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# A Note on the Spectrality of Moran-Type Bernoulli Convolutions by Deng and Li

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## Abstract

Let $\{p_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ be two sequences of integers such that $|p_n| > |d_n| > 0$ and $\{d_n\}_{n\geq 1}$ is bounded. It is proven by Deng and Li that the Moran-type Bernoulli convolution

$$\mu := \delta_{p_1^{-1}\{0,d_1\}} * \delta_{p_1^{-1}p_2^{-1}\{0,d_2\}} * \cdots * \delta_{p_1^{-1}\ldots p_n^{-1}\{0,d_n\}} * \cdots$$

is a spectral measure if and only if the numbers of factor 2 in the sequence $\left\{\frac{p_1p_2\ldots p_n}{2d_n}\right\}_{n\geq 1}$ are different from each other. Unfortunately, there is a gap in the proof of the sufficiency. Here we give a new proof to close the gap.

**Keywords** Moran-type · Bernoulli convolution · One dimension · Spectrality

**Mathematics Subject Classification** MSC: Primary 42C05 · 42A65; Secondary 28A78 · 28A80

## 1 Introduction

In the proof of [2, Theorem 4.3 (iii)], the inclusion relationship "$\{\gamma + b_\gamma : \gamma \in \Gamma\} \subset \sum_{j=1}^{\ell_n}(\{0\} \cup U_j)$" maybe wrong in some cases. Actually, this inclusion relationship need a precondition "$\ell_j \leq \ell_n$ for all $j < \ell_n$". The following example shows that,

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Extended author information available on the last page of the article there are examples that $\ell_j > \ell_n$ holds for at least one integer $j < \ell_n$ for all $n > 0$. Hence, the sufficiency of [2, Theorem 1.1] needs to be reproved.

**Example** Let $p_{2n-1} = 4,\ p_{2n} = 9$ and $d_{2n-1} = 1,\ d_{2n} = 8$ for all $n \geq 1$. Then, the definition of $k_n$ and $\ell_n$ shows

$$k_{2n} = v_2(\frac{p_1 p_2 \cdots p_{2n}}{2d_{2n}}) = v_2(\frac{36^n}{16}) = 2n - 4, \quad \forall\, n \geq 1,$$

$$k_{2n-1} = v_2(\frac{p_1 p_2 \cdots p_{2n-1}}{2d_{2n-1}}) = v_2(\frac{36^{n-1} \times 4}{2}) = 2n - 1, \quad \forall\, n \geq 1.$$

Also, $\ell_{2n-1} = 2n + 2$ and $\ell_{2n} = 2n$ for all $n \geq 1$. This means $\ell_{\ell_n - 1} > \ell_n$ for all $n \geq 1$.

We recall the definition of Moran-type Bernoulli convolution. Let $\{p_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ be two sequences of integers satisfying $|p_n| \geq 2,\ |d_n| \geq 1$ and

$$\sum_{n=1}^{+\infty} |p_1^{-1} p_2^{-1} \cdots p_n^{-1} d_n| < +\infty.$$

The weak limit of the following convolutions is called a Moran-type Bernoulli convolution

$$\mu_n = \delta_{p_1^{-1} D_1} * \delta_{p_1^{-1} p_2^{-1} D_2} * \cdots * \delta_{p_1^{-1} p_2^{-1} \cdots p_n^{-1} D_n}.$$

And we denote it by

$$\mu = \delta_{p_1^{-1} D_1} * \delta_{p_1^{-1} p_2^{-1} D_2} * \cdots * \delta_{p_1^{-1} p_2^{-1} \cdots p_n^{-1} D_n} \cdots . \tag{1.1}$$

We shall reprove the sufficiency of the following result (i.e. [2, Theorem 1.1]).

**Theorem 1.1** *For the measure $\mu$ defined by (1.1) with $|p_n| > |d_n|$ for all $n \geq 2$, assume that the sequence $\{|d_n|\}_{n=1}^{+\infty}$ is bounded. Then, $\mu$ is a spectral measure if and only if $k_j \neq k_i$ for all $j > i \geq 1$, where*

$$k_n = v_2\Big(\frac{p_1 p_2 \ldots p_n}{2d_n}\Big) = v_2(p_1 p_2 \ldots p_n) - v_2(2d_n), \quad n = 1, 2, 3, \ldots . \tag{1.2}$$

## 2 Proof of the Sufficiency of Theorem 1.1

In order to make the proof more readable, we first simplify our model.

**Proposition 2.1** *For the measure $\mu$ defined by (1.1), there exist two sequences of integers $\{c_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ such that for $n \geq 1$, we have $\gcd(q_n, c_n) = 1$ and*

$$q_1^{-1} q_2^{-1} \cdots q_n^{-1} c_n = p_1^{-1} p_2^{-1} \cdots p_n^{-1} d_n. \tag{2.1}$$

*Furthermore, we have $|q_n| > |c_n|$ when $|p_n| > |d_n|$ $(n = 1, 2, \ldots)$. Hence, we can rewrite $\mu$ as*

$$\mu = \delta_{q_1^{-1} C_1} * \delta_{q_1^{-1} q_2^{-1} C_2} * \delta_{q_1^{-1} q_2^{-1} q_3^{-1} C_3} \cdots * \delta_{q_1^{-1} q_2^{-1} \cdots q_n^{-1} C_n} * \cdots, \tag{2.2}$$

*where $C_n = \{0, c_n\}$.*

***Proof*** Write $g_0 = 1$, and define inductively

$$g_n = \gcd(|g_{n-1} p_n|,\ |d_n|),\ q_n = \frac{g_{n-1} p_n}{g_n} \text{ and } c_n = \frac{d_n}{g_n}, \quad \forall\, n \geq 1.$$

It is clear that for any $n \geq 1$, we have $\gcd(q_n,\ c_n) = 1$ and (2.1). By writing $C_n = \{0, c_n\}$, we have

$$\delta_{q_1^{-1} q_2^{-1} \cdots q_n^{-1} C_n} = \delta_{p_1^{-1} p_2^{-1} \cdots p_n^{-1} D_n},$$

which implies that (2.2) holds.

