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Robust Mathematical Programming Problems Involving Vanishing Constraints via Strongly Invex Functions

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Abstract

This manuscript demonstrates robust optimality conditions, Wolfe and Mond–Weir type robust dual models for a robust mathematical programming problem involving vanishing constraints (RMPVC). Further, the theorems of duality are examined based on the concept of generalized higher order invexity and strict invexity that establish relations between the primal and the Wolfe type robust dual problems. In addition, the duality results for a Mond–Weir type robust dual problem based on the concept of generalized higher order pseudoinvex, strict pseudoinvex and quasiinvex functions are also studied. Furthermore, numerical examples are provided to validate robust optimality conditions and duality theorems of Wolfe and Mond–Weir type dual problems.

Keywords Robust optimization · Mathematical programming problem · Vanishing constraints · Duality · Strong invexity

Mathematics Subject Classification 26A51 · 49J35 · 90C30

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1 Introduction

Mathematical optimization problems involving data uncertainty are being analyzed using the popular deterministic paradigm namely robust optimization. This is an upcoming field of research which enables scholars to resolve a host of optimization issues, especially in the face of real world scenarios where the data input for a mathematical model is often noisy or uncertain as a result of measurement inaccuracies and also in industrial settings. The constraint and objective functions are regarded as members of "uncertainty sets" in function space as part of this new approach. The readers may refer to [1–4] to gain more insights on robust optimization problems. Robust optimization problems have broad spectrum of applications in every day life situations namely forestry management [5], internet routing [6], agriculture [7] and scheduling of electric vehicles aggregator [8].

The mathematical programming problems involving vanishing constraints (MPVC) emerged as a challenging topic due to its applications in several frontiers of present day research viz., topology design problems [9], economic dispatch problems [10], robot motion planning problems [11], optimal control and structural optimization problems [12]. Recently, a number of good studies have spurred interest in this difficult class of optimization problems. Achtziger and Kanzow [13] introduced an appropriate enhancement of the standard Abadie constraint qualification and also a related optimality criteria and showed thereby that the enhanced constraint qualification satisfies the given moderate presumptions. Hoheisel et al. [14] developed a new MPVC-tailored penalty function that was precise in the given reasonable presumptions and was used to derive appropriate optimality conditions for MPVCs. Mishra et al. [15] constructed the Wolfe and the Mond–Weir type dual models for (MPVC) and examined the usual results of duality amongst the primal and the corresponding dual model based on the presumptions of convexity, strict convexity, pseudoconvexity, strict pseudoconvexity and quasiconvexity, respectively.

Kazemi and Kanzi [16] extended Achtziger and Kanzow's [13] work by proposing some constraint qualifications for a structure involving nonsmooth vanishing constraints. Furthermore, the applications of the above constraint qualifications for numerous types of stationary conditions to (MPVC) were also examined. For a single objective mathematical programming problem with vanishing constraints, Khare and Nath [17] studied the Fritz-John type stationary criteria to derive an enhanced Fritz-John type stationary criteria catering to the concept of enhanced M-stationarity with a modern constraint qualification. Later, Hu et al. [18] introduced the new Wolfe and Mond–Weir type duals to (MPVC), without computing the index set. Using the same presumptions of Mishra et al. [15], the duality results were proved with regard to the primal and its associated new dual models. Recently, Ahmad et al. [19] formulated a new mixed type dual model which unifies to the dual model of Hu et al. [18] for (MPVC) without the index set. Also, they discussed the duality results amongst the (MPVC) with its mixed type dual model based on generalized convexity. In recent times, lot of attention drew many researchers for developing modern ways to examine the solvability of the mathematical programming problems using some related vector optimization problems/modified objective function methods and the readers are advised to refer to [20–23].

Duality is an essential factor for optimization problems since the weak duality furnishes a lower bound to the objective function value of the primal problem. Wolfe proposed the conventional duality [24], whereas Mond and Weir pioneered the Mond–Weir duality [25] for scalar functions that are differentiable. By using the assumptions of generalized convexity, Tung [26] discussed Karush–Kuhn–Tucker optimality and duality of Wolfe and Mond–Weir to semi-infinite programming issues having vanishing constraints. Alternatively, Wang and Wang [27] studied Wolfe and Mond–Weir type theorems of duality for a nondifferentiable semi-infinite interval-valued optimization problem having vanishing constraints (IOPVC) based on generalized convexity presumptions. Later, Su [28] constructed Wolfe and Mond–Weir type duals with respect to contingent epiderivatives in real Banach spaces, for nonsmooth mathematical programming problems having equilibrium constraints (NMPEC). Recently, Antczak [29] discussed optimality and Mond–Weir duality results using invexity, for category of differentiable semi-infinite multi-objective programming problems with vanishing constraints.

The above works motivate in addressing a robust mathematical programming problem involving vanishing constraints via strongly invex functions. As per the authors knowledge, there is no work focusing on robust mathematical programming problems using vanishing constraints in the literature. Consequently, the current research study investigates the conditions of optimality and duality results for (RMPVC). This document specifies a few basic and fundamental concepts in Sect. 2. Section 3, establishes the results of duality amongst the primal (RMPVC) and its associated Wolfe type robust dual model based on the presumptions of generalized higher order invexity and strict invexity. The duality results amongst the primal (RMPVC) and its associated Mond–Weir type robust dual model based on the presumptions of generalized higher order pseudoinvex, strict pseudoinvex and quasiinvex functions are discussed in Sect. 4. Section 5, deals with special cases. Section 6, concludes the above analysis.

2 Preliminaries

Consider the below mentioned robust mathematical programming problem involving vanishing constraints (RMPVC):

$$
\begin{aligned}
\text{(RMPVC)}\quad \min_{\pi_0\in\mathbb{R}^n}\quad & f_0(\pi_0) \\
& \text{subject to} \\
& \psi_\varepsilon(\pi_0,\sigma_\varepsilon)\le 0,\ \forall\varepsilon\in\mathbb{Q},\ \forall\sigma_\varepsilon\in\Omega_\varepsilon, \\
& \Phi_\varepsilon(\pi_0)=0,\ \varepsilon\in\mathbb{Y}_0,\ \mathbb{Y}_0=\{1,2,\ldots,y\}, \\
& \varpi_\varepsilon(\pi_0)\ge 0,\ \varepsilon\in\mathbb{K}_0,\ \mathbb{K}_0=\{1,2,\ldots,k\}, \\
& \zeta_\varepsilon(\pi_0)\varpi_\varepsilon(\pi_0)\le 0,\ \varepsilon\in\mathbb{K}_0,\ \mathbb{K}_0=\{1,2,\ldots,k\},
\end{aligned}
$$

where $f_0:\mathbb{R}^n\to\mathbb{R}$ is a Lipschitz continuous function, $\psi_\varepsilon:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, $\varepsilon\in\mathbb{Q}$, where $\mathbb{Q}$ is an arbitrary index set (possibly infinite), $\Phi=(\Phi_1,\Phi_2,\ldots,\Phi_y):\mathbb{R}^n\to\mathbb{R}^y$, $\zeta=(\zeta_1,\zeta_2,\ldots,\zeta_k):\mathbb{R}^n\to\mathbb{R}^k$ and $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_k):\mathbb{R}^n\to\mathbb{R}^k$

are all continuously differentiable functions and $\sigma_\varepsilon \in \mathbb{R}^p$ is an uncertain parameter of the convex compact set $\Omega_\varepsilon \subset \mathbb{R}^p$, $\varepsilon \in \mathbb{Q}$. The uncertainty set-valued function $\Omega : \mathbb{Q} \rightrightarrows \mathbb{R}^p$, is given $\Omega(\varepsilon) := \Omega_\varepsilon, \forall \varepsilon \in \mathbb{Q}$, so,

$$\text{graph}(\Omega) = \{(\varepsilon, \sigma_\varepsilon) : \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \in \mathbb{Q}\}$$

and $\sigma \in \Omega$ means that σ is a choice of Ω that is, $\sigma : \mathbb{Q} \rightrightarrows \mathbb{R}^p$ and $\sigma_\varepsilon \in \Omega_\varepsilon, \forall \varepsilon \in \mathbb{Q}$.

All through this article, Δ represents the robust feasible region of the (RMPVC) and defined as:

$$\begin{aligned}\Delta = \{\pi_0 \in \mathbb{R}^n :\ & \psi_\varepsilon(\pi_0, \sigma_\varepsilon) \leq 0, \forall \varepsilon \in \mathbb{Q}, \forall \sigma_\varepsilon \in \Omega_\varepsilon, \\ & \Phi_\varepsilon(\pi_0) = 0, \varepsilon \in \mathbb{Y}_0, \\ & \varpi_\varepsilon(\pi_0) \geq 0, \varepsilon \in \mathbb{K}_0, \\ & \zeta_\varepsilon(\pi_0)\varpi_\varepsilon(\pi_0) \leq 0, \varepsilon \in \mathbb{K}_0\}.\end{aligned}$$

Definition 1 An n-dimensional open ball with radius r_0 is the set of points whose distance is less than r_0 from a fixed point in Euclidean n-space. It is clear that the open ball with centre π_0 and radius r_0 is described as follows:

$$\mathbb{A}_{r_0}(\pi_0) = \{\vartheta_0 : |\vartheta_0 - \pi_0| < r_0\}.$$

When $n = 1$ or $n = 2$, the open ball is an open interval or an open disk, respectively.

Definition 2 A point $\check{\pi}_0 \in \Delta$ is termed as a robust local minimum of the (RMPVC), if and only if there is an open ball $\mathbb{A}(\check{\pi}_0, r_0)$ with centre $\check{\pi}_0$ and radius $r_0 > 0$ such that

$$f_0(\check{\pi}_0) \leq f_0(\pi_0), \forall \pi_0 \in \Delta \cap \mathbb{A}(\check{\pi}_0, r_0).$$

A point $\check{\pi}_0 \in \Delta$ is termed as a robust global minimum of the (RMPVC), if and only if

$$f_0(\check{\pi}_0) \leq f_0(\pi_0), \forall \pi_0 \in \Delta.$$

Let $\check{\pi}_0 \in \Delta$ be any robust feasible point of (RMPVC). The sequel will make use of the following index sets.

$$\begin{aligned}\varphi_\psi(\check{\pi}_0) =& \{\varepsilon : \exists\, \sigma_\varepsilon \in \Omega_\varepsilon \text{ such that } \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) = 0\} \text{ and} \\ \Omega_\varepsilon(\check{\pi}_0) =& \{\sigma_\varepsilon \in \Omega_\varepsilon : \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) = 0\}. \\ \varphi_\Phi(\check{\pi}_0) =& \{1, 2, \ldots, y\}, \\ \varphi_+(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) > 0\}, \\ \varphi_0(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) = 0\}, \\ \varphi_{+0}(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) > 0, \zeta_\varepsilon(\check{\pi}_0) = 0\},\end{aligned}$$

$$\begin{aligned} \varphi_{+-}(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) > 0, \zeta_\varepsilon(\check{\pi}_0) < 0\},\\ \varphi_{0+}(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) = 0, \zeta_\varepsilon(\check{\pi}_0) > 0\},\\ \varphi_{00}(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) = 0, \zeta_\varepsilon(\check{\pi}_0) = 0\},\\ \varphi_{0-}(\check{\pi}_0) =& \{\varepsilon : \varpi_\varepsilon(\check{\pi}_0) = 0, \zeta_\varepsilon(\check{\pi}_0) < 0\}. \end{aligned}$$

The subsequent Lagrangian function and its gradient are used throughout this article:

$$\begin{aligned} \Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) \ =& f_0(\vartheta_0) + \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho^{\varpi}_\varepsilon \varpi_\varepsilon(\vartheta_0)\\ &+ \sum_{\varepsilon\in\mathbb{K}_0} \varrho^{\zeta}_\varepsilon \zeta_\varepsilon(\vartheta_0), \end{aligned}$$

where, $\upsilon = (\upsilon_\varepsilon)_{\varepsilon\in\mathbb{Q}} \in \mathbb{R}^{(\mathbb{Q})}_{+}$, $\sigma_\varepsilon \in \Omega_\varepsilon$, $\xi = (\xi_1, \xi_2, \ldots, \xi_y)$, $\varrho^{\varpi} = (\varrho^{\varpi}_1, \varrho^{\varpi}_2, \ldots, \varrho^{\varpi}_k)$, $\varrho^{\zeta} = (\varrho^{\zeta}_1, \varrho^{\zeta}_2, \ldots, \varrho^{\zeta}_k)$ and

$$\begin{aligned} \nabla\Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) \ =& \nabla f_0(\vartheta_0) + \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \nabla\psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \nabla\Phi_\varepsilon(\vartheta_0)\\ &- \sum_{\varepsilon\in\mathbb{K}_0} \varrho^{\varpi}_\varepsilon \nabla\varpi_\varepsilon(\vartheta_0) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho^{\zeta}_\varepsilon \nabla\zeta_\varepsilon(\vartheta_0). \end{aligned}$$

We define the subsequent index sets for $\pi_0 \in \Delta$:

$$\left.\begin{aligned} \varphi^{+}_{\psi}(\pi_0) =& \ \{\varepsilon \in \mathbb{Q} : \upsilon_\varepsilon > 0\},\\ \varphi^{+}_{\Phi}(\pi_0) =& \ \{\varepsilon \in \varphi_{\Phi}(\pi_0) : \xi_\varepsilon > 0\},\\ \varphi^{-}_{\Phi}(\pi_0) =& \ \{\varepsilon \in \varphi_{\Phi}(\pi_0) : \xi_\varepsilon < 0\},\\ \varphi^{+}_{0+}(\pi_0) =& \ \{\varepsilon \in \varphi_{0+}(\pi_0) : \varrho^{\varpi}_\varepsilon > 0\},\\ \varphi^{-}_{0+}(\pi_0) =& \ \{\varepsilon \in \varphi_{0+}(\pi_0) : \varrho^{\varpi}_\varepsilon < 0\},\\ \varphi^{+}_{00}(\pi_0) =& \ \{\varepsilon \in \varphi_{00}(\pi_0) : \varrho^{\varpi}_\varepsilon > 0\},\\ \varphi^{+}_{+0}(\pi_0) =& \ \{\varepsilon \in \varphi_{+0}(\pi_0) : \varrho^{\varpi}_\varepsilon > 0\},\\ \varphi^{+}_{+-}(\pi_0) =& \ \{\varepsilon \in \varphi_{+-}(\pi_0) : \varrho^{\varpi}_\varepsilon > 0\},\\ \varphi^{+}_{0-}(\pi_0) =& \ \{\varepsilon \in \varphi_{0-}(\pi_0) : \varrho^{\varpi}_\varepsilon > 0\},\\ \varphi^{++}_{+0}(\pi_0) =& \ \{\varepsilon \in \varphi_{+0}(\pi_0) : \varrho^{\zeta}_\varepsilon > 0\},\\ \varphi^{++}_{+-}(\pi_0) =& \ \{\varepsilon \in \varphi_{+-}(\pi_0) : \varrho^{\zeta}_\varepsilon > 0\}. \end{aligned}\right\} \quad (1)$$

The following Definitions 3 and 4, and Theorem 1 below are given on the lines of Achtziger and Kanzow [13] and Lee and Lee [2].

Definition 3 Let $\check{\pi}_0 \in \Delta$ be a robust feasible point of the (RMPVC). The Abadie constraint qualification, represented by (ACQ) is said to be fulfilled at $\check{\pi}_0$, iff $\mathbb{B}(\check{\pi}_0) = \Theta(\check{\pi}_0)$, where the standard tangent cone of (RMPVC) at $\check{\pi}_0$ is

$$\mathbb{B}(\check{\pi}_0) = \left\{ d_0 \in \mathbb{R}^n : \exists\{\pi^k_0\} \subseteq \Delta, \exists\{t_k\} \downarrow 0, \pi^k_0 \to \check{\pi}_0 \text{ and } \frac{\pi^k_0 - \check{\pi}_0}{t_k} \to d_0 \right\},$$

and the associated linearized cone of the (RMPVC) at $\check{\pi}_0$ is

$$\Theta(\check{\pi}_0) = \Big\{ d_0 \in \mathbb{R}^n : \nabla \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon)^T d_0 \leq 0, \varepsilon \in \varphi_\psi(\check{\pi}_0), \sigma_\varepsilon \in \Omega_\varepsilon,$$
$$\nabla \Phi_\varepsilon(\check{\pi}_0)^T d_0 = 0, \varepsilon \in \mathbb{Y}_0,$$
$$\nabla \varpi_\varepsilon(\check{\pi}_0)^T d_0 = 0, \varepsilon \in \varphi_{0+}(\check{\pi}_0),$$
$$\nabla \varpi_\varepsilon(\check{\pi}_0)^T d_0 \geq 0, \varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0),$$
$$\nabla \zeta_\varepsilon(\check{\pi}_0)^T d_0 \leq 0, \varepsilon \in \varphi_{+0}(\check{\pi}_0) \Big\}.$$

Definition 4 Let $\check{\pi}_0 \in \Delta$ be a robust feasible point of the (RMPVC). The modified Abadie constraint qualification (VC-ACQ) is said to be fulfilled at $\check{\pi}_0$, iff $\Theta^{VC}(\check{\pi}_0) \subseteq \mathbb{B}(\check{\pi}_0)$, where the associated VC-linearized cone of the (RMPVC) at $\check{\pi}_0$ is

$$\Theta^{VC}(\check{\pi}_0) = \Big\{ d_0 \in \mathbb{R}^n : \nabla \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon)^T d_0 \leq 0, \varepsilon \in \varphi_\psi(\check{\pi}_0), \sigma_\varepsilon \in \Omega_\varepsilon,$$
$$\nabla \Phi_\varepsilon(\check{\pi}_0)^T d_0 = 0, \varepsilon \in \mathbb{Y}_0,$$
$$\nabla \varpi_\varepsilon(\check{\pi}_0)^T d_0 = 0, \varepsilon \in \varphi_{0+}(\check{\pi}_0),$$
$$\nabla \varpi_\varepsilon(\check{\pi}_0)^T d_0 \geq 0, \varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0),$$
$$\nabla \zeta_\varepsilon(\check{\pi}_0)^T d_0 \leq 0, \varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0) \Big\}.$$

Theorem 1 *Let $\check{\pi}_0 \in \Delta$ be a robust local minimum of the (RMPVC) such that the (VC-ACQ) holds at $\check{\pi}_0$. Then one can find $(\upsilon_\varepsilon)_{\varepsilon \in \mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\xi_\varepsilon \in \mathbb{R}(\varepsilon \in \varphi_\psi)$, $\varrho_\varepsilon^\varpi, \varrho_\varepsilon^\zeta \in \mathbb{R}(\varepsilon \in \mathbb{K}_0)$ such that*

$$\nabla \Theta(\check{\pi}_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta) = 0 \tag{2}$$

and

$$\left.\begin{array}{l}
\upsilon_\varepsilon \geq 0, \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) \leq 0, \upsilon_\varepsilon \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) = 0, (\varepsilon \in \mathbb{Q}, \sigma_\varepsilon \in \Omega_\varepsilon), \\
\Phi_\varepsilon(\check{\pi}_0) = 0, (\varepsilon \in \varphi_\Phi(\check{\pi}_0)), \\
\varrho_\varepsilon^\varpi = 0, (\varepsilon \in \varphi_+(\check{\pi}_0)), \\
\varrho_\varepsilon^\varpi \geq 0, (\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0)), \\
\varrho_\varepsilon^\varpi \text{ is free}, (\varepsilon \in \varphi_{0+}(\check{\pi}_0)), \\
\varrho_\varepsilon^\zeta = 0, (\varepsilon \in \varphi_{0+}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0)), \\
\varrho_\varepsilon^\zeta \geq 0, (\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0)).
\end{array}\right\} \tag{3}$$

The conditions of optimality and the theorems of duality, among other areas of mathematical programming, heavily rely on the following generalized invexity notions. In Joshi's [30] lines, we state the following Definitions 5 to 7.

