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The Garden of Eden Theorem over Generalized Cellular Automata

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Abstract
The Garden of Eden theorem is a fundamental result in the theory of cellular automata, which establishes a necessary and sufficient condition for the surjectivity of a cellular automaton with a finite alphabet over an amenable group. Specifically, the theorem states that such an automaton is surjective if and only if it is pre-injective, where pre-injectivity requires that any two almost equal configurations with the same image under the automaton must be equal. This paper focuses on establishing the Garden of Eden theorem over a φ-cellular automaton by demonstrating both Moore theorem and Myhill theorem over φ-cellular automata are true. These results have significant implications for the theoretical framework of the Garden of Eden theorem and its applicability across diverse groups or altered versions of the same group. Overall, this paper provides a more comprehensive study of φ-cellular automata and extends the Garden of Eden theorem to a broader class of automata.

Keywords φ-Cellular automata · The Garden of Eden theorem · Surjectivity · Pre-injectivity

Mathematics Subject Classification 37B10 · 37B15 · 68Q80 · 43A07

1 Introduction

Cellular automata [4, Chapter 1] have been extensively studied in mathematics due to their various applications in fields such as complexity theory, symbolic dynamics,

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and modeling complex systems. These constructs can be described as maps between prodiscrete topological spaces, and possess important characteristics such as being determined by a finite memory set and a local defining function.

If G is a group and A is a finite set, known as an alphabet, the configuration space over G and A, denoted by A^G, is the set of all functions $x : G \to A$. We endow A^G with the prodiscrete topology, i.e. the product topology of the discrete topology of A, and with the shift action of G on A^G given by

$$g \cdot x(h) := x\left(g^{-1}h\right), \quad \forall x \in A^G, g, h \in G.$$

A cellular automaton over A^G is a function $\sigma : A^G \to A^G$ that satisfies the following conditions: there exists a finite subset $R \subseteq G$, referred to as the memory set of σ, and a local function $\mu : A^R \to A$ such that

$$\sigma(x)(g) = \mu\left(\left(g^{-1} \cdot x\right)\Big|_R\right) \text{ for all } x \in A^G \text{ and } g \in G.$$

In the beginning, the Garden of Eden theorem is a result in the theory of cellular automata which states that a cellular automaton is surjective if and only if it satisfies a weak form version of injectivity, called pre-injectivity. The theorem was originally proved by Moore and Myhil in the early 1960s for cellular automata with finite alphabet over the groups $\mathbb{Z}^d$. Indeed, the surjectivity implies pre-injectivity for such cellular automata was first proved by Moore in [7], and Myhill [8] obtained the converse implication shortly after. In 1993, Machì and Mignosi [9] obtained that the Garden of Eden theorem is still valid over any finitely generated groups with subexponential growth. Later, Ceccherini-Silberstein, Machì and Scarabotti [5] proved in 1999 that every amenable groups satisfies the Garden of Eden theorem. The recent results of Bartholdi [2] and [1] finally showed that the class of groups that satisfies the Garden of Eden theorem is precisely the amenable groups.

Theorem 1 (The Garden of Eden theorem) *Let G be an amenable group and let B be a finite set. Let $\sigma : B^G \to B^G$ be a cellular automaton. Then one has*

$$\sigma \text{ is surjective } \iff \sigma \text{ is pre-injective.}$$

In the pursuit of advancing our understanding of cellular automata, Castillo-Ramirez et al in [3] introduced a new concept known as Generalized Cellular Automata (GCA), taking a different approach than the so-called sliding block codes [6]. In their work, the focus is on defining a generalized cellular automaton $\tau : A^G \to A^H$, where H is an arbitrary group, through the utilization of a group homomorphism $\varphi : H \to G$. This interesting extension opens new avenues for studying cellular automata within the context of diverse mathematical structures.

Let T, G be two groups. A finite-to-one surjective homomorphism $\varphi : T \to G$ means that for a surjective homomorphism $\varphi : T \to G$, there exists a positive integer $k \geq 1$ such that the cardinality of each pre-image of φ is no more than k, that is, $\left|\varphi^{-1}(g)\right| \leq k$ for all $g \in G$.

In this paper, we delve into their definition of Generalized Cellular Automata, exploring its implications and applications. Our primary contribution lies in proving the Garden of Eden Theorem for the generalized cellular automata. The Garden of Eden Theorem holds significance as it sheds light on the fundamental properties and possibilities inherent in this extended class of cellular automata. We obtain:

Theorem 2 (The Moore theorem over a φ-cellular automaton) *Assume G is an amenable group, T is a group that is homomorphic to G, and B is a finite set. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton, where $\varphi : T \to G$ is a finite-to-one surjective homomorphism. Then, we have the following implication:*

$$\sigma \text{ is surjective } \Longrightarrow \sigma \text{ is pre-injective.}$$

Theorem 3 (The Myhill theorem over a φ-cellular automaton) *Assume G is an amenable group, T is a group that is isomorphic to G, and B is a finite set. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton, where $\varphi : T \to G$ is an isomorphic map. Then, we have the following implication:*

$$\sigma \text{ is pre-injective } \Longrightarrow \sigma \text{ is surjective.}$$

Corollary 4 (The Garden of Eden theorem over a φ-cellular automaton) *Assume G is an amenable group, T is a group that is isomorphic to G, and B is a finite set. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton, where $\varphi : T \to G$ is an isomorphic map. Then, we have the following equivalence:*

$$\sigma \text{ is surjective } \Longleftrightarrow \sigma \text{ is pre-injective.}$$

The structure of this paper is as follows. In Sect. 2, we provide a review of some definitions. Section 3 establishes the equivalence between the existence of Garden of Eden Configurations and Garden of Eden Patterns on a φ-cellular automaton. Section 4 introduces the property that a φ-cellular automaton $\sigma : B^G \to B^T$ satisfies, which states that if two configurations, x and x', coincide on U, then their mappings $\sigma(x)$ and $\sigma\left(x'\right)$ also coincide on U^{-R}. In Sect. 5, we focus on a key property of φ-cellular automata for a finite-to-one surjective homomorphism φ, which states that applying a φ-cellular automaton to a set of configurations does not increase the entropy of the set. Finally, in Sect. 6, we provide proofs of the Moore and Myhill theorems over a φ-cellular automaton by showing the equivalence between surjectivity, pre-injectivity, and the maximality of the entropy of the φ-cellular automaton's image.