If $|p_n| > |d_n|$, noting $|g_{n-1} p_n| \geq |p_n|$, it is obvious that $|q_n| > |c_n|$. □

The above Proposition 2.1 shows that, in order to prove the sufficiency of Theorem 1.1, without loss of generality, we can assume that $\gcd(d_n,\ p_n) = 1$. By the argument in [2], we will always assume that [2, (2.9)] holds without loss of generality. Therefore, we shall assume that the following conditions hold in the sequel:

$$p_n \geq 2,\ \ d_n \geq 1,\ \ \gcd(d_n,\ p_n) = 1,\ \frac{p_1 p_2 \cdots p_n}{2 d_n} \in \mathbb{N}, \qquad \forall\, n \geq 1. \tag{2.3}$$

The following Proposition 2.2 is obviously true.

**Proposition 2.2** *Let $\nu$ be a probability measure and its support has finite cardinality $N$. If $L^2(\nu)$ has an orthogonal set $\{e^{2\pi i \lambda x} : \lambda \in \Lambda\}$ and $\#\Lambda$ is at least $N$, then $\Lambda$ is a spectrum of $\nu$ and $\#\Lambda = N$.*

We will continue to use notations $\ell_n,\ k_n,\ U_n,\ t_n,\ r_n$ defined in [2] and the constant $c$ is defined in [2, Lemma 4.1 (i)]. Given a nonzero integer $n$, we denote by $\theta(n)$ the odd part of $n$, i.e. $\theta(n) = \frac{n}{2^{\nu_2(n)}}$. Then, we rewrite

$$U_n = 2^{k_n} \theta(p_1 p_2 \cdots p_{\ell_n})(2\mathbb{Z} + 1), \quad n \geq 1. \tag{2.4}$$

The following set will play an important role in the sequel

$$\mathcal{I} = \{n : \ell_n = n\}. \tag{2.5}$$

**Lemma 2.3** *Assume that* (2.3) *holds and* $k_i \neq k_j$ *for all* $i > j \geq 1$. *Then, we have the following statements.*

*(i).* $\#\mathcal{I} = +\infty$.

*(ii). Let* $B$ *be a finite nonempty subset of positive integers. Then,* $\Delta := \sum_{j\in B}\{0, \ a_j\}$ *is a spectrum (with cardinality* $2^{\#B}$*) of* $*_{j\in B}\delta_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}$ *for any* $a_j \in U_j$.

*(iii). Let* $B$ *be a finite nonempty subset of positive integers. Assume that* $\Lambda \subset \sum_{j\in B}(\{0\}\cup U_j)$ *is a spectrum of the probability measure* $*_{j\in B}\delta_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}$. *For any* $m \geq \max\{\ell_j : j \in B\}$ *and* $\lambda \in \Lambda$, *we take an integer* $b_\lambda \in p_1p_2\cdots p_m\mathbb{Z}$ *with* $b_0 = 0$. *Then, the set* $\{\lambda + b_\lambda : \ \gamma \in \Lambda\}$ *is also a spectrum of the probability measure* $*_{j\in B}\delta_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}$. *Furthermore, we have* $\{\gamma + b_\lambda : \ \gamma \in \Lambda\} \subset \sum_{j\in B}(\{0\}\cup U_j)$.

***Proof*** (i) It is sufficient to prove that for any integer $N > 0$, there exists a positive integer $n > N$ such that $\ell_n = n$.

Indeed, since $\{d_n\}_{n\geq 1}$ is bounded, there is an integer $z_0$ such that $k_n \geq z_0$ for all $n > 0$. Hence, there is an integer $n > N$ such that $k_n = \min\{k_j : \ j > N\}$. Since $k_i \neq k_j$ for all $i > j \geq 1$, we see that $k_n < k_j$ for all $j > n$. Hence, the definition of $\ell_n$ shows $\ell_n = n$. The conclusion is proven.

(ii) Suppose $B = \{j_1, \ j_2, \ \cdots, \ j_s\}$ with $k_{j_1} < k_{j_2} < \cdots < k_{j_s}$. From [2, Lemma 4.2], it follows that $k_i < k_j$ implies $\ell_i \leq \ell_j$. Then, the definition of $U_n$ shows

$$U_{j_t} + \sum_{t<i\leq s}(\{0\}\cup U_{j_i}) = U_{j_t}, \quad t = 1, 2, \ldots, s-1. \tag{2.6}$$

For any $\xi = \sum_{j\in B}\xi_j$ and $\eta = \sum_{j\in B}\eta_j \in \Delta$ with $\xi_j, \ \eta_j \in \{0, \ a_j\}$ and $\xi_j \neq \eta_j$ for at least one $j \in B$, it is easy to see $\xi - \eta \in \sum_{j\in B}\{0, \ \pm a_j\}$. Write $t = \min\{i : \ \xi_{j_i} \neq \eta_{j_i}, \ 1 \leq i \leq s\}$. Then, we have $\xi - \eta \in U_{j_t} + \sum_{t<i\leq s}(\{0\}\cup U_{j_i})$. From (2.6) it follows $\xi - \eta \in U_{j_t}$. This implies $\xi - \eta \in U_{j_t}$ and $\#\Delta = 2^s$. Furthermore, $\xi - \eta \in U_{j_t}$ shows that $\xi - \eta$ is a zero point of the Fourier transformation of the probability measure $*_{j\in B}\delta_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}$, i.e. $\prod_{j\in B}\widehat{\delta}_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}(\xi - \eta) = 0$.

It is easy to see that the support of the measure $*_{j\in B}\delta_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}$ has cardinality at most $2^s$. Proposition 2.2 shows that $\Delta$ is a spectrum of the measure $*_{j\in B}\delta_{p_1^{-1}p_2^{-1}\cdots p_j^{-1}D_j}$ and $\#\Delta = 2^s = 2^{\#B}$.

