Bull. Malays. Math. Sci. Soc. (2024) 47:129
https://doi.org/10.1007/s40840-024-01717-0

BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

A class of special formal triangular matrix rings

Lixin Mao[1]

Received: 11 August 2023 / Revised: 19 May 2024 / Accepted: 21 May 2024 /
Published online: 18 June 2024

Abstract

Let $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ be a formal triangular matrix ring with C a semidualizing (S, R)-bimodule. It is proven that (1) A left S-module M in Bass class is C-torsionless (resp. C-reflexive) if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a torsionless (resp. reflexive) left T-module; (2) A left S-module M in Bass class is C-Gorenstein projective if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a Gorenstein projective left T-module; (3) If C is a faithfully semidualizing (S, R)-bimodule, then a left S-module M is C-n-tilting if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}$ is an n-tilting left T-module.

Keywords Formal triangular matrix ring · Semidualizing module · C-torsionless module · C-Gorenstein projective module · C-n-tilting module

Mathematics Subject Classification 16D40 · 16D50 · 16E30

1 Introduction

Let R and S be rings and U an (S, R)-bimodule. $T = \begin{pmatrix} R & 0 \\ U & S \end{pmatrix}$ is known as a *formal (or generalized) triangular matrix ring* with usual matrix addition and multiplication. Formal triangular matrix rings play an important role in ring theory and the representation theory of algebra [3]. This kind of rings are noncommutative and are often used to construct examples and counterexamples. As a consequence of the classical results by Green [13], the module category over the formal triangular matrix ring T can be constructed from the categories of modules over R and S. So one can describe classes

Communicated by Shiping Liu.

✉ Lixin Mao
maolx2@hotmail.com

1 School of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China

of modules over T from the corresponding classes of modules over R and S. Thus the properties of formal triangular matrix rings and modules over them make the theory of rings and modules more abundant and concrete, and have deserved more and more interest (see [1, 3, 7, 9, 13–15, 19, 30, 31]).

On the other hand, semidualizing modules were studied independently by Foxby [10], Golod [12] and Vasconcelos [28] over a commutative Noetherian ring. Later, Holm and White [18] extended the definition of semidualizing modules to general associative rings. Recall that an (S, R)-bimodule ${}_SC_R$ is *semidualizing* [18] if

(1) ${}_SC$ admits a projective resolution $\cdots \to P_1 \to P_0 \to C \to 0$ such that each P_i is a finitely generated projective left S-module;
(2) C_R admits a projective resolution $\cdots \to Q_1 \to Q_0 \to C \to 0$ such that each Q_i is a finitely generated projective right R-module;
(3) The homothety map ${}_SS_S \to \mathrm{Hom}_R(C, C)$ is an isomorphism;
(4) The homothety map ${}_RR_R \to \mathrm{Hom}_S(C, C)$ is an isomorphism;
(5) $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$;
(6) $\mathrm{Ext}^i_S(C, C) = 0$ for all $i \geq 1$.

Examples of semidualizing modules can be found in [6, 18]. One basic subject on semidualizing modules is to extend the classical results in homological algebra to the relative setting with respect to a semidualizing module C. For example, C-torsionless (C-reflexive) modules, C-Gorenstein projective modules and C-tilting modules have been introduced and studied in [4, 17, 20, 25, 26, 29].

It is natural to ask what special properties the formal triangular matrix ring $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ admits when C is a semidualizing (S, R)-bimodule. In the present article, we will exhibit the connections between C-torsionless (resp. C-Gorenstein projective, C-tilting) left S-modules and torsionless (resp. Gorenstein projective, tilting) left T-modules. For example, we prove that (1) A left S-module M in the Bass class $\mathcal{B}_C(S)$ is C-torsionless (resp. C-reflexive) if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a torsionless (resp. reflexive) left T-module (see Theorem 2.1); (2) A left S-module M in the Bass class $\mathcal{B}_C(S)$ is C-Gorenstein projective if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a Gorenstein projective left T-module (see Theorem 2.3); (3) If C is a faithfully semidualizing (S, R)-bimodule, then a left S-module M is C-n-tilting if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}$ is an n-tilting left T-module (see Theorem 2.8).

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a module M is denoted by M^+. $pd(M)$ and $id(M)$ denote the projective and injective dimensions of a module M respectively. $\mathrm{Add}(M)$ (resp. $\mathrm{Prod}(M)$) denotes the class of modules isomorphic to direct summands of direct sums (resp. direct products) of copies of M.

Let $T = \begin{pmatrix} R & 0 \\ U & S \end{pmatrix}$ be a formal triangular matrix ring with R and S rings and U an (S, R)-bimodule. By [13, Theorem 1.5], the category T-Mod of left T-modules is equivalent to the category Ω whose objects are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1 \in R$-Mod, $M_2 \in S$-Mod and $\varphi^M : U \otimes_R M_1 \to M_2$ is an S-morphism, and whose morphisms from $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_R(M_1, N_1)$, $f_2 \in \mathrm{Hom}_S(M_2, N_2)$ and $\varphi^N(1 \otimes f_1) = f_2\varphi^M$. The regular module ${}_T T$ corresponds to $\begin{pmatrix} R \\ U \oplus S \end{pmatrix}_{\varphi^T}$, where $\varphi^T : U \otimes_R R \to U \oplus S$ is given by $\varphi^T(u \otimes r) = (ur, 0)$ for $u \in U, r \in R$. Note that a sequence $0 \to \begin{pmatrix} M_1' \\ M_2' \end{pmatrix}_{\varphi^{M'}} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} M_1'' \\ M_2'' \end{pmatrix}_{\varphi^{M''}} \to 0$ of left T-modules is exact if and only if both sequences $0 \to M_1' \to M_1 \to M_1'' \to 0$ and $0 \to M_2' \to M_2 \to M_2'' \to 0$ are exact. Analogously, the category Mod-T of right T-modules is equivalent to the category Γ whose objects are triples $W = (W_1, W_2)_{\varphi_W}$, where $W_1 \in$ Mod-R, $W_2 \in$ Mod-S and $\varphi_W : W_2 \otimes_S U \to W_1$ is an R-morphism, and whose morphisms from $(W_1, W_2)_{\varphi_W}$ to $(X_1, X_2)_{\varphi_X}$ are pairs (g_1, g_2) such that $g_1 \in \mathrm{Hom}_R(W_1, X_1)$, $g_2 \in \mathrm{Hom}_S(W_2, X_2)$ and $\varphi_X(g_2 \otimes 1) = g_1\varphi_W$. In the rest of the paper, we shall identify T-Mod (resp. Mod-T) with this category Ω (resp. Γ) and, whenever there is no possible confusion, we shall omit φ^M (resp. φ_W). For example, for the left T-module $\begin{pmatrix} M_1 \\ (U \otimes_R M_1) \oplus M_2 \end{pmatrix}$, the S-morphism $U \otimes_R M_1 \to (U \otimes_R M_1) \oplus M_2$ is just the injection and for the left T-module $\begin{pmatrix} M_1 \oplus \mathrm{Hom}_S(U, M_2) \\ M_2 \end{pmatrix}$, the R-morphism $M_1 \oplus \mathrm{Hom}_S(U, M_2) \to \mathrm{Hom}_S(U, M_2)$ is just the projection.

