma\|_X=1\right\}\\
&\quad=\sup\left\{|e^{i\theta}\left(D(f_1\circ\Phi)(0)\varsigma+e^{-2i\theta}\overline{D(f_2\circ\Phi)(0)\varsigma}\right)|:\theta\in\mathbb{R},\ \|\varsigma\|_X=1\right\}\\
&\quad=\sup\left\{|D(f_1\circ\Phi)(0)\varsigma+e^{-2i\theta}\overline{D(f_2\circ\Phi)(0)\varsigma}|:\theta\in\mathbb{R},\ \|\varsigma\|_X=1\right\}\\
&\quad=\sup\left\{|D(f_1\circ\Phi)(0)\varsigma|+|\overline{D(f_2\circ\Phi)(0)\varsigma}|:\|\varsigma\|_X=1\right\}. \qquad (3.2)
\end{aligned}$$

Combining (3.1) and (3.2) gives

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s}\leq\sup\{Q_f(z):z\in\mathbb{B}_X\}.$$

This completes the proof. □

**Proof of Proposition 1**

Let $z\in\mathbb{B}_X$ and $x\in X\setminus\{0\}$ be fixed. We only need to show that

$$\frac{|D(f_1\circ\phi)(z)x+\overline{D(f_2\circ\phi)(z)x}|}{\kappa_X(z,x)}\leq K_\phi Q_f(\phi(z)).$$

If $D\phi(z)x=0$, then $D(f_1\circ\phi)(z)x+\overline{D(f_2\circ\phi)(z)x}=0$. So the above inequality holds. If $D\phi(z)x\neq0$, then we have

$$\begin{aligned}
\frac{|D(f_1\circ\phi)(z)x+\overline{D(f_2\circ\phi)(z)x}|}{\kappa_X(z,x)}&=\frac{\kappa_Y(\phi(z),D\phi(z)x)}{\kappa_X(z,x)}\\
&\quad\times\frac{|Df_1(\phi(z))D\phi(z)x+\overline{Df_2(\phi(z))D\phi(z)x}|}{\kappa_Y(\phi(z),D\phi(z)x)}\\
&\leq K_\phi Q_f(\phi(z)).
\end{aligned}$$

This completes the proof. □

**Proof of Theorem 2**

The proof of (2)⇒(3) can be obtained by adapting the proof method of [8, Theorem 3.8]. (4)⇒(1) and (5)⇒(6) follow from Theorem 1 and (1)⇒(2), respectively. By using the similar reasoning as in the proof of (3)⇒(4), we can obtain (6)⇒(5). Hence we only need to prove (1)⇒(2), (3)⇒(4) and (4)⇔(5).

We first prove (1)⇒(2). Let $C=\sup\{Q_f(z):z\in\mathbb{B}_X\}$, and let $z_1,z_2\in\mathbb{B}_X$ be fixed. Suppose that $\gamma:[0,1]\to\mathbb{B}_X$ is an arbitrary piecewise $C^1$ smooth curve with

$\gamma(0) = z_1$ and $\gamma(1) = z_2$. Then

$$\begin{aligned} |f(z_1) - f(z_2)| &\le \int_0^1 |Df_1(\gamma(t))\gamma'(t) + \overline{Df_2(\gamma(t))\gamma'(t)}|dt \\ &\le \int_0^1 Q_f(\gamma(t))\kappa_X(\gamma(t), \gamma'(t))dt \\ &\le C\int_0^1 \kappa_X(\gamma(t), \gamma'(t))dt. \end{aligned}$$

This gives

$$|f(z_1) - f(z_2)| \le C\rho(z_1, z_2),$$

and proves uniform continuity of $f$ as a function between the metric spaces $(\mathbb{B}_X, \rho)$ and $(\mathbb{C}$, Euclidean distance).

Next, we prove (3)⇒(4). Let $r \in (0, 1)$ be fixed. Then there is a constant $M > 0$ such that

$$|f(\Phi(z)) - f(\Phi(0))| \le M$$

for $\|z\|_X < r$ and $\Phi \in \mathrm{Aut}(\mathbb{B}_X)$. By the Schwarz-Pick Lemma for pluriharmonic functions on bounded symmetric domains ([14, Theorem 4.2]), we have

$$\Lambda_{f\circ\Phi}(0) \le \frac{4}{\pi}\frac{M}{r}, \quad \Phi \in \mathrm{Aut}(\mathbb{B}_X),$$

which gives $\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \le (4M)/(\pi r)$.