(iii) Note a fact that for all $j \in B$, the integer $p_1 p_2 \cdots p_j$ is a period of $\widehat{\delta}_{p_1^{-1}\cdots p_j^{-1} D_j}$. For $m \geq \max\{\ell_j : j \in B\} \geq j$, we have $\widehat{\delta}_{p_1^{-1}\cdots p_j^{-1} D_j}(x+\lambda+b_\lambda) = \widehat{\delta}_{p_1^{-1}\cdots p_j^{-1} D_j}(x+\lambda)$ for all $x \in \mathbb{R}$, $j \in B$ and $\lambda \in \Lambda$. Hence

$$\sum_{\gamma \in \{\lambda+b_\lambda : \lambda \in \Lambda\}} \left| \prod_{j \in B} \widehat{\delta}_{p_1^{-1}\cdots p_j^{-1} D_j}(x+\gamma) \right|^2 = \sum_{\lambda \in \Lambda} \left| \prod_{j \in B} \widehat{\delta}_{p_1^{-1}\cdots p_j^{-1} D_j}(x+\lambda) \right|^2, \quad \forall\, x \in \mathbb{R}.$$

That means that the first conclusion in (iii) is proven by using [2, Proposition 2.3].

For any $\lambda \in \Lambda$, it can be written as $\lambda = \sum_{j \in B} b_j$ with $b_j \in (\{0\} \cup U_j)$. Let $B = \{j_1, j_2, \cdots, j_s\}$ with $k_{j_1} < k_{j_2} < \cdots < k_{j_s}$ as in (ii) and define $t = \min\{i : b_{j_i} \neq 0\}$. Then, (2.6) shows $\lambda \in U_{j_t}$. Since $m \geq \max\{\ell_j : j \in B\}$, the definition of $U_j$ implies that $U_j + b_\lambda = U_j$ for all $j \in B$, which implies $\lambda + b_\lambda \in U_{j_t}$. Hence, we have $\{\lambda + b_\lambda : \lambda \in \Lambda\} \subset \sum_{j \in B} (\{0\} \cup U_j)$ for any $b_\lambda \in p_1 p_2 \cdots p_m \mathbb{Z}$ with $b_0 = 0$. The second conclusion in (iii) is proven. □

The following two lemmas deal with the possible case that $\ell_j > \ell_n$ for some $j < \ell_n$.

**Lemma 2.4** *Assume that $k_n \neq k_m$ for all $n \neq m$ and (2.3) holds. Furthermore, assume that there exists a positive integer $n_0$ such that for any $n \geq n_0$ there exists an integer $j_n < \ell_n$ satisfying $\ell_{j_n} > \ell_n$.*

*(i) . For any $i \geq n_0$, there is at least one member of the group $p_i, p_{i+1}, \cdots, p_{i+c}$ which is an odd integer larger than or equal to 3.*

*(ii) . There exists a positive integer $N_1 \geq 0$ such that*

$$\frac{d_n}{\theta(p_{\ell_i+1} \cdots p_n)} \leq 1, \ n_0 \leq i \leq n - N_1. \tag{2.7}$$

*(iii) . For any $n \geq n_0 + c$, we have $v_2(p_n) < \max\{v_2(d_j) : j > 0\}$.*

***Proof*** (i) Given $i \geq n_0$, suppose $p_i, p_{i+1}, \cdots, p_{i+c}$ are all even. From the assumption $\gcd(p_n, d_n) = 1$, it is clear that $d_i, d_{i+1}, \cdots, d_{i+c}$ are all odd. Hence, $k_i < k_{i+1} < \cdots < k_{i+c}$, which implies $\ell_i = i$ since [2, Lemma 4.1 (i)] shows $i \leq \ell_i \leq i + c$. On the other hand, however, our assumption shows for the integer $i$, there is a positive integer $j < \ell_i$ such that $\ell_j > \ell_i$. Then, we have $k_j > k_{\ell_j} > k_i$. In virtue of $\ell_i = i$, we have $j < i$. Since $p_i$ is even and $d_i$ is odd, we have $k_i = v_2(p_1 p_2 \cdots p_i) - 1 > v_2(p_1 p_2 \cdots p_j) - 1 \geq k_j$, which leads to a contradiction. Therefore, at least one member of $p_i, p_{i+1}, \cdots, p_{i+c}$ is odd which is larger than or equal to 3. The conclusion (i) is proven.

(ii) Choose a positive integer $s > 0$ such that $3^s \geq \max\{d_n : n > 0\}$. Thus for $i \geq n_0$, we have

$$\frac{\max\{d_n : n > 0\}}{p_{\ell_i+1} p_{\ell_i+2} \cdots p_{\ell_i+sc}} \leq 1.$$

It is clear that we finish the proof by taking $N_1 = sc$.

(iii) Suppose $v_2(p_n) \geq \max\{v_2(d_j) : j > 0\}$ for some $n \geq n_0 + c$. For any $j$ and $m$ with $j < n \leq m$, we have $k_m = v_2(p_1 p_2 \cdots p_m) - 1 - v_2(d_m) \geq v_2(p_1 p_2 \cdots p_j) + v_2(p_n) - 1 - v_2(d_m) \geq v_2(p_1 p_2 \cdots p_j) - 1 \geq k_j$. In fact, by the assumption that $k_m \neq k_j$, we have $k_m > k_j$. Let $k_s = \max\{k_j : n_0 \leq j \leq n-1\}$. According to the definition of $\ell_s$, we have $\ell_s = n - 1$.