2 Modules over a special formal triangular matrix ring

A semidualizing (S, R)-bimodule C defines two important classes of modules. Following [18], the *Auslander class* with respect to C, denoted by $\mathcal{A}_C(R)$, consists of all left R-modules N satisfying

(1) $\mathrm{Tor}_i^R(C, N) = 0 = \mathrm{Ext}_S^i(C, C \otimes_R N)$ for all $i \geq 1$;
(2) The natural evaluation map $\mu_N : N \to \mathrm{Hom}_S(C, C \otimes_R N)$ is an isomorphism.

Dually, the *Bass class* with respect to C, denoted by $\mathcal{B}_C(S)$, consists of all left S-modules M satisfying

(1) $\mathrm{Ext}_S^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_S(C, M))$ for all $i \geq 1$;
(2) The natural evaluation map $\nu_M : C \otimes_R \mathrm{Hom}_S(C, M) \to M$ is an isomorphism.

Let C be a semidualizing (S, R)-bimodule. Recall that a left S-module M is C-*torsionless* (resp. C-*reflexive*) if the biduality map $\delta : M \to \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)$, defined as $\delta(x)(f) = f(x)$ for each $x \in M$ and $f \in \mathrm{Hom}_S(M, C)$, is a monomorphism (resp. isomorphism). If $C = R = S$, then a C-torsionless (resp. C-reflexive) module is exactly a *torsionless* (resp. *reflexive*) module.

We first exhibit the relation between C-torsionless (resp. C-reflexive) left S-modules and torsionless (resp. reflexive) left T-modules.

Theorem 2.1 *Let* $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ *be a formal triangular matrix ring with* C *a semidualizing* (S, R)*-bimodule and* $M \in \mathcal{B}_C(S)$*. Then*

(1) M *is a* C*-torsionless left* S*-module if and only if* $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ *is a torsionless left* T*-module.*

(2) M *is a* C*-reflexive left* S*-module if and only if* $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ *is a reflexive left* T*-module.*

Proof By [18, Theorem 6.4], we have

$$\mathrm{Hom}_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, C)) \cong \mathrm{Hom}_S(M, C).$$

So [22, Corollary 2.3] implies that

$$\begin{aligned} &\mathrm{Hom}_T\left(\mathrm{Hom}_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}, T\right), T\right) \\ &\cong \mathrm{Hom}_T((\mathrm{Hom}_R(\mathrm{Hom}_S(C, M), R), 0), T) \\ &\cong \mathrm{Hom}_T((\mathrm{Hom}_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, C)), 0), T) \\ &\cong \mathrm{Hom}_T((\mathrm{Hom}_S(M, C), 0), T) \\ &\cong \begin{pmatrix} \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), R) \\ \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C) \end{pmatrix}. \end{aligned}$$

(1) " $\Rightarrow$ " There is an exact sequence of left S-modules

$$0 \to M \to \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C),$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_S(C, M) \to \mathrm{Hom}_S(C, \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)).$$

Note that $\mathrm{Hom}_S(C, \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), \mathrm{Hom}_S(C, C))$

$$\cong \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), R).$$

Thus $\mathrm{Hom}_S(C, M) \to \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), R)$ is a monomorphism. Consequently, $\delta : \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix} \to \mathrm{Hom}_T\left(\mathrm{Hom}_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}, T\right), T\right)$ is a monomorphism. So $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a torsionless left T-module.

"⇐" Since $\delta : \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix} \to \mathrm{Hom}_T\left(\mathrm{Hom}_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}, T\right), T\right)$ is a monomorphism, $M \to \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)$ is a monomorphism. So M is a C-torsionless left S-module.

(2) "⇒" Since $M \cong \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)$, we get the isomorphisms

$$\mathrm{Hom}_S(C, M) \cong \mathrm{Hom}_S(C, \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), R).$$

Hence $\delta : \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix} \to \mathrm{Hom}_T\left(\mathrm{Hom}_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}, T\right), T\right)$ is an isomorphism. Thus $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a reflexive left T-module.

"⇐" Since $\delta : \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix} \to \mathrm{Hom}_T\left(\mathrm{Hom}_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}, T\right), T\right)$ is an isomorphism, $M \to \mathrm{Hom}_R(\mathrm{Hom}_S(M, C), C)$ is an isomorphism. So M is a C-reflexive left S-module. □

Let C be a semidualizing (S, R)-bimodule. According to [18], a left S-module is called *C-projective* (resp. *C-flat*) if it has the form $C \otimes_R M$ for some projective (resp. flat) left R-module M. A left R-module is called *C-injective* if it has the form $\mathrm{Hom}_S(C, Y)$ for some injective left S-module Y.

Proposition 2.2 *Let* $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ *be a formal triangular matrix ring with* C *a semidualizing* (S, R)*-bimodule,* $M \in \mathcal{B}_C(S)$ *and* $N \in \mathcal{A}_C(R)$*. Then*

(1) M *is a* C*-projective left* S*-module if and only if* $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ *is a projective left* T*-module.*

(2) M *is a* C*-flat left* S*-module if and only if* $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ *is a flat left* T*-module.*

(3) N *is a* C*-injective left* R*-module if and only if* $\begin{pmatrix} N \\ C \otimes_R N \end{pmatrix}$ *is an injective left* T*-module.*

Proof (1) follows from [15, Theorem 3.1]. (2) holds by [9, Proposition 1.14].
(3) follows from [14, Proposition 5.1]. □

The origin of Gorenstein homological algebra may date back to 1960s when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring [2]. In 1990s, Enochs and Jenda extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein projective and injective modules over arbitrary rings [8]. The notions of Gorenstein projective and injective modules with respect to a semidualizing module were first introduced by Holm and Jørgensen in [17] and White in [29] for commutative rings. The noncommutative versions of them were given by Liu et al. [20].

