




Shadowable Points of Free Semigroup Actions

Ritong Li¹ · Dongkui Ma¹ · Rui Kuang¹ · Xiaojiang Ye¹ 

Received: 11 November 2023 / Revised: 16 May 2024 / Accepted: 21 May 2024 /
Published online: 30 May 2024

© Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2024

Abstract

The shadowable points of dynamical systems have been well-studied by Morales in his recent paper (2016). This paper aims to generalize the main results obtained by Morales to free semigroup actions. To this end, we introduce the notion of shadowable points of a free semigroup action. Let G be a free semigroup generated by finite continuous self-maps acting on compact metric space. We will prove the following results for G on compact metric spaces. The set of shadowable points of G is a Borel set. G has the pseudo-orbit tracing property (POTP) if and only if every point is shadowable point of G . The chain recurrent and non-wandering sets of G coincide when every chain recurrent point is shadowable point of G . The space X is totally disconnected at every shadowable point of G under certain condition.

Keywords Free semigroup actions · Shadowable point · Pseudo-orbit

Mathematics Subject Classification Primary: 37B05 · Secondary: 37B20

1 Introduction

The shadowing theory is an important component of the qualitative theory of dynamical systems, producing many interesting and profound results. If any pseudo orbit with sufficient accuracy approaches a certain precise trajectory, then the dynamical system

Communicated by Rosihan M. Ali.

✉ Xiaojiang Ye
yexiaojiang12@163.com

Ritong Li
summerleslieli@163.com

Dongkui Ma
dkma@scut.edu.cn

Rui Kuang
kuangrui@scut.edu.cn

¹ School of Mathematics, South China University of Technology, Guangzhou 510641, People's Republic of China

has the shadowing property or pseudo-orbit tracing property (POTP). So far, many researchers have studied homeomorphisms with shadowing property, for examples [1, 17]. Recently, Morales [15] introduced the concept of shadowable points by decomposing shadowing property into points and studied the properties of the shadowable points set. Subsequently Kawaguchi [11] extended the notion for continuous maps and gave a quantitative version of a Morales' result in [15]. In addition, Kawaguchi [11] also obtained more and better results. Afterwards, Aponte and Villavicencio [2] transferred the notion of shadowable points for homeomorphisms to the context of continuous time flows. Jung, Lee and Morales [10] introduced the concept of the persistent shadowing property, leading to the derivation of several compelling theorems. Besides the classical theory of dynamical systems (which studies actions of \mathbb{Z} and \mathbb{R}), shadowing for actions of all kinds of groups and semigroups were studied. For instance, Pilyugin and Tikhomirov [18] studied the shadowing property for action of the higher rank groups $\mathbb{Z}^p \times \mathbb{R}^q$. Afterwards, Osipov and Tikhomirov [16] introduced the notion of shadowing property for actions of finitely generated groups and study its basic properties. Very recently Kim and Lee [12] introduced the notion of shadowable points for finitely generated group actions on compact metric space and extended the theorems of [15] to finitely generated group actions. Jung, Lee and etc. [9] defined robust expansivity and the shadowing property for random group actions and proved the rigidity of actions enjoying these two properties. Barzanouni [5] introduced the notion of persistent shadowing property for finitely generated group actions and studied it via measure theory. Bahabadi [3] introduced the definitions of shadowing and average shadowing properties for iterated function systems, and also proved that an iterated function system has the shadowing property if and only if the skew-product transformation corresponding to the iterated function system has the shadowing property. Hui and Ma [8], Zhu and Ma [21] introduced the definitions of δ -pseudo-orbit for free semigroup actions and the pseudo-orbit tracing property for free semigroup actions, where they considered the δ -pseudo-orbit under free semigroup actions similar to a random route, which is different from the case of finitely generated group in [12]. What's more, Huang, Wang and Qiu [7] studied the relation between the two different definitions of shadowing property.

In this paper, we study the shadowing property for free semigroup actions based on [8] and [21] and introduce the notion of shadowable points of free semigroup actions. The main work is to extend Theorem 1.1 and Theorem 1.2 of [15] to free semigroup actions, and improve the proof process of Lemma 2.1, Lemma 2.3 and Theorem 1.2 in [15] using Lemma 2.2 in [11] and new conditions. By studying relationship between free semigroup actions and skew-product transformation corresponding to the free semigroup action, we find out relation between the shadowable points of free semigroup actions and the totally disconnected points. Let X be a compact metric space, f_0, \dots, f_{m-1} continuous self-maps on X . Denote G the free semigroup generated by $G_1 = \{f_0, \dots, f_{m-1}\}$ on X and $F : \Sigma_m^+ \times X \rightarrow \Sigma_m^+ \times X$ a skew-product transformation, where Σ_m^+ denotes the one-side symbol space. The main results of this paper are:

Theorem 1.1 *Let (X, d) be a compact metric space. If f_0, f_1, \dots, f_{m-1} are continuous self-maps on X and G is the free semigroup generated by $\{f_0, f_1, \dots, f_{m-1}\}$, then*

- (1) G has the POTP if and only if $Sh(G) = X$;
- (2) $Sh(G)$ is a Borel set;
- (3) if $CR(G) \subset Sh(G)$, then $CR(G) = \Omega(G)$,

where POTP is the abbreviation of pseudo-orbit tracing property, $Sh(G)$ denotes the set of shadowable points of G , $CR(G)$ denotes the set of chain recurrent points of G , and $\Omega(G)$ denotes the set of non-wandering points of G .

Theorem 1.2 *Let f_0, \dots, f_{m-1} be continuous self-maps of a compact metric space X , G the free semigroup generated by $\{f_0, \dots, f_{m-1}\}$ and F the skew-product transformation corresponding to $\{f_0, \dots, f_{m-1}\}$. If there exists $\delta > 0$ satisfying $B(\gamma(Sh(F)), \delta) \subset R(F)$, then $Sh(G) \subset X^{deg}$,*

where $\gamma(Sh(F))$ denotes the union of all connected components of X whose intersection with $Sh(F)$ is non-empty, X^{deg} denotes the set of all totally disconnected points of X , $B(\gamma(Sh(F)), \delta)$ denotes the δ neighborhood of set $\gamma(Sh(F))$, and $R(F)$ denotes the set of recurrent points of F .

This paper is organized as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we prove our main results. These results extend and improve the work of Morales [15].