On the other hand, however, for the integer $s$, there exits a positive integer $j_0$ with $j_0 < \ell_s$ such that $\ell_{j_0} > \ell_s$, which implies $\ell_{j_0} \geq n$. Noting that $j_0 < n$, according to the above argument, we get $k_{\ell_{j_0}} > k_{j_0}$, which is a contradiction to the definition of $\ell_{j_0}$. The statement (iii) is proven. □

**Lemma 2.5** *Assume that $k_n \neq k_m$ for all $n \neq m$ and (2.3) hold. Furthermore, assume that there exists a positive integer $n_0$ such that for any $n \geq n_0$ there exists an integer $j_n < \ell_n$ satisfying $\ell_{j_n} > \ell_n$. Then, there are small constants $\varepsilon > 0$ and $\theta_0 > 0$ such that for any $n_1$ and $n_2 \in \mathcal{I}$ with $n_2 > n_1 + N_1$, there exists a spectrum $\Lambda = \sum_{i=n_1+1}^{n_2}\{0, a_i\}$ of $*_{i=n_1+1}^{n_2} \delta_{p_1^{-1} p_2^{-1} \cdots p_i^{-1} D_i}$ such that*

$$\inf_{\lambda \in \Lambda,\ |y| \leq \theta_0} \left\{ \prod_{j=1}^{c} \left| m_{p_{n_2+1}^{-1} \cdots p_{n_2+j}^{-1} D_{n_2+j}} \left( \frac{\lambda}{p_1 p_2 \cdots p_{n_2}} + y \right) \right| \right\} > \varepsilon. \tag{2.8}$$

***Proof*** We first construct the spectrum $\Lambda$. Write $D = \max\{d_n : n \geq 1\}$ and $\mathcal{S} = \{i : n_1 + 1 \leq i \leq n_2\}$. We divide the set $\mathcal{S}$ into two parts $\mathcal{S}_1$ and $\mathcal{S}_2$, where

$$\mathcal{S}_1 = \left\{ i \in \mathcal{S} : \frac{D\theta(p_1 \cdots p_{\ell_i})}{\theta(p_1 \cdots p_{n_2+1})} \geq 1 \right\} \text{ and } \mathcal{S}_2 = \left\{ i \in \mathcal{S} : \frac{D\theta(p_1 \cdots p_{\ell_i})}{\theta(p_1 \cdots p_{n_2+1})} < 1 \right\}.$$

Take

$$a_i = \begin{cases} 2^{k_i} \theta(p_1 \cdots p_{n_2+c}), & i \in \mathcal{S}_1, \\ 2^{k_i} \theta(p_1 \cdots p_{\ell_i}), & i \in \mathcal{S}_2. \end{cases} \tag{2.9}$$

[2, Lemma 4.1 (i)] shows that $n_2 + c \geq \ell_i$ for all $i \in \mathcal{S}_1$. By the definition of $U_n$, it is clear that $a_i \in U_i$ for all $i \in \mathcal{S}$. Then, Lemma 2.3 shows that $\Lambda = \sum_{i=n_1+1}^{n_2}\{0, a_i\}$ is a spectrum of $*_{i=n_1+1}^{n_2} \delta_{p_1^{-1} p_2^{-1} \cdots p_i^{-1} D_i}$. By the definition of the function $\theta$, for $i \in \mathcal{S}_1$ we have

$$\frac{d_{n_2+j} a_i}{p_1 \cdots p_{n_2+j}} = \frac{\theta(d_{n_2+j}) \theta(p_{n_2+j+1} \cdots p_{n_2+c})}{2^{k_{n_2+j}+1-k_i}} \in 2^{k_i - k_{n_2+j} - 1}(2\mathbb{Z} + 1). \tag{2.10}$$

According to the definition of $\mathcal{S}_2$, we have $\ell_i \le n_2$ for any $i \in \mathcal{S}_2$. Thus we have

$$\frac{d_{n_2+j}a_i}{p_1\cdots p_{n_2+j}} = \frac{2^{k_i-k_{n_2+j}-1}\theta(d_{n_2+j})}{\theta(p_{\ell_i+1}\cdots p_{n_2+j})},\ i \in \mathcal{S}_2. \tag{2.11}$$

Given $\lambda = \sum_{i=n_1+1}^{n_2} b_i \in \Lambda$ with $b_i \in \{0,\ a_i\}(n_1+1 \le i \le n_2)$, write $k_{i_1} = \min\{k_i : i \in \mathcal{S}_1,\ b_i \ne 0\}$ when the set $\{i : i \in \mathcal{S}_1,\ b_i \ne 0\}$ is not empty. According to (2.10) and the assumption that $k_i \ne k_j$ for any $i \ne j$, we have

$$\begin{cases} \sum_{i\in\mathcal{S}_1} \frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} \in 2^{k_{i_1}-1-k_{n_2+j}}(2\mathbb{Z}+1), & \{i : i \in \mathcal{S}_1,\ b_i \ne 0\} \ne \emptyset, \\ \sum_{i\in\mathcal{S}_1} \frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} = 0, & \{i : i \in \mathcal{S}_1,\ b_i \ne 0\} = \emptyset. \end{cases} \tag{2.12}$$

Let $k_{i_2} = \max\{k_i : i \in \mathcal{S}_2, b_i \ne 0\}$ when the set $\{i : i \in \mathcal{S}_2, b_i \ne 0\}$ is not empty. According to the definition of $\mathcal{S}_2$, we have $\theta(d_{n_2+j}) \le D < \theta(p_{\ell_i+1}\cdots p_{n_2+1})$ for $i \in \mathcal{S}_2$. Thus we have $\theta(d_{n_2+j}) + 2 \le \theta(p_{\ell_i+1}\cdots p_{n_2+1})$. Hence, $\frac{\theta(d_{n_2+j})}{\theta(p_{\ell_i+1}\cdots p_{n_2+1})} \le \frac{\theta(d_{n_2+j})}{\theta(d_{n_2+j})+2} \le \frac{D}{D+2}$ for any $i \in \mathcal{S}_2$. Also by (2.11) and the assumption that $k_n \ne k_m$ for any $n \ne m$, we get

$$0 \le \sum_{i\in\mathcal{S}_2} \frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} < \frac{D}{D+2}\sum_{s\ge 0} 2^{k_{i_2}-1-k_{n_2+j}-s} = \frac{D}{D+2}2^{k_{i_2}-k_{n_2+j}}. \tag{2.13}$$

Given $1 \le j \le c$, we consider

$$m_{p_{n_2+1}^{-1}\cdots p_{n_2+j}^{-1}D_{n_2+j}}\left(\frac{\lambda}{p_1p_2\cdots p_{n_2}}\right) = m_{\{0,1\}}\left(\frac{\lambda d_{n_2+j}}{p_1p_2\cdots p_{n_2+j}}\right),\ \lambda \in \Lambda. \tag{2.14}$$

And then we will deal with two cases.