Let C be a semidualizing (S, R)-bimodule. Recall that a left S-module M is *C-Gorenstein projective* [20] if there exists an exact sequence of left S-modules

$$\cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

such that all P_i are projective left S-modules and P^i are projective left R-modules, $M \cong \mathrm{coker}(P_1 \to P_0)$ and $\mathrm{Hom}_S(-, C \otimes_R X)$ leaves the sequence exact whenever X is a projective left R-module.

A left R-module N is called *C-Gorenstein injective* [20] if there exists an exact sequence of left R-modules

$$\cdots \to \mathrm{Hom}_S(C, E_1) \to \mathrm{Hom}_S(C, E_0) \to E^0 \to E^1 \to \cdots$$

such that all E_i are injective left S-modules and E^i are injective left R-modules, $N \cong \ker(E^0 \to E^1)$ and $\mathrm{Hom}_R(\mathrm{Hom}_S(C, Y), -)$ leaves the sequence exact whenever Y is an injective left S-module.

If $C = R = S$ in the above definitions, then we recover the categories of ordinary Gorenstein projective and injective modules in the sense of [8].

Next we give the Gorenstein version of Proposition 2.2.

Theorem 2.3 *Let* $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ *be a formal triangular matrix ring with C a semidualizing (S, R)-bimodule. If $M \in \mathcal{B}_C(S)$ and $N \in \mathcal{A}_C(R)$, then*

(1) M is a C-Gorenstein projective left S-module if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ *is a Gorenstein projective left T-module.*

(2) N is a C-Gorenstein injective left R-module if and only if $\begin{pmatrix} N \\ C \otimes_R N \end{pmatrix}$ *is a Gorenstein injective left T-module.*

Proof (1) "⇒" There exists an exact sequence of left S-modules

$$\Theta : 0 \to M \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

such that all P^i are projective left R-modules and the sequence $\mathrm{Hom}_S(\Theta, C \otimes_R X)$ is exact whenever X is a projective left R-module.

Since $M \in \mathcal{B}_C(S)$, $\mathrm{Ext}^i_S(C, M) = 0$ for all $i \geq 1$. Since $P^i \in \mathcal{A}_C(R)$ by [18, Lemma 4.1], $\mathrm{Hom}_S(C, C \otimes_R P^i) \cong P^i$ and $\mathrm{Ext}^i_S(C, C \otimes_R P^i) = 0$ for all $i \geq 1$. So we get the exact sequence of left T-modules

$$\Upsilon : 0 \to \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix} \to \begin{pmatrix} P^0 \\ C \otimes_R P^0 \end{pmatrix} \to \begin{pmatrix} P^1 \\ C \otimes_R P^1 \end{pmatrix} \to \cdots .$$

On the one hand, for any projective left T-module $\begin{pmatrix} A_1 \\ (C \otimes_R A_1) \oplus A_2 \end{pmatrix}$, where A_1 is a projective left R-module and A_2 is a projective left S-module, by [18, Theorem 6.4], we have

$$\begin{aligned} &\mathrm{Hom}_T\left(\begin{pmatrix} P^i \\ C \otimes_R P^i \end{pmatrix}, \begin{pmatrix} A_1 \\ (C \otimes_R A_1) \oplus A_2 \end{pmatrix}\right) \cong \mathrm{Hom}_R(P^i, A_1) \\ &\cong \mathrm{Hom}_S(C \otimes_R P^i, C \otimes_R A_1), \end{aligned}$$

$$\mathrm{Hom}_T(\begin{pmatrix}\mathrm{Hom}_S(C,M)\\ M\end{pmatrix}, \begin{pmatrix}A_1\\ (C\otimes_R A_1)\oplus A_2\end{pmatrix}) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(C,M),A_1)$$
$$\cong \mathrm{Hom}_S(M, C\otimes_R A_1).$$

So the sequence $\mathrm{Hom}_T(\Upsilon, \begin{pmatrix}A_1\\ (C\otimes_R A_1)\oplus A_2\end{pmatrix}) \cong \mathrm{Hom}_S(\Theta, C\otimes_R A_1)$ is exact.

On the other hand, by [21, Lemma 3.2], for any $i \geq 1$, we have

$$\mathrm{Ext}^i_T\left(\begin{pmatrix}\mathrm{Hom}_S(C,M)\\ M\end{pmatrix}, \begin{pmatrix}A_1\\ (C\otimes_R A_1)\oplus A_2\end{pmatrix}\right) \cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(C,M),A_1)$$
$$\cong \mathrm{Ext}^i_S(M, C\otimes_R A_1) = 0.$$

It follows that $\begin{pmatrix}\mathrm{Hom}_S(C,M)\\ M\end{pmatrix}$ is a Gorenstein projective left T-module by [16, Proposition 2.3].

"$\Leftarrow$" By [15, Theorem 3.1], there exists an exact sequence of left T-modules

$$\Delta : \cdots \to \begin{pmatrix}P_0\\ (C\otimes_R P_0)\oplus Q_0\end{pmatrix} \to \begin{pmatrix}P^0\\ (C\otimes_R P^0)\oplus Q^0\end{pmatrix} \xrightarrow{\begin{pmatrix}f^0\\ g^0\end{pmatrix}}$$
$$\begin{pmatrix}P^1\\ (C\otimes_R P^1)\oplus Q^1\end{pmatrix} \to \cdots$$

such that $\begin{pmatrix}\mathrm{Hom}_S(C,M)\\ M\end{pmatrix} \cong \ker\begin{pmatrix}f^0\\ g^0\end{pmatrix}$, all P_i and P^i are projective left R-modules, Q_i and Q^i are projective left S-modules and the sequence $\mathrm{Hom}_T(\Delta, W)$ is exact whenever W is a projective left T-module.

Therefore we get the exact sequence of left S-modules

$$\Lambda : \cdots \to (C\otimes_R P_0)\oplus Q_0 \to (C\otimes_R P^0)\oplus Q^0 \xrightarrow{g^0} (C\otimes_R P^1)\oplus Q^1 \to \cdots$$

such that $M \cong \ker(g^0)$. For any projective left R-module X, $\begin{pmatrix}X\\ C\otimes_R X\end{pmatrix}$ is a projective left T-module. By [18, Theorem 6.4], the sequence $\mathrm{Hom}_S(\Lambda, C\otimes_R X) \cong \mathrm{Hom}_T(\Delta, \begin{pmatrix}X\\ C\otimes_R X\end{pmatrix})$ is exact. Thus M is a C-Gorenstein projective left S-module by [20, Theorem 2.9].