2 Preliminaries

We first introduce some basic notions. Let (X, d) be a compact metric space and $f : X \rightarrow X$ be a continuous map. A sequence $\{x_i\}_{i=0}^\infty$ is called a δ -pseudo-orbit of f if for each $i \geq 0$,

$$d(f(x_i), x_{i+1}) \leq \delta.$$

The continuous map f is said to have the pseudo-orbit tracing property(POTP) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that every δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ is ε -shadowed by the orbit of some point $y \in X$, i.e. for all $i \geq 0$,

$$d(f^i(y), x_i) \leq \varepsilon.$$

We say that a point $x \in X$ is *shadowable* point of f if for every $\varepsilon > 0$, there is $\delta(x, \varepsilon) > 0$ such that every δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for f with $x_0 = x$ can be ε -shadowed. We denote by $Sh(f)$ the set of shadowable points of f . In [11], the author defined quantitative shadowable points. For $b > 0$, a point $x \in X$ is called *b-shadowable* point of f if there exists $\delta > 0$ for which every δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for f with $x_0 = x$ is b -shadowed by some point of X . Denote by $Sh_b^+(f)$ the set of b -shadowable points of f . Then, for $c \geq 0$, defined

$$Sh_{c+}^+(f) = \bigcap_{b>c} Sh_b^+(f).$$

Denote by F_m^+ the set of all finite words of symbols $0, 1, \dots, m - 1$. For any $w \in F_m^+, |w|$ stands for the length of w , that is, the number of symbols in w . We write $w \leq w'$ if there exists a word $w'' \in F_m^+$ such that $w' = w''w$.

Denote by Σ_m^+ the set of all one-side infinite sequences of symbols $0, 1, \dots, m - 1$, i.e.

$$\Sigma_m^+ = \{\omega = (w_0, w_1, \dots) \mid w_i \in \{0, 1, \dots, m - 1\} \text{ for all } i \geq 0\}.$$

Let $\omega \in \Sigma_m^+, w \in F_m^+, a, b \geq 0$, and $a \geq b$. We write $\omega|_{[a,b]} = w$ if $w = w_a w_{a+1} \dots w_{b-1} w_b$. For $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$, denote $(\omega)_i = w_i, i \geq 0$.

Let $w \in F_m^+, w = w_1 w_2 \dots w_k$, where $w_i \in \{0, 1, \dots, m - 1\}$ for all $i \in \{1, \dots, k\}$, and denote $\bar{w} = w_k w_{k-1} \dots w_1$. Let $f_w = f_{w_1} \circ f_{w_2} \circ \dots \circ f_{w_k}$. Obviously, $f_{ww'} = f_w \circ f_{w'}$, and $f_{\bar{w}} = f_{w_k} \circ f_{w_{k-1}} \circ \dots \circ f_{w_1}$.

In [3], the author introduced the shadowing property of the semigroup action generated by two continuous maps on compact metric space X . Here we can generalize it to general case. Let (X, d) be a compact metric space and f_0, f_1, \dots, f_{m-1} be continuous self maps on X . Denote G the free semigroup acting on the space X generated by $G_1 = \{f_0, \dots, f_{m-1}\}$. For $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$, an orbit of $x \in X$ under G for ω is a sequence $\{f_\omega^n(x)\}_{n=0}^\infty$, where

$$\begin{cases} f_\omega^n(x) = f_{w_{n-1}} \circ f_{w_{n-2}} \circ \dots \circ f_{w_1} \circ f_{w_0}(x). & n \geq 1 \\ f_\omega^0(x) = x. & n = 0 \end{cases}$$

The following definition of pseudo-orbit and pseudo-orbit tracing property for G are refer to [3] and [8].

Definition 2.1 ([8]) Given $\delta > 0$, a δ -pseudo-orbit(δ -chain) $\{x_i\}_{i=0}^\infty$ for G is defined as

$$\min\{d(f_0(x_i), x_{i+1}), \dots, d(f_{m-1}(x_i), x_{i+1})\} < \delta \text{ for every } i \geq 0. \tag{1}$$

Also, we can express the above expression as follows: there exists $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$ such that

$$d(f_{w_i}(x_i), x_{i+1}) < \delta \text{ for every } i \geq 0. \tag{2}$$

We say that a δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for G is ε -shadowed by a point $z \in X$ if for some ω satisfying (1) or (2), it holds that

$$d(f_\omega^n(z), x_n) < \varepsilon \text{ for all } n \geq 0. \tag{3}$$

Definition 2.2 ([8]) The free semigroup action G has the pseudo-orbit tracing property(POTP) if for any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that every δ -pseudo-orbit for G can be ε -shadowed by some point of X .

Remark 2.3 The strict inequality in (2) and (3) can be relaxed to an inequality without affecting the proof of the following lemmas and theorems except for Lemma 3.5. So the following default definitions are strict inequalities, except for Lemma 3.5.

Remark 2.4 If G has the POTP, then maps f_0, f_1, \dots, f_{m-1} all have POTP by Theorem 1.4 of [3], but the converse is not true. Please refer to Example 1.5 of [3] for specific example.

Example 2.5 ([3] Example 1.2) Define two continuous maps f_0, f_1 on Σ_2^+ as follows:

$$f_0(s_0s_1s_2 \dots) = 0s_0s_1s_2 \dots, f_1(s_0s_1s_2 \dots) = 1s_0s_1s_2 \dots.$$

Denote G the free semigroup generated by $G_1 = \{f_0, f_1\}$, then G has the POTP.

Next, we introduce two new definitions based on [15] in this paper.

Definition 2.6 A point $x \in X$ is called *shadowable* point of G if for every $\varepsilon > 0$, there is $\delta(x, \varepsilon) > 0$ such that every δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for G with $x_0 = x$ can be ε -shadowed. We denote by $Sh(G)$ the set of shadowable points of G .

Remark 2.7 Clearly, if G has the POTP, then $Sh(G) = X$ (i.e. every point is shadowable point of G). The converse is true on compact metric spaces by Theorem 1.1.

Let X be a compact metric space. We say that a sequence $\{x_i\}_{i=0}^\infty$ is *through* subset $K \subset X$ if $x_0 \in K$.

Definition 2.8 The free semigroup action G has the POTP *through* K if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ -pseudo-orbit for G *through* K can be ε -shadowed.