Now we prove (4)⇒(5). By Proposition 1, we have

$$Q_{f\circ\psi}(\zeta) \le Q_f(\psi(\zeta)), \quad \zeta \in \mathbb{D},$$

which, together with Theorem 1, yields that $\|f \circ \psi\|_{\mathscr{B}(\mathbb{D}),s} \le \|f\|_{\mathscr{B}(\mathbb{B}_X),s}$.

At last, we show (5)⇒(4). For any $\Phi \in \mathrm{Aut}(\mathbb{B}_X)$ and $x \in X$ with $\|x\|_X = 1$, let $\psi_0(\zeta) = \Phi(\zeta x)$, $\zeta \in \mathbb{D}$. Then $\psi_0 \in H(\mathbb{D}, \mathbb{B}_X)$ and

$$\begin{aligned} \left|D(f_1 \circ \Phi)(0)x + \overline{D(f_2 \circ \Phi)(0)x}\right| &= |(f_1 \circ \psi_0)'(0) + \overline{(f_2 \circ \psi_0)'(0)}| \\ &\le \|f \circ \psi_0\|_{\mathscr{B}(\mathbb{D}),s} \\ &\le \sup\{\|f \circ \psi\|_{\mathscr{B}(\mathbb{D}),s} : \psi \in H(\mathbb{D}, \mathbb{B}_X)\} \\ &< \infty, \end{aligned}$$

which implies that

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \le \sup\{\|f \circ \psi\|_{\mathscr{B}(\mathbb{D}),s} : \psi \in H(\mathbb{D}, \mathbb{B}_X)\} < \infty.$$

The proof of this theorem is finished. □

**Proof of Theorem 3**

Since $\mathscr{L}_f \leq \|f\|_{\mathscr{B}(\mathbb{B}_X),s}$ follows from the proof of Theorem 2, we only need to prove $\mathscr{L}_f \geq \|f\|_{\mathscr{B}(\mathbb{B}_X),s}$. For $\Phi \in \mathrm{Aut}(\mathbb{B}_X)$, $w \in \mathbb{B}_X$ and any $\lambda \in (0, 1)$, we have

$$\begin{aligned}\frac{|f(\Phi(\lambda w)) - f(\Phi(0))|}{\lambda} &\leq \mathscr{L}_f \frac{\rho(\Phi(\lambda w), \Phi(0))}{\lambda}\\ &= \mathscr{L}_f \frac{\rho(\lambda w, 0)}{\lambda}\\ &= \mathscr{L}_f \frac{\tanh^{-1}(\lambda\|w\|_X)}{\lambda}.\end{aligned}$$

Letting $\lambda \to 0^+$ in the above inequality, we have

$$|D(f_1 \circ \Phi)(0)w + \overline{D(f_2 \circ \Phi)(0)w}| \leq \mathscr{L}_f \|w\|_X$$

and hence $\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \leq \mathscr{L}_f$. This completes the proof. □

**Proof of Theorem 4**

We may assume that $\|f\|_\infty = \sup_{z\in\mathbb{B}_X} |f(z)| \leq 1$. Then, for any $\Phi \in \mathrm{Aut}(\mathbb{B}_X)$, we have $\|f \circ \Phi\|_\infty \leq 1$. It follows from [14, Theorem 4.2] that

$$\Lambda_{f\circ\Phi}(0) \leq \frac{4}{\pi}\|f\|_\infty \leq \frac{4}{\pi},$$

which, together with Theorem 3, implies that

$$\mathscr{L}_f = \|f\|_{\mathscr{B}(\mathbb{B}_X),s} \leq \frac{4}{\pi}.$$

Finally, we show the sharpness part. Without loss of generality, we assume that $\|f\|_\infty = 1$. For a fixed point $z_0 \in \mathbb{B}_X\backslash\{0\}$, let $w_0 = z_0/\|z_0\|$ and $\ell_{w_0} \in T(w_0)$ be fixed. For $z \in \mathbb{B}_X$, set

$$f(z) = \frac{2}{\pi} \arg\left(\frac{1 + \ell_{w_0}(z)}{1 - \ell_{w_0}(z)}\right).$$