(2) " $\Rightarrow$ " There exists an exact sequence of left R-modules

$$\Psi : \cdots \to \mathrm{Hom}_S(C,E_2) \to \mathrm{Hom}_S(C,E_1) \to \mathrm{Hom}_S(C,E_0) \to N \to 0$$

such that all E_i are injective left S-modules and the sequence $\mathrm{Hom}_R(\mathrm{Hom}_S(C,Y),\Psi)$ is exact whenever Y is an injective left S-module.

Since $N \in \mathcal{A}_C(R)$, $\mathrm{Tor}_i^R(C, N) = 0$ for all $i \geq 1$. Since $E_i \in \mathcal{B}_C(S)$ by [18, Lemma 4.1], $C \otimes_R \mathrm{Hom}_S(C, E_i) \cong E_i$ and $\mathrm{Tor}_i^R(C, \mathrm{Hom}_S(C, E_i)) = 0$ for all $i \geq 1$. So we get the exact sequence of left T-modules

$$\Sigma : \cdots \to \begin{pmatrix} \mathrm{Hom}_S(C, E_2) \\ E_2 \end{pmatrix} \to \begin{pmatrix} \mathrm{Hom}_S(C, E_1) \\ E_1 \end{pmatrix} \to \begin{pmatrix} \mathrm{Hom}_S(C, E_0) \\ E_0 \end{pmatrix} \to \begin{pmatrix} N \\ C \otimes_R N \end{pmatrix} \to 0.$$

On the one hand, for any injective left T-module $\begin{pmatrix} B_1 \oplus \mathrm{Hom}_S(C, B_2) \\ B_2 \end{pmatrix}$, where B_1 is an injective left R-module and B_2 is an injective left S-module, by [18, Theorem 6.4], we have

$$\begin{aligned} &\mathrm{Hom}_T\left(\begin{pmatrix} B_1 \oplus \mathrm{Hom}_S(C, B_2) \\ B_2 \end{pmatrix}, \begin{pmatrix} \mathrm{Hom}_S(C, E_i) \\ E_i \end{pmatrix}\right) \cong \mathrm{Hom}_S(B_2, E_i) \\ &\quad \cong \mathrm{Hom}_R(\mathrm{Hom}_S(C, B_2), \mathrm{Hom}_S(C, E_i)), \\ &\mathrm{Hom}_T\left(\begin{pmatrix} B_1 \oplus \mathrm{Hom}_S(C, B_2) \\ B_2 \end{pmatrix}, \begin{pmatrix} N \\ C \otimes_R N \end{pmatrix}\right) \cong \mathrm{Hom}_S(B_2, C \otimes_R N) \\ &\quad \cong \mathrm{Hom}_R(\mathrm{Hom}_S(C, B_2), N). \end{aligned}$$

Therefore the sequence $\mathrm{Hom}_T(\begin{pmatrix} B_1 \oplus \mathrm{Hom}_S(C, B_2) \\ B_2 \end{pmatrix}, \Sigma) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(C, B_2), \Psi)$ is exact.

On the other hand, by [21, Lemma 3.2], for any $i \geq 1$, we have

$$\begin{aligned} &\mathrm{Ext}_T^i\left(\begin{pmatrix} B_1 \oplus \mathrm{Hom}_S(C, B_2) \\ B_2 \end{pmatrix}, \begin{pmatrix} N \\ C \otimes_R N \end{pmatrix}\right) \cong \mathrm{Ext}_S^i(B_2, C \otimes_R N) \\ &\quad \cong \mathrm{Ext}_R^i(\mathrm{Hom}_S(C, B_2), N) = 0. \end{aligned}$$

Therefore $\begin{pmatrix} N \\ C \otimes_R N \end{pmatrix}$ is a Gorenstein injective left T-module.

"⇐" By [14, Proposition 5.1], there exists an exact sequence of left T-modules

$$\Xi : \cdots \to \begin{pmatrix} G_0 \oplus \mathrm{Hom}_S(C, H_0) \\ H_0 \end{pmatrix} \to \begin{pmatrix} G^0 \oplus \mathrm{Hom}_S(C, H^0) \\ H^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} f^0 \\ g^0 \end{pmatrix}} \begin{pmatrix} G^1 \oplus \mathrm{Hom}_S(C, H^1) \\ H^1 \end{pmatrix} \to \cdots$$

such that $\begin{pmatrix} N \\ C \otimes_R N \end{pmatrix} \cong \ker\begin{pmatrix} f^0 \\ g^0 \end{pmatrix}$, all G_i and G^i are injective left R-modules, H_i and H^i are injective left S-modules and the sequence $\mathrm{Hom}_T(K, \Xi)$ is exact whenever

K is an injective left T-module. So we get the exact sequence of left R-modules

$$\Phi : \cdots \to G_0 \oplus \mathrm{Hom}_S(C, H_0) \to G^0 \oplus \mathrm{Hom}_S(C, H^0) \xrightarrow{f^0} G^1 \oplus \mathrm{Hom}_S(C, H^1) \to \cdots$$

such that $N \cong \ker(f^0)$. For any injective left S-module Y, $\begin{pmatrix} \mathrm{Hom}_S(C, Y) \\ Y \end{pmatrix}$ is an injective left T-module. By [18, Theorem 6.4], the sequence

$$\mathrm{Hom}_R(\mathrm{Hom}_S(C, Y), \Phi) \cong \mathrm{Hom}_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, Y) \\ Y \end{pmatrix}, \Xi\right)$$

is exact. Thus N is a C-Gorenstein injective left R-module by the dual of [20, Theorem 2.9]. □

Let C be a semidualizing (S, R)-bimodule. A left S-module M is said to be *C-Gorenstein flat* if there exists an exact sequence of left S-modules

$$\cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

such that all F_i are flat left S-modules and F^i are flat left R-modules, $M \cong \mathrm{coker}(F_1 \to F_0)$ and $\mathrm{Hom}_R(C, Y) \otimes_S -$ leaves the sequence exact whenever Y is an injective right R-module.

Recall that a semidualizing (S, R)-bimodule C is *faithfully semidualizing* [18] if it satisfies the following conditions for all modules ${}_SM$ and N_R: (1) If $\mathrm{Hom}_S(C, M) = 0$, then $M = 0$; (2) If $\mathrm{Hom}_R(C, N) = 0$, then $N = 0$.