Definition 5 Let $f_0 : \mathbb{M} \to \mathbb{R}$ be continuously differentiable function, where $\mathbb{M} \subseteq \mathbb{R}^n$ is any nonempty set. Then f_0 is termed as higher order strongly (strict) invex function at $\check{\pi}_0 \in \mathbb{M}$ with regard to the kernel function $\eta_0 : \mathbb{M}X\mathbb{M} \to \mathbb{R}^n$ on $\mathbb{M}$, if for any $\pi_0 \in \mathbb{M}$, there exist some $\tilde{h} > 0$ such that $\forall \iota > 0$, we have

$$f_0(\pi_0) - f_0(\check{\pi}_0) \geq (>) \left\langle \nabla f_0(\check{\pi}_0), \eta_0(\pi_0, \check{\pi}_0) \right\rangle + \tilde{h} \| \eta_0(\pi_0, \check{\pi}_0) \|^{\iota}.$$

Definition 6 Let $f_0 : \mathbb{M} \to \mathbb{R}$ be continuously differentiable function, where $\mathbb{M} \subseteq \mathbb{R}^n$ is any nonempty set. Then f_0 is termed as higher order strongly (strict) pseudoinvex function at $\check{\pi}_0 \in \mathbb{M}$ with regard to the kernel function $\eta_0 : \mathbb{M}X\mathbb{M} \to \mathbb{R}^n$ on $\mathbb{M}$, if for any $\pi_0 \in \mathbb{M}$, there exist some $\tilde{h} > 0$ such that $\forall \iota > 0$, we have

$$\left\langle \nabla f_0(\check{\pi}_0), \eta_0(\pi_0, \check{\pi}_0) \right\rangle + \tilde{h} \| \eta_0(\pi_0, \check{\pi}_0) \|^{\iota} \geq 0 \implies f_0(\pi_0) \geq (>) f_0(\check{\pi}_0).$$

Definition 7 Let $f_0 : \mathbb{M} \to \mathbb{R}$ be continuously differentiable function, where $\mathbb{M} \subseteq \mathbb{R}^n$ is any nonempty set. Then f_0 is termed as higher order strongly quasiinvex function at $\check{\pi}_0 \in \mathbb{M}$ with regard to the kernel function $\eta_0 : \mathbb{M}X\mathbb{M} \to \mathbb{R}^n$ on $\mathbb{M}$, if for any $\pi_0 \in \mathbb{M}$, there exist some $\tilde{h} > 0$ such that $\forall \iota > 0$, we have

$$f_0(\pi_0) \leq f_0(\check{\pi}_0) \implies \left\langle \nabla f_0(\check{\pi}_0), \eta_0(\pi_0, \check{\pi}_0) \right\rangle + \tilde{h} \| \eta_0(\pi_0, \check{\pi}_0) \|^{\iota} \leq 0.$$

Theorem 2 (Robust sufficient optimality conditions) *Suppose that $\check{\pi}_0$ is a robust feasible point of the (RMPVC), there exist $(\upsilon_\varepsilon)_{\varepsilon \in \mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\xi_\varepsilon \in \mathbb{R}(\varepsilon \in \varphi_\psi)$, $\varrho_\varepsilon^{\varpi}, \varrho_\varepsilon^{\zeta} \in \mathbb{R}(\varepsilon \in \mathbb{K}_0)$ such that conditions (2) and (3) hold at $\check{\pi}_0$. Assume that f_0, $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(.))$, $\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(.))$, $-\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(.))$, $-\varpi_\varepsilon(\varepsilon \in \varphi_{+0}(.) \cup \varphi_{+-}(.) \cup \varphi_{00}(.) \cup \varphi_{0-}(.) \cup \varphi_{0+}^+(.))$, $\varpi_\varepsilon(\varepsilon \in \varphi_{0+}^-(.))$, $-\zeta_\varepsilon(\varepsilon \in \varphi_{0+}(.))$, $\zeta_\varepsilon(\varepsilon \in \varphi_{00}(.) \cup \varphi_{+0}(.) \cup \varphi_{0-}(.) \cup \varphi_{+-}(.))$ are higher order strongly invex functions at $\check{\pi}_0 \in \Delta$ with regard to the common kernel function η_0. Then $\check{\pi}_0$ is a robust local minimum of the (RMPVC).*

Proof Suppose that conditions (2) and (3) hold at $\check{\pi}_0$ with $(\upsilon_\varepsilon)_{\varepsilon \in \mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\xi_\varepsilon \in \mathbb{R}(\varepsilon \in \varphi_\psi)$, $\varrho_\varepsilon^{\varpi}, \varrho_\varepsilon^{\zeta} \in \mathbb{R}(\varepsilon \in \mathbb{K}_0)$. It follows from (2) that

$$\nabla \Theta(\check{\pi}_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) = \nabla f_0(\check{\pi}_0) + \sum_{\varepsilon \in \mathbb{Q}} \upsilon_\varepsilon \nabla \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \xi_\varepsilon \nabla \Phi_\varepsilon(\check{\pi}_0)$$

$$- \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\varpi} \nabla \varpi_\varepsilon(\check{\pi}_0) + \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\zeta} \nabla \zeta_\varepsilon(\check{\pi}_0) = 0. \tag{4}$$

Suppose that $\check{\pi}_0$ is not a robust local minimum of the (RMPVC), that is, there exists $\check{\vartheta}_0$ such that

$$f_0(\check{\vartheta}_0) < f_0(\check{\pi}_0). \tag{5}$$

By using the higher order strong invexity of $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(.))$, $\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(.))$, $-\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(.))$, $-\varpi_\varepsilon(\varepsilon \in \varphi_{+0}(.) \cup \varphi_{+-}(.) \cup \varphi_{00}(.) \cup \varphi_{0-}(.) \cup \varphi_{0+}^+(.))$, $\varpi_\varepsilon(\varepsilon \in \varphi_{0+}^-(.))$, $-\zeta_\varepsilon(\varepsilon \in \varphi_{0+}(.))$, $\zeta_\varepsilon(\varepsilon \in \varphi_{00}(.) \cup \varphi_{+0}(.) \cup \varphi_{0-}(.) \cup \varphi_{+-}(.))$, with regard to the common kernel function η_0, at $\check{\pi}_0 \in \Delta$, which leads to

$$
\begin{aligned}
&\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \left\langle \nabla\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \\
&\le \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) \le 0, \\
&\tilde{h}_\varepsilon > 0, \upsilon_\varepsilon > 0, \varepsilon \in \varphi_\psi^+(\check{\vartheta}_0), \\
&\Phi_\varepsilon(\check{\pi}_0) + \left\langle \nabla\Phi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \le \Phi_\varepsilon(\check{\vartheta}_0) = 0, \tilde{h}_\varepsilon > 0, \\
&\xi_\varepsilon > 0, \varepsilon \in \varphi_\Phi^+(\check{\vartheta}_0), \\
&\Phi_\varepsilon(\check{\pi}_0) + \left\langle \nabla\Phi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \ge \Phi_\varepsilon(\check{\vartheta}_0) = 0, \tilde{h}_\varepsilon > 0, \\
&\xi_\varepsilon < 0, \varepsilon \in \varphi_\Phi^-(\check{\vartheta}_0), \\
&\quad - \varpi_\varepsilon(\check{\pi}_0) - \left\langle \nabla\varpi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \le -\varpi_\varepsilon(\check{\vartheta}_0) \le 0, \\
&\quad (\tilde{h}_\varepsilon)^\varpi > 0, \\
&\varrho_\varepsilon^\varpi \ge 0, \varepsilon \in \varphi_{+0}(\check{\vartheta}_0) \cup \varphi_{+-}(\check{\vartheta}_0) \cup \varphi_{00}(\check{\vartheta}_0) \cup \varphi_{0-}(\check{\vartheta}_0) \cup \varphi_{0+}^+(\check{\vartheta}_0), \\
&\quad - \varpi_\varepsilon(\check{\pi}_0) - \langle \nabla\varpi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \le -\varpi_\varepsilon(\check{\vartheta}_0) = 0, \\
&\quad (\tilde{h}_\varepsilon)^\varpi > 0, \\
&\varrho_\varepsilon^\varpi < 0, \varepsilon \in \varphi_{0+}^-(\check{\vartheta}_0), \\
&\zeta_\varepsilon(\check{\pi}_0) + \left\langle \nabla\zeta_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \ge \zeta_\varepsilon(\check{\vartheta}_0) > 0, (\tilde{h}_\varepsilon)^\zeta > 0, \\
&\varrho_\varepsilon^\zeta = 0, \varepsilon \in \varphi_{0+}(\check{\vartheta}_0), \\
&\zeta_\varepsilon(\check{\pi}_0) + \left\langle \nabla\zeta_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \le \zeta_\varepsilon(\check{\vartheta}_0) = 0, (\tilde{h}_\varepsilon)^\zeta > 0, \\
&\varrho_\varepsilon^\zeta \ge 0, \varepsilon \in \varphi_{+0}(\check{\vartheta}_0) \cup \varphi_{00}(\check{\vartheta}_0), \\
&\zeta_\varepsilon(\check{\pi}_0) + \left\langle \nabla\zeta_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0) \right\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \le \zeta_\varepsilon(\check{\vartheta}_0) < 0, (\tilde{h}_\varepsilon)^\zeta > 0, \\
&\varrho_\varepsilon^\zeta \ge 0, \varepsilon \in \varphi_{0-}(\check{\vartheta}_0) \cup \varphi_{+-}(\check{\vartheta}_0),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\check{\pi}_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\varpi \varpi_\varepsilon(\check{\pi}_0) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\zeta \zeta_\varepsilon(\check{\pi}_0) \\
&+\langle \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \nabla\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \nabla\Phi_\varepsilon(\check{\pi}_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\varpi \nabla\varpi_\varepsilon(\check{\pi}_0) \\
&\quad + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\zeta \nabla\zeta_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\rangle \\
&\mathit{which\ yields}\ \upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota
\end{aligned}
$$

$$+\xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota - \varrho_\varepsilon^\varpi (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota$$
$$+\varrho_\varepsilon^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \leq 0. \tag{6}$$

By using the higher order strong invexity of f_0 at $\check{\pi}_0$, with regard to the kernel function η_0, we get

$$f_0(\check{\pi}_0) + \langle \nabla f_0(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\rangle + \tilde{h}\|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \leq f_0(\check{\vartheta}_0). \tag{7}$$

On adding (6) and (7), we have

$$\Theta(\check{\pi}_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta) + \langle \nabla\Theta(\check{\pi}_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\rangle + h^\circ\|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota$$
$$\leq f_0(\check{\vartheta}_0),$$

where

$$h^\circ\|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota = \upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota + \xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota - \varrho_\varepsilon^\varpi (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota$$
$$+\varrho_\varepsilon^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota + \tilde{h}\|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota.$$

From (4), it follows that

$$\Theta(\check{\pi}_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta) \leq f_0(\check{\vartheta}_0),$$

which leads to

$$f_0(\check{\vartheta}_0) \geq f_0(\check{\pi}_0),$$

which contradicts (5). Hence, $\check{\pi}_0$ is a robust local minimum of the (RMPVC). □

The Theorem 2 (robust sufficient optimality conditions) is justified with an illustration mentioned below (Fig. 1).

Example 1

$$\text{(RMPVC-1)} \quad \min_{\pi_0 \in \mathbb{R}} \quad f_0(\pi_0) = \frac{1}{2}\pi_0 - 2$$

subject to

$$\psi_\varepsilon(\pi_0, \sigma_\varepsilon) = -\varepsilon\pi_0^2 + \sigma_\varepsilon\pi_0 \leq 0, \forall\varepsilon \in \mathbb{Q} = [0, 1],$$
$$\forall\sigma_\varepsilon \in [-\varepsilon + 2, \varepsilon + 2],$$
$$\varpi_1(\pi_0) = \pi_0 \geq 0,$$
$$\zeta_1(\pi_0)\varpi_1(\pi_0) = (\pi_0 + \pi_0^2)\pi_0 \leq 0,$$

with $n = 1, \varepsilon \in \mathbb{Q} = [0, 1], y = 0, k = 1$. Clearly, $f_0(\pi_0) = \frac{1}{2}\pi_0 - 2$ is Lipschitz continuous in $\mathbb{R}$. The robust feasible solution set of the (RMPVC-1) is represented by Δ, where

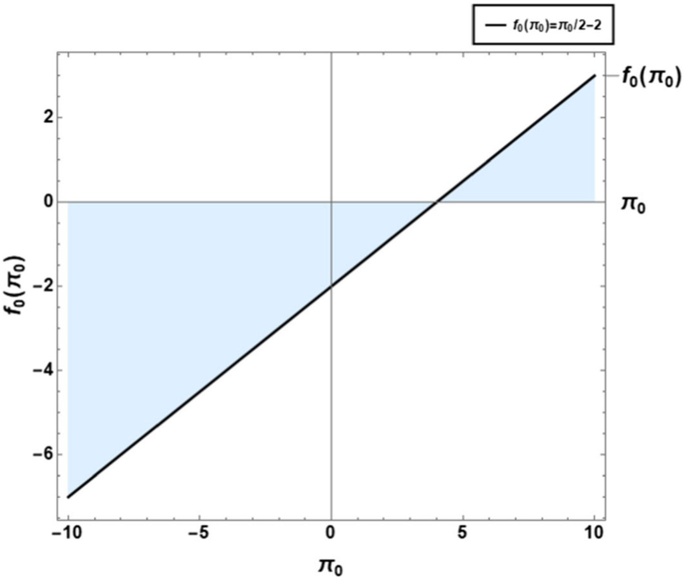

Fig. 1 Graphical view of the objective function of the problem (RMPVC-1)

$$\Delta = \{\pi_0 \in \mathbb{R} : -\varepsilon\pi_0^2 + \sigma_\varepsilon\pi_0 \leq 0, \pi_0 \geq 0, \pi_0(\pi_0 + \pi_0^2) \leq 0\}.$$

Therefore, the robust feasible solution of the (RMPVC-1) is $\check{\pi}_0 = 0$. By straightforward calculations, we obtain $\varphi_\psi = \mathbb{Q}$, $\varphi_+ = \varphi_{0+} = \varphi_{0-} = \varnothing$, $\varphi_{00} = 1$, $\nabla\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) = \sigma_\varepsilon$, where $\sigma_\varepsilon = 1$, for $\varepsilon = 1$ and $\sigma_\varepsilon = 0$, for $0 \leq \varepsilon < 1$, $\nabla\varpi_1(\check{\pi}_0) = \{1\}$, $\nabla\zeta_1(\check{\pi}_0) = \{1\}$, $\eta_0(\pi_0, \check{\pi}_0) = (\pi_0 - \pi_0^2)$. It is seen that $\check{\pi}_0 = 0$ satisfies (VC-ACQ) of the (RMPVC-1). There exist $\upsilon_1 = 0$, $\varrho_1^\varpi = \frac{1}{2}$, $\varrho_1^\zeta = 0$ such that conditions (2) and (3) of Theorem 1 are satisfied at $\check{\pi}_0 = 0$. Also, the assumptions of Theorem 2 hold at $\check{\pi}_0 = 0$. Therefore, $\check{\pi}_0 = 0$ is a robust local minimum of the (RMPVC-1).