**Case A**. $\{i : i \in \mathcal{S}_1,\ b_i \ne 0\} = \emptyset$ or $k_{i_1} > k_{n_2+j}$.

From (2.12) it is clear that

$$\sum_{i\in\mathcal{S}_1} \frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} \in \mathbb{Z}. \tag{2.15}$$

Noting that $i_2 \in \mathcal{S}_2$, we see $\ell_{i_2} \le n_2$, which implies $k_{n_2+j} > k_{i_2}$. In virtue of (2.13), we get

$$\sum_{i\in\mathcal{S}_2} \frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} \in \left(0,\ \frac{1}{2}\frac{D}{D+2}\right).$$

Since $m_{\{0,1\}}$ has period 1, we have

$$\begin{aligned} m_{\{0,1\}}\left(\frac{\lambda d_{n_2+j}}{p_1p_2\cdots p_{n_2+j}}\right) &= m_{\{0,1\}}\left(\sum_{i=n_1+1}^{n_2}\frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}}\right) \\ &= m_{\{0,1\}}\left(\sum_{i\in\mathcal{S}_2}\frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}}\right) \\ &= m_{\{0,1\}}(\alpha) \end{aligned} \tag{2.16}$$

for some $\alpha \in \left(0,\ \frac{1}{2}\frac{D}{D+2}\right)$.

**Case B**. $k_{i_1} < k_{n_2+j}$.

Without loss of generality, we assume that the set $\{i : i \in \mathcal{S}_2, b_i \neq 0\}$ is not empty. Otherwise, we have $\sum_{i\in\mathcal{S}_2}\frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} = 0$. From the definitions of $i_1$ and $i_2$, it follows that $k_{i_1} > k_{i_2}$. By (2.13), we get

$$0 \le \sum_{i\in\mathcal{S}_2}\frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} < \frac{D}{D+2}2^{k_{i_2}-k_{n_2+j}} \le \frac{D}{D+2}2^{k_{i_1}-k_{n_2+j}-1}.$$

Combining (2.12), this shows there exists an integer $z$ such that

$$\sum_{i\in\mathcal{S}_1\cup\mathcal{S}_2}\frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}} = 2^{k_{i_1}-k_{n_2+j}-1}(2z+1+\eta)$$

for some $\eta \in \left(0,\ \frac{D}{D+2}\right)$.

Given a real number $r \in \mathbb{R}$, we denote by $||r||_{\frac{1}{2}}$ the distance between $r$ and it's nearest middle point of two neighboring integer points, i.e.

$$||r||_{\frac{1}{2}} = \min_{z\in\mathbb{Z}}\left\{\left|r-z-\frac{1}{2}\right|, \left|r-z+\frac{1}{2}\right|\right\}.$$

Thus the assumption $k_{i_1} < k_{n_2+j}$ implies

$$\left|\left|2^{k_{i_1}-k_{n_2+j}-1}(2z+1)\right|\right|_{\frac{1}{2}} \ge 2^{k_{i_1}-k_{n_2+j}-1}.$$

By noting that $\left|2^{k_{i_1}-k_{n_2+j}-1}\eta\right| < 2^{k_{i_1}-k_{n_2+j}-1}\frac{D}{D+2}$, we get

$$\begin{aligned} \left|\left|\frac{\lambda d_{n_2+j}}{p_1p_2\cdots p_{n_2+j}}\right|\right|_{\frac{1}{2}} &= \left|\left|\sum\nolimits_{i\in\mathcal{S}_1\cup\mathcal{S}_2}\frac{d_{n_2+j}b_i}{p_1\cdots p_{n_2+j}}\right|\right|_{\frac{1}{2}} \\ &= \left|\left|2^{k_{i_1}-k_{n_2+j}-1}(2z+1+\eta)\right|\right|_{\frac{1}{2}} \\ &\ge 2^{k_{i_1}-k_{n_2+j}-1}\frac{2}{D+2}. \end{aligned} \tag{2.17}$$

Since $i_1 \in \mathcal{S}_1$, from Lemma 2.4 (i) it follows that $i_1 > n_2 - N_1$. By Lemma 2.4 (iii), we have

$$\begin{aligned} k_{n_2+j} - k_{i_1} &= \nu_2(p_1 \cdots p_{n_2+j}) - \nu_2(2d_{n_2+j}) - \nu_2(p_1 \cdots p_{i_1}) + \nu_2(2d_{i_1}) \\ &\le \nu_2(p_{i_1+1} \cdots p_{n_2+j}) + \nu_2(d_{i_1}) \\ &\le (n_2 + j - i_1 + 1) \max\{\nu_2(d_n) : n \ge 1\} \\ &\le (N_1 + c + 2) \max\{\nu_2(d_n) : n \ge 1\}. \end{aligned} \tag{2.18}$$

Together with (2.17) and the boundedness of $\{d_n\}_{n=1}^{\infty}$, we conclude that there is a positive constant $0 < \theta < \frac{1}{2}$ such that

$$\left\|\frac{\lambda d_{n_2+j}}{p_1 p_2 \cdots p_{n_2+j}}\right\|_{\frac{1}{2}} = \left\|\sum_{i \in \mathcal{S}_1 \cup \mathcal{S}_2} \frac{d_{n_2+j} b_i}{p_1 \cdots p_{n_2+j}}\right\|_{\frac{1}{2}} > \theta.$$