Lemma 2.4 *Let R be a right coherent ring and C a faithfully semidualizing (S, R)-bimodule. Then a left S-module M is C-Gorenstein flat if and only if M^+ is C-Gorenstein injective.*

Proof "⇒" There exists an exact sequence of left S-modules

$$\cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

such that all F_i are flat left S-modules and F^i are flat left R-modules, $M \cong \mathrm{coker}(F_1 \to F_0)$ and $\mathrm{Hom}_R(C, Y) \otimes_S -$ leaves the sequence exact whenever Y is an injective right R-module. Then we get the exact sequence

$$\cdots \to (C \otimes_R F^1)^+ \to (C \otimes_R F^0)^+ \to F_0^+ \to F_1^+ \to \cdots \quad (*)$$

such that $M^+ \cong \ker(F_0^+ \to F_1^+)$ and $\mathrm{Hom}_S(\mathrm{Hom}_R(C, Y), -)$ leaves the sequence exact. Since $(*)$ is equivalent to the exact sequence

$$\cdots \to \mathrm{Hom}_R(C, (F^1)^+) \to \mathrm{Hom}_R(C, (F^0)^+) \to F_0^+ \to F_1^+ \to \cdots$$

and all F_i^+ and $(F^i)^+$ are injective, M^+ is C-Gorenstein injective.

"$\Leftarrow$" For any injective right R-module Y and any $i \geq 1$,

$$\mathrm{Tor}_i^S(\mathrm{Hom}_R(C, Y), M)^+ \cong \mathrm{Ext}_S^i(\mathrm{Hom}_R(C, Y), M^+) = 0.$$

Thus $\mathrm{Tor}_i^S(\mathrm{Hom}_R(C, Y), M) = 0$.

There exists an epimorphism $\mathrm{Hom}_R(C, E_0) \to M^+$ with E_0 an injective right R-module, which yields the monomorphism $M^{++} \to \mathrm{Hom}_R(C, E_0)^+ \cong C \otimes_R E_0^+$ by [24, Lemma 3.60]. Since there is a monomorphism $M \to M^{++}$, one gets a monomorphism $M \to C \otimes_R E_0^+$. Since E_0^+ is flat by [5, Theorem 1], M has a monic C-flat preenvelope $M \to C \otimes_R F^0$ with F^0 a flat left R-module by [18, Proposition 5.3(d)]. Thus we get the exact sequence

$$0 \to M \to C \otimes_R F^0 \to L^1 \to 0.$$

For any injective right R-module Q, Q^+ is flat by [5, Theorem 1]. So we get the exact sequence

$$0 \to \mathrm{Hom}_S(L^1, C \otimes_R Q^+) \to \mathrm{Hom}_S(C \otimes_R F^0, C \otimes_R Q^+) \to \mathrm{Hom}_S(M, C \otimes_R Q^+) \to 0.$$

Since $\mathrm{Hom}_R(C, Q)^+ \cong C \otimes_R Q^+$, one gets the exact sequence

$$0 \to \mathrm{Hom}_R(C, Q) \otimes_S M \to \mathrm{Hom}_R(C, Q) \otimes_S (C \otimes_R F^0) \to \mathrm{Hom}_R(C, Q) \otimes_S L^1 \to 0.$$

[18, Theorem 6.4(c)] implies that

$$\mathrm{Tor}_1^S(\mathrm{Hom}_R(C, Q), C \otimes_R F^0) \cong \mathrm{Tor}_1^R(Q, F^0) = 0.$$

Thus we have $\mathrm{Tor}_1^S(\mathrm{Hom}_R(C, Q), L^1) = 0$. Note that

$$\mathrm{Ext}_S^1(\mathrm{Hom}_R(C, Q), (L^1)^+) \cong \mathrm{Tor}_1^S(\mathrm{Hom}_R(C, Q), L^1)^+ = 0.$$

Since $(C \otimes_R F^0)^+$ and M^+ are C-Gorenstein injective, the induced exact sequence

$$0 \to (L^1)^+ \to (C \otimes_R F^0)^+ \to M^+ \to 0$$

implies that $(L^1)^+$ is C-Gorenstein injective from the dual version of [29, Corollary 3.8]. Thus the above proof gives rise to the exact sequence

$$0 \to L^1 \to C \otimes_R F^1 \to L^2 \to 0$$

such that $\mathrm{Hom}_R(C, Q) \otimes_S -$ leaves it exact, F^1 is flat and $(L^2)^+$ is C-Gorenstein injective. Continuing this process yields the exact sequence

$$0 \to M \to C \otimes_R F^0 \to C \otimes_R F^1 \to C \otimes_R F^2 \to \cdots$$

such that $\mathrm{Hom}_R(C, Q) \otimes_S -$ leaves it exact and each F^i is flat.

It follows that M is C-Gorenstein flat. □

Theorem 2.5 *Let $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ be a formal triangular matrix ring with R and S right coherent rings and C a faithfully semidualizing (S, R)-bimodule. If $M \in \mathcal{B}_C(S)$, then M is a C-Gorenstein flat left S-module if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a Gorenstein flat left T-module.*

Proof Since R and S are right coherent rings and C_R is finitely presented, T is a right coherent ring by [9, Corollary 2.3]. So M is a C-Gorenstein flat left S-module if and only if M^+ is a C-Gorenstein injective right S-module by Lemma 2.4 if and only if $(M^+ \otimes_S C, M^+)$ is a Gorenstein injective right T-module by Theorem 2.3 if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ M \end{pmatrix}$ is a Gorenstein flat left T-module by [16, Theorem 3.6]. □

The following lemma is a noncommutative version of [27, Theorem 2.8].

Lemma 2.6 *Let C be a faithfully semidualizing (S, R)-bimodule. Then*

(1) $M \in \mathcal{B}_C(S)$ if and only if $\mathrm{Hom}_S(C, M) \in \mathcal{A}_C(R)$.
(2) $N \in \mathcal{A}_C(R)$ if and only if $C \otimes_R N \in \mathcal{B}_C(S)$.

Proof (1) "⇒" It follows from [18, Proposition 4.1].

"⇐" Since $\mathrm{Hom}_S(C, M) \in \mathcal{A}_C(R)$, $\mu_{\mathrm{Hom}_S(C,M)} : \mathrm{Hom}_S(C, M) \to \mathrm{Hom}_S(C, C \otimes_R \mathrm{Hom}_S(C, M))$ is an isomorphism. Note that

$$\mathrm{Hom}_S(C, \nu_M)\mu_{\mathrm{Hom}_S(C,M)} = id_{\mathrm{Hom}_S(C,M)}.$$

So $\mathrm{Hom}_S(C, \nu_M)$ is an isomorphism. The exact sequence

$$0 \to \ker(\nu_M) \to C \otimes_R \mathrm{Hom}_S(C, M) \stackrel{\nu_M}{\to} M$$

induces the exact sequence

$$0 \to \mathrm{Hom}_S(C, \ker(\nu_M)) \to \mathrm{Hom}_S(C, C \otimes_R \mathrm{Hom}_S(C, M)) \stackrel{\mathrm{Hom}_S(C,\nu_M)}{\to} \mathrm{Hom}_S(C, M).$$