Then $f \in \mathscr{PH}(\mathbb{B}_X)$ with $\sup_{z\in\mathbb{B}_X} |f(z)| \leq 1$. Let $r \in (0, 1)$. Since

$$\frac{|f(irw_0) - f(0)|}{\rho(irw_0, 0)} = \frac{2}{\pi}\frac{\tan^{-1}\frac{2r}{1-r^2}}{\frac{1}{2}\log\frac{1+r}{1-r}},$$

we have

$$\lim_{r\to 0^+}\frac{|f(irw_0)-f(0)|}{\rho(irw_0,0)}=\lim_{r\to 0^+}\frac{2}{\pi}\frac{\left(\tan^{-1}\frac{2r}{1-r^2}\right)'}{\left(\frac{1}{2}\log\frac{1+r}{1-r}\right)'}$$
$$=\frac{4}{\pi}.$$

This completes the proof. □

**Proof of Theorem 5**

Assume that $f$ is a pluriharmonic Bloch function in $\mathbb{B}_X$. Since

$$\lim_{\nu\to 0^+}\frac{(1-\nu^2)^{1/2}}{2\nu}\log\frac{1+\nu}{1-\nu}=1$$

and

$$\lim_{\nu\to 1^-}\frac{(1-\nu^2)^{1/2}}{2\nu}\log\frac{1+\nu}{1-\nu}=0,$$

we have

$$C=\sup_{\nu\in(0,1)}\frac{(1-\nu^2)^{1/2}}{2\nu}\log\frac{1+\nu}{1-\nu}<\infty.$$

Then, by using Theorem 3, we obtain that

$$\begin{aligned}(1-\|\Phi_z(w)\|_X^2)^{1/2}\frac{|f(z)-f(w)|}{\|\Phi_z(w)\|_X}&\leq C\frac{|f(z)-f(w)|}{\rho(\Phi_z(w),0)}\\&=C\frac{|f(z)-f(w)|}{\rho(z,w)}\\&\leq C\mathscr{L}_f\\&=C\|f\|_{\mathscr{B}(\mathbb{B}_X),s}\end{aligned}$$

for any $z, w\in\mathbb{B}_X$ with $z\neq w$. Therefore, we have $S(f)\leq C\|f\|_{\mathscr{B}(\mathbb{B}_X),s}<\infty$.

Next, assume that $S(f)<\infty$. Then, for any $\Phi\in \mathrm{Aut}(\mathbb{B}_X)$, $w\in X$ with $\|w\|_X=1$ and $\lambda>0$, we have

$$\begin{aligned}\|\Phi_{\Phi(0)}(\Phi(\lambda w))\|&=\tanh\rho(\Phi_{\Phi(0)}(\Phi(\lambda w)),0)\\&=\tanh\rho(\Phi_{\Phi(0)}(\Phi(\lambda w)),\Phi_{\Phi(0)}(\Phi(0)))\\&=\tanh\rho(\lambda w,0)\\&=\lambda.\end{aligned}$$

Therefore, we have

$$\frac{|f(\Phi(\lambda w)) - f(\Phi(0))|}{\lambda} \leq \frac{S(f)}{(1-\lambda^2)^{1/2}}.$$

Letting $\lambda \to 0^+$ and using the arbitrariness of $w \in \partial \mathbb{B}_X$, we have

$$\Lambda_{f\circ\Phi}(0) \leq S(f).$$

Since $\Phi \in \mathrm{Aut}(\mathbb{B}_X)$ is arbitrary, we have

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \leq S(f).$$

This completes the proof. □

## 4 Composition Operators

### Proof of Theorem 6

Let $f \in \mathscr{B}(\mathbb{B}_Y)$, and let $\rho_Y$ be the Kobayashi metric on $\mathbb{B}_Y$. Then, by Theorem 3, we have

$$\begin{aligned} |f(\phi(0))| &\leq |f(0)| + |f(\phi(0)) - f(0)| \\ &\leq |f(0)| + \|f\|_{\mathscr{B}(\mathbb{B}_Y),s}\,\rho_Y(\phi(0), 0). \end{aligned}$$

By Proposition 1, we have

$$\|f \circ \phi\|_{\mathscr{B}(\mathbb{B}_X),s} \leq K_\phi \|f\|_{\mathscr{B}(\mathbb{B}_Y),s},$$

which implies that $C_\phi$ is bounded and

$$\begin{aligned} \|C_\phi(f)\|_{\mathscr{B}(\mathbb{B}_X)} &\leq |f(0)| + (\rho_Y(\phi(0), 0) + K_\phi)\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \\ &\leq \max\{1, \rho_Y(\phi(0), 0) + K_\phi\}\|f\|_{\mathscr{B}(\mathbb{B}_Y)}. \end{aligned}$$

The lower estimate follows from an argument similar to that in the holomorphic case in [9, Theorem 3.2]. This completes the proof. □

We recall that a sequence $\{f_n\}$ of functions on a domain $D \subset X$ locally uniformly converges to a function $f$ if and only if it uniformly converges on every closed ball strictly contained in $D$ (cf. [13]), where $X$ is a complex Banach space. By using Theorem 3, we obtain the following result. For the related investigations of holomorphic functions, see [1, 8].