2 Preliminaries

In this section, we will build upon prior knowledge of amenable group theory, topology, classical cellular automata over groups, generalized cellular automata over groups, and the classical Garden of Eden theorem.

We have utilized a number of definitions and theorems proposed by Castillo-Ramirez et al. in their paper [3]. Among these, we pay special attention to Definitions 1, 2, and Lemmas 1, 2, 4, which lay important theoretical foundations for our subsequent research work. We have compiled these definitions and theorems in this section.

Let us begin by introducing some definitions and notations. Throughout the rest of this paper, let B denote a finite set, while G and T denote groups. We will assume that $|B|$ is greater than or equal to 2, and that the set $\{0, 1\}$ is included in B. This assumption is important, as the case where $|B| = 1$ is trivial and lacks interest.

Definition 1 [4, Chapter 1] A cellular automaton over the group G and the alphabet B is a mapping $\sigma : B^G \to B^G$ that satisfies the following properties: there exists a finite subset $R \subset G$ and a mapping $\mu : B^R \to B$ such that for all $x \in B^G$ and $g \in G$, we have

$$\sigma(x)(g) = \mu\left(\left(g^{-1}x\right)\Big|_R\right)$$

Here, $\left(g^{-1}x\right)\big|_R$ represents the restriction of the configuration $g^{-1}x$ to R. The set R is referred to as the memory set, and μ is known as the local defining mapping for σ.

Definition 2 [3, Definition 1] Let $\varphi : T \to G$ be a homomorphism. A φ-cellular automaton over two homomorphic groups G, T and the alphabet B is a map $\sigma : B^G \to B^T$ satisfying the following property: there exist a finite subset $R \subset G$ and a map $\mu : B^R \to B$ such that for all $x \in B^G$ and $t \in T$, the equation

$$\sigma(x)(t) = \mu\left(\left(\varphi\left(t^{-1}\right)x\right)\Big|_R\right) \tag{1}$$

holds, where $\left(\varphi\left(t^{-1}\right)x\right)\big|_R$ denotes the restriction of the configuration $\varphi\left(t^{-1}\right)x$ to R.

The finite subset R mentioned above is commonly referred to as a "memory set," and the map μ is known as a "local defining map" for the φ-cellular automaton.

Example 1 Let's set $G = 2\mathbb{Z}$, $T = \mathbb{Z}$, and $B = \mathbb{Z}/2\mathbb{Z}$. The map $\varphi : T \to G$ is defined as $\varphi(n) = 2n$ for all $n \in T$. Now, consider the φ-cellular automaton $\sigma : B^G \to B^T$, which is defined by

$$\sigma(x)(n) = x(2n) + x(2n+2)$$

for all $x \in B^G$ and $n \in T$. This φ-cellular automaton operates over $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, with a memory set $R = \{0, 2\}$, and a local defining map $\mu : B^R \to B$ given by

$$\mu(y) = y(0) + y(2) \quad \text{for all } y \in B^R$$

Example 2 Let's set $G = \mathbb{Z}$, $T = \{\bar{0}, \bar{1}\} \times \mathbb{Z}$, and $B = \{0, 1\}$. The map $\varphi : \{\bar{0}, \bar{1}\} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined as $\varphi(a, n) = n$ for all $(a, n) \in T$. Now, consider the φ-cellular automaton $\sigma : B^G \to B^T$, which is defined by

$$\sigma(x)(a, n) = x(n) = x(\varphi(a, n)) = \varphi\left((a, n)^{-1}\right)x(0)$$

for all $x \in B^G$ and $(a, n) \in T$. Then σ is a φ-cellular automaton with memory set $R = \{0\}$ and local defining map $\mu = id_R$.

Lemma 5 *Let G, T be two groups. Suppose B is a finite set and $\varphi : T \to G$ is a group homomorphim. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton with memory set R, and let $t \in T$. Then $\sigma(x)(t)$ only depends on the restriction of x to $\varphi(t)R$.*

Proof This follows directly from (1) because $\left(\varphi\left(t^{-1}\right)x\right)(r) = x(\varphi(t)r)$ for all $r \in R$. □

Definition 3 [3, Definition 2] Let G, T be two groups and suppose $\varphi : T \to G$ is a group homomorphim. Let B be a finite set. A map $\sigma : B^G \to B^T$ is said to be φ-equivariant if

$$t\sigma(x) = \sigma(\varphi(t)x) \text{ for all } x \in B^G, t \in T.$$

Lemma 6 [3, Lemma 1] *Each φ-cellular automaton is φ-equivariant, where $\varphi : T \to S$ is a homomorphic map.*

Lemma 7 [3, Lemma 2] *Each φ-cellular automaton is continuous.*

If V is a subgroup of G, we can define a configuration in B^G to be V-periodic if $v \cdot x = x$ for all $v \in V$. The set of all V-periodic configurations in B^G is denoted by Fix (V)

Lemma 8 [3, Lemma 4] *Let $\sigma : B^G \to B^T$ be a surjective φ-cellular automaton, with $\varphi : T \to G$ being a homomorphism, then φ is injective.*

Suppose G and T are homomorphic groups, and B is a finite set. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton, with R as its memory set and $\mu : B^R \to B$ as the associated defining map. If R' is a finite subset of G such that $R \subset R'$, then R' is also a memory set for σ, and the local defining map associated with R' is the map μ': $B^{R'} \to B$, given by $\mu' = \mu \circ \pi$, where $\pi : B^{R'} \to B^R$ is the canonical projection (restriction map). This demonstrates that a finite subset of G containing the memory set is also a memory set for a φ-cellular automaton.