For G , we assign the following skew-product transformation $F : \Sigma_m^+ \times X \rightarrow \Sigma_m^+ \times X$ defined as

$$F(\omega, x) = (\sigma\omega, f_{w_0}(x)),$$

where $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$ and σ is the shift map. Here f_{w_0} stands for f_0 if $w_0 = 0$, and for f_1 if $w_0 = 1$, and so on. Let $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$, then

$$\begin{aligned} F^n(\omega, x) &= (\sigma^n\omega, f_{w_{n-1}} \circ f_{w_{n-2}} \circ \dots \circ f_{w_0}(x)) \\ &= (\sigma^n\omega, f_\omega^n(x)). \end{aligned}$$

The metric on Σ_m^+ is defined as

$$d_1(\omega, \omega') = \frac{1}{2^k},$$

where $k = \min\{j : w_j \neq w'_j\}$, $\omega = (w_0, w_1, \dots)$, $\omega' = (w'_0, w'_1, \dots) \in \Sigma_m^+$. A metric on $\Sigma_m^+ \times X$ is defined as follows:

$$D((\omega, x), (\omega', x')) = \max\{d_1(\omega, \omega'), d(x, x')\}$$

for $(\omega, x), (\omega', x') \in \Sigma_m^+ \times X$, where d_1 and d are metrics on Σ_m^+ and X respectively.

The following definitions refer to [19] and [21].

We say that a point $x \in X$ is a *non-wandering* point of G if for every neighbourhood U of x there is $\omega \in \Sigma_m^+$ and $k \in \mathbb{N}$ such that $(f_\omega^k)^{-1}(U) \cap U \neq \emptyset$.

For $w = w_0w_1 \cdots w_{n-1} \in F_m^+$, a (w, ε) -chain (or (w, ε) -pseudo-orbit) of G from x to y is a sequence $\{x_0 = x, x_1, \dots, x_n = y\}$ such that $d(f_{w_i}(x_i), x_{i+1}) < \varepsilon$ for $i \in \{0, \dots, n-1\}$. We say that x is a *chain recurrent* point of G if for every $\varepsilon > 0$, there is a (w, ε) -chain from x to itself for some $w \in F_m^+$.

A point $x \in X$ is called a *recurrent* point of G if there exist $\omega \in \Sigma_m^+$ and an increasing sequence $\{n_i\}_{i \geq 0}$ of positive integers such that $\lim_{i \rightarrow +\infty} f_\omega^{n_i}(x) = x$.

Denote by $\Omega(G)$, $CR(G)$ and $R(G)$ the set of non-wandering, chain recurrent and recurrent points of G , respectively. Clearly, we have $R(G) \subset \Omega(G)$, $\Omega(G) \subset CR(G)$.

On the other hand, the space X is *totally disconnected* at $p \in X$ if the connected component of X containing p is $\{p\}$. As in [4], we denote

$$X^{deg} = \{p \in X : X \text{ is totally disconnected at } p\}.$$

Recall that X is *totally disconnected* if it is totally disconnected at any point (i.e. $X = X^{deg}$). We denote $\gamma(A)$ is the union of all connected components of X whose intersection A is non-empty.

The following lemmas and theorems are all from references that need to be used in the proof process of this paper.

Lemma 2.9 ([11] Lemma 2.2) *Let $f : X \rightarrow X$ be a continuous map. If $x \in Sh_{c+}^+(f)$ with $c \geq 0$, then for every $b > c$, there exists $\delta = \delta(x, b) > 0$ such that every δ -pseudo orbit $\{x_i\}_{i=0}^\infty$ with $d(x, x_0) < \delta$ is b -shadowed by some point of X .*

Lemma 2.10 ([15] Lemma 2.1) *A homeomorphism of a compact metric space has the POTP through a subset K if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ -pseudo-orbit of f through K can be ε -shadowed.*

Lemma 2.11 ([15] Lemma 2.2) *Let f be a homeomorphism of a compact metric space. Then, for every $z \in \Omega(f) \cap Sh(f)$ and every $\varepsilon > 0$ there are $k \in \mathbb{N}^+$ and $y \in X$ such that $f^{pk}(y) \in B[z, \varepsilon]$ for every $p \in \mathbb{Z}$.*

Lemma 2.12 ([15] Lemma 2.6) *If $f : X \rightarrow X$ is a homeomorphism of a compact metric space X , then $Sh(f) = Sh(f^k)$ for every $k \in \mathbb{Z} \setminus \{0\}$.*

Theorem 2.13 ([15] Theorem 1.2) *If $f : X \rightarrow X$ is a pointwise-recurrent homeomorphism of a compact metric space X , then $Sh(f) \subset X^{deg}$.*

Lemma 2.14 ([6] Chapter IV Lemma 25) *If $f : X \rightarrow X$ is a homeomorphism of a compact metric space X , then for any positive integer m , we have $R(f) = R(f^m)$.*

3 The Proof of the Main Result

In this section, we give the proof of Theorem 1.1 and Theorem 1.2. Before our proof, we must give the following lemmas.

Lemma 3.1 *If $x \in Sh(G)$, then for every $\varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$, such that every δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for G with $d(x_0, x) < \delta$ is ε -shadowed by some point of X .*

Proof Since $x \in Sh(G)$, then take $\varepsilon > 0$. For $\frac{\varepsilon}{2} > 0$, there exists $0 < \delta_1(x, \varepsilon) < \varepsilon$, such that every δ_1 -pseudo-orbit $\{z_i\}_{i=0}^\infty$ for G with $z_0 = x$ can be $\frac{\varepsilon}{2}$ -shadowed. Take $\delta < \frac{\delta_1}{2}$ such that $d(a, b) < \delta$ implies $d(f_j(a), f_j(b)) < \frac{\delta_1}{2}$ for every $a, b \in X$, $j \in \{0, 1, \dots, m - 1\}$. Given δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ with $d(x_0, x) < \delta$. Define $\{y_i\}_{i=0}^\infty$ by $y_0 = x$ and $y_i = x_i$ for $i \geq 1$. Assume $d(f_{w_i}(x_i), x_{i+1}) < \delta$, $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$, then

$$d(f_{w_i}(y_i), y_{i+1}) = \begin{cases} d(f_{w_i}(x_i), x_{i+1}) & i \geq 1 \\ d(f_{w_0}(x), x_1) & i = 0 \end{cases}$$

and

$$d(f_{w_i}(x_i), x_{i+1}) < \delta < \delta_1, \\ d(f_{w_0}(x), x_1) \leq d(f_{w_0}(x), f_{w_0}(x_0)) + d(f_{w_0}(x_0), x_1) < \frac{\delta_1}{2} + \delta < \delta_1.$$

so $\{y_i\}_{i=0}^\infty$ is a δ_1 -pseudo-orbit for G with $y_0 = x$, then there exists $y \in X$, such that $d(f_\omega^n(y), y_n) < \frac{\varepsilon}{2}$ for every $n \geq 0$. Then, we have

$$d(y, x_0) \leq d(y, x) + d(x, x_0) < \frac{\varepsilon}{2} + \delta < \varepsilon,$$

and

$$d(f_\omega^n(y), x_n) = d(f_\omega^n(y), y_n) < \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geq 1$. Hence, y is a ε -shadowing point of $\{x_i\}_{i=0}^\infty$. □