Thus $\mathrm{Hom}_S(C, \ker(\nu_M)) = 0$. Since C a faithfully semidualizing (S, R)-bimodule, $\ker(\nu_M) = 0$. So we get the exact sequence

$$0 \to C \otimes_R \mathrm{Hom}_S(C, M) \stackrel{\nu_M}{\to} M \to \mathrm{coker}(\nu_M) \to 0,$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_S(C, C \otimes_R \mathrm{Hom}_S(C, M)) \stackrel{\mathrm{Hom}_S(C,\nu_M)}{\to} \mathrm{Hom}_S(C, M) \to$$

$$\mathrm{Hom}_S(C, \mathrm{coker}(\nu_M)) \to 0.$$

Thus $\mathrm{Hom}_S(C, \mathrm{coker}(\nu_M)) = 0$. So $\mathrm{coker}(\nu_M) = 0$. Hence ν_M is an isomorphism. Therefore $M \in \mathcal{B}_C(S)$.

The proof of (2) is similar. □

Recall that a left R-module M is *n-tilting* [11] if $pd(M) \leq n$, $\mathrm{Ext}^i_R(M, M^{(\kappa)}) = 0$ for any $i \geq 1$ and any cardinal κ, there exist an integer $r \geq 0$ and an exact sequence of left R-modules $0 \to {}_RR \to X^0 \to X^1 \to \cdots \to X^r \to 0$ with each $X^i \in \mathrm{Add}(M)$. Dually we have the concept of an *n-cotilting module*.

Recently, the notion of tilting modules with respect to a semidualizing module over a commutative ring was introduced by Salimi in [25]. Here we extend it to the noncommutative setting.

Definition 2.7 Let C be a semidualizing (S, R)-bimodule.

A left S-module M is called *C-n-tilting* if it satisfies the following conditions:

(1) $pd_R(\mathrm{Hom}_S(C, M)) \leq n$;
(2) $\mathrm{Ext}^i_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, M)^{(\kappa)}) = 0$ for any $i \geq 1$ and any cardinal κ;
(3) There exist an integer $r \geq 0$ and an exact sequence of left S-modules $0 \to {}_SC \to X^0 \to X^1 \to \cdots \to X^r \to 0$ with each $X^i \in \mathrm{Add}(M)$.

Dually, a left R-module N is called *C-n-cotilting* if it satisfies the following conditions:

(1) $id_S(C \otimes_R N) \leq n$;
(2) $\mathrm{Ext}^i_S((C \otimes_R N)^\kappa, C \otimes_R N) = 0$ for any $i \geq 1$ and any cardinal κ;
(3) There exist an integer $r \geq 0$ and an exact sequence of left R-modules $0 \to X_r \to \cdots \to X_1 \to X_0 \to C^+ \to 0$ with each $X_i \in \mathrm{Prod}(N)$.

The following theorem establishes the connection between C-tilting left S-modules (resp. C-cotilting left R-modules) and tilting (resp. cotilting) left T-modules.

Theorem 2.8 *Let* $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ *be a formal triangular matrix ring with C a faithfully semidualizing* (S, R)*-bimodule. Then*

(1) M is a C-n-tilting left S-module if and only if $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}$ *is an n-tilting left T-module.*
(2) N is a C-n-cotilting left R-module if and only if $\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}$ *is an n-cotilting left T-module.*

Proof (1) "⇒" Since $pd_R(\mathrm{Hom}_S(C, M)) \leq n$, $\mathrm{Hom}_S(C, M) \in \mathcal{A}_C(R)$ by [18, Corollary 6.2]. So $M \in \mathcal{B}_C(S)$ by Lemma 2.6.

Note that $pd_T\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix} = pd_R(\mathrm{Hom}_S(C, M)) \leq n$ by [23, Theorem 2.4].

By [21, Lemma 3.2], for any $i \geq 1$ and any cardinal κ, we have

$$\mathrm{Ext}^i_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}, \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}^{(\kappa)}\right)$$

$$\cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, M)^{(\kappa)}) = 0.$$

There exist an integer $r \geq 0$ and an exact sequence of left S-modules

$$0 \to C \to X^0 \to X^1 \to \cdots \to X^r \to 0$$

with each $X^i \in \mathrm{Add}(M)$. Since $\mathrm{Ext}^i_S(C, M) = 0 = \mathrm{Ext}^i_S(C, C)$, we get the exact sequence of left R-modules

$$0 \to \mathrm{Hom}_S(C, C) \to \mathrm{Hom}_S(C, X^0) \to \mathrm{Hom}_S(C, X^1) \to \cdots \to \mathrm{Hom}_S(C, X^r) \to 0.$$

Thus there is an exact sequence of left T-modules

$$0 \to \begin{pmatrix} R \\ S \oplus C \end{pmatrix} \to \begin{pmatrix} \mathrm{Hom}_S(C, X^0) \\ S \oplus X^0 \end{pmatrix} \to \begin{pmatrix} \mathrm{Hom}_S(C, X^1) \\ X^1 \end{pmatrix} \to \cdots \to \begin{pmatrix} \mathrm{Hom}_S(C, X^r) \\ X^r \end{pmatrix} \to 0$$

with $\begin{pmatrix} \mathrm{Hom}_S(C, X^0) \\ S \oplus X^0 \end{pmatrix}, \begin{pmatrix} \mathrm{Hom}_S(C, X^i) \\ X^i \end{pmatrix} \in \mathrm{Add}\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}$, $1 \leq i \leq r$.

It follows that $\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}$ is an n-tilting left T-module.

"$\Leftarrow$" Note that $pd_R(\mathrm{Hom}_S(C, M)) \leq pd_T\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix} \leq n$. Thus $\mathrm{Hom}_S(C, M) \in \mathcal{A}_C(R)$ by [18, Corollary 6.2]. So $M \in \mathcal{B}_C(S)$ by Lemma 2.6.