3 Garden of Eden Configurations

Suppose we have two groups, G and T, a homomorphic map $\varphi : T \to G$, and a set B. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton. We define a configuration $y \in B^T$ as a "Garden of Eden" configuration for σ if it does not belong to the image of σ. Hence, when σ is surjective, there are no Garden of Eden configurations.

Next, let $\Omega \subset T$ be a finite subset. A pattern $\pi : \Omega \to B$ is considered a Garden of Eden pattern for σ if there is no configuration $x \in B^G$ such that $\sigma(x)|_\Omega = \pi$. According to this definition, if $\pi : \Omega \to B$ is a Garden of Eden pattern for σ, then any configuration $y \in B^T$ such that $y|_\Omega = \pi$ is also a Garden of Eden configuration for σ. Therefore, the existence of a Garden of Eden pattern implies the existence of Garden of Eden configurations, indicating the non-surjectivity of σ. We can show that the converse is also true when the alphabet set is finite.

Proposition 9 *Consider groups G and T, with $\varphi : T \to G$ being a homomorphism, and let B be a finite set. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton. Assuming that σ is not surjective, then it follows that σ contains a Garden of Eden pattern.*

Proof It is a known fact that the set $\sigma\left(B^G\right)$ is closed in B^T under the prodiscrete topology. Consequently, the set $B^T \backslash \sigma\left(B^G\right)$ is open in B^T. Hence, for a Garden of Eden configuration $y \in B^T$ with respect to σ, we can identify a finite subset $\Omega \subset G$ such that

$$V(y, \Omega) = \left\{x \in B^T : x|_\Omega = y|_\Omega\right\} \subset B^T \backslash \sigma\left(B^G\right).$$

In simpler terms, any configuration that extends $y|_\Omega$ does not belong to $\sigma\left(B^G\right)$, implying that $y|_\Omega$ serves as a Garden of Eden pattern for σ. □

Suppose G is a group and B is a set. We define two configurations $x_1, x_2 \in B^G$ to be almost equal if the set $\{g \in G : x_1(g) \neq x_2(g)\}$ is finite. It is evident that this definition establishes an equivalence relation on the set B^G.

A map $p : B^G \to B^T$ is termed pre-injective if it satisfies the condition that if two configurations $x_1, x_2 \in B^S$ are almost equal and $p(x_1) = p(x_2)$, then $x_1 = x_2$. It is evident from this definition that injectivity implies pre-injectivity. When the group G is finite, the converse is obviously true. However, in the case of an infinite group G, a pre-injective map $p : B^G \to B^T$ may not be injective.

Example 3 Let's consider $G = 2\mathbb{Z}$, $T = \mathbb{Z}$, and $B = \mathbb{Z}/2\mathbb{Z}$. Define $\varphi : T \to G$ as $\varphi(n) = 2n$ for all $n \in T$. Now, let's look at the φ-cellular automaton $\sigma : B^G \to B^T$ defined by $\sigma(x)(n) = x(2n) + x(2n+2)$ for all $x \in B^G$ and $n \in T$.

We claim that σ is pre-injective. Suppose $x_1, x_2 \in B^G$ are two configurations such that the set $\Omega = \{n \in G : x_1(n) \neq x_2(n)\}$ is a nonempty finite subset of $2\mathbb{Z}$. Let n_0 be the largest even element in Ω. Then $\sigma(x_1)(n_0/2) \neq \sigma(x_2)(n_0/2)$, and thus $\sigma(x_1) \neq \sigma(x_2)$. This demonstrates that σ is pre-injective. However, σ is not injective, since the constant configurations $c_0, c_1 \in B^G$ given by $c_0(n) = 0$ and $c_1(n) = 1$ for all $n \in 2\mathbb{Z}$ have the same image c_0 under σ.

4 Interiors, Closures, and Boundaries

In this part of a group G, given subsets A and W, the A-interior W^{-A} and the A-closure W^{+A} of W are defined as subsets of G, where

$$W^{-A} = \{g \in G : gA \subset W\} \quad \text{and}$$
$$W^{+A} = \{g \in G : gA \cap W \neq \varnothing\}$$

Take notice that

$$W^{-A} = \bigcap_{a \in A} Wa^{-1} \tag{2}$$

and

$$W^{+A} = \bigcup_{a \in A} Wa^{-1} = WA^{-1} \tag{3}$$

The A-boundary of W is the subset $\partial_A(W)$ of S defined as

$$\partial_A(W) = W^{+A} \backslash W^{-A}.$$

Below are some general properties of the sets W^{-A}, W^{+A}, and $\partial_A(W)$ that we will frequently utilize in the following discussions.

Proposition 10 [4, Chapter 5] *Let G be a group. Let A, A_1, A_2 and W be subsets of G. Then the following hold:*

(i) $(G\backslash W)^{-A} = G\backslash W^{+A}$*;*
(ii) $(G\backslash W)^{+A} = G\backslash W^{-A}$*;*
(iii) *if $a \in A$, then $W^{-A} \subset Wa^{-1} \subset W^{+A}$;*
(iv) *if $1_G \in A$, then $W^{-A} \subset W \subset W^{+A}$;*
(v) *if A is nonempty and W is finite, then W^{-A} is finite;*
(vi) *if A and W are both finite, then W^{+A} and $\partial_A(W)$ are finite;*
(vii) *if $A_1 \subset A_2$, then $W^{-A_2} \subset W^{-A_1}$, $W^{+A_1} \subset W^{+A_2}$ and $\partial_{A_1}(W) \subset \partial_{A_2}(W)$;*
(viii) *if $g \in G$, then $g\left(W^{-A}\right) = (gW)^{-A}$, $g\left(W^{+A}\right) = (gW)^{+A}$ and $g\left(\partial_A(W)\right) = \partial_A(gW)$.*

Let G be a group. Recall that if there exists a net $\left(F_j\right)_{j \in J}$ of nonempty finite subsets of G such that $\lim_j \frac{|F_j \backslash F_j g|}{|F_j|} = 0$ for all $g \in G$, then $\left(F_j\right)_{j \in J}$ is a right Følner net for G.