Lemma 3.2 *If f_0, \dots, f_{m-1} are continuous self-maps of a compact metric space and G is the free semigroup generated by $\{f_0, \dots, f_{m-1}\}$, then G has the POTP through a compact subset K if and only if every point in K is a shadowable point of G .*

Proof By the previous remark we only have to prove the sufficiency. Now we use the finite cover theorem to prove the conclusion. Take $\varepsilon > 0$, $x \in K$. There exists $\delta_x > 0$ by Lemma 3.1. Consider the open cover $\{B(x, \delta_x) \mid x \in K\}$, obviously it is the open cover of K . Since K is compact, there exists a finite open cover $\{B(x_1, \delta_1), \dots, B(x_n, \delta_n)\}$ such that $K \subset \bigcup_{j=1}^n B(x_j, \delta_j)$. Let $\delta = \min_{1 \leq j \leq n} \{\delta_j\}$. Take δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for G with $x_0 \in K$, then there exists $1 \leq k \leq n$ such that $x_0 \in B(x_k, \delta_k)$. Obviously $\{x_i\}_{i=0}^\infty$ is also a δ_k -pseudo-orbit. Then $\{x_i\}_{i=0}^\infty$ can be ε -shadowed by Lemma 3.1. So G has the POTP through K . □

Lemma 3.3 *If f_0, \dots, f_{m-1} are continuous self-maps of a compact metric space and G is the free semigroup generated by $\{f_0, \dots, f_{m-1}\}$, then $CR(G) \cap Sh(G) \subset \Omega(G)$.*

Proof Take $x \in CR(G) \cap Sh(G)$. Since $x \in Sh(G)$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ -pseudo-orbit $\{z_i\}_{i=0}^\infty$ for G with $z_0 = x$ can be ε -shadowed. Since $x \in CR(G)$ and $\delta > 0$, there is a (w, δ) -chain from x to itself for some $w \in F_m^+$, assume the (w, δ) -chain is $\{x_0 = x, x_1, \dots, x_n = x\}$ with $d(f_{w_i}(x_i), x_{i+1}) < \delta$ for $0 \leq i \leq n - 1$ and $w = (w_0, w_1, \dots, w_{n-1})$. Define the sequence $\xi_{kn+r} = x_r$, where $0 \leq r \leq n - 1$ and $k \in \mathbb{N}$. Let $v = (w_0, w_1, \dots, w_{n-1})^\infty$. Then we obtain $d(f_{v_i}(\xi_i), \xi_{i+1}) < \delta$. So $\{\xi_i\}_{i=0}^\infty$ is a δ -pseudo-orbit for G with $\xi_0 = x_0 = x$, then there exists $y \in X$ such that $d(f_v^j(y), \xi_j) < \varepsilon$ for every $j \in \mathbb{N}$. Then we have

$$d(y, \xi_0) = d(y, x) < \varepsilon,$$

and

$$d(f_v^n(y), \xi_n) = d(f_v^n(y), x) < \varepsilon$$

that is $(f_v^n)^{-1}B(x, \varepsilon) \cap B(x, \varepsilon) \neq \emptyset$, so $x \in \Omega(G)$. We obtain $CR(G) \cap Sh(G) \subset \Omega(G)$ by the arbitrariness of x . □

Remark 3.4 From the proof process of Lemma 3.3, we can know that the point y satisfies $d(f_v^{kn}(y), x) < \varepsilon$ for every $k \in \mathbb{N}$, so the orbit of point y will periodically enter the neighbourhood of x .

Lemma 3.5 *Let f_0, \dots, f_{m-1} be continuous self-maps of a compact metric space and G the free semigroup generated by $\{f_0, \dots, f_{m-1}\}$, then $Sh(G)$ is a Borel set.*

Proof Given $\varepsilon > 0$, let $Sh_\varepsilon(G)$ be the set of points $x \in X$ satisfying for $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that every δ_ε -pseudo-orbit $\{z_i\}_{i=0}^\infty$ with $z_0 = x$ can be ε -shadowed. Also, given $\varepsilon > 0$ and $\delta > 0$, let $S_{\delta, \varepsilon}(G)$ be the set of points $x \in X$ satisfying for every pseudo-orbit $\{y_i\}_{i=0}^\infty$ with $y_0 = x$ and $d(f_{w_i}(y_i), y_{i+1}) < \delta$ for all $i \geq 0$ where $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$, there exists $y \in X$ such that $d(f_\omega^i(y), y_i) \leq \varepsilon$ for all $i \geq 0$. Clearly from the definition of $Sh(G)$ we easily have

$$Sh(G) = \bigcap_{k \in \mathbb{N}^+} Sh_{\frac{1}{k}}(G), \tag{4}$$

and for every $k \in \mathbb{N}^+$

$$Sh_{\frac{1}{k}}(G) = \bigcup_{m \in \mathbb{N}^+} S_{\frac{1}{m}, \frac{1}{k}}(G). \tag{5}$$

For any given $k, m \in \mathbb{N}^+$, we prove that $S_{\frac{1}{m}, \frac{1}{k}}(G)$ is a closed subset of X for all $m \in \mathbb{N}^+$. Assume $\{z_n\}_{n \in \mathbb{N}} \subset S_{\frac{1}{m}, \frac{1}{k}}(G)$ and $\lim_{n \rightarrow \infty} z_n = z$ for some $z \in X$. Next we show $z \in S_{\frac{1}{m}, \frac{1}{k}}(G)$. Given $\frac{1}{m}$ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for G with $x_0 = z$ and $d(f_{w_i}(x_i), x_{i+1}) < \frac{1}{m}$