**Lemma 2** *For a JB*-triple $X$, let $\{f_k\}$ be a sequence of pluriharmonic Bloch functions in a bounded symmetric domain $\mathbb{B}_X$ converging locally uniformly to some*

*$f \in \mathscr{PH}(\mathbb{B}_X)$. If the sequence $\{\|f_k\|_{\mathscr{B}(\mathbb{B}_X),s}\}$ is bounded, then $f$ is a pluriharmonic Bloch function and*

$$\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \leq \liminf_{k\to\infty} \|f_k\|_{\mathscr{B}(\mathbb{B}_X),s}.$$

**Proof of Lemma 2**

Since $\{\|f_k\|_{\mathscr{B}(\mathbb{B}_X),s}\}$ is bounded,

$$C = \liminf_{k\to\infty} \|f_k\|_{\mathscr{B}(\mathbb{B}_X),s}$$

exists and is finite. There exists a subsequence $\{f_{k_l}\}$ of $\{f_k\}$ such that

$$C = \lim_{l\to\infty} \|f_{k_l}\|_{\mathscr{B}(\mathbb{B}_X),s}.$$

Let $z, w \in \mathbb{B}_X$ and $\varepsilon > 0$ be fixed. There exists an integer $k_l$ such that

$$|f(z) - f_{k_l}(z)| < \frac{\varepsilon}{2}, \quad |f(w) - f_{k_l}(w)| < \frac{\varepsilon}{2}, \quad \|f_{k_l}\|_{\mathscr{B}(\mathbb{B}_X),s} < C + \varepsilon.$$

Then, by Theorem 3, we have

$$|f(z) - f(w)| < \varepsilon + |f_{k_l}(z) - f_{k_l}(w)| \leq \varepsilon + (C + \varepsilon)\rho(z, w).$$

Letting $\varepsilon \to 0$, we have

$$|f(z) - f(w)| \leq C\rho(z, w).$$

This implies that $\mathscr{L}_f \leq C$. By Theorem 3, we obtain that $f \in \mathscr{B}(\mathbb{B}_X)$ and $\|f\|_{\mathscr{B}(\mathbb{B}_X),s} \leq \liminf_{k\to\infty} \|f_k\|_{\mathscr{B}(\mathbb{B}_X),s}$. This completes the proof. □

**Proof of Theorem 7**

If $f \in \mathscr{B}(\mathbb{B}_X)$, then $\mathscr{L}_f < \infty$ by Theorem 3 and

$$|f(\phi(S_j(z))) - f(z)| \leq \mathscr{L}_f \rho(\phi(S_j(z)), z),$$

which implies that $f\circ\phi\circ S_j \to f$ locally uniformly on $\mathbb{B}_X$. Since $\|f\circ\phi\circ S_j\|_{\mathscr{B}(\mathbb{B}_X),s} = \|f \circ \phi\|_{\mathscr{B}(\mathbb{B}_X),s}$, applying Lemma 2 to $f \circ \phi \circ S_j$, we have

$$\begin{aligned}
\|f\|_{\mathscr{B}(\mathbb{B}_X),s} &\leq \liminf_{j\to\infty} \|f \circ \phi \circ S_j\|_{\mathscr{B}(\mathbb{B}_X),s} \\
&= \|f \circ \phi\|_{\mathscr{B}(\mathbb{B}_X),s} \\
&\leq \|f\|_{\mathscr{B}(\mathbb{B}_X),s}.
\end{aligned}$$

Therefore, we have

$$\|f \circ \phi\|_{\mathscr{B}(\mathbb{B}_X),s} = \|f\|_{\mathscr{B}(\mathbb{B}_X),s}.$$

Since $\phi(0) = 0$, this implies that

$$\|f \circ \phi\|_{\mathscr{B}(\mathbb{B}_X)} = \|f\|_{\mathscr{B}(\mathbb{B}_X)}.$$

This completes the proof. □

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## Declartion

**Conflict of interest** The authors declare that they have no Conflict of interest.

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