Theorem 3 (Robust sufficient optimality conditions) *Suppose that $\check{\pi}_0$ is a robust feasible point of the (RMPVC), there exist $(\upsilon_\varepsilon)_{\varepsilon\in\mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\xi_\varepsilon \in \mathbb{R}(\varepsilon \in \varphi_\psi)$, $\varrho_\varepsilon^\varpi, \varrho_\varepsilon^\zeta \in \mathbb{R}(\varepsilon \in \mathbb{K}_0)$ such that*

$$\begin{aligned}\nabla\Theta(\check{\pi}_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta) &= \nabla f_0(\check{\pi}_0) + \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \nabla\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \nabla\Phi_\varepsilon(\check{\pi}_0) \\ &\quad - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\varpi \nabla\varpi_\varepsilon(\check{\pi}_0) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\zeta \nabla\zeta_\varepsilon(\check{\pi}_0) = 0, \end{aligned} \tag{8}$$

$$\upsilon_\varepsilon \geq 0, \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) \leq 0, \upsilon_\varepsilon\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) = 0, (\varepsilon \in \mathbb{Q}, \sigma_\varepsilon \in \Omega_\varepsilon), \tag{9}$$

$$\Phi_\varepsilon(\check{\pi}_0) = 0, (\varepsilon \in \varphi_\Phi(\check{\pi}_0)), \tag{10}$$

$$\varrho_\varepsilon^{\varpi} = 0, (\varepsilon \in \varphi_+(\check{\pi}_0)), \varrho_\varepsilon^{\varpi} \geq 0, (\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0)), \varrho_\varepsilon^{\varpi} \text{ is free}, (\varepsilon \in \varphi_{0+}(\check{\pi}_0)), \tag{11}$$

$$\varrho_\varepsilon^{\zeta} = 0, (\varepsilon \in \varphi_{0+}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0)), \varrho_\varepsilon^{\zeta} \geq 0, (\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0)), \tag{12}$$

$$\varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\check{\pi}_0) = 0,\ \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\check{\pi}_0) = 0, \varepsilon \in \mathbb{K}_0. \tag{13}$$

Further, assume that $f_0(.)$ is higher order strongly pseudoinvex function and $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(.))$, $\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(.))$, $-\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(.))$, $-\varpi_\varepsilon(\varepsilon \in \varphi_{+0}^+(.) \cup \varphi_{+-}^+(.) \cup \varphi_{00}^+(.) \cup \varphi_{0-}^+(.) \cup \varphi_{0+}^+(.))$, $\varpi_\varepsilon(\varepsilon \in \varphi_{0+}^-(.))$, $\zeta_\varepsilon(\varepsilon \in \varphi_{+0}^{++}(.) \cup \varphi_{+-}^{++}(.))$ are higher order strongly quasiinvex functions at $\check{\pi}_0 \in \Delta$ with regard to the common kernel function η_0. Then $\check{\pi}_0$ is a robust local minimum of the (RMPVC).

Proof Suppose that $\check{\pi}_0$ is not a robust local minimum of the (RMPVC), then there exists $\check{\vartheta}_0$ such that

$$f_0(\check{\vartheta}_0) < f_0(\check{\pi}_0). \tag{14}$$

For $\check{\pi}_0 \in \Delta$, $(\upsilon_\varepsilon)_{\varepsilon\in\mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\varepsilon \in \mathbb{Q}$, we have, $\upsilon_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) \leq 0$, $\varepsilon \in \mathbb{Q}$, $\sigma_\varepsilon \in \Omega_\varepsilon$ which in view of (9) implies that

$$\upsilon_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) \leq \upsilon_\varepsilon \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon),$$

by using the higher order strong quasiinvexity of $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(.))$ at $\check{\pi}_0 \in \Delta$, we get

$$\Big\langle \nabla\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \leq 0, \tilde{h}_\varepsilon > 0, \upsilon_\varepsilon > 0, \varepsilon \in \varphi_\psi^+(\check{\vartheta}_0). \tag{15}$$

By similar arguments, we have

$$\Big\langle \nabla\Phi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \leq 0, \tilde{h}_\varepsilon > 0, \xi_\varepsilon > 0, \varepsilon \in \varphi_\Phi^+(\check{\vartheta}_0),$$
$$\Big\langle \nabla\Phi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \geq 0, \tilde{h}_\varepsilon > 0, \xi_\varepsilon < 0, \varepsilon \in \varphi_\Phi^-(\check{\vartheta}_0),$$
$$-\Big\langle \nabla\varpi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle - (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^\iota \leq 0, (\tilde{h}_\varepsilon)^{\varpi} > 0, \varrho_\varepsilon^{\varpi} \geq 0, \varepsilon \in \varphi_{+0}^+(\check{\vartheta}_0) \cup \varphi_{+-}^+(\check{\vartheta}_0) \cup \varphi_{00}^+(\check{\vartheta}_0)$$

$$\cup\, \varphi_{0-}^{+}(\check{\vartheta}_0) \cup \varphi_{0+}^{+}(\check{\vartheta}_0),$$

$$-\Big\langle \nabla \varpi_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle - (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} \geq 0,\, (\tilde{h}_\varepsilon)^{\varpi} > 0,\, \varrho_\varepsilon^{\varpi} \leq 0,\, \varepsilon \in \varphi_{+0}^{-}(\check{\vartheta}_0),$$

$$\Big\langle \nabla \zeta_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle + (\tilde{h}_\varepsilon)^{\zeta} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} \leq 0,\, (\tilde{h}_\varepsilon)^{\zeta} > 0,\, \varrho_\varepsilon^{\zeta} \geq 0,\, \varepsilon \in \varphi_{+0}^{++}(\check{\vartheta}_0) \cup \varphi_{+-}^{++}(\check{\vartheta}_0),$$

which by definition of index set along with the inequality (15), yields

$$\begin{aligned}
&\langle \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \nabla \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \nabla \Phi_\varepsilon(\check{\pi}_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\varpi} \nabla \varpi_\varepsilon(\check{\pi}_0) \\
&\quad + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\zeta} \nabla \zeta_\varepsilon(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\rangle \\
&\quad + \upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} + \xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} - \varrho_\varepsilon^{\varpi} (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} \\
&\quad + \varrho_\varepsilon^{\zeta} (\tilde{h}_\varepsilon)^{\zeta} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} \leq 0.
\end{aligned}$$

Using the above inequality and (8), it follows that

$$\Big\langle \nabla f_0(\check{\pi}_0), \eta_0(\check{\vartheta}_0, \check{\pi}_0)\Big\rangle + \tilde{h}\|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} \geq 0,$$

where

$$\begin{aligned}
\tilde{h}\|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} &= -\upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} - \xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} \\
&\quad + \varrho_\varepsilon^{\varpi} (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota} - \varrho_\varepsilon^{\zeta} (\tilde{h}_\varepsilon)^{\zeta} \|\eta_0(\check{\vartheta}_0, \check{\pi}_0)\|^{\iota}.
\end{aligned}$$

By using higher order strong pseudoinvexity of f_0, with regard to the kernel function η_0, we obtain

$$f_0(\check{\vartheta}_0) \geq f_0(\check{\pi}_0),$$

which contradicts (14). Hence, $\check{\pi}_0$ is a robust local minimum of the (RMPVC). □

In the next section, we discuss a Wolfe type robust dual model and prove the duality theorems. The dual model used here is based on the lines of Hu et al. [18].

3 Wolfe type robust dual model

The Wolfe type robust dual model of the (RMPVC) depending on a robust feasible point $\check{\pi}_0 \in \Delta$, represented by (VC-RWD)($\check{\pi}_0$), is provided in this section. The details are as follows:

$$\max \quad \Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta})$$
$$\text{subject to}$$

$$\left.\begin{array}{l}\nabla\Theta(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta})=0,\\ \upsilon_\varepsilon\geq 0,\ \forall\,\varepsilon\ \in\mathbb{Q},\\ \varrho_\varepsilon{}^{\zeta}=\delta_\varepsilon\varpi_\varepsilon(\pi_0),\,\delta_\varepsilon\geq 0,\forall\varepsilon\in\mathbb{K}_0,\\ \varrho_\varepsilon{}^{\varpi}=\chi_\varepsilon-\delta_\varepsilon\zeta_\varepsilon(\pi_0),\,\chi_\varepsilon\geq 0,\forall\varepsilon\in\mathbb{K}_0.\end{array}\right\}\tag{16}$$

Let $\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0)$ represents the set of all robust feasible solutions of the problem (VC-RWD)(π_0) where $\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0)=\{(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta},\chi,\delta):\nabla\Theta(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta})=0,$

$$\begin{aligned}&\upsilon_\varepsilon\geq 0,\ \forall\,\varepsilon\ \in\mathbb{Q},\\ &\varrho_\varepsilon{}^{\zeta}=\delta_\varepsilon\varpi_\varepsilon(\pi_0),\,\delta_\varepsilon\geq 0,\forall\varepsilon\in\mathbb{K}_0,\\ &\varrho_\varepsilon{}^{\varpi}=\chi_\varepsilon-\delta_\varepsilon\zeta_\varepsilon(\pi_0),\,\chi_\varepsilon\geq 0,\forall\varepsilon\in\mathbb{K}_0\}.\end{aligned}\tag{17}$$

We represent the projection of the set $\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0)$ on $\mathbb{R}^n$ by

$$pr\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0)=\{\vartheta_0\in\mathbb{R}^n:(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta},\chi,\delta)\in\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0)\}.$$

For $\pi_0\in\Delta$, the new Wolfe type robust dual is independent of the (RMPVC), we consider the subsequent Wolfe type robust dual problem:

$$\begin{aligned}\max\quad&\Theta(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta})\\ \text{such that}\quad&(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta},\chi,\delta)\in\bigcap_{\pi_0\in\Delta}\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0).\end{aligned}\tag{18}$$

The set of all robust feasible points of the (VC-RWD) is represented by $\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}=\bigcap_{\pi_0\in\Delta}\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\pi_0)$ and the projection of the set $\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}$ on $\mathbb{R}^n$ is represented by $pr\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}$.

Remark 1 Wolfe type dual model exists in literature for a mathematical programming problem with vanishing constraints (See Mishra et al. [15]) using index sets. These models are not suitable for numerical solutions to dual problems since they need to calculate index sets. As a result, Hu et al. [18] recently proposed new Wolfe type dual model for a mathematical programming problem with vanishing constraints and established duality outcomes under generalized convexity assumptions that do not require index set calculations.

Theorem 4 (Weak robust duality theorem) *Let* $\pi_0\in\Delta$, $(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta},\chi,\delta)\in\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}$ *be robust feasible points of the (RMPVC) and the (VC-RWD) respectively. Suppose one of the subsequent cases occurs:*

(i) $\Theta(.,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta})$ *is higher order strongly invex function at* $\vartheta_0\in\Delta\cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}$ *with regard to the kernel function* η_0*,*

(ii) $f_0,\psi_\varepsilon(\varepsilon\in\varphi^+_\psi(\pi_0)),\Phi_\varepsilon(\varepsilon\in\varphi^+_\Phi(\pi_0)),-\Phi_\varepsilon(\varepsilon\in\varphi^-_\Phi(\pi_0)),-\varpi_\varepsilon(\varepsilon\in\varphi_{+0}(\pi_0)\cup\varphi_{+-}(\pi_0)\cup\varphi_{00}(\pi_0)\cup\varphi_{0-}(\pi_0)\cup\varphi^+_{0+}(\pi_0)),\ \varpi_\varepsilon(\varepsilon\in\varphi^-_{0+}(\pi_0)),\ -\zeta_\varepsilon(\varepsilon\in\varphi_{0+}(\pi_0)),$

$\zeta_\varepsilon(\varepsilon \in \varphi_{00}(\pi_0) \cup \varphi_{+0}(\pi_0) \cup \varphi_{0-}(\pi_0) \cup \varphi_{+-}(\pi_0))$ *are higher order strongly invex functions at* $\vartheta_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$ *with regard to the common kernel function* η_0. *Then*

$$f_0(\pi_0) \geq \Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}).$$

Proof (i) Suppose that

$$f_0(\pi_0) < \Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}),$$

that is,

$$f_0(\pi_0) < f_0(\vartheta_0) + \sum_{\varepsilon \in \mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\vartheta_0)$$

$$+ \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\vartheta_0). \tag{19}$$

Since $\pi_0 \in \Delta$, it follows that

$$\begin{aligned}
\psi_\varepsilon(\pi_0, \sigma_\varepsilon) &< 0, \upsilon_\varepsilon \geq 0, \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \notin \varphi_\psi(\pi_0), \\
\psi_\varepsilon(\pi_0, \sigma_\varepsilon) &= 0, \upsilon_\varepsilon \geq 0, \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \in \varphi_\psi(\pi_0), \\
\Phi_\varepsilon(\pi_0) &= 0, \xi_\varepsilon \in \mathbb{R}, \varepsilon \in \varphi_\Phi, \\
-\varpi_\varepsilon(\pi_0) &< 0, \varrho_\varepsilon^{\varpi} \geq 0, \varepsilon \in \varphi_+(\pi_0), \\
-\varpi_\varepsilon(\pi_0) &= 0, \varrho_\varepsilon^{\varpi} \in \mathbb{R}, \varepsilon \in \varphi_0(\pi_0), \\
\zeta_\varepsilon(\pi_0) &> 0, \varrho_\varepsilon^{\zeta} = 0, \varepsilon \in \varphi_{0+}(\pi_0), \\
\zeta_\varepsilon(\pi_0) &= 0, \varrho_\varepsilon^{\zeta} \geq 0, \varepsilon \in \varphi_{00}(\pi_0) \cup \varphi_{+0}(\pi_0), \\
\zeta_\varepsilon(\pi_0) &< 0, \varrho_\varepsilon^{\zeta} \geq 0, \varepsilon \in \varphi_{0-}(\pi_0) \cup \varphi_{+-}(\pi_0),
\end{aligned}$$

which leads to,

$$\sum_{\varepsilon \in \mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\pi_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\pi_0) - \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\pi_0) + \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\pi_0) \leq 0. \tag{20}$$

On adding (19) and (20), we have

$$f_0(\pi_0) + \sum_{\varepsilon \in \mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\pi_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\pi_0) - \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\pi_0) + \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\pi_0)$$

$$< f_0(\vartheta_0) + \sum_{\varepsilon \in \mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\vartheta_0) + \sum_{\varepsilon \in \mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\vartheta_0). \tag{21}$$

That is,

$$\Theta(\pi_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) < \Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}). \tag{22}$$

By using the higher order strong invexity of $\Theta(., \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta})$ with regard to the kernel function η_0, we get

$$\Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varphi^{\zeta}) + \left\langle \nabla\Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}), \eta_0(\pi_0, \vartheta_0) \right\rangle$$
$$+\tilde{h}\|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \leq \Theta(\pi_0, \upsilon, \xi, \varrho^{\varpi}, \varphi^{\zeta}).$$

In view of the first equation in (16), we obtain

$$\Theta(\pi_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) \geq \Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}),$$

which contradicts (22). The theorem is therefore validated.

(ii) By using higher order strong invexity of $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(\pi_0))$, $\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(\pi_0))$, $-\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(\pi_0))$, $-\varpi_\varepsilon(\varepsilon \in \varphi_{+0}(\pi_0) \cup \varphi_{+-}(\pi_0) \cup \varphi_{00}(\pi_0) \cup \varphi_{0-}(\pi_0) \cup \varphi_{0+}^+(\pi_0))$, $\varpi_\varepsilon(\varepsilon \in \varphi_{0+}^-(\pi_0))$, $-\zeta_\varepsilon(\varepsilon \in \varphi_{0+}(\pi_0))$, $\zeta_\varepsilon(\varepsilon \in \varphi_{00}(\pi_0) \cup \varphi_{+0}(\pi_0) \cup \varphi_{0-}(\pi_0) \cup \varphi_{+-}(\pi_0))$, with regard to the common kernel function η_0, at $\vartheta_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$, $\pi_0 \in \Delta$ and $(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}, \chi, \delta) \in \mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$, we have

$$\begin{aligned}
&\psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \langle \nabla\psi_\varepsilon(\vartheta_0, \sigma_\varepsilon), \eta_0(\pi_0, \vartheta_0)\rangle + \tilde{h}_\varepsilon\|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \\
&\leq \psi_\varepsilon(\pi_0, \sigma_\varepsilon) \leq 0, \\
&\tilde{h}_\varepsilon > 0, \upsilon_\varepsilon > 0, \\
&\varepsilon \in \varphi_\psi^+(\pi_0), \\
&\Phi_\varepsilon(\vartheta_0) + \langle \nabla\Phi_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + \tilde{h}_\varepsilon\|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \leq \Phi_\varepsilon(\pi_0) = 0, \\
&\tilde{h}_\varepsilon > 0, \xi_\varepsilon > 0, \\
&\varepsilon \in \varphi_\Phi^+(\pi_0), \\
&\Phi_\varepsilon(\vartheta_0) + \langle \nabla\Phi_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + \tilde{h}_\varepsilon\|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \geq \Phi_\varepsilon(\pi_0) = 0, \\
&\tilde{h}_\varepsilon > 0, \xi_\varepsilon < 0, \\
&\varepsilon \in \varphi_\Phi^-(\pi_0), \\
&\quad -\varpi_\varepsilon(\vartheta_0) - \langle \nabla\varpi_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle - (\tilde{h}_\varepsilon)^{\varpi}\|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \leq -\varpi_\varepsilon(\pi_0) \leq 0, \\
&(\tilde{h}_\varepsilon)^{\varpi} > 0, \varrho_\varepsilon^{\varpi} \geq 0, \\
&\varepsilon \in \varphi_{+0}(\pi_0) \\
&\cup\varphi_{+-}(\pi_0) \cup \varphi_{00}(\pi_0) \\
&\cup\varphi_{0-}(\pi_0) \cup \varphi_{0+}^+(\pi_0), \\
&\quad -\varpi_\varepsilon(\vartheta_0) - \langle \nabla\varpi_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle - (\tilde{h}_\varepsilon)^{\varpi}\|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \leq -\varpi_\varepsilon(\pi_0) = 0, \\
&(\tilde{h}_\varepsilon)^{\varpi} > 0, \varrho_\varepsilon^{\varpi} < 0, \\
&\varepsilon \in \varphi_{0+}^-(\pi_0),
\end{aligned}$$

$$
\begin{aligned}
&\zeta_\varepsilon(\vartheta_0) + \langle \nabla \zeta_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\pi_0, \vartheta_0)\|^\iota \geq \zeta_\varepsilon(\pi_0) > 0, (\tilde{h}_\varepsilon)^\zeta > 0,\\
&\varrho_\varepsilon^\zeta = 0, \varepsilon \in \varphi_{0+}(\pi_0),\\
&\zeta_\varepsilon(\vartheta_0) + \langle \nabla \zeta_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\pi_0, \vartheta_0)\|^\iota \leq \zeta_\varepsilon(\pi_0) = 0,\\
&(\tilde{h}_\varepsilon)^\zeta > 0, \varrho_\varepsilon^\zeta \geq 0,\\
&\varepsilon \in \varphi_{+0}(\pi_0) \cup \varphi_{00}(\pi_0),\\
&\zeta_\varepsilon(\vartheta_0) + \langle \nabla \zeta_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\pi_0, \vartheta_0)\|^\iota \leq \zeta_\varepsilon(\pi_0) < 0,\\
&(\tilde{h}_\varepsilon)^\zeta > 0, \varrho_\varepsilon^\zeta \geq 0,\\
&\varepsilon \in \varphi_{0-}(\pi_0) \cup \varphi_{+-}(\pi_0),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\varpi \varpi_\varepsilon(\vartheta_0) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\zeta \zeta_\varepsilon(\vartheta_0) +\\
&\langle \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \nabla \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \nabla \Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\varpi \nabla \varpi_\varepsilon(\vartheta_0)\\
&\quad + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^\zeta \nabla \zeta_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle\\
&\quad + \upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^\iota + \xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^\iota - \varrho_\varepsilon^\varpi (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\pi_0, \vartheta_0)\|^\iota\\
&\quad + \varrho_\varepsilon^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\pi_0, \vartheta_0)\|^\iota \leq 0.
\end{aligned}
\tag{23}
$$