where $\omega = (w_0, w_1, \dots) \in \Sigma_m^+$. For every $n \in \mathbb{N}$, consider sequence $\{x_i^{(n)}\}_{i=0}^\infty$ satisfying $x_0^{(n)} = z_n$ and $x_i^{(n)} = x_i$ for $i > 0$. Since $d(f_{w_0}(x_0), x_1) < \frac{1}{m}$, there exists $t > 0$ such that $d(f_{w_0}(x_0), x_1) < t < \frac{1}{m}$. We can choose $\delta > 0$ such that $d(a, b) < \delta$ implies $d(f_j(a), f_j(b)) < \frac{1}{m} - t$ for every $j \in \{0, 1, \dots, m - 1\}$ and every $a, b \in X$. Since $\lim_{n \rightarrow \infty} z_n = z$, we have $d(z_n, z) < \delta$ for sufficiently large $n \in \mathbb{N}$, so $d(f_{w_0}(z_n), f_{w_0}(z)) < \frac{1}{m} - t$. Then we have

$$\begin{aligned} d(f_{w_0}(x_0^{(n)}), x_1^{(n)}) &= d(f_{w_0}(z_n), x_1) \leq d(f_{w_0}(z_n), f_{w_0}(z)) + d(f_{w_0}(z), x_1) \\ &< \frac{1}{m} - t + t = \frac{1}{m}, \end{aligned}$$

and for $i > 0$

$$d(f_{w_i}(x_i^{(n)}), x_{i+1}^{(n)}) = d(f_{w_i}(x_i), x_{i+1}) < \frac{1}{m},$$

when n is large enough. For such $n \in \mathbb{N}$, the pseudo-orbit $\{x_i^{(n)}\}_{i=0}^\infty$ is a $\frac{1}{m}$ -pseudo-orbit for G with $x_0^{(n)} = z_n \in S_{\frac{1}{m}, \frac{1}{k}}(G)$ and $d(f_{w_i}(x_i^{(n)}), x_{i+1}^{(n)}) < \frac{1}{m}$, so there exists $y_n \in X$ such that $d(f_\omega^i(y_n), x_i^{(n)}) < \frac{1}{k}$ for every $i \geq 0$. Take a subsequence $\{y_{n_j}\}_{j \in \mathbb{N}}$ and assume $\lim_{j \rightarrow \infty} y_{n_j} = y$ for some $y \in X$. Then we have

$$d(y, x_0) = d(y, z) = \lim_{j \rightarrow \infty} d(z_{n_j}, y_{n_j}) = \lim_{j \rightarrow \infty} d(x_0^{(n_j)}, y_{n_j}) \leq \frac{1}{k},$$

and for every $i > 0$

$$d(f_\omega^i(y), x_i) = \lim_{j \rightarrow \infty} d(f_\omega^i(y_{n_j}), x_i^{(n_j)}) \leq \frac{1}{k}.$$

Hence $\{x_i\}_{i=0}^\infty$ can be $\frac{1}{k}$ -shadowed, that is $z \in S_{\frac{1}{m}, \frac{1}{k}}(G)$. From this we know that $S_{\frac{1}{m}, \frac{1}{k}}(G)$ is a closed subset of X , and therefore we obtain $Sh_{\frac{1}{k}}(G)$ is a Borel set in X for every $k \in \mathbb{N}^+$ by (5). Thus $Sh(G)$ is a Borel set in X by (4). □

Lemma 3.6 *If f_0, \dots, f_{m-1} are continuous self-maps of a compact metric space X and G is the free semigroup generated by $\{f_0, \dots, f_{m-1}\}$, F is the skew-product transformation corresponding to $\{f_0, \dots, f_{m-1}\}$ which satisfies $Sh(F) = (\Sigma_m^+ \times X)^{deg}$, then $Sh(G) = X^{deg}$.*

Proof Firstly we prove $Sh(G) \subset X^{deg}$. Take $x \in Sh(G)$ and $\varepsilon > 0$. There exists $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$ and $\delta' > 0$ such that every δ' -pseudo-orbit $\{z_i\}_{i=0}^\infty$ for G with $z_0 = x$ can be ε -shadowed. Let $\delta = \min\{\frac{1}{2^{k+1}}, \delta'\}$, $\omega \in \Sigma_m^+$. Now take a δ -pseudo-orbit $\{\xi_i\}_{i=0}^\infty$ of F with $\xi_i = (\omega^{(i)}, x_i)$ and $\xi_0 = (\omega, x)$, where $\omega^{(i)} = (w_0^{(i)}, w_1^{(i)}, \dots)$.

Therefore

$$D(F(\xi_i), \xi_{i+1}) = \max\{d_1(\sigma\omega^{(i)}, \omega^{(i+1)}), d(f_{w_0^{(i)}}(x_i), x_{i+1})\} < \delta \text{ for all } i \geq 0.$$

So we have

$$d_1(\sigma\omega^{(i)}, \omega^{(i+1)}) < \delta \leq \frac{1}{2^{k+1}}.$$

That is $w_{j+1}^{(i)} = w_j^{(i+1)}$ for every $0 \leq j \leq k + 1, i \geq 0$. Consider $v = (w_0^{(0)}, w_0^{(1)}, w_0^{(2)}, \dots)$. Since $d(f_{w_0^{(i)}}(x_i), x_{i+1}) < \delta < \delta'$ by $D(F(\xi_i), \xi_{i+1}) < \delta$, and $x_0 = x \in Sh(G)$, there exists $y \in X$, such that $d(f_v^n(y), x_n) < \varepsilon$ for all $n \geq 0$. In addition for this v , one can see that $d_1(\sigma^n(v), \omega^{(n)}) \leq \frac{1}{2^{k+3}} < \varepsilon$ for all $n \geq 0$. Therefore we have

$$D(F^n(v, y), \xi_n) < \varepsilon \text{ for all } n \geq 0.$$

Hence $(\omega, x) \in Sh(F) = (\Sigma_m^+ \times X)^{deg}$. Next, we will prove that $x \in X^{deg}$.

Suppose by contradiction that $x \notin X^{deg}$. Then the connected component E of X containing x (which is compact) has positive diameter $\text{diam}(E) > 0$. Obviously $\{\omega\} \times F$ is also a connected component of $\Sigma_m^+ \times X$ containing (ω, x) . This contradicts with $(\omega, x) \in (\Sigma_m^+ \times X)^{deg}$. Hence $Sh(G) \subset X^{deg}$.