Therefore, combining the conclusions of Case A and Case B we see the modulus of (2.14) has a positive lower bound. Furthermore, there is a constant $\varepsilon > 0$ such that

$$\prod_{j=1}^{c} \left| m_{p_{\ell_n+1}^{-1} \cdots p_{\ell_n+j}^{-1} D_{\ell_n+j}} \left(\frac{\lambda}{p_1 p_2 \cdots p_{n_2}}\right)\right| > 2\varepsilon, \quad \forall\, \lambda \in \Lambda.$$

Finally, note that

$$\{\delta_{p_{n+1}^{-1} D_{n+1}} * \delta_{p_{n+1}^{-1} p_{n+2}^{-1} D_{n+2}} * \cdots * \delta_{p_{n+1}^{-1} \cdots p_{n+c}^{-1} D_{n+c}} \;:\; n > 0\}$$

is a family of probability measures supported on subsets of $[0,\ 1]$. Hence, their Fourier transformations are equi-continuous (cf [2, Definition 4.4 (iii)]). Thus we see that there is a small positive number $\theta_0 > 0$ such that (2.8) holds for some constant $\varepsilon > 0$. The proof is completed. □

Furthermore, we have the following Lemma 2.6. For $k \ge 1$, we write

$$\mu_{>k} := \delta_{p_{k+1}^{-1} D_{k+1}} * \delta_{p_{k+1}^{-1} p_{k+2}^{-1} D_{k+2}} * \cdots .$$

**Lemma 2.6** *Assume that $k_n \neq k_m$ for all $n \neq m$ and (2.3) holds. Furthermore, assume that there exists a positive integer $n_0$ such that for any $n \ge n_0$, there exists an integer $j_n < \ell_n$ satisfying $\ell_{j_n} > \ell_n$. Consider the set $\Lambda$ defined in Lemma 2.5 for $n_1,\ n_2 \in \mathcal{I}$ satisfying $n_2 > n_1 + N_1$. There are small positive constants $\varepsilon_1 > 0$ and $\theta_1 > 0$ such that for any $\lambda \in \Lambda$, there exists an integer $b_\lambda \in \mathbb{Z}$ with $b_0 = 0$ such that*

$$\left|\widehat{\mu_{>n_2}}(y + \frac{\lambda + p_1 p_2 \cdots p_{n_2+c} b_\lambda}{p_1 p_2 \cdots p_{n_2}})\right| > \varepsilon_1, \quad \forall\, y \in [-\theta_1,\ \theta_1],\ \lambda \in \Lambda. \tag{2.19}$$

***Proof*** By [2, Lemma 4.5], there are small positive constants $\varepsilon' > 0$ and $\theta_0 > \theta_1 > 0$ such that for any $\lambda \in \Lambda$, there exists an integer $b_\lambda$ with $b_0 = 0$ such that

$$\left|\widehat{\mu_{>n_2+c}}(y + b_\lambda + \frac{\lambda}{p_1 p_2 \cdots p_{n_2+c}})\right| > \varepsilon', \quad \forall\, y \in [-\theta_1,\ \theta_1]. \tag{2.20}$$

Recall a fact that the mask function $m_{\{0,1\}}(x)$ has period 1. For any $\lambda \in \Lambda$, we have

$$\begin{aligned}
&\left|\widehat{\mu_{>n_2}}(y + \tfrac{\lambda + p_1 p_2 \cdots p_{n_2+c} b_\lambda}{p_1 p_2 \cdots p_{n_2}})\right| \\
&= \left|\widehat{\mu_{>n_2+c}}(y + \tfrac{\lambda + p_1 p_2 \cdots p_{n_2+c} b_\lambda}{p_1 p_2 \cdots p_{n_2+c}})\right| \cdot \textstyle\prod_{j=1}^{c} \left| m_{p_{n_2+1}^{-1} \cdots p_{n_2+j}^{-1} D_{n_2+j}} \big(y + \tfrac{\lambda + p_1 p_2 \cdots p_{n_2+c} b_\lambda}{p_1 p_2 \cdots p_{n_2}}\big)\right| \\
&= \left|\widehat{\mu_{>n_2+c}}(y + b_\lambda + \tfrac{\lambda}{p_1 p_2 \cdots p_{n_2+c}})\right| \cdot \textstyle\prod_{j=1}^{c} \left| m_{p_{n_2+1}^{-1} \cdots p_{n_2+j}^{-1} D_{n_2+j}} \big(y + \tfrac{\lambda}{p_1 p_2 \cdots p_{n_2}}\big)\right|.
\end{aligned} \tag{2.21}$$

Lemma 2.5 shows there are small constants $\varepsilon > 0$ and $\theta_0$ such that

$$\prod_{j=1}^{c} \left| m_{p_{n_2+1}^{-1} \cdots p_{n_2+j}^{-1} D_{n_2+j}} \big(y + \frac{\lambda}{p_1 p_2 \cdots p_{n_2}}\big)\right| > \varepsilon, \quad \forall\, y \in [-\theta_0,\ \theta_0]. \tag{2.22}$$

Letting $\varepsilon_1 = \varepsilon\varepsilon'$, the inequality (2.19) follows from (2.20), (2.21) and (2.22). The proof is completed. □

Now we are in the place to reprove the sufficiency of [2, Theorem 1.1].

**Proof of the sufficiency of** [2, Theorem 1.1].

We shall deal with two cases.

(A) If there is an infinite subset $\mathcal{I}_0 \subset \mathcal{I}$ ($\mathcal{I}$ is defined in (2.5)) such that $\ell_i \le n$ for any $i \le n$ and $n \in \mathcal{I}_0$. Then, the proof in [2] works by replacing $\mathcal{B}$ by $\mathcal{I}_0$.