By using the higher order strong invexity of f_0 at $\vartheta_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$, with regard to the kernel function η_0, we get

$$
f_0(\vartheta_0) + \langle \nabla f_0(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + \tilde{h}\|\eta_0(\pi_0, \vartheta_0)\|^\iota \leq f_0(\pi_0). \tag{24}
$$

On adding (23) and (24), we have

$$
\begin{aligned}
&\Theta(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta) + \langle \nabla\Theta(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta), \eta_0(\pi_0, \vartheta_0)\rangle + h^\circ \|\eta_0(\pi_0, \vartheta_0)\|^\iota\\
&\quad \leq f_0(\pi_0),
\end{aligned}
$$

where

$$
\begin{aligned}
h^\circ \|\eta_0(\pi_0, \vartheta_0)\|^\iota \; &= \upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^\iota + \xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^\iota - \varrho_\varepsilon^\varpi (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\pi_0, \vartheta_0)\|^\iota\\
&\quad + \varrho_\varepsilon^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\pi_0, \vartheta_0)\|^\iota + \tilde{h}\|\eta_0(\pi_0, \vartheta_0)\|^\iota.
\end{aligned}
$$

In view of the first equation in (16), we obtain

$$
\Theta(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta) \leq f_0(\pi_0).
$$

The theorem is therefore validated. □

Theorem 5 (Strong robust duality theorem) *Let $\check{\pi}_0 \in \Delta$ be a robust local minimum of the (RMPVC) such that the (VC-ACQ) is fulfilled at $\check{\pi}_0$. Then there exist $\check{\upsilon} = (\check{\upsilon}_\varepsilon)_{\varepsilon \in \mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\check{\xi} \in \mathbb{R}^y$, $(\check{\varrho})^\varpi$, $(\check{\varrho})^\zeta$, $\check{\chi}$, $\check{\delta} \in \mathbb{R}^k$ such that $(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ is a robust feasible point of the (VC-RWD)($\check{\pi}_0$) and*

$$\sum_{\varepsilon \in \mathbb{Q}} \check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \check{\xi}_\varepsilon \Phi_\varepsilon(\check{\pi}_0) - \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^\varpi \varpi_\varepsilon(\check{\pi}_0) + \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^\zeta \zeta_\varepsilon(\check{\pi}_0) = 0. \tag{25}$$

Suppose one of the subsequent cases occurs:

(i) $\Theta(., \upsilon, \xi, \varrho^\varpi, \varrho^\zeta)$ is higher order strongly invex function at $\vartheta_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}(\check{\pi}_0)$ with regard to the kernel function η_0,

(ii) f_0, $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(\check{\pi}_0))$, $\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(\check{\pi}_0))$, $-\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(\check{\pi}_0))$, $-\varpi_\varepsilon(\varepsilon \in \varphi_{+0}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0) \cup \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi_{0+}^+(\check{\pi}_0))$, $\varpi_\varepsilon(\varepsilon \in \varphi_{0+}^-(\check{\pi}_0))$, $-\zeta_\varepsilon(\varepsilon \in \varphi_{0+}(\check{\pi}_0))$, $\zeta_\varepsilon(\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0))$ are higher order strongly invex functions at $\vartheta_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}(\check{\pi}_0)$ with regard to the common kernel function η_0. Then $(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ is a robust global maximum of (VC-RWD)($\check{\pi}_0$), that is, $\Theta(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta) \geq \Theta(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta), \forall(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta, \chi, \delta) \in \mathbb{S}_{\mathbb{W}}^{\mathbb{R}}(\check{\pi}_0)$ and

$$f_0(\check{\pi}_0) = \Theta(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta).$$

Proof Since $\check{\pi}_0$ is a robust local minimum of the (RMPVC) and the (VC-ACQ) holds at $\check{\pi}_0$, from Theorem 1, there exist $\check{\upsilon} = (\check{\upsilon}_\varepsilon)_{\varepsilon \in \mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\check{\xi} \in \mathbb{R}^y$, $(\check{\varrho})^\varpi$, $(\check{\varrho})^\zeta$, $\check{\chi}$, $\check{\delta} \in \mathbb{R}^k$ such that the conditions (2) and (3) hold and hence $(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ is a robust feasible point of the (VC-RWD)($\check{\pi}_0$). By Theorem 4, we get

$$f_0(\check{\pi}_0) \geq \Theta(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta), \forall(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta, \chi, \delta) \in \mathbb{S}_{\mathbb{W}}^{\mathbb{R}}(\check{\pi}_0). \tag{26}$$

On adding (25) and (26), we have

$$\begin{aligned} \Theta(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta) \; &\geq \Theta(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta), \forall(\vartheta_0, \upsilon, \xi, \varrho^\varpi, \varrho^\zeta, \chi, \delta) \\ &\in \mathbb{S}_{\mathbb{W}}^{\mathbb{R}}(\check{\pi}_0), \end{aligned} \tag{27}$$

that is, $(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ is a robust global maximum of the (VC-RWD)($\check{\pi}_0$). Also, the robust local minimum of the (RMPVC) and the robust global minimum of the (VC-RWD)($\check{\pi}_0$) are equal. □

Theorem 6 (Converse robust duality theorem) *Let $\pi_0 \in \Delta$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ be robust feasible points of the (RMPVC) and the (VC-RWD), respectively such that*

$$\check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) \geq 0, \forall \varepsilon \in \mathbb{Q}, \forall \sigma_\varepsilon \in \Omega_\varepsilon,$$
$$\check{\xi}_\varepsilon \Phi_\varepsilon(\check{\vartheta}_0) = 0, \varepsilon \in \mathbb{Y}_0,$$

$$-(\check{\varrho}_\varepsilon)^{\varpi}\varpi_\varepsilon(\check{\vartheta}_0) \geq 0, \varepsilon \in \mathbb{K}_0,$$
$$(\check{\varrho}_\varepsilon)^{\zeta}\zeta_\varepsilon(\check{\vartheta}_0) \geq 0, \varepsilon \in \mathbb{K}_0,$$

Suppose one of the subsequent cases occurs:

(i) $\Theta(., \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta})$ *is higher order strongly invex function at* $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$ *with regard to the kernel function* η_0,

(ii) $f_0, \psi_\varepsilon(\varepsilon \in \varphi_\psi^+(\pi_0)), \Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(\pi_0)), -\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(\pi_0)), -\varpi_\varepsilon(\varepsilon \in \varphi_{+0}^+(\pi_0)\cup \varphi_{+-}^+(\pi_0) \cup \varphi_{00}^+(\pi_0) \cup \varphi_{0-}^+(\pi_0) \cup \varphi_{0+}^+(\pi_0)), \varpi_\varepsilon(\varepsilon \in \varphi_{0+}^-(\pi_0)), \zeta_\varepsilon(\varepsilon \in \varphi_{+0}^{++}(\pi_0)) \cup \varphi_{+-}^{++}(\pi_0))$ *are higher order strongly invex functions at* $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$ *with regard to the common kernel function* η_0. *Then* $\check{\vartheta}_0$ *is a robust global minimum of the (RMPVC).*

Proof Suppose that $\check{\vartheta}_0$ is not a robust global minimum of (RMPVC), then there exists $\tilde{\pi}_0 \in \Delta$ such that

$$f_0(\tilde{\pi}_0) < f_0(\check{\vartheta}_0). \tag{28}$$

(i) Since $\tilde{\pi}_0$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta})$ are the robust feasible points of the (RMPVC) and the (VC-RWD), respectively. Based on the assumption in the theorem, we arrive at the following inequality

$$\sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon\psi_\varepsilon(\tilde{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon\Phi_\varepsilon(\tilde{\pi}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi}\varpi_\varepsilon(\tilde{\pi}_0) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta}\zeta_\varepsilon(\tilde{\pi}_0)$$

$$\leq 0 \leq \sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon)$$

$$+ \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon\Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi}\varpi_\varepsilon(\check{\vartheta}_0) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta}\zeta_\varepsilon(\check{\vartheta}_0). \tag{29}$$

On adding (28) and (29), we have

$$\Theta(\tilde{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}) < \Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}).$$

By using the higher order strong invexity of $\Theta(., \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta})$ with regard to the kernel function η_0 at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$, we get

$$\langle\nabla\Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle + h\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} < 0,$$

which contradicts the dual constraint of the (VC-RWD)(π_0). The theorem is therefore validated.

(ii) Since $\tilde{\pi}_0$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta})$ are robust feasible points of the (RMPVC) and the (VC-RWD), respectively. Based on the assumptions in the theorem, we arrive at the following inequalities

$$
\begin{aligned}
\psi_\varepsilon(\tilde{\pi}_0, \sigma_\varepsilon) &\le \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \varepsilon \in \varphi_\psi^+(\tilde{\pi}_0),\\
\Phi_\varepsilon(\tilde{\pi}_0) &= \Phi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi_\Phi^+(\tilde{\pi}_0) \cup \varphi_\Phi^-(\tilde{\pi}_0),\\
-\varpi_\varepsilon(\tilde{\pi}_0) &\le -\varpi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi_{+0}^+(\tilde{\pi}_0) \cup \varphi_{+-}^+(\tilde{\pi}_0) \cup \varphi_{00}^+(\tilde{\pi}_0) \cup \varphi_{0-}^+(\tilde{\pi}_0) \cup \varphi_{0+}^+(\tilde{\pi}_0),\\
-\varpi_\varepsilon(\tilde{\pi}_0) &\ge -\varpi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi_{0+}^-(\tilde{\pi}_0),\\
\zeta_\varepsilon(\tilde{\pi}_0) &\le \zeta_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi_{+0}^{++}(\tilde{\pi}_0) \cup \varphi_{+-}^{++}(\tilde{\pi}_0).
\end{aligned}
$$

By using the higher order strong invexity of the function with regard to the common kernel η_0, we have

$$
\begin{aligned}
&\langle \nabla \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle + \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \le 0, \tilde{h}_\varepsilon > 0, \check{\upsilon}_\varepsilon > 0, \varepsilon \in \varphi_\psi^+(\tilde{\pi}_0),\\
&\langle \nabla \Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle + \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \le 0, \tilde{h}_\varepsilon > 0, \check{\xi}_\varepsilon > 0, \varepsilon \in \varphi_\Phi^+(\tilde{\pi}_0),\\
&\langle \nabla \Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle + \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \ge 0, \tilde{h}_\varepsilon > 0, \check{\xi}_\varepsilon < 0, \varepsilon \in \varphi_\Phi^-(\tilde{\pi}_0),\\
&\quad - \langle \nabla \varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \le 0, (\tilde{h}_\varepsilon)^\varpi > 0, (\check{\varrho}_\varepsilon)^\varpi \ge 0,\\
&\quad \varepsilon \in \varphi_{+0}^+(\tilde{\pi}_0)\\
&\cup \varphi_{+-}^+(\tilde{\pi}_0) \cup \varphi_{00}^+(\tilde{\pi}_0) \cup \varphi_{0-}^+(\tilde{\pi}_0)\\
&\cup \varphi_{0+}^+(\tilde{\pi}_0),\\
&\quad - \langle \nabla \varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \ge 0, (\tilde{h}_\varepsilon)^\varpi > 0, (\check{\varrho}_\varepsilon)^\varpi \le 0,\\
&\quad \varepsilon \in \varphi_{+0}^+(\tilde{\pi}_0),\\
&\langle \nabla \zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \le 0, (\tilde{h}_\varepsilon)^\zeta > 0, (\check{\varrho}_\varepsilon)^\zeta \ge 0, \varepsilon \in \varphi_{+0}^{++}(\tilde{\pi}_0)\\
&\cup \varphi_{+-}^{++}(\tilde{\pi}_0),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\langle \sum_{\varepsilon \in \mathbb{Q}} \check{\upsilon}_\varepsilon \nabla \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \check{\xi}_\varepsilon \nabla \Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^\varpi \nabla \varpi_\varepsilon(\check{\vartheta}_0)\\
&\quad + \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^\zeta \nabla \zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle\\
&\quad + \check{\upsilon}_\varepsilon h_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota + \check{\xi}_\varepsilon \check{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota - (\check{\varrho}_\varepsilon)^\varpi (\check{h}_\varepsilon)^\varpi \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota\\
&\quad + (\check{\varrho}_\varepsilon)^\zeta (\check{h}_\varepsilon)^\zeta \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \le 0.
\end{aligned}
$$

Using the above inequality and the first equation of (16), it follows that

$$
\langle \nabla f_0(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle + h^\circ \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \ge 0,
$$

where,

$$h^{\circ}\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} = -\check{\upsilon}_{\varepsilon}\check{h}_{\varepsilon}\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} - \check{\xi}_{\varepsilon}\check{h}_{\varepsilon}\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota}$$
$$+(\check{\varrho}_{\varepsilon})^{\varpi}(\check{h}_{\varepsilon})^{\varpi}\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} - (\check{\varrho}_{\varepsilon})^{\zeta}(\check{h}_{\varepsilon})^{\zeta}\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota}.$$

By using the higher order strong invexity of f_0, with regard to the kernel function η_0, we get

$$f_0(\tilde{\pi}_0) \geq f_0(\check{\vartheta}_0),$$

which contradicts (28). The theorem is therefore validated. □

Theorem 7 (Restricted converse robust duality theorem) *Let $\check{\pi}_0 \in \Delta$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta})$ be robust feasible points of the (RMPVC) and the (VC-RWD), respectively, such that*

$$f_0(\check{\pi}_0) = \Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}).$$

Suppose one of the subsequent cases occurs:

(i) $\Theta(., \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta})$ is higher order strongly invex function at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathscr{W}}$ with regard to the kernel function η_0,
(ii) f_0, $\psi_{\varepsilon}(\varepsilon \in \varphi^{+}_{\psi}(\check{\pi}_0))$, $\Phi_{\varepsilon}(\varepsilon \in \varphi^{+}_{\Phi}(\check{\pi}_0))$, $-\Phi_{\varepsilon}(\varepsilon \in \varphi^{-}_{\Phi}(\check{\pi}_0))$, $-\varpi_{\varepsilon}(\varepsilon \in \varphi^{+}_{+0}(\check{\pi}_0) \cup \varphi^{+}_{+-}(\check{\pi}_0) \cup \varphi^{+}_{00}(\check{\pi}_0) \cup \varphi^{+}_{0-}(\check{\pi}_0) \cup \varphi^{+}_{0+}(\check{\pi}_0))$, $\varpi_{\varepsilon}(\varepsilon \in \varphi^{-}_{0+}(\check{\pi}_0))$, $\zeta_{\varepsilon}(\varepsilon \in \varphi^{++}_{+0}(\check{\pi}_0)) \cup \varphi^{++}_{+-}(\check{\pi}_0))$ are higher order strongly invex functions at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}$ with regard to the common kernel function η_0. Then $\check{\pi}_0$ is a robust global minimum of the (RMPVC).