Proposition 11 [4, Chapter 5] *Suppose G is a group and $\left(F_j\right)_{j \in J}$ is a net consisting of nonempty finite subsets of G. Then the following conditions are equivalent:*

(a) *The net $\left(F_j\right)_{j \in J}$ is a right Følner net for G.*
(b) *It holds that $\lim_j \frac{|\partial_A(F_j)|}{|F_j|} = 0$ for every finite subset $A \subset G$.*

Corollary 12 [4, Chapter 5] *For a group G, the following conditions are equivalent:*

(a) *G is amenable;*
(b) *for every finite subset $A \subset G$ and every real number $\varepsilon > 0$, there exists a nonempty finite subset $F \subset G$ such that*

$$\frac{|\partial_A(F)|}{|F|} < \varepsilon \tag{4}$$

One significant property of φ-cellular automata, as stated in the aforementioned papers, is the following:

One important characteristic of φ-cellular automata is that a φ-cellular automaton $\sigma : B^G \to B^T$ satisfies, which states that if two configurations, x and x', coincide on U, then their mappings $\sigma(x)$ and $\sigma\left(x'\right)$ also coincide on U^{-R}. In more precise terms, the following statement holds.

Proposition 13 *Given groups G and T, a homomorphism $\varphi : T \to G$, and a set B, let $\sigma : B^G \to B^T$ be a φ-cellular automaton with memory set R. If x and x' are elements of B^G and there exists a subset Ω of G such that x and x' are equal on Ω (or $G\backslash\Omega$), then the outcomes of $\sigma(x)$ and $\sigma\left(x'\right)$ will also be equal on the corresponding sets determined as the preimages of Ω^{-R} (or $G\backslash\left(\Omega^{+R}\right)$).*

Proof Suppose that x and x' coincide on Ω. If $t \in \varphi^{-1}\left(\Omega^{-R}\right)$, then $\varphi(t)R \subset \Omega$ and therefore $\sigma(x)(t) = \sigma\left(x'\right)(t)$ by Lemma 1. It follows that $\tau(x)$ and $\tau\left(x'\right)$ coincide on $\varphi^{-1}\left(\Omega^{-R}\right)$.

Suppose now x and x' coincide on $S\backslash\Omega$. Then $\sigma(x)$ and $\sigma\left(x'\right)$ coincide on $\varphi^{-1}\left((G\backslash\Omega)^{-R}\right) = \varphi^{-1}\left(G\backslash\Omega^{+R}\right) = T\backslash\varphi^{-1}\left(\Omega^{+R}\right)$ by the first part of the proof and Proposition 10(i). □

In simpler terms, this property ensures that if two initial configurations have the same values within a certain region, their resulting configurations after applying the φ-cellular automaton will also have the same values within the corresponding shifted region, but in the opposite direction.

Note that this property plays a significant role in understanding the behavior and dynamics of φ-cellular automata, providing insights into how local interactions can affect the global evolution of these systems.

5 Tilings and Entropy

Let G be a group and let F and F' be subsets of G. A subset $L \subset G$ is an $\left(F, F'\right)$-tiling of G if the sets lF, $l \in L$, are pairwise disjoint, and the sets lF' cover G. In other words, $L \subset G$ is an $\left(F, F'\right)$-tiling if the following conditions hold:

(T1) $l_1F \cap l_2F = \varnothing$ for all $l_1, l_2 \in L$ where $l_1 \neq l_2$;

(T2) $G = \bigcup_{l\in L} lF'$.

Remark 1 Let G be a group and let F and F' be subsets of G. If L is an $\left(F, F'\right)$-tiling of G and if F_1 and F_1' are subsets of G such that $F_1 \subset F$ and $F' \subset F_1'$, then it is clear that L is also an $\left(F_1, F_1'\right)$-tiling of G.

The Zorn lemma may be used to prove the existence of $\left(F, F'\right)$-tilings for any subset F of G and for $F' \subset G$ "large enough". More precisely, we have the following:

Proposition 14 [4, Chapter 5] *Let G be a group and let F be a nonempty subset of G. Let $F' = \left\{g_1g_2^{-1} : g_1, g_2 \in F\right\}$. Then there is an $\left(F, F'\right)$-tiling $L \subset G$.*

Proposition 15 [4, Chapter 5] *Let G be an amenable group and let $\left(F_j\right)_{j\in J}$ be a right Følner net for G. Let E and E' be finite subsets of G and suppose that $L \subset G$ is an $\left(E, E'\right)$-tiling of G. Let us set, for each $j \in J$,*

$$L_j = L \cap F_j^{-E} = \left\{l \in L : lE \subset F_j\right\}.$$

Then there exist a real number $\alpha > 0$ *and an element* $j_0 \in J$ *such that*

$$\left|L_j\right| \geq \alpha \left|F_j\right| \quad \textit{for all } j \geq j_0.$$

Let G be an amenable group, $\mathcal{F} = \left(F_j\right)_{j\in J}$ be a right Følner net for G, and B be a finite set.

For $E \subset G$, we denote by $\pi_E : B^G \to B^E$ the canonical projection (restriction map). We thus have $\pi_E(x) = x|_E$ for all $x \in B^G$.