On the other hand, for any $x \in X^{deg}$, we claim that for any $\omega \in \Sigma_m^+$ we have $(\omega, x) \in (\Sigma_m^+ \times X)^{deg}$. If not, there exists $v \in \Sigma_m^+$, such that $(v, x) \notin (\Sigma_m^+ \times X)^{deg}$. Then we obtain a connected component E of $\Sigma_m^+ \times X$ containing (v, x) and $E \setminus (v, x) \neq \emptyset$. We divide into two cases:

Case 1: If $E|_X = \{x\}$, we have $E = A \times \{x\}$ where $A \subset \Sigma_m^+$. Since Σ_m^+ is totally disconnected, there exists open sets $U_1, U_2 \subset \Sigma_m^+$ such that $(U_1 \cap A) \cap (U_2 \cap A) = \emptyset$ and $A \subset U_1 \cup U_2$, then we have $[(U_1 \times \{x\}) \cap E] \cap [(U_2 \times \{x\}) \cap E] = \emptyset$ and $E \subset (U_1 \times \{x\}) \cup (U_2 \times \{x\})$. According to the definition of connectivity we know that E is not connected. It's wrong.

Case 2: If $E|_X \setminus \{x\} \neq \emptyset$, $E|_X$ is not connected by $x \in X^{deg}$. There exists open sets $V_1, V_2 \subset X$ such that $(V_1 \cap E|_X) \cap (V_2 \cap E|_X) = \emptyset$ and $E|_X \subset V_1 \cup V_2$, then we have $[(\Sigma_m^+ \times V_1) \cap E] \cap [(\Sigma_m^+ \times V_2) \cap E] = \emptyset$ and $E \subset \Sigma_m^+ \times V_1 \cup \Sigma_m^+ \times V_2$. So E is not connected. This contradicts the fact that E is connected.

In all, we obtain $E = \{(v, x)\}$, therefore $(v, x) \in (\Sigma_m^+ \times X)^{deg}$. It is conflicts with $(v, x) \notin (\Sigma_m^+ \times X)^{deg}$, so we have $(\omega, x) \in (\Sigma_m^+ \times X)^{deg} = Sh(F)$ for every $\omega \in \Sigma_m^+$.

Next, we show $x \in Sh(G)$. Take $\omega \in \Sigma_m^+$ and $0 < \varepsilon < \frac{1}{2}$. we have $(\omega, x) \in Sh(F)$ according to the above proof. For $0 < \varepsilon < \frac{1}{2}$, there exists $\delta_\omega > 0$ by Lemma 2.9, such that

$$\Sigma_m^+ \subset \bigcup_{\omega \in \Sigma_m^+} B(\omega, \delta_\omega).$$

Actually Σ_m^+ is compact with respect to metric d_1 by Tychonoff theorem. So there exists finite open covers such that

$$\Sigma_m^+ \subset \bigcup_{i=1}^n B(\omega^{(i)}, \delta_i),$$

where $\omega^{(i)} \in \Sigma_m^+$ and $(\omega^{(i)}, x) \in Sh(F)$. Let $\delta = \min_{1 \leq i \leq n} \{\delta_i\}$, given δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ for G with $x_0 = x$ and $d(f_{v_i}(x_i), x_{i+1}) < \delta$ for all $i \geq 0$ where $v = (v_0, v_1, \dots) \in \Sigma_m^+$. Consider sequence $\{\xi_i\}_{i=0}^\infty$ with $\xi_i = (\sigma^i v, x_i)$, $\xi_0 = (v, x)$. Obviously $\{\xi_i\}_{i=0}^\infty$ is a δ -pseudo-orbit of F . Since $v \in \Sigma_m^+$, there exists $1 \leq k \leq n$, such that $v \in B(\omega^{(k)}, \delta_k)$. Also we have

$$D(\xi_0, (\omega^{(k)}, x)) = \max\{d(x, x), d_1(v, \omega^{(k)})\} < \delta_k.$$

For above ε and Lemma 2.9, we know $\{\xi_i\}_{i=0}^\infty$ can be ε -shadowed. There exists $(u, y) \in \Sigma_m^+ \times X$ such that

$$D(F^i(u, y), \xi_i) < \varepsilon,$$

that is $d_1(\sigma^i u, \sigma^i v) < \varepsilon$ and $d(f_u^i(y), x_i) < \varepsilon$ for every $i \geq 0$. Since $d_1(\sigma^i u, \sigma^i v) < \varepsilon < \frac{1}{2}$ we obtain $u = v$. Hence $d(f_u^i(y), x_i) = d(f_v^i(y), x_i) < \varepsilon$ for all $i \geq 0$, that is $x \in Sh(G)$. Therefore $X^{deg} \subset Sh(G)$. \square

The following lemma is an improvement of Theorem 2.13 and with some modifications on the proof process of Theorem 2.13.

Lemma 3.7 *Let $f : X \rightarrow X$ be a continuous map of a compact metric space X . If there exists $\delta > 0$ satisfying $B(\gamma(Sh(f)), \delta) \subset R(f)$, then $Sh(f) \subset X^{deg}$.*

Proof Take $z \in Sh(f)$ and suppose by contradiction that $z \notin X^{deg}$. Then the connected component $\gamma(\{z\})$, denoted by E , is compact and $\text{diam}(E) > 0$. Take $0 < \varepsilon < \min\{\frac{1}{11}\text{diam}(E), \delta\}$. Obviously $z \in B(\gamma(Sh(f)), \delta)$, so $z \in R(f) \subset \Omega(f)$. There exists $k \in \mathbb{N}^+$ and $y \in X$ such that $f^{nk}(y) \in B(z, \varepsilon)$ for every $n \geq 0$ by Lemma 2.11. Define $g = f^k$. Then

$$g^n(y) \in B(z, \varepsilon) \text{ for all } n \geq 0. \tag{6}$$

On the other hand $z \in Sh(g)$ by Lemma 2.12. Then for above ε , there exists $\delta' > 0$ by Lemma 3.1. We can assume $\delta' < \varepsilon$. Since E is compact and connected, we can choose a sequence $y = p_1, p_2, \dots, p_N \in E$ such that $d(p_i, p_{i+1}) \leq \frac{\delta'}{2}$ for $1 \leq i \leq N - 1$ and

$$E \subset \bigcup_{i=1}^N B(p_i, \delta'). \tag{7}$$

Also $R(g) = R(f^k) = R(f)$ by Lemma 2.14, and $p_i \in E \subset B(\gamma(Sh(f)), \delta) \subset R(f) = R(g)$. From this we can find positive integers $c(i)$ ($1 \leq i \leq N$) such that $d(p_i, g^{c(i)}(p_i)) \leq \frac{\delta'}{2}$ for all $1 \leq i \leq N$. Since $z \in E \subset \bigcup_{i=1}^N B(p_i, \delta')$, there exists p_{i_z} ($1 \leq i_z \leq N$) such that $z \in B(p_{i_z}, \delta')$. We define the sequence $\{\xi_i\}_{i=0}^\infty$ as follow