Proof Suppose that $\check{\pi}_0$ is not a robust global minimum of the (RMPVC), then there exists $\tilde{\pi}_0 \in \Delta$ such that

$$f_0(\tilde{\pi}_0) < f_0(\check{\pi}_0).$$

Based on the assumptions in the theorem, we arrive at the following inequality

$$f_0(\tilde{\pi}_0) < \Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}),$$

which contradicts Theorem 4. The theorem is therefore validated. □

Theorem 8 (Strict converse robust duality theorem) *Let $\check{\pi}_0 \in \Delta$ be a robust local minimum of the (RMPVC) such that the (VC-ACQ) is fulfilled at $\check{\pi}_0$. Assume the conditions of* Theorem 5 *hold and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta})$ is a robust global maximum point of the (VC-RWD)$(\check{\pi}_0)$. Suppose one of the subsequent cases occurs:*

(i) $\Theta(., \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta})$ is strictly higher order strongly invex function at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}$ with regard to the kernel function η_0,
(ii) f_0 is strictly higher order strongly invex function and $\psi_{\varepsilon}(\varepsilon \in \varphi^{+}_{\psi}(\check{\pi}_0))$, $\Phi_{\varepsilon}(\varepsilon \in \varphi^{+}_{\Phi}(\check{\pi}_0))$, $-\Phi_{\varepsilon}(\varepsilon \in \varphi^{-}_{\Phi}(\check{\pi}_0))$, $-\varpi_{\varepsilon}(\varepsilon \in \varphi_{+0}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0) \cup \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi^{+}_{0+}(\check{\pi}_0))$, $\varpi_{\varepsilon}(\varepsilon \in \varphi^{-}_{0+}(\check{\pi}_0))$, $-\zeta_{\varepsilon}(\varepsilon \in \varphi_{0+}(\check{\pi}_0))$, $\zeta_{\varepsilon}(\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0) \cup$

$\varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0))$ *are invex functions at* $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$ *with regard to the common kernel function* η_0. *Then* $\check{\pi}_0 = \check{\vartheta}_0$.

Proof (i) Suppose that $\check{\pi}_0 \neq \check{\vartheta}_0$. From Theorem 5, we can find $\check{\upsilon} = (\check{\upsilon}_\varepsilon)_{\varepsilon \in \mathbb{Q}} \in \mathbb{R}_+^{(\mathbb{Q})}$, $\check{\xi} \in \mathbb{R}^y$, $(\check{\varrho})^\varpi$, $(\check{\varrho})^\zeta$, $\check{\chi}$, $\check{\delta} \in \mathbb{R}^k$ such that $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ is a robust global maximum point of the (VC-RWD)($\check{\pi}_0$). Thus,

$$f_0(\check{\pi}_0) = \Theta(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta) = \Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta). \tag{30}$$

By using the robust feasibility of $\check{\pi}_0$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta})$ for the (RMPVC) and (VC-RWD)($\check{\pi}_0$), respectively, we obtain

$$\begin{aligned}
&\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) < 0, \check{\upsilon}_\varepsilon \geq 0, \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \notin \varphi_\psi(\check{\pi}_0),\\
&\psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) = 0, \check{\upsilon}_\varepsilon \geq 0, \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \in \varphi_\psi(\check{\pi}_0),\\
&\Phi_\varepsilon(\check{\pi}_0) = 0, \check{\xi}_\varepsilon \in \mathbb{R}, \varepsilon \in \varphi_\Phi(\check{\pi}_0),\\
&\quad -\varpi_\varepsilon(\check{\pi}_0) < 0, (\check{\varrho})_\varepsilon^\varpi \geq 0, \varepsilon \in \varphi_+(\check{\pi}_0),\\
&\quad -\varpi_\varepsilon(\check{\pi}_0) = 0, (\check{\varrho})_\varepsilon^\varpi \in \mathbb{R}, \varepsilon \in \varphi_0(\check{\pi}_0),\\
&\zeta_\varepsilon(\check{\pi}_0) > 0, (\check{\varrho})_\varepsilon^\zeta = 0, \varepsilon \in \varphi_{0+}(\check{\pi}_0),\\
&\zeta_\varepsilon(\check{\pi}_0) = 0, (\check{\varrho})_\varepsilon^\zeta \geq 0, \varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0),\\
&\zeta_\varepsilon(\check{\pi}_0) < 0, (\check{\varrho})_\varepsilon^\zeta \geq 0, \varepsilon \in \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0),
\end{aligned}$$

which leads to,

$$\sum_{\varepsilon \in \mathbb{Q}} \check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \check{\xi}_\varepsilon \Phi_\varepsilon(\check{\pi}_0) - \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^\varpi \varpi_\varepsilon(\check{\pi}_0) + \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^\zeta \zeta_\varepsilon(\check{\pi}_0) \leq 0. \tag{31}$$

On adding (30) and (31), we have

$$\Theta(\check{\pi}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta) \leq \Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta). \tag{32}$$

By using the strict higher order strong invexity of $\Theta(., \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta)$ with regard to the common kernel function η_0, leads to

$$\langle \nabla\Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\rangle + \tilde{h}\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota < 0,$$

which contradicts the first equation in (16). The theorem is therefore validated.

(ii) By using the strict higher order strong invexity of f_0 at $\check{\vartheta}_0$ with regard to the kernal function η_0, we get

$$f_0(\check{\pi}_0) - f_0(\check{\vartheta}_0) > \Big\langle \nabla f_0(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\Big\rangle + \tilde{h}\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota. \tag{33}$$

By using the higher order strong invexity of $\psi_\varepsilon(\varepsilon \in \varphi_\psi^+(\check{\pi}_0))$, $\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^+(\check{\pi}_0))$, $-\Phi_\varepsilon(\varepsilon \in \varphi_\Phi^-(\check{\pi}_0))$, $-\varpi_\varepsilon(\varepsilon \in \varphi_{+0}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0) \cup \varphi_{00}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi_{0+}^+(\check{\pi}_0))$,

$\varpi_\varepsilon(\varepsilon \in \varphi^-_{0+}(\check{\pi}_0))$, $-\zeta_\varepsilon(\varepsilon \in \varphi_{0+}(\check{\pi}_0))$, $\zeta_\varepsilon(\varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0) \cup \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0))$ at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\check{\pi}_0)$, $\check{\pi}_0 \in \Delta$, with regard to common kernel function η_0 and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta}) \in \mathbb{S}^{\mathbb{R}}_{\mathbb{W}}(\check{\pi}_0)$, which leads to

$$
\begin{aligned}
&\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \left\langle \nabla\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \\
&\leq \psi_\varepsilon(\check{\pi}_0, \sigma_\varepsilon) \leq 0, \tilde{h}_\varepsilon > 0, \\
&\check{\upsilon}_\varepsilon > 0, \varepsilon \in \varphi^+_\psi(\check{\pi}_0), \\
&\Phi_\varepsilon(\check{\vartheta}_0) + \left\langle \nabla\Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq \Phi_\varepsilon(\check{\pi}_0) = 0, \tilde{h}_\varepsilon > 0, \\
&\check{\xi}_\varepsilon > 0, \varepsilon \in \varphi^+_\Phi(\check{\pi}_0), \\
&\Phi_\varepsilon(\check{\vartheta}_0) + \left\langle \nabla\Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + \tilde{h}_\varepsilon \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \geq \Phi_\varepsilon(\check{\pi}_0) = 0, \tilde{h}_\varepsilon > 0, \\
&\check{\xi}_\varepsilon < 0, \varepsilon \in \varphi^-_\Phi(\check{\pi}_0), \\
&\quad -\varpi_\varepsilon(\check{\vartheta}_0) - \left\langle \nabla\varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq -\varpi_\varepsilon(\check{\pi}_0) \leq 0, (\tilde{h}_\varepsilon)^\varpi > 0, \\
&(\check{\varrho}_\varepsilon)^\varpi \geq 0, \varepsilon \in \varphi_{+0}(\check{\pi}_0) \\
&\cup \varphi_{+-}(\check{\pi}_0) \cup \varphi_{00}(\check{\pi}_0) \\
&\cup \varphi_{0-}(\check{\pi}_0) \cup \varphi^+_{0+}(\check{\pi}_0), \\
&\quad -\varpi_\varepsilon(\check{\vartheta}_0) - \left\langle \nabla\varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq -\varpi_\varepsilon(\check{\pi}_0) = 0, (\tilde{h}_\varepsilon)^\varpi > 0, \\
&(\check{\varrho}_\varepsilon)^\varpi < 0, \varepsilon \in \varphi^-_{0+}(\check{\pi}_0), \\
&\zeta_\varepsilon(\check{\vartheta}_0) + \left\langle \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \geq \zeta_\varepsilon(\check{\pi}_0) > 0, (\tilde{h}_\varepsilon)^\zeta > 0, \\
&(\check{\varrho}_\varepsilon)^\zeta = 0, \varepsilon \in \varphi_{0+}(\check{\pi}_0), \\
&\zeta_\varepsilon(\check{\vartheta}_0) + \left\langle \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq \zeta_\varepsilon(\check{\pi}_0) = 0, (\tilde{h}_\varepsilon)^\zeta > 0, \\
&(\check{\varrho}_\varepsilon)^\zeta \geq 0, \\
&\varepsilon \in \varphi_{+0}(\check{\pi}_0) \cup \varphi_{00}(\check{\pi}_0), \\
&\zeta_\varepsilon(\check{\vartheta}_0) + \left\langle \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq \zeta_\varepsilon(\check{\pi}_0) < 0, (\tilde{h}_\varepsilon)^\zeta > 0, \\
&(\check{\varrho}_\varepsilon)^\zeta \geq 0, \\
&\varepsilon \in \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^\varpi \varpi_\varepsilon(\check{\vartheta}_0) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^\zeta \zeta_\varepsilon(\check{\vartheta}_0) + \\
&\langle \sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \nabla\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \nabla\Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^\varpi \nabla\varpi_\varepsilon(\check{\vartheta}_0) \\
&+ \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^\zeta \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\rangle \\
&+\check{\upsilon}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota + \check{\xi}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota - (\check{\varrho}_\varepsilon)^\varpi (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota
\end{aligned}
$$

$$+ (\check{\varrho}_\varepsilon)^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0. \tag{34}$$

On adding (33) and (34), we have

$$\Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta) < f_0(\check{\pi}_0),$$

which contradicts (30). The theorem is therefore validated. □

We now re-explore Example 1 to verify the above Theorems.

Example 2

$$(\text{RMPVC-1}) \quad \min_{\pi_0 \in \mathbb{R}} \quad f_0(\pi_0) = \frac{1}{2}\pi_0 - 2$$
$$\text{subject to}$$

$$\left.\begin{array}{l} \psi_\varepsilon(\pi_0, \sigma_\varepsilon) = -\varepsilon\pi_0^2 + \sigma_\varepsilon\pi_0 \leq 0, \forall \varepsilon \in \mathbb{Q} = [0,1], \\ \forall \sigma_\varepsilon \in [-\varepsilon+2, \varepsilon+2], \\ \varpi_1(\pi_0) = \pi_0 \geq 0, \\ \zeta_1(\pi_0)\varpi_1(\pi_0) = \pi_0(\pi_0 + \pi_0^2) \leq 0, \end{array}\right\} \tag{35}$$

with $n = 1$, $\varepsilon \in \mathbb{Q} = [0, 1]$, $y = 0$, $k = 1$. Clearly, $f_0(\pi_0) = \frac{1}{2}\pi_0 - 2$ is Lipschitz continuous in $\mathbb{R}$. Let $\sigma_\varepsilon = 1$, for $\varepsilon = 1$ and $\sigma_\varepsilon = 0$, for $0 \leq \varepsilon < 1$. For any robust feasible solution $\pi_0 \in \Delta$, the Wolfe type robust dual model to condition (35) is shown as

$$\max \quad \Theta(\vartheta_0, \upsilon_1, \varrho_1^\varpi, \varrho_1^\zeta) = \frac{1}{2}\vartheta_0 - 2 - \upsilon_1(-\vartheta_0{}^2 + \vartheta_0) - \varrho_1^\varpi\vartheta_0 + \varrho_1^\zeta(\vartheta_0 + \vartheta_0{}^2)$$
$$\text{subject to}$$

$$\left.\begin{array}{l} \nabla\Theta(\vartheta_0, \upsilon_1, \varrho_1^\varpi, \varrho_1^\zeta) = \frac{1}{2} - \upsilon_1(-2\vartheta_0 + 1) - \varrho_1^\varpi + \varrho_1^\zeta(1 + 2\vartheta_0) = 0, \\ \varrho_1^\zeta = \delta_1\pi_0, \delta_1 \geq 0, \\ \varrho_1^\varpi = \chi_1 - \delta_1(\pi_0 + \pi_0^2), \chi_1 \geq 0. \end{array}\right\} \tag{36}$$

(i) The robust feasible set $\mathbb{S}_{\mathbb{W}}^{\mathbb{R}}$, of the VC-RWD is given by $\{(\vartheta_0, \upsilon_1, \varrho_1^\varpi, \varrho_1^\zeta, \chi_1, \delta_1) : 2\vartheta_0(\upsilon_1 + \varrho_1^\varpi) = 0,$

$$\upsilon_1 = \frac{1}{2} - \varrho_1^\varpi + \varrho_1^\zeta,$$
$$\varrho_1^\zeta = \delta_1\varpi_1(\pi_0), \delta_1 \geq 0,$$
$$\varrho_1{}^\varpi = \chi_1 - \delta_1\zeta_1(\pi_0), \chi_1 \geq 0\}.$$

Also, from condition (35), we get $\check{\gamma}_0 = 0$ as a robust feasible solution and from condition (36), we have $\varrho_1^{\zeta} = 0$, ${\varrho_1}^{\varpi} \geq 0$ and we obtain

$$\Theta(\vartheta_0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}) = -2 < 0$$

and it is easy to see that $f_0(\check{\pi}_0) = -2 = \Theta(\vartheta_0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta})$. It is proved that the hypothesis of Theorem 6 is fulfilled. From the condition (35), $\check{\pi}_0$ is a robust global minimum of the (RMPVC). Therefore, Theorem 6 is verified.

(ii) From condition (35), we get $\vartheta_0 = 0$, we get

$$\Theta(\vartheta_0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}) = -2 < 0.$$

Since $\check{\pi}_0 = 0$ is the robust feasible point of the (RMPVC) and VC-ACQ holds at $\check{\pi}_0$. We get $f_0(\check{\gamma}_0) \geq \Theta(\vartheta_0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta})$. Hence, Theorem 4 is verified.

(iii) Clearly, VC-ACQ is fulfilled at $\check{\pi}_0 = 0$. By Theorem 1, there exist $\upsilon_1 \in \mathbb{R}_+$, $\varrho_1^{\varpi}, \varrho_1^{\zeta} \in \mathbb{R}$ such that $(0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}, \chi_1, \delta_1)$ is a robust feasible point of the VC-ACQ(0) and

$$\upsilon_1(\psi_1(0, \sigma_1)) - \varrho_1^{\varpi} \varpi_1(0) + \varrho_1^{\zeta}(0) = 0.$$

So, $(0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}, \chi_1, \delta_1)$ is a robust global maximum of the VC-RWD(0) and $f_0(0) = -2 = \Theta(0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta})$. Theorem 5 is justified.

In the following section, we discuss the Mond–Weir type robust dual model and prove the duality theorems. The dual model used here is based on the lines of Hu et al. [18].

4 Mond–Weir type robust dual model

The Mond–Weir type robust dual of the (RMPVC) depending on a robust feasible point $\pi_0 \in \Delta$, represented by the (VC-RMWD)(π_0), is provided in this section. The details are as follows:

$$\max \quad f_0(\vartheta_0)$$
$$\text{subject to}$$

$$\left.\begin{array}{l}
\nabla\Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) = 0,\\
\upsilon_\varepsilon \geq 0, \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) \geq 0, \forall \varepsilon \in \mathbb{Q}, \forall \sigma_\varepsilon \in \Omega_\varepsilon,\\
\xi_\varepsilon \Phi_\varepsilon(\vartheta_0) = 0, \forall \varepsilon \in \mathbb{Y}_0,\\
{\varrho_\varepsilon}^{\zeta} \zeta_\varepsilon(\vartheta_0) \geq 0, \forall \varepsilon \in \mathbb{K}_0,\\
{\varrho_\varepsilon}^{\zeta} = \delta_\varepsilon \varpi_\varepsilon(\pi_0), \delta_\varepsilon \geq 0, \forall \varepsilon \in \mathbb{K}_0,\\
\quad -{\varrho_\varepsilon}^{\varpi} \varpi_\varepsilon(\vartheta_0) \geq 0, \forall \varepsilon \in \mathbb{K}_0,\\
{\varrho_\varepsilon}^{\varpi} = \chi_\varepsilon - \delta_\varepsilon \zeta_\varepsilon(\vartheta_0), \chi_\varepsilon \geq 0, \forall \varepsilon \in \mathbb{K}_0.
\end{array}\right\} \tag{37}$$

Let $\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0)$ represents the set of all robust feasible points of the problem (VC-RMWD)(π_0) where $\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0) = \{(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}, \chi, \delta) : \nabla\Theta(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}) = 0,$

$$\begin{aligned} &\upsilon_\varepsilon \geq 0, \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) \geq 0, \forall \varepsilon \in \mathbb{Q}, \forall \sigma_\varepsilon \in \Omega_\varepsilon, \\ &\xi_\varepsilon \Phi_\varepsilon(\vartheta_0) = 0, \forall \varepsilon \in \mathbb{Y}_0, \\ &{\varrho_\varepsilon}^{\zeta} \zeta_\varepsilon(\vartheta_0) \geq 0, \forall \varepsilon \in \mathbb{K}_0, \\ &{\varrho_\varepsilon}^{\zeta} = \delta_\varepsilon \varpi_\varepsilon(\pi_0), \delta_\varepsilon \geq 0, \forall \varepsilon \in \mathbb{K}_0, \\ &\quad - {\varrho_\varepsilon}^{\varpi} \varpi_\varepsilon(\vartheta_0) \geq 0, \forall \varepsilon \in \mathbb{K}_0, \\ &{\varrho_\varepsilon}^{\varpi} = \chi_\varepsilon - \delta_\varepsilon \zeta_\varepsilon(\pi_0), \chi_\varepsilon \geq 0, \forall \varepsilon \in \mathbb{K}_0\}. \end{aligned} \tag{38}$$

We represent the projection of the set $\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0)$ on $\mathbb{R}^n$ by

$$pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0) = \{\vartheta_0 \in \mathbb{R}^n : (\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}, \chi, \delta) \in \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0)\}.$$

For $\pi_0 \in \Delta$, the new Mond–Weir type robust dual is independent of the (RMPVC), we consider the subsequent Mond–Weir type robust dual problem:

$$\begin{aligned} \max \quad & f_0(\vartheta_0) \\ \text{such that} \quad & (\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}, \chi, \delta) \in \bigcap_{\pi_0 \in \Delta} \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0). \end{aligned} \tag{39}$$

The set of all robust feasible points of (VC-RMWD) is represented by $\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}} = \bigcap_{\pi_0 \in \Delta} \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}(\pi_0)$ and the projection of the set $\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ on $\mathbb{R}^n$ is represented by $pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$.