(B) If there are only finitely many $n \in \mathcal{I}$ such that $\ell_i \le n$ for any $i \le n$. Then, there is an integer $n_0 > 0$ such that for any $n \in \mathcal{I}$ with $n \ge n_0$, there exists at least one integer $j_n < \ell_n$ satisfying $\ell_{j_n} > \ell_n$. Also, as stated in the beginning of this section, all conditions in (2.3) can be assumed without loss of generality.

Then, we extend the idea of [1, Lemma 2.6] and [1, Theorem 2.7] to construct a spectrum of $\mu$. This spectrum is different from the one in [2]. Let $\mathcal{I}_1 = \{n \in \mathcal{I} :\ n > n_0\}$

We first choose $n_1 \in \mathcal{I}_1$ and define

$$\Lambda_1 = \{0,\ a_1\} + \{0,\ a_2\} + \cdots + \{0,\ a_{n_1}\},$$

where $a_i = 2^{k_i}\theta(p_1 \cdots p_{\ell_i}) \in U_i$ for $1 \le i \le n_1$. Since $\mathcal{I}_1$ is infinite and $p_n \ge 2$, we can find a sufficiently large integer $n_2 \in \mathcal{I}_1$ such that $n_2 > n_1 + N_1$ and

$$(p_1 p_2 \cdots p_{n_2})^{-1}\Lambda_1 \subset \Big[-\frac{\theta_1}{2^2},\ \frac{\theta_1}{2^2}\Big],$$

where $N_1$ and $\theta_1$ are defined in Lemma 2.4 and 2.6, respectively. Let $\epsilon_1$ be the constant in Lemma 2.6 and $\Lambda_{1,2}$ be a spectrum of $*_{i=n_1+1}^{n_2}\delta_{p_1^{-1}p_2^{-1}\cdots p_i^{-1}D_i}$ as stated in Lemma 2.5, i.e.

$$\Lambda_{1,2}=\{0,\ a_{n_1+1}\}+\{0,\ a_{n_1+2}\}+\cdots+\{0,\ a_{n_2}\},$$

where

$$a_i=\begin{cases}2^{k_i}\theta(p_1\cdots p_{n_2+c}), & \text{if } \frac{D\theta(p_1\cdots p_{\ell_i})}{\theta(p_1\cdots p_{n_2+1})}\geq 1,\\ 2^{k_i}\theta(p_1\cdots p_{\ell_i}), & \text{if } \frac{D\theta(p_1\cdots p_{\ell_i})}{\theta(p_1\cdots p_{n_2+1})}<1.\end{cases}\tag{2.23}$$

According to Lemma 2.6, for any $\lambda\in\Lambda_{1,2}$, there exits an integer $k_{1,\lambda}\in\mathbb{Z}$ with $k_{1,\,0}=0$ such that

$$\left|\widehat{\mu_{>n_2}}\Big(\frac{\gamma}{p_1p_2\cdots p_{n_2}}+\frac{\lambda+p_1\cdots p_{n_2+c}k_{1,\,\lambda}}{p_1p_2\cdots p_{n_2}}\Big)\right|>\varepsilon_1,\quad\forall\,\gamma\in\Lambda_1,\ \lambda\in\Lambda_{1,2}.$$

Lemma 2.3 (ii) and (iii) show that $\Lambda_2:=\{\gamma+\lambda+p_1p_2\cdots p_{n_2+c}k_{1,\,\lambda}:\ \gamma\in\Lambda_1,\ \lambda\in\Lambda_{1,2}\}$ is a spectrum of the probability measure $\delta_{p_1^{-1}D_1}*\delta_{p_1^{-1}p_2^{-1}D_2}*\cdots*\delta_{p_1^{-1}p_2^{-1}\cdots p_{n_2}^{-1}D_{n_2}}$. Furthermore, [2, Lemma 4.1] and the definitions of $U_i$ show that $U_i+p_1p_2\cdots p_{n_2+c}k_{1,\,\lambda}=U_i$ for all $i\leq n_2$. Hence, by $k_{1,\,0}=0$ and the definitions of $\Lambda_1$ and $\Lambda_2$, we see $\Lambda_1\subset\Lambda_2\subset\sum_{j=1}^{n_2}(\{0\}\cup U_j)$. In a word, we have

$$\left|\widehat{\mu_{>n_2}}\Big(\frac{\lambda}{p_1p_2\cdots p_{n_2}}\Big)\right|>\varepsilon_1,\quad\forall\,\lambda\in\Lambda_2.\tag{2.24}$$

Continuing in this way, we can find a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}\subset\mathcal{I}_1$ and $\Lambda_k$ such that the following properties (2.25), (2.26), (2.27) and claim hold.

$$0\in\Lambda_k\subset\Lambda_{k+1}\subset\sum_{j=1}^{n_{k+1}}(\{0\}\cup U_j),\quad k=1,\ 2,\cdots,\tag{2.25}$$

$$(p_1p_2\cdots p_{n_{k+1}})^{-1}\Lambda_k\subset\Big[-\frac{\theta_1}{2^{k+1}},\ \frac{\theta_1}{2^{k+1}}\Big],\quad k=1,\ 2,\cdots,\tag{2.26}$$

$$\left|\widehat{\mu_{>n_k}}\Big(\frac{\lambda}{p_1p_2\cdots p_{n_k}}\Big)\right|>\varepsilon_1,\quad\forall\,\lambda\in\Lambda_k,\ k=2,\ 3,\cdots.\tag{2.27}$$

**Claim**. The set $\Lambda_k$ is a spectrum of the probability measure $\delta_{p_1^{-1}D_1}*\delta_{p_1^{-1}p_2^{-1}D_2}*\cdots*\delta_{p_1^{-1}p_2^{-1}\cdots p_{n_k}^{-1}D_{n_k}}$ for all $k=1,\ 2,\cdots$.

Let $\Gamma=\bigcup_{k=1}^{\infty}\Lambda_k$. We shall prove $\Gamma$ is a spectrum of $\mu$.