$$\begin{aligned} \xi_i &= g^i(p_{i_z}) && \text{if } 0 \leq i \leq c(i_z) - 1, \\ \xi_{c(i_z)+i} &= g^i(p_{i_z+1}) && \text{if } 0 \leq i \leq c(i_z + 1) - 1, \\ &\vdots \\ \xi_{c(i_z)+\dots+c(N-2)+i} &= g^i(p_{N-1}) && \text{if } 0 \leq i \leq c(N - 1) - 1, \\ \xi_{c(i_z)+\dots+c(N-1)+i} &= g^i(p_N) && \text{if } 0 \leq i \leq c(N) - 1, \\ \xi_{c(i_z)+\dots+c(N)+i} &= g^i(p_{N-1}) && \text{if } 0 \leq i \leq c(N - 1) - 1, \\ &\vdots \\ \xi_{c(1)+\dots+c(i_z-1)+2\{c(i_z)+\dots+c(N-1)\}+c(N)+i} &= g^i(p_1) && \text{if } i \geq 0. \end{aligned}$$

Obviously $\{\xi_i\}_{i=0}^\infty$ is a δ' -pseudo-orbit of g with $\xi_0 = p_{i_z} \in B(z, \delta')$. Then there exists $x \in X$ such that $d(g^n(x), \xi_n) < \varepsilon$ for every $n \geq 0$ by Lemma 2.10. From the definition of $\{\xi_i\}_{i=0}^\infty$ we conclude that there exists integers n_1, \dots, n_N satisfying $d(g^{n_i}(x), p_i) < \varepsilon$ for every $1 \leq i \leq N$. Take $c = c(1) + \dots + c(i_z - 1) + 2\{c(i_z) + \dots + c(N - 1)\} + c(N)$, we obtain

$$d(g^{i+c}(x), g^i(y)) < \varepsilon \quad (\text{for } i \geq 0).$$

This combined with (6) yields

$$g^i(x) \in B(z, 2\varepsilon) \text{ whenever } i \geq c.$$

Since $d(g^{n_i}(x), p_i) < \varepsilon \leq \delta$ and $p_i \in E$ for every $1 \leq i \leq N$, we have $g^{n_i}(x) \in B(\gamma(sh(f)), \delta) \subset R(g)$. Thus for $g^{n_i}(x)$, there exists $k_i \geq c$ such that $d(g^{k_i}(x), g^{n_i}(x)) < \varepsilon$. Now take $w \in E$. It follows from (7) that $d(w, p_i) < \delta'$ for some $1 \leq i \leq N$. Then $d(g^{n_i}(x), w) \leq d(g^{n_i}(x), p_i) + d(p_i, w) < \varepsilon + \delta' < 2\varepsilon$. Now we have two cases:

Case 1: If $0 \leq n_i < c$, then

$$\begin{aligned} d(w, z) &\leq d(w, g^{n_i}(x)) + d(g^{n_i}(x), g^{k_i}(x)) + d(g^{k_i}(x), g^{k_i-c}(p_1)) + d(g^{k_i-c}(p_1), z) \\ &< 2\varepsilon + \varepsilon + \varepsilon + \varepsilon \\ &= 5\varepsilon. \end{aligned}$$

Case 2: If $n_i \geq c$, then

$$d(w, z) \leq d(w, g^{n_i}(x)) + d(g^{n_i}(x), g^{n_i-c}(p_1)) + d(g^{n_i-c}(p_1), z)$$

$$\begin{aligned} &< 72\varepsilon + \varepsilon + \varepsilon \\ &= 4\varepsilon. \end{aligned}$$

We conclude that $E \subset B(z, 5\varepsilon)$ by the arbitrariness of w and adove cases, so $\text{diam}(E) < 10\varepsilon$. This contradicts the choice of ε therefore $z \in X^{deg}$. As $z \in Sh(f)$ is arbitrary, we obtain $Sh(f) \subset X^{deg}$. □

Remark 3.8 Since F is not pointwise recurrent, we need to change the condition to Lemma 3.7, which only focuses on the local properties of the shadowable points, and we can prove that if f is a pointwise recurrent continuous map, then there exists $\delta > 0$ such that $B(\gamma(Sh(f)), \delta) \subset R(f)$. But the inverse may not necessarily hold true. For example, let $a, b, c, p \in \mathbb{R}$ satisfying $a \neq p$ and $a, p \notin [b, c]$. Consider compact metric space $([b, c] \cup \{a\} \cup \{p\}, d)$, where d is Euclidean metric. Define map:

$$f(x) = \begin{cases} p & x = p \\ b & x = a \\ x & x \in [b, c] \end{cases}$$

Then we have $Sh(f) = \{p\}$ by Theorem 2.13, so it holds for $B(\gamma(Sh(f)), \delta) \subset R(f)$, but f is not recurrent by $f^n a = b$ for every $n \geq 1$.

Proof of Theorem 1.1 Let f_0, f_1, \dots, f_{m-1} be continuous self-maps on compact metric space X and G the free semigroup generated by $\{f_0, f_1, \dots, f_{m-1}\}$. We have that $Sh(G)$ is a Borel set by Lemma 3.5. Take $K = X$ in Lemma 3.2, then we obtain that G has the POTP if and only if $Sh(G) = X$. Finally, since $\Omega(G) \subset CR(G)$ we have that if $CR(G) \subset Sh(G)$, then $CR(G) = \Omega(G)$ by Lemma 3.3. □

Proof of Theorem 1.2 Let f_0, f_1, \dots, f_{m-1} be continuous self-maps on compact metric space X , G the free semigroup generated by $\{f_0, f_1, \dots, f_{m-1}\}$ and F the skew-product transformation corresponding to $\{f_0, \dots, f_{m-1}\}$. If there exists $\delta > 0$ such that $B(\gamma(Sh(F)), \delta) \subset R(F)$, we obtain $Sh(F) \subset (\Sigma_m^+ \times X)^{deg}$ by Lemma 3.7. From the proof process of Lemma 3.6, it can be seen that $Sh(G) \subset X^{deg}$. □

The following example will state the rationality of the condition in Theorem 1.2.