Remark 2 Mond–Weir type dual model exists in literature for a mathematical programming problem with vanishing constraints (See, Mishra et al. [15] and Ahmad et al. [21]) using index sets. These models are not suitable for numerical solutions to dual problems since they need to calculate index sets. As a result, Hu et al. [18] recently proposed new Mond–Weir type dual model for a mathematical programming problem with vanishing constraints and established the results of duality under generalized convexity assumptions that do not require index set calculations.

Theorem 9 (Weak robust duality theorem) *Let* $\pi_0 \in \Delta$, $(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}, \chi, \delta) \in \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ *be robust feasible points of the (RMPVC) and the (VC-RMWD), respectively. Suppose one of the subsequent cases occurs:*

(i) $f_0(.)$ *is higher order strongly pseudoinvex function and* $\sum_{\varepsilon \in \mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(., \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(.) - \sum_{\varepsilon \in \mathbb{K}_0} \varrho^{\varpi}_\varepsilon \varpi_\varepsilon(.) + \sum_{\varepsilon \in \mathbb{K}_0} \varrho^{\zeta}_\varepsilon \zeta_\varepsilon(.)$ *is quasiinvex function at* $\vartheta_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ *with regard to the kernel function* η_0,

(ii) $f_0(.)$ *is higher order strongly pseudoinvex function and* $\psi_\varepsilon(\varepsilon \in \varphi^{+}_{\psi}(\pi_0))$, $\Phi_\varepsilon(\varepsilon \in \varphi^{+}_{\Phi}(\pi_0))$, $-\Phi_\varepsilon(\varepsilon \in \varphi^{-}_{\Phi}(\pi_0))$, $-\varpi_\varepsilon(\varepsilon \in \varphi^{+}_{+0}(\pi_0) \cup \varphi^{+}_{+-}(\pi_0) \cup \varphi^{+}_{00}(\pi_0) \cup$

$\varphi_{0-}^{+}(\pi_0) \cup \varphi_{0+}^{+}(\pi_0))$, $\varpi_\varepsilon(\varepsilon \in \varphi_{0+}^{-}(\pi_0))$, $\zeta_\varepsilon(\varphi_{+0}^{++}(\pi_0) \cup \varphi_{+-}^{++}(\pi_0))$ *are higher order strongly quasiinvex functions at* $\vartheta_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$ *with regard to the common kernel function* η_0. *Then*

$$f_0(\pi_0) \geq f_0(\vartheta_0).$$

Proof (i) Since $\pi_0 \in \Delta$ and $(\vartheta_0, \upsilon, \xi, \varrho^{\varpi}, \varrho^{\zeta}, \chi, \delta) \in \mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$, it follows that

$$\begin{aligned}
\psi_\varepsilon(\pi_0, \sigma_\varepsilon) \leq 0, \upsilon_\varepsilon \geq 0, \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \in \mathbb{Q},\\
\Phi_\varepsilon(\pi_0) = 0, \xi_\varepsilon \in \mathbb{R}, \varepsilon \in \varphi_\Phi,\\
-\varpi_\varepsilon(\pi_0) < 0, \varrho_\varepsilon^{\varpi} \geq 0, \varepsilon \in \varphi_+(\pi_0),\\
-\varpi_\varepsilon(\pi_0) = 0, \varrho_\varepsilon^{\varpi} \in \mathbb{R}, \varepsilon \in \varphi_0(\pi_0),\\
\zeta_\varepsilon(\pi_0) > 0, \varrho_\varepsilon^{\zeta} = 0, \varepsilon \in \varphi_{0+}(\pi_0),\\
\zeta_\varepsilon(\pi_0) = 0, \varrho_\varepsilon^{\zeta} \geq 0, \varepsilon \in \varphi_{00}(\pi_0) \cup \varphi_{+0}(\pi_0),\\
\zeta_\varepsilon(\pi_0) < 0, \varrho_\varepsilon^{\zeta} \geq 0, \varepsilon \in \varphi_{0-}(\pi_0) \cup \varphi_{+-}(\pi_0),
\end{aligned}$$

By (37), it implies that

$$\begin{aligned}
&\sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\pi_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\pi_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\pi_0) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\pi_0)\\
&\leq \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(\vartheta_0) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(\vartheta_0).
\end{aligned}$$

Combining the higher order strongly quasiinvexity of $\sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \psi_\varepsilon(., \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \Phi_\varepsilon(.) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\varpi} \varpi_\varepsilon(.) + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\zeta} \zeta_\varepsilon(.)$ with regard to the kernel function η_0, we have

$$\begin{aligned}
&\langle \sum_{\varepsilon\in\mathbb{Q}} \upsilon_\varepsilon \nabla\psi_\varepsilon(\vartheta_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \xi_\varepsilon \nabla\Phi_\varepsilon(\vartheta_0) - \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\varpi} \nabla\varpi_\varepsilon(\vartheta_0)\\
&\quad + \sum_{\varepsilon\in\mathbb{K}_0} \varrho_\varepsilon^{\zeta} \nabla\zeta_\varepsilon(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle\\
&\quad + \upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^{\iota} +\\
&\xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^{\iota} - \varrho_\varepsilon^{\varpi} (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\pi_0, \vartheta_0)\|^{\iota}\\
&\quad + \varrho_\varepsilon^{\zeta} (\tilde{h}_\varepsilon)^{\zeta} \|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \leq 0.
\end{aligned}$$

By using the above inequality and the first equation in (37), we get

$$\langle \nabla f_0(\vartheta_0), \eta_0(\pi_0, \vartheta_0)\rangle + h^{\circ} \|\eta_0(\pi_0, \vartheta_0)\|^{\iota} \geq 0,$$

where,

$$h^{\circ} \|\eta_0(\pi_0, \vartheta_0)\|^{\iota} = -\upsilon_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \vartheta_0)\|^{\iota} - \xi_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\pi_0, \upsilon)\|^{\iota}$$

$$+\varrho_\varepsilon^{\varpi}(\tilde{h}_\varepsilon)^{\varpi}\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}-\varrho_\varepsilon^{\zeta}(\tilde{h}_\varepsilon)^{\zeta}\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}.$$

By using the higher order strong pseudoinvexity of f_0 with regard to the kernel function η_0, we get

$$f_0(\pi_0)\geq f_0(\vartheta_0).$$

The theorem is therefore validated.

(ii) Since $\pi_0\in\Delta$ and $(\vartheta_0,\upsilon,\xi,\varrho^{\varpi},\varrho^{\zeta},\chi,\delta)\in\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$, it follows that

$$\begin{aligned}
&\psi_\varepsilon(\pi_0,\sigma_\varepsilon)\leq\psi_\varepsilon(\vartheta_0,\sigma_\varepsilon),\ \varepsilon\in\varphi_\psi^{+}(\pi_0),\ \sigma_\varepsilon\in\Omega_\varepsilon,\\
&\Phi_\varepsilon(\pi_0)=\Phi_\varepsilon(\vartheta_0),\ \varepsilon\in\varphi_\Phi^{+}(\pi_0)\cup\varphi_\Phi^{-}(\pi_0),\\
&\quad-\varpi_\varepsilon(\pi_0)\leq-\varpi_\varepsilon(\vartheta_0),\ \varepsilon\in\varphi_{+0}^{+}(\pi_0)\cup\varphi_{+-}^{+}(\pi_0)\cup\varphi_{00}^{+}(\pi_0)\cup\varphi_{0-}^{+}(\pi_0)\cup\varphi_{0+}^{+}(\pi_0),\\
&\quad-\varpi_\varepsilon(\pi_0)\geq-\varpi_\varepsilon(\vartheta_0),\ \varepsilon\in\varphi_{0+}^{-}(\pi_0),\\
&\zeta_\varepsilon(\pi_0)\leq\zeta_\varepsilon(\vartheta_0),\ \varepsilon\in\varphi_{+0}^{++}(\pi_0)\cup\varphi_{+-}^{++}(\pi_0).
\end{aligned}$$

By using the higher order strong quasiinvexity of $\psi_\varepsilon(\varepsilon\in\varphi_\psi^{+}(\pi_0))$, $\Phi_\varepsilon(\varepsilon\in\varphi_\Phi^{+}(\pi_0))$, $-\Phi_\varepsilon(\varepsilon\in\varphi_\Phi^{-}(\pi_0))$, $-\varpi_\varepsilon(\varepsilon\in\varphi_{+0}^{+}(\pi_0)\cup\varphi_{+-}^{+}(\pi_0)\cup\varphi_{00}^{+}(\pi_0)\cup\varphi_{0-}^{+}(\pi_0)\cup\varphi_{0+}^{+}(\pi_0))$, $\varpi_\varepsilon(\varepsilon\in\varphi_{0+}^{-}(\pi_0))$, $\zeta_\varepsilon(\varphi_{+0}^{++}(\pi_0)\cup\varphi_{+-}^{++}(\pi_0))$ with regard to the common kernel function η_0, we get

$$\begin{aligned}
&\langle\nabla\psi_\varepsilon(\vartheta_0,\sigma_\varepsilon),\eta_0(\pi_0,\vartheta_0)\rangle+\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}\leq 0,\ \tilde{h}_\varepsilon>0,\ \upsilon_\varepsilon>0,\\
&\varepsilon\in\varphi_\psi^{+}(\pi_0),\\
&\langle\nabla\Phi_\varepsilon(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle+\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}\leq 0,\ \tilde{h}_\varepsilon>0,\ \xi_\varepsilon>0,\\
&\varepsilon\in\varphi_\Phi^{+}(\pi_0),\\
&\langle\nabla\Phi_\varepsilon(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle+\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}\geq 0,\ \tilde{h}_\varepsilon>0,\ \xi_\varepsilon<0,\\
&\varepsilon\in\varphi_\Phi^{-}(\pi_0),\\
&\quad-\langle\nabla\varpi_\varepsilon(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle-(\tilde{h}_\varepsilon)^{\varpi}\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}\leq 0,\ (\tilde{h}_\varepsilon)^{\varpi}>0,\ \varrho_\varepsilon^{\varpi}\geq 0,\\
&\varepsilon\in\varphi_{+0}^{+}(\pi_0)\cup\varphi_{+-}^{+}(\pi_0)\\
&\cup\varphi_{00}^{+}(\pi_0)\cup\varphi_{0-}^{+}(\pi_0)\cup\varphi_{0+}^{+}(\pi_0),\\
&\quad-\langle\nabla\varpi_\varepsilon(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle-(\tilde{h}_\varepsilon)^{\varpi}\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}\geq 0,\ (\tilde{h}_\varepsilon)^{\varpi}>0,\ \varrho_\varepsilon^{\varpi}\leq 0,\\
&\varepsilon\in\varphi_{+0}^{-}(\pi_0),\\
&\langle\nabla\zeta_\varepsilon(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle+(\tilde{h}_\varepsilon)^{\zeta}\|\eta_0(\pi_0,\vartheta_0)\|^{\iota}\leq 0,\ (\tilde{h}_\varepsilon)^{\zeta}>0,\ \varrho_\varepsilon^{\zeta}\geq 0,\\
&\varepsilon\in\varphi_{+0}^{++}(\pi_0)\cup\varphi_{+-}^{++}(\pi_0).
\end{aligned}$$

By using the definition of index set in the above inequalities, we get

$$\langle\sum_{\varepsilon\in\mathbb{Q}}\upsilon_\varepsilon\nabla\psi_\varepsilon(\vartheta_0,\sigma_\varepsilon)+\sum_{\varepsilon\in\mathbb{Y}_0}\xi_\varepsilon\nabla\Phi_\varepsilon(\vartheta_0)-\sum_{\varepsilon\in\mathbb{K}_0}\varrho_\varepsilon^{\varpi}\nabla\varpi_\varepsilon(\vartheta_0)$$

$$+\sum_{\varepsilon\in\mathbb{K}_0}\varrho_\varepsilon^\zeta\nabla\zeta_\varepsilon(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle$$
$$+\upsilon_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^\iota+\xi_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^\iota-\varrho_\varepsilon^\varpi(\tilde{h}_\varepsilon)^\varpi\|\eta_0(\pi_0,\vartheta_0)\|^\iota$$
$$+\varrho_\varepsilon^\zeta(\tilde{h}_\varepsilon)^\zeta\|\eta_0(\pi_0,\vartheta_0)\|^\iota\leq 0.$$

By using the above inequality and the first equation in (37), we have

$$\langle\nabla f_0(\vartheta_0),\eta_0(\pi_0,\vartheta_0)\rangle+h^\circ\|\eta_0(\pi_0,\vartheta_0)\|^\iota\geq 0,$$

where,

$$\begin{aligned}h^\circ\|\eta_0(\pi_0,\vartheta_0)\|^\iota &= -\upsilon_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^\iota-\xi_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\pi_0,\vartheta_0)\|^\iota\\ &\quad+\varrho_\varepsilon^\varpi(\tilde{h}_\varepsilon)^\varpi\|\eta_0(\pi_0,\vartheta_0)\|^\iota-\varrho_\varepsilon^\zeta(\tilde{h}_\varepsilon)^\zeta\|\eta_0(\pi_0,\vartheta_0)\|^\iota.\end{aligned}$$

By using the higher order strong pseudoinvexity of f_0, with regard to the kernel function η_0, we get

$$f_0(\pi_0)\geq f_0(\vartheta_0).$$

The theorem is therefore validated. □

Theorem 10 (Strong robust duality theorem) *Let $\check{\pi}_0\in\Delta$ be a robust local minimum of the (RMPVC) such that the (VC-ACQ) is fulfilled at $\check{\pi}_0$. Then there exist $\check{\upsilon}=(\check{\upsilon}_\varepsilon)_{\varepsilon\in\mathbb{Q}}\in\mathbb{R}_+^{(\mathbb{Q})},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta}\in\mathbb{R}^k$ such that $(\check{\pi}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta})$ is a robust feasible point of the (VC-RMWD)$(\check{\pi}_0)$, that is, $(\check{\pi}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta})\in\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}(\check{\pi}_0)$. Moreover,* Theorem 9 *holds, then $(\check{\pi}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta)$ is a robust global maximum of the (VC-RMWD)$(\check{\pi}_0)$.*

Proof Since $\check{\pi}_0\in\Delta$ is a robust local minimum of the (RMPVC) such that the (VC-ACQ) holds at $\check{\pi}_0$. From Theorem 1, there exist $\check{\upsilon}=(\check{\upsilon}_\varepsilon)_{\varepsilon\in\mathbb{Q}}\in\mathbb{R}_+^{(\mathbb{Q})},\check{\xi}\in\mathbb{R}^y,(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta}\in\mathbb{R}^k$ such that the conditions (2) and (3) are satisfied and hence $(\check{\pi}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta})$ is a robust feasible point of (VC-RMWD)$(\check{\pi}_0)$. From Theorem 9, we get

$$f_0(\check{\pi}_0)\geq f_0(\vartheta_0),\ \forall(\vartheta_0,\upsilon,\xi,\varrho^\varpi,\varrho^\zeta,\chi,\delta)\in\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}(\check{\pi}_0)$$

and hence $(\check{\pi}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta})$ is a robust global maximum of (VC-RMWD) $(\check{\pi}_0)$. □

Theorem 11 (Converse robust duality theorem) *Let $\check{\pi}_0\in\Delta$ and $(\check{\vartheta}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^\varpi,(\check{\varrho})^\zeta,\check{\chi},\check{\delta})\in\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$ be robust feasible points of the (RMPVC) and the (VC-RMWD), respectively. Suppose one of the subsequent cases occurs:*
(i) $f_0(.)$ is higher order strongly pseudoinvex function and $\sum_{\varepsilon\in\mathbb{Q}}\check{\upsilon}_\varepsilon\psi_\varepsilon(.,\sigma_\varepsilon)+\sum_{\varepsilon\in\mathbb{Y}_0}\check{\xi}_\varepsilon\Phi_\varepsilon(.)-\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^\varpi\varpi_\varepsilon(.)+\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^\zeta\zeta_\varepsilon(.)$ is quasiinvex function at $\check{\vartheta}_0\in$

$\Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ with regard to the common kernel function η_0,
(ii) $f_0(.)$ is higher order strongly pseudoinvex function and $\psi_\varepsilon(\varepsilon \in \varphi^{+}_{\psi}(\pi_0))$, $\Phi_\varepsilon(\varepsilon \in \varphi^{+}_{\Phi}(\pi_0))$, $-\Phi_\varepsilon(\varepsilon \in \varphi^{-}_{\Phi}(\pi_0))$, $-\varpi_\varepsilon(\varepsilon \in \varphi^{+}_{+0}(\pi_0) \cup \varphi^{+}_{+-}(\pi_0) \cup \varphi^{+}_{00}(\pi_0) \cup \varphi^{+}_{0-}(\pi_0) \cup \varphi^{+}_{0+}(\pi_0))$, $\varpi_\varepsilon(\varepsilon \in \varphi^{-}_{0+}(\pi_0))$, $\zeta_\varepsilon(\varphi^{++}_{+0}(\pi_0) \cup \varphi^{++}_{+-}(\pi_0))$ are higher order strongly quasiinvex functions at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ with regard to the common kernel function η_0. Then $\check{\vartheta}_0$ is a robust global minimum of the (RMPVC).