Definition 4 [4, Chapter 5] Let $X \subset B^G$. The entropy $\text{ent}_{\mathcal{F}}(X)$ of X with respect to the right Følner net $\mathcal{F} = \left(F_j\right)_{j\in J}$ is defined by

$$\text{ent}_{\mathcal{F}}(X) = \limsup_j \frac{\log \left|\pi_{F_j}(X)\right|}{\left|F_j\right|}.$$

Here are some immediate properties of entropy.

Proposition 16 [4, Chapter 5] *One has*

(i) $\text{ent}_{\mathcal{F}}\left(B^G\right) = \log|B|$*;*
(ii) $\text{ent}_{\mathcal{F}}(X) \leq \text{ent}_{\mathcal{F}}(Y)$ *if* $X \subset Y \subset B^G$*;*
(iii) $\text{ent}_{\mathcal{F}}(X) \leq \log|B|$ *for all* $X \subset B^G$.

A finite-to-one surjective homomorphism can preserve the property of being amenable groups as the following states:

Proposition 17 *Let* $\varphi : T \to G$ *be a finite-to-one surjective homomorphism,* G *be an amenable group,* $\mathcal{F} = \left(F_j\right)_{j\in J}$ *be a right Følner net for* G*. Then,* T *is also an amenable group, and* $\varphi^{-1}(\mathcal{F}) = \left(\varphi^{-1}\left(F_j\right)\right)_{j\in J}$ *is a right Følner net for* T.

Proof There exist a positive integer $k \geq 1$, for each subset $F \subset S$, we have

$$|F| \leq \left|\varphi^{-1}(F)\right| \leq k|F|. \tag{5}$$

This implies we can choose for each subset $k_F \in [1, k]$, such that $\left|\varphi^{-1}(F)\right| = k_F|F|$.

For arbitrary $t \in T$, it is obvious that $\varphi\left(\varphi^{-1}\left(F_j\right)\backslash\varphi^{-1}\left(F_j\right)t\right) \subseteq F_j\backslash F_j\varphi(t)$. Then we have $\varphi^{-1}\left(F_j\right)\backslash\varphi^{-1}\left(F_j\right)t \subseteq \varphi^{-1}\left(F_j\backslash F_j\varphi(t)\right)$. Hence

$$\begin{aligned}
\frac{\left|\varphi^{-1}\left(F_j\right)\backslash\varphi^{-1}\left(F_j\right)t\right|}{\left|\varphi^{-1}\left(F_j\right)\right|} &\leq \frac{\left|\varphi^{-1}\left(F_j\backslash F_j\varphi\left(t\right)\right)\right|}{\left|\varphi^{-1}\left(F_j\right)\right|}\\
&\leq \frac{k\left|F_j\backslash F_j\varphi\left(t\right)\right|}{\left|\varphi^{-1}\left(F_j\right)\right|}\\
&= \frac{k\left|F_j\backslash F_j\varphi\left(t\right)\right|}{k_{F_j}\left|F_j\right|} \quad (by(5))\\
&\leq \frac{k\left|F_j\backslash F_j\varphi\left(t\right)\right|}{\left|F_j\right|}.
\end{aligned}$$

Since

$$\lim_j \frac{\left|F_j \backslash F_j \varphi(t)\right|}{\left|F_j\right|} = 0,$$

we get

$$\lim_j \frac{\left|\varphi^{-1}\left(F_j\right) \backslash \varphi^{-1}\left(F_j\right) t\right|}{\left|\varphi^{-1}\left(F_j\right)\right|} = 0.$$

This shows T satisfies the Følner conditions, with $\varphi^{-1}(\mathcal{F}) = \left(\varphi^{-1}\left(F_j\right)\right)_{j\in J}$ being a right Følner net for it. Thus T is amenable by virtue of the Tarski-Følner theorem. □

Understanding and analyzing the entropy behavior in φ-cellular automata is crucial for studying their information processing capabilities, complexity, and potential applications in various fields such as cryptography, pattern recognition, and computational modeling.

Recall that if R is a memory set for a φ-cellular automaton $\sigma : B^G \to B^T$, then every finite subset R' of G such that $R \subset R'$ is also a memory set for σ. One important characteristic of φ-cellular automata is that the application of a φ-cellular automaton to a set of configurations does not lead to an increase in the entropy of the set. In more precise terms, the following statement holds:

Proposition 18 *Suppose $\varphi : T \to G$ is a finite-to-one surjective homomorphism, where G is an amenable group, and $\mathcal{F} = \left(F_j\right)_{j\in J}$ is a right Følner net for G. Let $\sigma : B^G \to B^T$ be a φ-cellular automaton, and let $X \subset B^G$. Then the following holds:*

$$\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}(\sigma(X)) \leq \mathrm{ent}_{\mathcal{F}}(X).$$

Proof Let $Y = \sigma(X)$. Suppose $R \subset G$ is a memory set for σ. Upon including 1_G in R by replacing R with $R \cup \{1_G\}$, we may assume that $1_G \in R$. Let Ω be a finite subset of G. First observe that σ induces a map

$$\sigma_\Omega : \pi_\Omega(X) \to \pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y)$$

defined as follows. If $u \in \pi_\Omega(X)$, then

$$\sigma_\Omega(u) = (\tau(x))|_{\varphi^{-1}\left(\Omega^{-R}\right)},$$

where x is an element of X such that $x|_\Omega = u$. Note that the fact that $\sigma_\Omega(u)$ does not depend on the choice of such an x follows from Proposition 13.