By [21, Lemma 3.2], for any $i \geq 1$ and any cardinal κ, we have

$$\begin{aligned} &\mathrm{Ext}^i_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, M)^{(\kappa)}) \\ &\cong \mathrm{Ext}^i_T\left(\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}, \begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}^{(\kappa)}\right) = 0. \end{aligned}$$

There exist an integer $r \geq 0$ and an exact sequence of left T-modules

$$0 \to \begin{pmatrix} R \\ S \oplus C \end{pmatrix} \to \begin{pmatrix} X_1^0 \\ X_2^0 \end{pmatrix}_{\varphi^0} \to \begin{pmatrix} X_1^1 \\ X_2^1 \end{pmatrix}_{\varphi^1} \to \cdots \to \begin{pmatrix} X_1^r \\ X_2^r \end{pmatrix}_{\varphi^r} \to 0$$

with each $\begin{pmatrix} X_1^i \\ X_2^i \end{pmatrix}_{\varphi^i} \in \mathrm{Add}\begin{pmatrix} \mathrm{Hom}_S(C, M) \\ S \oplus M \end{pmatrix}$. So we get the exact sequence of left R-modules

$$0 \to R \to X_1^0 \to X_1^1 \to \cdots \to X_1^r \to 0$$

with each $X_1^i \in \mathrm{Add}(\mathrm{Hom}_S(C, M))$. Since $\mathrm{Tor}_i^R(C, \mathrm{Hom}_S(C, M)) = 0$ for all $i \geq 1$, we obtain the exact sequence of left S-modules

$$0 \to C \to C \otimes_R X_1^0 \to C \otimes_R X_1^1 \to \cdots \to C \otimes_R X_1^r \to 0$$

with each $C \otimes_R X_1^i \in \mathrm{Add}(M)$.

Thus M is a C-n-tilting left S-module.

(2) "$\Rightarrow$" Since $id_S(C \otimes_R N) \leq n$, $C \otimes_R N \in \mathcal{B}_C(S)$ by [18, Corollary 6.2]. So $N \in \mathcal{A}_C(R)$ by Lemma 2.6.

Note that $id_T\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix} = id_S(C \otimes_R N) \leq n$ by [23, Theorem 2.4].

By [21, Lemma 3.2], for any $i \geq 1$ and any cardinal κ,

$$\mathrm{Ext}_T^i\left(\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}^\kappa, \begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}\right) \cong \mathrm{Ext}_S^i((C \otimes_R N)^\kappa, C \otimes_R N) = 0.$$

There exist an integer $r \geq 0$ and an exact sequence of left R-modules

$$0 \to X_r \to \cdots \to X_1 \to X_0 \to C^+ \to 0$$

with each $X_i \in \mathrm{Prod}(N)$. Since $\mathrm{Tor}_i^R(C, N) = 0 = \mathrm{Tor}_i^R(C, C^+)$ for any $i \geq 1$ by [18, Theorem 6.4], we get the exact sequence of left S-modules

$$0 \to C \otimes_R X_r \to \cdots \to C \otimes_R X_1 \to C \otimes_R X_0 \to C \otimes_R C^+ \to 0.$$

Since $C \otimes_R C^+ \cong \mathrm{Hom}_R(C, C)^+ \cong S^+$, we get the exact sequence of left T-modules

$$0 \to \begin{pmatrix} X_r \\ C \otimes_R X_r \end{pmatrix} \to \cdots \to \begin{pmatrix} X_1 \\ C \otimes_R X_1 \end{pmatrix} \to \begin{pmatrix} R^+ \oplus X_0 \\ C \otimes_R X_0 \end{pmatrix} \to \begin{pmatrix} R^+ \oplus C^+ \\ S^+ \end{pmatrix} \to 0$$

with $\begin{pmatrix} R^+ \oplus X_0 \\ C \otimes_R X_0 \end{pmatrix}, \begin{pmatrix} X_i \\ C \otimes_R X_i \end{pmatrix} \in \mathrm{Prod}\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}, 1 \leq i \leq r$.

Since $\begin{pmatrix} R^+ \oplus C^+ \\ S^+ \end{pmatrix} \cong (R \oplus C, S)^+ = (T_T)^+$, $\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}$ is an n-cotilting left T-module.

"$\Leftarrow$" Note that $id_S(C \otimes_R N) \leq id_T\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix} \leq n$. Thus $C \otimes_R N \in \mathcal{B}_C(S)$ by [18, Corollary 6.2], which implies that $N \in \mathcal{A}_C(R)$ by Lemma 2.6.

By [21, Lemma 3.2], for any $i \geq 1$ and any cardinal κ, we have

$$\mathrm{Ext}_S^i((C \otimes_R N)^\kappa, C \otimes_R N) \cong \mathrm{Ext}_T^i\left(\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}^\kappa, \begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}\right) = 0.$$

There exist an integer $r \geq 0$ and an exact sequence of left T-modules

$$0 \to \begin{pmatrix} X_1^r \\ X_2^r \end{pmatrix}_{\varphi^r} \to \cdots \to \begin{pmatrix} X_1^1 \\ X_2^1 \end{pmatrix}_{\varphi^1} \to \begin{pmatrix} X_1^0 \\ X_2^0 \end{pmatrix}_{\varphi^0} \to \begin{pmatrix} R^+ \oplus C^+ \\ S^+ \end{pmatrix} \to 0$$

with each $\begin{pmatrix} X_1^i \\ X_2^i \end{pmatrix}_{\varphi^i} \in \mathrm{Prod}\begin{pmatrix} R^+ \oplus N \\ C \otimes_R N \end{pmatrix}$. So we get the exact sequence of left S-modules

$$0 \to X_2^r \to \cdots \to X_2^1 \to X_2^0 \to S^+ \to 0$$

with each $X_2^i \in \mathrm{Prod}(C \otimes_R N)$. Since $\mathrm{Ext}_S^i(C, C \otimes_R N) = 0$ for all $i \geq 1$, we get the exact sequence of left R-modules

$$0 \to \mathrm{Hom}_S(C, X_2^r) \to \cdots \to \mathrm{Hom}_S(C, X_2^1) \to \mathrm{Hom}_S(C, X_2^0) \to \mathrm{Hom}_S(C, S^+) \to 0$$

with each $\mathrm{Hom}_S(C, X_2^i) \in \mathrm{Prod}(N)$.

Since $\mathrm{Hom}_S(C, S^+) \cong (S \otimes_S C)^+ \cong C^+$, N is a C-n-cotilting left R-module. □

Finally, we give a computing formula of the self-injective dimension of a formal triangular matrix ring $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ with C a faithfully semidualizing (S, R)-bimodule.

Proposition 2.9 *Let* $T = \begin{pmatrix} R & 0 \\ C & S \end{pmatrix}$ *be a formal triangular matrix ring with C a faithfully semidualizing* (S, R)*-bimodule. Then*

(1) $id_T\begin{pmatrix} 0 \\ C \end{pmatrix} = max\{id({}_SC), id({}_RR) + 1\}$.
(2) $id({}_TT) = max\{id({}_SC), id({}_SS), id_R(\mathrm{Hom}_S(C, S)) + 1\}$.

Proof (1) follows from [23, Theorem 2.4].