Proof Suppose that $\check{\vartheta}_0$ is not a robust global minimum of the (RMPVC), that is, there exists $\tilde{\pi}_0 \in \Delta$ such that

$$f_0(\tilde{\pi}_0) < f_0(\check{\vartheta}_0). \tag{40}$$

(i) By using the higher order strongly pseudoinvexity of $f_0(.)$ with respect to the common kernel function η_0, we have

$$\left\langle \nabla f_0(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + \tilde{h}\|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} < 0. \tag{41}$$

Since $\tilde{\pi}_0 \in \Delta$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta}) \in \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$, it follows that

$$\begin{aligned} \check{\upsilon}_\varepsilon \psi_\varepsilon(\tilde{\pi}_0, \sigma_\varepsilon) &\le \check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \varepsilon \in \mathbb{Q}, \sigma_\varepsilon \in \Omega_\varepsilon, \\ \check{\xi}_\varepsilon \Phi_\varepsilon(\tilde{\pi}_0) &= \check{\xi}_\varepsilon \Phi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \mathbb{Y}_0, \\ -(\check{\varrho}_\varepsilon)^{\varpi} \varpi_\varepsilon(\tilde{\pi}_0) &\le -(\check{\varrho}_\varepsilon)^{\varpi} \varpi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \mathbb{K}_0, \\ (\check{\varrho}_\varepsilon)^{\zeta} \zeta_\varepsilon(\tilde{\pi}_0) &\le (\check{\varrho}_\varepsilon)^{\zeta} \zeta_\varepsilon(\check{\vartheta}_0), \varepsilon \in \mathbb{K}_0, \end{aligned}$$

which leads to

$$\begin{aligned} &\sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \psi_\varepsilon(\tilde{\pi}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \Phi_\varepsilon(\tilde{\pi}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi} \varpi_\varepsilon(\tilde{\pi}_0) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta} \zeta_\varepsilon(\tilde{\pi}_0) \\ &\le \sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi} \varpi_\varepsilon(\check{\vartheta}_0) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta} \zeta_\varepsilon(\check{\vartheta}_0). \end{aligned}$$

By using the higher order strong quasiinvexity of $\sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \psi_\varepsilon(., \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \Phi_\varepsilon(.) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi} \varpi_\varepsilon(.) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta} \zeta_\varepsilon(.)$ with regard to the common kernel function η_0, we get

$$\begin{aligned} &\langle \sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \nabla\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \nabla\Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi} \nabla\varpi_\varepsilon(\check{\vartheta}_0) \\ &+ \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta} \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle \\ &+\check{\upsilon}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} + \check{\xi}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} - (\check{\varrho}_\varepsilon)^{\varpi} (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \end{aligned}$$

$$+ (\check{\varrho}_\varepsilon)^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0. \tag{42}$$

On adding the inequalities (41) and (42), we have

$$\left\langle \nabla\Theta(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + h^\circ \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota < 0,$$

where,

$$\begin{aligned} h^\circ \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota &= \check{\upsilon}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota + \check{\xi}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota - (\check{\varrho}_\varepsilon)^\varpi (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \\ &\quad + (\check{\varrho}_\varepsilon)^\zeta (\tilde{h}_\varepsilon)^\zeta \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota + \tilde{h} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota, \end{aligned}$$

which contradicts the first equation in (37). The theorem is therefore validated.

(ii) Since $\tilde{\pi}_0 \in \Delta$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^\varpi, (\check{\varrho})^\zeta, \check{\chi}, \check{\delta}) \in \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$, it follows that

$$\begin{aligned} &\check{\upsilon}_\varepsilon \psi_\varepsilon(\tilde{\pi}_0, \sigma_\varepsilon) \leq \check{\upsilon}_\varepsilon \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \varepsilon \in \mathbb{Q}, \sigma_\varepsilon \in \Omega_\varepsilon, \\ &\check{\xi}_\varepsilon \Phi_\varepsilon(\tilde{\pi}_0) = \check{\xi}_\varepsilon \Phi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \mathbb{Y}_0, \\ &\quad - (\check{\varrho}_\varepsilon)^\varpi \varpi_\varepsilon(\tilde{\pi}_0) \leq -(\check{\varrho}_\varepsilon)^\varpi \varpi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \mathbb{K}_0, \\ &\check{\varrho}^\zeta_\varepsilon \zeta_\varepsilon(\tilde{\pi}_0) \leq \check{\varrho}^\zeta_\varepsilon \zeta_\varepsilon(\check{\vartheta}_0), \varepsilon \in \mathbb{K}_0. \end{aligned}$$

By using the definition of index set in the above inequalities, we have

$$\begin{aligned} &\psi_\varepsilon(\tilde{\pi}_0, \sigma_\varepsilon) \leq \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \sigma_\varepsilon \in \Omega_\varepsilon, \varepsilon \in \varphi^+_\psi(\tilde{\pi}_0), \\ &\Phi_\varepsilon(\tilde{\pi}_0) = \Phi_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi^+_\Phi(\tilde{\pi}_0) \cup \varphi^-_\Phi(\tilde{\pi}_0), \\ &\quad - \zeta_\varepsilon(\tilde{\pi}_0) \leq -\zeta_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi^+_{+0}(\tilde{\pi}_0) \cup \varphi^+_{+-}(\tilde{\pi}_0) \cup \varphi^+_{00}(\tilde{\pi}_0) \cup \varphi^+_{0-}(\tilde{\pi}_0) \cup \varphi^+_{0+}(\tilde{\pi}_0), \\ &\quad - \zeta_\varepsilon(\tilde{\pi}_0) \geq -\zeta_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi^-_{0+}(\tilde{\pi}_0), \\ &\zeta_\varepsilon(\tilde{\pi}_0) \leq \zeta_\varepsilon(\check{\vartheta}_0), \varepsilon \in \varphi^{++}_{+0}(\tilde{\pi}_0) \cup \varphi^{++}_{+-}(\tilde{\pi}_0). \end{aligned}$$

By using the higher order strong quasiinvexity of $\psi_\varepsilon(\varepsilon \in \varphi^+_\psi(\tilde{\pi}_0))$, $\Phi_\varepsilon(\varepsilon \in \varphi^+_\Phi(\tilde{\pi}_0))$, $-\Phi_\varepsilon(\varepsilon \in \varphi^-_\Phi(\tilde{\pi}_0))$, $-\varpi_\varepsilon(\varepsilon \in \varphi^+_{+0}(\tilde{\pi}_0) \cup \varphi^+_{+-}(\tilde{\pi}_0) \cup \varphi^+_{00}(\tilde{\pi}_0) \cup \varphi^+_{0-}(\tilde{\pi}_0) \cup \varphi^+_{0+}(\tilde{\pi}_0))$, $\varpi_\varepsilon(\varepsilon \in \varphi^-_{0+}(\tilde{\pi}_0))$, $\zeta_\varepsilon(\varphi^{++}_{+0}(\tilde{\pi}_0) \cup \varphi^{++}_{+-}(\tilde{\pi}_0))$ with regard to the common kernel function η_0, we get

$$\begin{aligned} &\left\langle \nabla\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, \tilde{h}_\varepsilon > 0, \check{\upsilon}_\varepsilon > 0, \varepsilon \in \varphi^+_\psi(\tilde{\pi}_0), \\ &\left\langle \nabla\Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, \tilde{h}_\varepsilon > 0, \check{\xi}_\varepsilon > 0, \varepsilon \in \varphi^+_\Phi(\tilde{\pi}_0), \\ &\left\langle \nabla\Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \geq 0, \tilde{h}_\varepsilon > 0, \check{\xi}_\varepsilon < 0, \varepsilon \in \varphi^-_\Phi(\tilde{\pi}_0), \\ &\quad - \left\langle \nabla\varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle - (\tilde{h}_\varepsilon)^\varpi \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, (\tilde{h}_\varepsilon)^\varpi > 0, (\check{\varrho}_\varepsilon)^\varpi \geq 0, \end{aligned}$$

$$
\begin{aligned}
&\varepsilon \in \varphi_{+0}^{+}(\tilde{\pi}_0) \cup \varphi_{+-}^{+}(\tilde{\pi}_0) \\
&\cup \varphi_{00}^{+}(\tilde{\pi}_0) \cup \varphi_{0-}^{+}(\tilde{\pi}_0) \\
&\cup \varphi_{0+}^{+}(\tilde{\pi}_0), \\
&\quad -\left\langle \nabla \varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle - (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \geq 0, (\tilde{h}_\varepsilon)^{\varpi} > 0, (\check{\varrho}_\varepsilon)^{\varpi} \leq 0, \\
&\varepsilon \in \varphi_{+0}^{-}(\tilde{\pi}_0), \\
&\left\langle \nabla \zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + (\tilde{h}_\varepsilon)^{\zeta} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \leq 0, (\tilde{h}_\varepsilon)^{\zeta} > 0, (\check{\varrho}_\varepsilon)^{\zeta} \geq 0, \\
&\varepsilon \in \varphi_{+0}^{++}(\tilde{\pi}_0) \cup \varphi_{+-}^{++}(\tilde{\pi}_0).
\end{aligned}
$$

By using the definition of index set in the above inequalities, we obtain

$$
\begin{aligned}
&\langle \sum_{\varepsilon \in \mathbb{Q}} \check{\upsilon}_\varepsilon \nabla \psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon \in \mathbb{Y}_0} \check{\xi}_\varepsilon \nabla \Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\varpi} \nabla \varpi_\varepsilon(\check{\vartheta}_0) \\
&\quad + \sum_{\varepsilon \in \mathbb{K}_0} (\check{\varrho}_\varepsilon)^{\zeta} \nabla \zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\rangle \\
&+ \check{\upsilon}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} + \\
&\check{\xi}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} - (\check{\varrho}_\varepsilon)^{\varpi} (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \\
&+ (\check{\varrho}_\varepsilon)^{\zeta} (\tilde{h}_\varepsilon)^{\zeta} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \leq 0.
\end{aligned}
$$

By using the above inequality and the first equation in (37), it follows that

$$
\left\langle \nabla f_0(\check{\vartheta}_0), \eta_0(\tilde{\pi}_0, \check{\vartheta}_0) \right\rangle + h^{\circ} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \geq 0,
$$

where,

$$
\begin{aligned}
h^{\circ} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} &= -\check{\upsilon}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} - \check{\xi}_\varepsilon \tilde{h}_\varepsilon \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} \\
&\quad + (\check{\varrho}_\varepsilon)^{\varpi} (\tilde{h}_\varepsilon)^{\varpi} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota} - (\check{\varrho}_\varepsilon)^{\zeta} \tilde{h}_\varepsilon^{\zeta} \|\eta_0(\tilde{\pi}_0, \check{\vartheta}_0)\|^{\iota}.
\end{aligned}
$$

By using the higher order strong pseudoinvexity of f_0, with regard to the kernel function η_0, we get

$$
f_0(\tilde{\pi}_0) \geq f_0(\check{\vartheta}_0),
$$

which contradicts (40). The theorem is therefore validated. □

Theorem 12 (Restricted converse robust duality theorem) *Let $\check{\pi}_0 \in \Delta$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta}) \in \mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$ be robust feasible points of the (RMPVC) and the (VC-RMWD), respectively, such that*

$$
f_0(\check{\pi}_0) = f_0(\check{\vartheta}_0).
$$

If the hypothesis of Theorem 9 *holds at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$, then $\check{\pi}_0$ is a robust global minimum of the (RMPVC).*

Proof Suppose that $\check{\pi}_0 \in \Delta$ is not a robust global minimum of the (RMPVC), then there exists $\tilde{\pi}_0 \in \Delta$ such that

$$f_0(\tilde{\pi}_0) \leq f_0(\check{\pi}_0).$$

Based on the assumption in the theorem, we arrive at the following inequality

$$f_0(\tilde{\pi}_0) \leq f_0(\check{\vartheta}_0),$$

which contradicts Theorem 9. The theorem is therefore validated. □

Theorem 13 (Strict converse robust duality theorem) *Let $\check{\pi}_0 \in \Delta$ be a robust local minimum of the (RMPVC) such that the (VC-ACQ) holds at $\check{\pi}_0$. Assume the conditions of Theorem 10 hold and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta})$ be a robust global maximum of the (VC-RWD)($\check{\pi}_0$). Suppose one of the subsequent cases occurs:*

(i) $f_0(.)$ is strictly higher order strongly pseudoinvex function and $\sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_{\varepsilon}\psi_{\varepsilon}(., \sigma_{\varepsilon}) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_{\varepsilon}\Phi_{\varepsilon}(.) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_{\varepsilon})^{\varpi}\varpi_{\varepsilon}(.) + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_{\varepsilon})^{\zeta}\zeta_{\varepsilon}(.)$ is quasiinvex function at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ with regard to the common kernel function η_0,

(ii) $f_0(.)$ is strictly higher order strongly pseudoinvex function and $\psi_{\varepsilon}(\varepsilon \in \varphi^{+}_{\psi}(\check{\pi}_0))$, $\Phi_{\varepsilon}(\varepsilon \in \varphi^{+}_{\Phi}(\check{\pi}_0))$, $-\Phi_{\varepsilon}(\varepsilon \in \varphi^{-}_{\Phi}(\check{\pi}_0))$, $-\varpi_{\varepsilon}(\varepsilon \in \varphi^{+}_{+0}(\check{\pi}_0) \cup \varphi^{+}_{+-}(\check{\pi}_0) \cup \varphi^{+}_{00}(\check{\pi}_0) \cup \varphi^{+}_{0-}(\check{\pi}_0) \cup \varphi^{+}_{0+}(\check{\pi}_0))$, $\varpi_{\varepsilon}(\varepsilon \in \varphi^{-}_{0+}(\check{\pi}_0))$, $\zeta_{\varepsilon}(\varphi^{++}_{+0}(\check{\pi}_0) \cup \varphi^{++}_{+-}(\check{\pi}_0))$ are strictly higher order strongly quasiinvex functions at $\check{\vartheta}_0 \in \Delta \cup pr\mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$ with regard to the common kernel function η_0.

Then $\check{\pi}_0 = \check{\vartheta}_0$.

Proof (i). Suppose that $\check{\pi}_0 \neq \check{\vartheta}_0$. By Theorem 10, there exist $\check{\upsilon} = (\check{\upsilon}_{\varepsilon})_{\varepsilon\in\mathbb{Q}} \in \mathbb{R}^{(\mathbb{Q})}_{+}$, $\check{\xi} \in \mathbb{R}^{y}$, $(\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta} \in \mathbb{R}^{k}$ such that $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta})$ is a robust global maximum of (VC-RMWD)($\check{\pi}_0$). Thus,

$$f(\check{\pi}_0) = f(\check{\vartheta}_0). \tag{43}$$

Since $\check{\pi}_0 \in \Delta$ and $(\check{\vartheta}_0, \check{\upsilon}, \check{\xi}, (\check{\varrho})^{\varpi}, (\check{\varrho})^{\zeta}, \check{\chi}, \check{\delta}) \in \mathbb{S}^{\mathbb{R}}_{\mathbb{MW}}$, it follows that

$$\begin{aligned}
\psi_{\varepsilon}(\pi_0, \sigma_{\varepsilon}) \leq 0, \upsilon_{\varepsilon} \geq 0, \sigma_{\varepsilon} \in \Omega_{\varepsilon}, \varepsilon \in \mathbb{Q},\\
\Phi_{\varepsilon}(\check{\pi}_0) = 0, \check{\xi}_{\varepsilon} \in \mathbb{R}, \varepsilon \in \mathbb{Y}_0,\\
-\varpi_{\varepsilon}(\check{\pi}_0) < 0, (\check{\varrho}_{\varepsilon})^{\varpi} \geq 0, \varepsilon \in \varphi_{+}(\check{\pi}_0),\\
-\varpi_{\varepsilon}(\check{\pi}_0) = 0, (\check{\varrho}_{\varepsilon})^{\varpi} \in \mathbb{R}, \varepsilon \in \varphi_{0}(\check{\pi}_0),\\
\zeta_{\varepsilon}(\check{\pi}_0) > 0, (\check{\varrho}_{\varepsilon})^{\zeta} = 0, \varepsilon \in \varphi_{0+}(\check{\pi}_0),\\
\zeta_{\varepsilon}(\check{\pi}_0) = 0, (\check{\varrho}_{\varepsilon})^{\zeta} \geq 0, \varepsilon \in \varphi_{00}(\check{\pi}_0) \cup \varphi_{+0}(\check{\pi}_0),\\
\zeta_{\varepsilon}(\check{\pi}_0) < 0, (\check{\varrho}_{\varepsilon})^{\zeta} \geq 0, \varepsilon \in \varphi_{0-}(\check{\pi}_0) \cup \varphi_{+-}(\check{\pi}_0),
\end{aligned}$$