Clearly σ_Ω is surjective. Indeed, if $v \in \pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y)$, then there exists $x \in X$ such that $(\sigma(x))|_{\varphi^{-1}\left(\Omega^{-R}\right)} = v$. Then, setting $u = \pi_\Omega(x)$ we have, by construction, $\sigma_\Omega(u) = v$. Therefore, we have

$$\left|\pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y)\right| \leq \left|\pi_\Omega(X)\right|. \tag{6}$$

Observe now that $\Omega^{-R} \subset \Omega$, since $1_G \in R$ (cf. Proposition 10(iv)). Thus $\varphi^{-1}\left(\Omega^{-R}\right) \subset \varphi^{-1}(\Omega)$, and $\pi_{\varphi^{-1}(\Omega)}(Y) \subset \pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y) \times B^{\varphi^{-1}(\Omega)\backslash\varphi^{-1}\left(\Omega^{-R}\right)}$. This implies

$$\begin{aligned}
\log\left|\pi_{\varphi^{-1}(\Omega)}(Y)\right| &\leq \log\left|\pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y) \times B^{\varphi^{-1}(\Omega)\backslash\varphi^{-1}\left(\Omega^{-R}\right)}\right| \\
&= \log\left|\pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y)\right| + \log\left|B^{\varphi^{-1}(\Omega)\backslash\varphi^{-1}\left(\Omega^{-R}\right)}\right| \\
&= \log\left|\pi_{\varphi^{-1}\left(\Omega^{-R}\right)}(Y)\right| + \left|\varphi^{-1}(\Omega)\backslash\varphi^{-1}\left(\Omega^{-R}\right)\right| \log|B| \\
&\leq \log|\pi_\Omega(X)| + \left|\varphi^{-1}(\Omega)\backslash\varphi^{-1}\left(\Omega^{-R}\right)\right| \log|B| \quad (by (6)).
\end{aligned}$$

As $\Omega\backslash\Omega^{-R} \subset \partial_R(\Omega)$, we have

$$\varphi^{-1}(\Omega)\backslash\varphi^{-1}\left(\Omega^{-R}\right) = \varphi^{-1}\left(\Omega\backslash\Omega^{-R}\right) \subset \varphi^{-1}\left(\partial_R(\Omega)\right),$$

so we can deduce that

$$\log\left|\pi_{\varphi^{-1}(\Omega)}(Y)\right| \leq \log|\pi_\Omega(X)| + \left|\varphi^{-1}\left(\partial_R(\Omega)\right)\right| \log|B|.$$

By taking $\Omega = F_j$, this gives us

$$\begin{aligned}
\frac{\log\left|\pi_{\varphi_{F_j}^{-1}}(Y)\right|}{\left|\varphi^{-1}\left(F_j\right)\right|} &\leq \frac{\log\left|\pi_{F_j}(X)\right|}{\left|\varphi^{-1}\left(F_j\right)\right|} + \frac{\left|\varphi^{-1}\left(\partial_R(F_j)\right)\right|}{\left|\varphi^{-1}\left(F_j\right)\right|} \log|B| \\
&\leq \frac{\log\left|\pi_{F_j}(X)\right|}{\left|F_j\right|} + \frac{k\left|\partial_R(F_j)\right|}{\left|F_j\right|} \log|B| \quad (by (5)).
\end{aligned}$$

Since

$$\lim_j \frac{\left|\partial_R\left(F_j\right)\right|}{\left|F_j\right|} = 0$$

by Proposition 11, we finally get

$$\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}(Y) = \limsup_j \frac{\log\left|\pi_{\varphi^{-1}(F_j)}(Y)\right|}{\left|\varphi^{-1}(F_j)\right|} \leq \limsup_j \frac{\log\left|\pi_{F_j}(X)\right|}{\left|F_j\right|} = \mathrm{ent}_{\mathcal{F}}(X).$$

□

The proposition in Reference 16 implies that the maximum entropy value for a subset $X \subset B^G$ is $\log|B|$. The subsequent result provides a condition on X that guarantees its entropy is strictly less than $\log|B|$.

Proposition 19 [4, Chapter 5] *If $X \subset B^G$, and there exist finite subsets E and E' of G, along with an $\left(E, E'\right)$-tiling $L \subset G$ such that $\pi_{lE}(X) \subsetneqq B^{lE}$ for all $l \in L$, then we have*

$$\mathrm{ent}_{\mathcal{F}}(X) < \log |B|.$$

Recall that G acts on the left on B^G by the shift $(g, x) \mapsto gx$ defined by $gx\left(g'\right) = x\left(g^{-1}g'\right)$ for $g, g' \in G$ and $x \in B^G$.

Corollary 20 [4, Chapter 5] *Let X be a G-invariant subset of B^G. Suppose that there exists a finite subset $E \subset G$ such that $\pi_E(X) \subsetneqq B^E$. Then one has* $\mathrm{ent}_{\mathcal{F}}(X) < \log |B|$.

6 Proof of The Garden of Eden Theorem on a φ-Cellular Automaton

The purpose of this section is to establish the following:

Theorem 21 *Let G be an amenable group, T be a group isomorphic to G, B be a finite set, $\mathcal{F} = \left(F_j\right)_{j \in J}$ be a right Følner net for G, and $\sigma \colon B^G \to B^T$ be a φ-cellular automaton with $\varphi : T \to G$ being an isomorphic map. Then the following are equivalent:*

(a) *σ is surjective;*
(b) $\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(\sigma\left(B^G\right)\right) = \log |B|$;
(c) *σ is pre-injective.*

We divide the proof of Theorem 21 into several lemmas. In these lemmas, it is assumed that the hypotheses of Theorem 21 are satisfied: G is an amenable group, T is a group isomorphic to G, B is a finite set, $\mathcal{F} = \left(F_j\right)_{j \in J}$ is a right Følner net for G, and $\sigma : B^G \to B^T$ is a φ-cellular automaton, where $\varphi : T \to G$ is a isomorphic map.

Lemma 22 *Suppose that σ is not surjective. Then one has* $\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(\sigma\left(B^G\right)\right) < \log |B|$.