Example 3.9 Let $X = \mathbb{S}^1$ is one-dimensional torus, f_0, f_1 are all irrational rotation with irrational independence. Assume $f_0(x) = x + \alpha_0 \pmod{1}$ and $f_1(x) = x + \alpha_1 \pmod{1}$ with $\alpha_0 + \alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$. The periodic point $\omega = (0, 1)^\infty$ is a minimal point in the dynamical system (Σ_2^+, σ) . For minimal sub-system $(\overline{Orb(\omega, \sigma)}, \sigma)$, define skew-product transformation $F : \overline{Orb(\omega, \sigma)} \times \mathbb{S}^1 \rightarrow \overline{Orb(\omega, \sigma)} \times \mathbb{S}^1$ corresponding to $\{f_0, f_1\}$. There exists $\delta > 0$ such that $B(\gamma(Sh(F)), \delta) \subset R(F)$ according to the following proof.

Clearly $\overline{Orb(\omega, \sigma)} = \{\omega, \sigma(\omega)\}$, then we prove that $R(F) = \overline{Orb(\omega, \sigma)} \times \mathbb{S}^1$. Take $x \in \mathbb{S}^1$. For $(\omega, x) \in \overline{Orb(\omega, \sigma)} \times \mathbb{S}^1$, define $g_0(x) = f_1 \circ f_0(x) = x + \alpha_0 + \alpha_1 \pmod{1}$. Since $\alpha_0 + \alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$, we obtain g_0 is minimal, that is $R(g_0) = \mathbb{S}^1$. For $x \in R(g_0)$, there exists an increasing sequence $\{n_k\}_{k \geq 0}$ of positive integers such that

$\lim_{k \rightarrow \infty} g_0^{n_k}(x) = x$, then we have

$$\lim_{k \rightarrow \infty} F^{2n_k}(\omega, x) = \lim_{k \rightarrow \infty} (\sigma^{2n_k}(\omega), f_\omega^{2n_k}(x)) = \lim_{k \rightarrow \infty} (\omega, g_0^{n_k}(x)) = (\omega, x),$$

that is $(\omega, x) \in R(F)$. For $(\sigma(\omega), x)$, define $g_1(x) = f_0 \circ f_1(x) = x + \alpha_1 + \alpha_1 \pmod{1}$. Similarly we have $R(g_1) = \mathbb{S}^1$. For $x \in R(g_1)$, there exists an increasing sequence $\{n_i\}_{i \geq 0}$ of positive integers such that $\lim_{i \rightarrow \infty} g_1^{n_i}(x) = x$, then we have

$$\lim_{i \rightarrow \infty} F^{2n_i}(\sigma(\omega), x) = \lim_{i \rightarrow \infty} (\sigma^{2n_i+1}(\omega), f_{\sigma(\omega)}^{2n_i}(x)) = \lim_{i \rightarrow \infty} (\sigma(\omega), g_1^{n_i}(x)) = (\sigma(\omega), x),$$

that is $(\sigma(\omega), x) \in R(F)$. In all we have $R(F) = \overline{Orb(\omega, \sigma)} \times \mathbb{S}^1$, so there exists $\delta > 0$ such that $B(\gamma(Sh(F)), \delta) \subset R(F)$.

Acknowledgements The authors really appreciate the referees' valuable remarks and suggestions that help a lot.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled, "Shadowable points of free semigroup actions".

References

1. Aoki, N., Hiraide, K.: Topological Theory of Dynamical Systems: Recent Advances, vol. 52. North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam (1994)
2. Aponte, J., Villavicencio, H.: Shadowable points for flows. *J. Dyn. Control Syst.* **24**(4), 701–719 (2018)
3. Bahabadi, A.: Shadowing and average shadowing properties for iterated function systems. *Georgian Math. J.* **22**(2), 179–184 (2015)
4. Balibrea, F., Downarowicz, T., Hric, R., Snoha, L., Špitalský, V.: Almost totally disconnected minimal systems. *Ergod. Theory Dyn. Syst.* **29**(3), 737–766 (2009)
5. Barzanouni, A.: Persistent shadowing for actions of some finitely generated groups and related measures (2023). [arXiv:2301.12384](https://arxiv.org/abs/2301.12384)
6. Block, L.S., Coppel, W.A.: Dynamics in One Dimension. Lecture Notes in Mathematics, vol. 1513. Springer, Berlin (1992)
7. Huang, X., Wang, X., Qiu, L.: Shadowing property of hyperspace for free semigroup actions. *J. Dyn. Control Syst.* **29**(2), 501–519 (2023)
8. Hui, H., Ma, D.: Some dynamical properties for free semigroup actions. *Stoch. Dyn.* **18**(4), 1850032 (2018)
9. Jung, W., Lee, K., Morales, C., Oh, J.: Rigidity of random group actions. *Discret. Contin. Dyn. Syst.* **40**(12), 6845–6854 (2020)
10. Jung, W., Lee, K., Morales, C.A.: Pointwise persistence and shadowing. *Monatshefte Math.* **192**(1), 111–123 (2020)
11. Kawaguchi, N.: Quantitative shadowable points. *Dyn. Syst.* **32**(4), 504–518 (2017)

12. Kim, S.J., Lee, K.: Shadowable points for finitely generated group actions. *J. Chungcheong Math. Soc.* **31**(4), 411–420 (2018)
13. Mai, J.: Pointwise-recurrent graph maps. *Ergod. Theory Dyn. Syst.* **25**(2), 629–637 (2005)
14. Moothathu, T.K.S.: Implications of pseudo-orbit tracing property for continuous maps on compacta. *Topol. Appl.* **158**(16), 2232–2239 (2011)
15. Morales, C.A.: Shadowable points. *Dyn. Syst.* **31**(3), 347–356 (2016)
16. Osipov, A.V., Tikhomirov, S.B.: Shadowing for actions of some finitely generated groups. *Dyn. Syst.* **29**(3), 337–351 (2014)
17. Pilyugin, S.Y.: *Shadowing in Dynamical Systems*. Lecture Notes in Mathematics, vol. 1706. Springer, Berlin (1999)
18. Pilyugin, S.Y., Tikhomirov, S.B.: Shadowing in actions of some abelian groups. *Fund. Math.* **179**(1), 83–96 (2003)
19. Tang, Y., Ye, X., Ma, D.: Chain recurrence rates and topological entropy of free semigroup actions. *Taiwan. J. Math.* **28**(1), 139–161 (2024)
20. Walters, P.: *An Introduction to Ergodic Theory*. Springer, New York (1982)
21. Zhu, L., Ma, D.: The upper capacity topological entropy of free semigroup actions for certain non-compact sets. *J. Stat. Phys.* **182**(1), 19, 22 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.