By (37), it implies that

$$\sum_{\varepsilon\in\mathbb{Q}}\check{\upsilon}_\varepsilon\psi_\varepsilon(\check{\pi}_0,\check{\sigma}_\varepsilon)+\sum_{\varepsilon\in\mathbb{Y}_0}\check{\xi}_\varepsilon\Phi_\varepsilon(\check{\pi}_0)-\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\varpi}\varpi_\varepsilon(\check{\pi}_0)+\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\zeta}\zeta_\varepsilon(\check{\pi}_0)$$
$$\leq\sum_{\varepsilon\in\mathbb{Q}}\check{\upsilon}_\varepsilon\psi_\varepsilon(\check{\vartheta}_0,\check{\sigma}_\varepsilon)+\sum_{\varepsilon\in\mathbb{Y}_0}\check{\xi}_\varepsilon\Phi_\varepsilon(\check{\vartheta}_0)-\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\varpi}\varpi_\varepsilon(\check{\vartheta}_0)+\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\zeta}\zeta_\varepsilon(\check{\vartheta}_0).$$

Combining the higher order strongly quasiinvexity of $\sum_{\varepsilon\in\mathbb{Q}}\check{\upsilon}_\varepsilon\psi_\varepsilon(.,\check{\sigma}_\varepsilon)+\sum_{\varepsilon\in\mathbb{Y}_0}\check{\xi}_\varepsilon\Phi_\varepsilon(.)-\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\varpi}\varpi_\varepsilon(.)+\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\zeta}\zeta_\varepsilon(.)$ with regard to the kernel function η_0, we get

$$\langle\sum_{\varepsilon\in\mathbb{Q}}\check{\upsilon}_\varepsilon\nabla\psi_\varepsilon(\check{\vartheta}_0,\sigma_\varepsilon)+\sum_{\varepsilon\in\mathbb{Y}_0}\check{\xi}_\varepsilon\nabla\Phi_\varepsilon(\check{\vartheta}_0)-\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\varpi}\nabla\varpi_\varepsilon(\check{\vartheta}_0)$$
$$+\sum_{\varepsilon\in\mathbb{K}_0}(\check{\varrho}_\varepsilon)^{\zeta}\nabla\zeta_\varepsilon(\check{\vartheta}_0),\eta_0(\check{\pi}_0,\check{\vartheta}_0)\rangle$$
$$+\check{\upsilon}_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0,\check{\vartheta}_0)\|^{\iota}+$$
$$\check{\xi}_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0,\check{\vartheta}_0)\|^{\iota}-(\check{\varrho}_\varepsilon)^{\varpi}(\tilde{h}_\varepsilon)^{\varpi}\|\eta_0(\check{\pi}_0,\check{\vartheta}_0)\|^{\iota}$$
$$+(\check{\varrho}_\varepsilon)^{\zeta}(\tilde{h}_\varepsilon)^{\zeta}\|\eta_0(\check{\pi}_0,\check{\vartheta}_0)\|^{\iota}\leq 0.$$

By using the above inequality and the first equation in (37), we have

$$\Big\langle\nabla f_0(\check{\vartheta}_0),\eta_0(\check{\pi}_0,\check{\vartheta}_0)\Big\rangle+\tilde{h}\|\eta_0(\check{\pi}_0,\check{\vartheta}_0)\|^{\iota}\geq 0,$$

By using the strict higher order strong pseudoinvexity of f_0 with regard to the kernel function η_0, one gets

$$f_0(\check{\pi}_0)>f_0(\check{\vartheta}_0),$$

which contradicts (43). The theorem is therefore validated.

(ii). Since $\check{\pi}_0\in\Delta$ and $(\check{\vartheta}_0,\check{\upsilon},\check{\xi},(\check{\varrho})^{\varpi},(\check{\varrho})^{\zeta},\check{\chi},\check{\delta})\in\mathbb{S}^{\mathbb{R}}_{\mathrm{MW}}$, it follows that

$$\psi_\varepsilon(\check{\pi}_0,\sigma_\varepsilon)\leq\psi_\varepsilon(\check{\vartheta}_0,\sigma_\varepsilon),\ \sigma_\varepsilon\in\Omega_\varepsilon,\ \varepsilon\in\varphi^+_\psi(\check{\pi}_0),$$
$$\Phi_\varepsilon(\check{\pi}_0)=\Phi_\varepsilon(\check{\vartheta}_0),\ \varepsilon\in\varphi^+_\Phi(\check{\pi}_0)\cup\varphi^-_\Phi(\check{\pi}_0),$$
$$-\varpi_\varepsilon(\check{\pi}_0)\leq-\varpi_\varepsilon(\check{\vartheta}_0),\ \varepsilon\in\varphi^+_{+0}(\check{\pi}_0)\cup\varphi^+_{+-}(\check{\pi}_0)\cup\varphi^+_{00}(\check{\pi}_0)\cup\varphi^+_{0-}(\check{\pi}_0)\cup\varphi^+_{0+}(\check{\pi}_0),$$
$$-\varpi_\varepsilon(\check{\pi}_0)\geq-\varpi_\varepsilon(\check{\vartheta}_0),\ \varepsilon\in\varphi^-_{0+}(\check{\pi}_0),$$
$$\zeta_\varepsilon(\check{\pi}_0)\leq\zeta_\varepsilon(\check{\vartheta}_0),\ \varepsilon\in\varphi^{++}_{+0}(\check{\pi}_0)\cup\varphi^{++}_{+-}(\check{\pi}_0).$$

By using the higher order strong quasiinvexity of $\psi_\varepsilon(\varepsilon\in\varphi^+_\psi(\check{\pi}_0))$, $\Phi_\varepsilon(\varepsilon\in\varphi^+_\Phi(\check{\pi}_0))$, $-\Phi_\varepsilon(\varepsilon\in\varphi^-_\Phi(\check{\pi}_0))$, $-\varpi_\varepsilon(\varepsilon\in\varphi^+_{+0}(\check{\pi}_0)\cup\varphi^+_{+-}(\check{\pi}_0)\cup\varphi^+_{00}(\check{\pi}_0)\cup\varphi^+_{0-}(\check{\pi}_0)\cup\varphi^+_{0+}(\check{\pi}_0))$,

$\varpi_\varepsilon(\varepsilon \in \varphi^{-}_{0+}(\check{\pi}_0)), \zeta_\varepsilon(\varphi^{++}_{+0}(\check{\pi}_0) \cup \varphi^{++}_{+-}(\check{\pi}_0))$ with regard to the common kernel function η_0, we get

$$\begin{aligned}
&\left\langle \nabla\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + \tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, \tilde{h}_\varepsilon > 0, \check{\upsilon}_\varepsilon > 0, \varepsilon \in \varphi^{+}_{\psi}(\check{\pi}_0),\\
&\left\langle \nabla\Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + \tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, \tilde{h}_\varepsilon > 0, \check{\xi}_\varepsilon > 0, \varepsilon \in \varphi^{+}_{\Phi}(\check{\pi}_0),\\
&\left\langle \nabla\Phi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + \tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \geq 0, \tilde{h}_\varepsilon > 0, \check{\xi}_\varepsilon < 0, \varepsilon \in \varphi^{-}_{\Phi}(\check{\pi}_0),\\
&\quad -\left\langle \nabla\varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle - (\tilde{h}_\varepsilon)^\varpi\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, (\tilde{h}_\varepsilon)^\varpi > 0, (\check{\varrho}_\varepsilon)^\varpi \geq 0,\\
&\varepsilon \in \varphi^{+}_{+0}(\check{\pi}_0) \cup \varphi^{+}_{+-}(\check{\pi}_0)\\
&\cup\, \varphi^{+}_{00}(\check{\pi}_0) \cup \varphi^{+}_{0-}(\check{\pi}_0) \cup \varphi^{+}_{0+}(\check{\pi}_0),\\
&\quad -\left\langle \nabla\varpi_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle - (\tilde{h}_\varepsilon)^\varpi\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \geq 0, (\tilde{h}_\varepsilon)^\varpi > 0, (\check{\varrho}_\varepsilon)^\varpi \leq 0,\\
&\varepsilon \in \varphi^{+}_{+0}(\check{\pi}_0),\\
&\left\langle \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + (\tilde{h}_\varepsilon)^\zeta\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0, (\tilde{h}_\varepsilon)^\zeta > 0, (\check{\varrho}_\varepsilon)^\zeta \geq 0,\\
&\varepsilon \in \varphi^{++}_{+0}(\check{\pi}_0) \cup \varphi^{++}_{+-}(\check{\pi}_0).
\end{aligned}$$

By using the definition of index set in the above inequalities, we have

$$\begin{aligned}
&\langle \sum_{\varepsilon\in\mathbb{Q}} \check{\upsilon}_\varepsilon \nabla\psi_\varepsilon(\check{\vartheta}_0, \sigma_\varepsilon) + \sum_{\varepsilon\in\mathbb{Y}_0} \check{\xi}_\varepsilon \nabla\Phi_\varepsilon(\check{\vartheta}_0) - \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^\varpi \nabla\varpi_\varepsilon(\check{\vartheta}_0)\\
&\quad + \sum_{\varepsilon\in\mathbb{K}_0} (\check{\varrho}_\varepsilon)^\zeta \nabla\zeta_\varepsilon(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\rangle\\
&\quad + \check{\upsilon}_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota +\\
&\check{\xi}_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota - (\check{\varrho}_\varepsilon)^\varpi(\tilde{h}_\varepsilon)^\varpi\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota\\
&\quad + (\check{\varrho}_\varepsilon)^\zeta(\tilde{h}_\varepsilon)^\zeta\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \leq 0.
\end{aligned}$$

By using the above inequalities and (37), it follows that

$$\left\langle \nabla f_0(\check{\vartheta}_0), \eta_0(\check{\pi}_0, \check{\vartheta}_0)\right\rangle + h^\circ\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota \geq 0,$$

where,

$$\begin{aligned}
h^\circ\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota &= -\check{\upsilon}_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota - \check{\xi}_\varepsilon\tilde{h}_\varepsilon\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota\\
&\quad + (\check{\varrho}_\varepsilon)^\varpi(\tilde{h}_\varepsilon)^\varpi\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota - (\check{\varrho}_\varepsilon)^\zeta(\tilde{h}_\varepsilon)^\zeta\|\eta_0(\check{\pi}_0, \check{\vartheta}_0)\|^\iota.
\end{aligned}$$

By using the strict higher order strong pseudoinvexity of f_0, with regard to the kernel function η_0, one gets

$$f_0(\check{\pi}_0) > f_0(\check{\vartheta}_0)$$

which contradicts (43). The theorem is therefore validated. □

Let us re-explore Example 2 to verify the above theorems.

Example 3 For any robust feasible $\pi_0 \in \Delta$, the VC-RMWD(π_0) to the (RMPVC) is shown as:

$$\max \quad f_0(\vartheta_0) = \frac{1}{2}\vartheta_0 - 2$$
$$\text{subject to}$$

$$\left.\begin{array}{l}\nabla\Theta(\vartheta_0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}) = \frac{1}{2} - \upsilon_1(-2\vartheta_0 + 1) - \varrho_1^{\varpi} + \varrho_1^{\zeta}(1 + 2\vartheta_0) = 0,\\ \upsilon_1 \geq 0, \upsilon_1\psi_1(\vartheta_0, \sigma_1) \geq 0,\\ \varrho_1^{\zeta}\zeta_1(\vartheta_0) = \varrho_1^{\zeta}\vartheta_0 \geq 0,\\ \varrho_1^{\zeta} = \delta_1\pi_0, \delta_1 \geq 0,\\ \quad -\varrho_1^{\varpi}\varpi_1(\vartheta_0) = -\varrho_1^{\varpi}\vartheta_0 \geq 0,\\ \varrho_1^{\varpi} = \chi_1 - \delta_1(\pi_0 + \pi_0^2), \chi_1 \geq 0.\end{array}\right\} \quad (44)$$

(i) Let $\check{\pi}_0 = 0$ and $(\vartheta_0, \upsilon_1, \xi_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}, \chi_1, \delta_1) = (0, \frac{1}{2} - \varrho_1^{\varpi}, \varrho_1^{\varpi}, 0, \chi_1, \delta_1) \in \mathbb{S}_{\mathbb{W}}^{\mathbb{R}}(\check{\pi}_0)$, that is, $\check{\vartheta}_0 = 0 \in pr\mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}$. This implies $(0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}, 0, \chi_1, \delta_1) \in \mathbb{S}_{\mathbb{MW}}^{\mathbb{R}}(\pi_0)$. we have, $f_0(\check{\pi}_0) = -2 = f_0(\check{\vartheta}_0)$. The hypotheses of Theorem 11 are easily verified. From condition (35), $\check{\gamma}_0$ is a robust global minimum of (RMPVC).

(ii) We get, $\check{\vartheta}_0 = 0$. From condition (44), we get $\varrho_1^{\zeta} = 0, \varrho_1^{\varpi} \geq 0$, that is,

$$\Theta(\vartheta_0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}) = f_0(\vartheta_0) + \upsilon_1\psi_1(\vartheta_0, \sigma_1) - \varrho_1^{\varpi}\varpi_1(\vartheta_0) + \varrho_1^{\zeta}\zeta_1(\vartheta_0) < 0.$$
$$f_0(\vartheta_0) < -\upsilon_1\psi_1(\vartheta_0, \sigma_1) + \varrho_1^{\varpi}\varpi_1(\vartheta_0) - \varrho_1^{\zeta}\zeta_1(\vartheta_0).$$

From condition (44), we get $f_0(\vartheta_0) \leq 0$. So, we obtain $f_0(\pi_0) \geq f_0(\vartheta_0)$, Theorem 9, is verified.

(iii) Since $\check{\pi}_0 = 0$ is the unique solution of (RMPVC) and $\nabla\varpi_1 = \{1\}, \nabla\zeta_1 = \{1\}$. It is easy to see that condition (35) satisfies (VC-ACQ). By Theorem 1, there exist Lagrange multipliers $\upsilon_1 \in \mathbb{R}_+, \varrho^{\varpi}, \varrho^{\zeta}, \delta_1, \chi_1 \in \mathbb{R}$ such that $(0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}, \delta_1, \chi_1)$ is a robust feasible solution of (VC-RWD)(0). Taking into account $f_0(\vartheta_0) \leq 0$, we get $(0, \upsilon_1, \varrho_1^{\varpi}, \varrho_1^{\zeta}, \delta_1, \chi_1)$ is a robust global maximum of (VC-RMWD)(0) and thus, Theorem 10 is validated.

5 Special cases

(i) In a scenario, lacking uncertain parameter σ and index set in the constraints the (RMPVC) model reduces to (MVPC1) model of Achtziger and Kanzow [13] and Joshi [30].

$$\text{(MPVC1)} \quad \min_{\pi_0 \in \mathbb{R}^n} \quad f_0(\pi_0)$$

$$\begin{aligned}
&\text{subject to}\\
&\psi_{\varepsilon}(\pi_0) \leq 0, \varepsilon \in \mathbb{Q}, \mathbb{Q} = \{1, 2, \ldots, q\},\\
&\Phi_{\varepsilon}(\pi_0) = 0, \varepsilon \in \mathbb{Y}_0, \mathbb{Y}_0 = \{1, 2, \ldots, y\},\\
&\varpi_{\varepsilon}(\pi_0) \geq 0, \varepsilon \in \mathbb{K}_0, \mathbb{K}_0 = \{1, 2, \ldots, k\},\\
&\zeta_{\varepsilon}(\pi_0)\varpi_{\varepsilon}(\pi_0) \leq 0, \varepsilon \in \mathbb{K}_0, \mathbb{K}_0 = \{1, 2, \ldots, k\}.
\end{aligned}$$

(ii) In the absence of uncertain parameter σ in the constraints and the objective function is single valued, the (RMPVC) model takes the form of (MVPC2) model of Tung [23].

$$\begin{aligned}
\text{(MPVC2)} \quad \min_{\pi_0 \in \mathbb{R}^n} \quad & f_0(\pi_0)\\
&\text{subject to}\\
&\psi_{\varepsilon}(\pi_0) \leq 0, \forall \varepsilon \in \mathbb{Q},\\
&\Phi_{\varepsilon}(\pi_0) = 0, \varepsilon \in \mathbb{Y}_0, \mathbb{Y}_0 = \{1, 2, \ldots, y\},\\
&\varpi_{\varepsilon}(\pi_0) \geq 0, \varepsilon \in \mathbb{K}_0, \mathbb{K}_0 = \{1, 2, \ldots, k\},\\
&\zeta_{\varepsilon}(\pi_0)\varpi_{\varepsilon}(\pi_0) \leq 0, \varepsilon \in \mathbb{K}_0, \mathbb{K}_0 = \{1, 2, \ldots, k\}.
\end{aligned}$$

(iii) In the absence of uncertain parameter σ in the constraints, the (VC-RWD) and (VC-RMWD) reduces to (VC-WD) and (VC-MWD) models respectively of Joshi [30].

6 Conclusion

This manuscript demonstrates robust optimality conditions, Wolfe and Mond–Weir type robust duals for a robust mathematical programming problem involving vanishing constraints (RMPVC). The results of duality are examined based on the concept of generalized higher order invexity and strict invexity amongst the primal and the Wolfe type robust dual problems. In addition, the duality results amongst the primal and the Mond–Weir type robust dual problems based on the concept of generalized higher order pseudoinvex, strict pseudoinvex and quasiinvex functions are also studied. Furthermore, numerical examples are provided to validate robust optimality criteria and duality theorems of Wolfe and Mond–Weir type duals. Also, by employing the univexity and generalized univexity presumptions while deriving results of duality for the mixed type robust dual model of (RMPVC) would be our subsequent study.

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