For any $a\neq b\in\Gamma$, from (2.25) it follows that $a\neq b\in\Lambda_k$ for some $k>0$. Hence, $a-b$ is a zero point of the Fourier transform of $\delta_{p_1^{-1}D_1}*\delta_{p_1^{-1}p_2^{-1}D_2}*\cdots*$

$\delta_{p_1^{-1}p_2^{-1}\cdots p_{n_k}^{-1}D_{n_k}}$. Hence, $\widehat{\mu}(a-b)=0$, which implies the exponential function set $E_\Gamma=\{e^{2\pi i\gamma x}:\gamma\in\Gamma\}$ is an orthogonal family of $L^2(\mu)$.

Assume, on the contrary, that $\Gamma$ is not a spectrum of $\mu$. Then, [2, Proposition 2.3] shows that $Q_{\mu,\Gamma}(x_0)<1$ for some $x_0\in\mathbb{R}$.

Recall that $\lim\limits_{k\to\infty}(p_1p_2\cdots p_{n_k})^{-1}x_0=0$ and $\widehat{\Phi}:=\{\widehat{\nu}:\nu\in\Phi\}$ (here $\Phi=\{\mu_{>n}:n\geq 1\}$) is equi-continuous. From (2.26) it follows that

$$\beta_k:=\inf_{\lambda\in\Lambda_k}|\widehat{\mu_{>n_k}}((p_1p_2\cdots p_{n_{k+1}})^{-1}(\lambda+x_0))|\to 1 \text{ as } k\to\infty. \tag{2.28}$$

Furthermore, from (2.27) it follows that there exists a positive integer $k_0>0$ such that for any $k\geq k_0$ and $\lambda\in\Lambda_k$, we have

$$|\widehat{\mu_{>n_k}}((p_1p_2\cdots p_{n_k})^{-1}(\lambda+x_0))|\geq\frac{1}{2}\varepsilon_1. \tag{2.29}$$

Let

$$Q_k(x_0)=\sum_{\lambda\in\Lambda_k}|\widehat{\mu}(\lambda+x_0)|^2,\quad k=1,2,\cdots.$$

According to [2, (2.2)] and (2.29), for $k\geq k_0$ we have

$$\begin{aligned}
&Q_{k+1}(x_0)-Q_k(x_0)\\
&=\sum_{\lambda\in\Lambda_{k+1}\setminus\Lambda_k}\prod_{n=1}^{\infty}|m_{D_n}((p_1\cdots p_n)^{-1}(\lambda+x_0))|^2\\
&=\sum_{\lambda\in\Lambda_{k+1}\setminus\Lambda_k}\prod_{n=1}^{n_{k+1}}|m_{D_n}((p_1\cdots p_n)^{-1}(\lambda+x_0)|^2|\widehat{\mu_{>n_{k+1}}}((p_1\cdots p_{n_{k+1}})^{-1}(\lambda+x_0))|^2\\
&\geq\tfrac{1}{4}\varepsilon_1^2\sum_{\lambda\in\Lambda_{k+1}\setminus\Lambda_k}\prod_{n=1}^{n_{k+1}}|m_{D_n}((p_1\cdots p_n)^{-1}(\lambda+x_0))|^2.
\end{aligned} \tag{2.30}$$

The above claim shows

$$\sum_{\lambda\in\Lambda_{k+1}}\prod_{n=1}^{n_{k+1}}\left|m_{D_n}(p_1^{-1}\cdots p_n^{-1}(\lambda+x_0))\right|^2=1,\quad k=1,\ 2,\cdots.$$

Thus (2.30) implies that for any $k\geq k_0$, we have

$$Q_{k+1}(x_0)-Q_k(x_0)\geq\frac{1}{4}\varepsilon_1^2\left(1-\sum_{\lambda\in\Lambda_k}\prod_{n=1}^{n_{k+1}}\left|m_{D_n}((p_1\cdots p_n)^{-1}(\lambda+x_0))\right|^2\right).$$

On the other hand, by the definition of $\beta_k$ in (2.28) for $k \geq 1$, we have

$$\begin{aligned} Q_k(x_0) &= \sum_{\lambda\in\Lambda_k} \prod_{n=1}^{\infty} |m_{D_n}((p_1\cdots p_n)^{-1}(\lambda + x_0)|^2 \\ &\geq \beta_k^2 \sum_{\lambda\in\Lambda_k} \prod_{n=1}^{n_{k+1}} \left|m_{D_n}((p_1\cdots p_n)^{-1}(\lambda + x_0))\right|^2. \end{aligned}$$

By the above inequality, we have

$$Q_{k+1}(x_0) - Q_k(x_0) \geq \frac{1}{4}\varepsilon_1^2 \left(1 - \beta_k^{-2} Q_k(x_0)\right), \quad \forall\, k \geq k_0.$$

Therefore, the limit property in (2.28) shows

$$\begin{aligned} \liminf_{k\to\infty}(Q_{k+1}(x_0) - Q_k(x_0)) &\geq \tfrac{1}{4}\varepsilon_1^2 \left(1 - \lim_{k\to\infty} \beta_k^{-2} Q_k(x_0)\right) \\ &= \tfrac{1}{4}\varepsilon_1^2 \left(1 - Q_{\mu,\Gamma}(x_0)\right) > 0. \end{aligned}$$

Together with (2.25), the above inequalities imply

$$1 > Q_{\mu,\Gamma}(x_0) = \lim_{k\to\infty} Q_k(x_0) \geq \sum_{k=1}^{\infty} (Q_{k+1}(x_0) - Q_k(x_0)) = +\infty,$$

which is impossible. Hence, $\Gamma$ is a spectrum of $\mu$. The sufficiency of [2, Theorem 1.1] is proven. □

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## Declarations

**Conflict of interest** We declare that we do not have any commercial or associative interest that represents a Conflict of interest in connection with the work submitted.

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