Proof By Proposition 9, σ admits a Garden of Eden pattern. This means that there is a finite subset $E \subset T$ such that $\pi_E\left(\sigma\left(B^G\right)\right) \subsetneqq B^E$. The set $\sigma\left(B^G\right)$ is T-invariant since σ is φ-equivariant by Lemma 6. We deduce that $\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(\sigma\left(B^G\right)\right) < \log |B|$ by applying Corollary 20. □

Lemma 23 *Suppose that*

$$\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(\sigma\left(B^G\right)\right) < \log |B|.$$

Then σ is not pre-injective.

Proof Let R be a memory set for σ such that $1_G \in R$. Let $Y = \sigma\left(B^G\right)$. We have $F_j^{-R} \subset F_j \subset F_j^{+R}$ by Proposition 10 (iv) and therefore $F_j^{+R} \backslash F_j \subset \partial_R\left(F_j\right)$. As $\pi_{\varphi^{-1}(F_j^{+R})}(Y) \subset \pi_{\varphi^{-1}(F_j)}(Y) \times B^{\varphi^{-1}(F_j^{+R}) \backslash \varphi^{-1}(F_j)}$, it follows that

$$\begin{aligned}\log\left|\pi_{\varphi^{-1}(F_j^{+R})}(Y)\right| &\leq \log\left|\pi_{\varphi^{-1}(F_j)}(Y)\right| + \left|\varphi^{-1}(F_j^{+R}) \backslash \varphi^{-1}(F_j)\right| \log|B| \\ &\leq \log\left|\pi_{\varphi^{-1}(F_j)}(Y)\right| + \left|\varphi^{-1}(\partial_R\left(F_j\right))\right| \log|B|.\end{aligned}$$

Since φ is a isomorphic map, we have

$$\begin{aligned}\frac{\log\left|\pi_{\varphi^{-1}(F_j^{+R})}(Y)\right|}{\left|\varphi^{-1}(F_j)\right|} &\leq \frac{\log\left|\pi_{\varphi^{-1}(F_j)}(Y)\right|}{\left|\varphi^{-1}(F_j)\right|} + \frac{\left|\varphi^{-1}(\partial_R\left(F_j\right))\right|}{\left|\varphi^{-1}(F_j)\right|} \log|B| \\ &= \frac{\log\left|\pi_{\varphi^{-1}(F_j)}(Y)\right|}{\left|\varphi^{-1}(F_j)\right|} + \frac{\left|\partial_R\left(F_j\right)\right|}{\left|F_j\right|} \log|B|.\end{aligned} \tag{7}$$

As

$$\operatorname{ent}(Y) = \limsup_j \frac{\log\left|\pi_{\varphi^{-1}(F_j)}(Y)\right|}{\left|\varphi^{-1}(F_j)\right|} < \log|B|$$

by hypothesis, and

$$\lim_j \frac{\left|\partial_R\left(F_j\right)\right|}{\left|F_j\right|} = 0$$

by Proposition 11, we deduce from inequality (7) that there exists $j_0 \in J$ such that

$$\frac{\log\left|\pi_{\varphi^{-1}(F_{j_0}^{+R})}(Y)\right|}{\left|\varphi^{-1}(F_{j_0})\right|} < \log|B|. \tag{8}$$

Let us fix an arbitrary element $a_0 \in B$ and denote by Z the finite set of configurations $z \in B^G$ such that $z(g) = a_0$ for all $g \in G \backslash F_{j_0}$. Inequality (8) gives us

$$\left|\pi_{\varphi^{-1}(F_{j_0}^{+R})}(Y)\right| < |B|^{\left|\varphi^{-1}(F_{j_0})\right|} = |B|^{\left|F_{j_0}\right|} = |Z|.$$

Observe that $\sigma\,(z_1)$ and $\sigma\,(z_2)$ coincide outside $\varphi^{-1}(F_{j_0}^{+R})$ for all $z_1, z_2 \in Z$. Thus

$$|\sigma(Z)| = \left|\pi_{\varphi^{-1}(F_{j_0}^{+R})}(\sigma(Z))\right| \leq \left|\pi_{\varphi^{-1}(F_{j_0}^{+R})}(Y)\right| < |Z|.$$

This implies that we may find distinct configurations $z_1, z_2 \in Z$ such that $\sigma(z_1)$ and $\sigma(z_2)$. Since z_1 and z_2 coincide outside the finite set F_{j_0}, this shows that σ is not pre-injective. □

Lemma 24 *Suppose that σ is not pre-injective. Then one has*

$$\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(\sigma\left(B^G\right)\right) < \log|B|. \tag{9}$$

Proof Since σ is not pre-injective, we may find two configurations $x_1, x_2 \in B^G$ satisfying $\sigma(x_1) = \sigma(x_2)$ such that the set

$$\Omega = \{g \in G : x_1(g) \neq x_2(g)\}$$

is a nonempty finite subset of G. Observe that, for each $l \in G$, the configurations lx_1 and lx_2 satisfy $\sigma(lx_1) = \sigma(lx_2)$ (since σ is φ-equivariant by Lemma 6) and $\{g \in G : lx_1(g) \neq lx_2(g)\} = l\Omega$. Let R be a memory set for σ such that $1_G \in R$. Then the set

$$U = \left\{r^{-1}r' : r, r' \in R\right\}$$

is finite and we have $1_G \in U$. Let $E = \Omega^{+U}$. By Proposition 14, we may find a finite subset $E' \subset G$ and an $\left(E, E'\right)$-tiling $L \subset G$. Consider the subset $Z \subset B^G$ consisting of all configurations $z \in B^G$ such that

$$z|_{lE} \neq (lx_1)|_{lE} \quad \text{for all } l \in L.$$

Observe that, for each $l \in L$, we have

$$\pi_{lE}(Z) \subsetneqq B^{lE}$$

since $(lx_1)|_{lE} \notin \pi_{lE}(Z)$. We deduce that $\mathrm{ent}_{\varphi_{-1}(\mathcal{F})}(Z) < \log|B|$ by applying Proposition 19. As $\mathrm{ent}_{\varphi_{-1}(\mathcal{F})}(\sigma(Z)) \leq \mathrm{ent}_{\varphi_{-1}(\mathcal{F})}(Z)$ by Proposition 18, this implies

$$\mathrm{ent}_{\varphi_{-1}(\mathcal{F})}(\sigma(Z)) < \log|B|. \tag{10}$$

Thus, to establish inequality (9), it suffices to prove that $\sigma\left(B^G\right) = \sigma(Z)$. To see this, consider an arbitrary configuration $x \in B^G$ and let us show that there is a configuration $z \in Z$ such that $\sigma(x) = \sigma(z)$. Let

$$L' = \left\{l \in L : x|_{lE} = (lx_1)|_{lE}\right\}.$$

Let $z \in B^G$ be the configuration defined by

$$z(g) = \begin{cases} lx_2(g) & \text{if there is } l \in L' \text{ such that } g \in lE, \\ x(g) & \text{otherwise.} \end{cases}$$

Notice that the configuration z is obtained from x by modifying the values taken by x only on the subsets of the form $l\Omega$, where $l \in L'$ (since, as we have seen above, lx_1 and lx_2 coincide outside $l\Omega$). By construction, we have $z \in Z$. Let $t \in T$. Let us shows that $\sigma(x)(t) = \tau(z)(t)$. Suppose first that $\varphi(t)R$ does not meet any of the sets $l\Omega, l \in L'$. Then we have $z|_{\varphi(t)R} = x|_{\varphi(t)R}$. We deduce that $\sigma(z)(t) = \sigma(x)(t)$ by applying Lemma 1.

Suppose now that there is an element $l \in L'$ such that $\varphi(t)R$ meets $l\Omega$. This means that there exists an element $r_0 \in R$ such that $\varphi(t)r_0 \in l\Omega$. For each $r \in R$, we have $\varphi(t)rr^{-1}r_0 = \varphi(t)r_0 \in l\Omega$. As $r^{-1}r_0 \in U$, this implies

$$\begin{aligned} \varphi(t)r \in (l\Omega)^{+U} &= l\Omega^{+U} \quad \text{(by Proposition 10 (viii))} \\ &= lE. \end{aligned}$$

We deduce that $\varphi(t)R \subset lE$. Thus we have $\sigma(x)(t) = \sigma(lx_1)(t)$ since $x|_{lE} = lx_1|_{lE}$. Similarly, by applying Lemma 1, we get $\tau(z)(t) = \tau(lx_2)(t)$, since z and lx_2 coincide on lE. As $\sigma(lx_1) = \sigma(lx_2)$, we deduce that $\sigma(x)(t) = \sigma(z)(t)$.

Thus $\sigma(z) = \sigma(x)$. This shows that $\sigma\left(B^G\right) = \sigma(Z)$ and completes the proof of the lemma. □

Proof of Theorem 21 If σ is surjective, then $\sigma\left(B^G\right) = B^T$ and hence $\mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(\sigma\left(B^G\right)\right) = \mathrm{ent}_{\varphi^{-1}(\mathcal{F})}\left(B^T\right) = \log|B|$. Thus (a) implies (b). Since the converse implication follows from Lemma 22, we deduce that conditions (a) and (b) are equivalent. The fact that (c) implies (b) follows from Lemma 23 and the converse implication follows from Lemma 24. Thus, conditions (b) and (c) are also equivalent.

Proof of Theorem 2 If σ is surjective, we deduce that φ is injective by Lemma 8. Thus φ is isomorphic. Thus σ is surjective implies that σ is pre-injective by Theorem 21.

Proofs of Theorem 3 and Carollary 4 These results follows from Theorem 21 immediately.

Remark 2 $\varphi : T \to G$ is a finite-to-one surjective homomorphism in the case of Moore theorem over φ-cellular automata, whereas φ is an isomorphic mapping in the case of the Garden of Eden theorem over φ-cellular automata and Myhill theorem over φ-cellular automata. And the Myhill theorem over φ-cellular automata is invalid when φ is a finite-to-one surjective homomorphism.

Example 4 Let's consider $G = 2\mathbb{Z}$, $T = \mathbb{Z}$, and $B = \mathbb{Z}/2\mathbb{Z}$. Define $\varphi : T \to G$ as $\varphi(n) = 2n$ for all $n \in T$. We have seen in Example 3 that the φ-cellular automaton $\sigma : B^G \to B^T$ defined by $\sigma(x)(n) = x(2n) + x(2n+2)$ is pre-injective. Since $G = 2\mathbb{Z}$ is amenable,, it follows from the Garden of Eden theorem for a φ-cellular automaton that σ is surjective.

Example 5 Let's set $G = \mathbb{Z}$, $T = \{\bar{0}, \bar{1}\} \times \mathbb{Z}$, and $B = \{0, 1\}$. The map $\varphi : \{\bar{0}, \bar{1}\} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined as $\varphi(a, n) = n$ for all $(a, n) \in T$. Now, consider the φ-cellular automaton in Example 2, $\sigma : B^G \to B^T$ which is defined by

$$\sigma(x)(a, n) = x(n)$$

for all $x \in B^G$ and $(a, n) \in T$.

It is clear that φ is a finite-to-one surjective homomorhic map. We claim that σ is injective. Suppose $x_1, x_2 \in B^G$ are two configurations such that $\sigma(x_1) = \sigma(x_2)$. Then we have $x_1(n) = x_2(n)$ for all $n \in G$, which gives $x_1 = x_2$. Thus, σ is injective. And we can get that σ is pre-injective immediately. However, σ is not surjective, since the configurations $y \in B^T$ given by

$$y(t) = \begin{cases} 0 & \text{if } t = (\bar{0}, 0), \\ 1 & \text{otherwise} \end{cases}$$

is a Garden of Eden configuration for σ. This demonstrates that the Myhill theorem does not hold true for φ-cellular automata when φ represents a finite-to-one surjective homomorphism.

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