(2) By [23, Theorem 2.4], $id_T\begin{pmatrix} R \\ C \end{pmatrix} = id({}_SC)$.

Next we prove that $id_T\begin{pmatrix} 0 \\ S \end{pmatrix} = \max\{id({}_SS), id_R(\mathrm{Hom}_S(C, S)) + 1\}$.

(i) If $id({}_SS) < \infty$, then $S \in \mathcal{B}_C(S)$ by [18, Corollary 6.2]. Therefore [23, Theorem 2.4] implies that $id_T\begin{pmatrix} 0 \\ S \end{pmatrix} = \max\{id({}_SS), id_R(\mathrm{Hom}_S(C, S)) + 1\}$.

(ii) If $id({}_SS) = \infty$, then it is clear that $id_T\begin{pmatrix} 0 \\ S \end{pmatrix} = \infty$.

It follows that $id({}_TT) = id_T\begin{pmatrix} R \\ C \oplus S \end{pmatrix} = \max\{id_T\begin{pmatrix} R \\ C \end{pmatrix}, id_T\begin{pmatrix} 0 \\ S \end{pmatrix}\}$
$= \max\{id({}_SC), id({}_SS), id_R(\mathrm{Hom}_S(C, S)) + 1\}$. □

Remark 2.10 Recall that a semidualizing (S, R)-bimodule C is *dualizing* if $id({}_SC) < \infty$ and $id(C_R) < \infty$. By Proposition 2.9, if $id({}_TT) < \infty$ and $id(T_T) < \infty$, then C is dualizing.

Acknowledgements This research was supported by NSFC (12271249, 12171230) and NSF of Jiangsu Province of China (BK20211358). The author wants to express his gratitude to the referee for the very helpful comments and suggestions.

Data availability Data are available from the corresponding author on reasonable request.

Declarations

Conflict of interest There are no conflict of interest.

References

1. Asadollahi, J., Salarian, S.: On the vanishing of Ext over formal triangular matrix rings. Forum Math. **18**, 951–966 (2006)
2. Auslander, M., Bridge, M.: Stable Module Theory. American Mathematical Soc, Providence, RI (1969)
3. Auslander, M., Reiten, I., Smalø, S.O.: Representation Theory of Artin Algebras. Cambridge Stud. Adv. Math. 36, Cambridge University Press, Cambridge (1995)
4. Bennis, D., El Maaouy, R., García Rozas, J.R., Oyonarte, L.: On relative counterpart of Auslander's conditions. J. Algebra Appl. **22**, 2350015 (2023)
5. Cheatham, T.J., Stone, D.R.: Flat and projective character modules. Proc. Am. Math. Soc. **81**, 175–177 (1981)
6. Christensen, L.W.: Semi-dualizing complexes and their Auslander categories. Trans. Am. Math. Soc. **353**, 1839–1883 (2001)
7. Enochs, E.E., Izurdiaga, M.C., Torrecillas, B.: Gorenstein conditions over triangular matrix rings. J. Pure Appl. Algebra **218**, 1544–1554 (2014)
8. Enochs, E.E., Jenda, O.M.G.: Relative Homological Algebra. Walter de Gruyter, Berlin, New York (2000)
9. Fossum, R.M., Griffith, P., Reiten, I.: Trivial Extensions of Abelian Categories, Homological Algebra of Trivial Extensions of Abelian Categories with Applications to Ring Theory. Lect. Notes in Math. 456, Springer, Berlin (1975)
10. Foxby, H.B.: Gorenstein modules and related modules. Math. Scand. **31**, 267–284 (1972)
11. Göbel, R., Trlifaj, J.: Approximations and Endomorphism Algebras of Modules, GEM 41. Walter de Gruyter, Berlin, New York (2006)
12. Golod, E.S.: G-dimension and generalized perfect ideals. Trudy Mat. Inst. Steklov. **165**, 62–66 (1984)
13. Green, E.L.: On the representation theory of rings in matrix form. Pac. J. Math. **100**, 123–138 (1982)
14. Haghany, A., Varadarajan, K.: Study of formal triangular matrix rings. Commun. Algebra **27**, 5507–5525 (1999)
15. Haghany, A., Varadarajan, K.: Study of modules over formal triangular matrix rings. J. Pure Appl. Algebra **147**, 41–58 (2000)
16. Holm, H.: Gorenstein homological dimensions. J. Pure Appl. Algebra **189**, 167–193 (2004)
17. Holm, H., Jørgensen, H.: Semi-dualizing modules and related Gorenstein homological dimensions. J. Pure Appl. Algebra **205**, 423–445 (2006)
18. Holm, H., White, D.: Foxby equivalence over associative rings. J. Math. Kyoto Univ. **47**, 781–808 (2007)
19. Krylov, P., Tuganbaev, A.: Formal Matrices. Springer, Switzerland (2017)
20. Liu, Z.F., Huang, Z.Y., Xu, A.M.: Gorenstein projective dimension relative to a semidualizing bimodule. Commun. Algebra **41**, 1–18 (2013)
21. Mao, L.X.: Cotorsion pairs and approximation classes over formal triangular matrix rings. J. Pure Appl. Algebra **224**(106271), 1–21 (2020)
22. Mao, L.X.: The structures of dual modules over formal triangular matrix rings. Publ. Math. Debrecen **97**, 367–380 (2020)
23. Mao, L.X.: Homological dimensions of special modules over formal triangular matrix rings. J. Algebra Appl. **21**, 2250146 (2022)
24. Rotman, J.J.: An Introduction to Homological Algebra. Academic Press, New York (1979)

25. Salimi, M.: Relative tilting modules with respect to a semidualizing module. Czech. Math. J. **69**, 781–800 (2019)
26. Salimi, M., Tavasoli, E., Moradifar, P., Yassemi, S.: Syzygy and torsionless modules with respect to a semidualizing module. Algebr. Represent. Theory **17**, 1217–1234 (2014)
27. Takahashi, R., White, D.: Homological aspects of semidualizing modules. Math. Scand. **106**, 5–22 (2010)
28. Vasconcelos, W.V.: Divisor Theory in Module Categories. North-Holland Mathematics Studies, 14, North-Holland Publishing, Amsterdam (1974)
29. White, D.: Gorenstein projective dimension with respect to a semidualizing module. J. Commut. Algebra **2**, 111–137 (2010)
30. Zhang, P.: Gorenstein-projective modules and symmetric recollements. J. Algebra **388**, 65–80 (2013)
31. Zhu, R.M., Liu, Z.K., Wang, Z.P.: Gorenstein homological dimensions of modules over triangular matrix rings. Turk. J. Math. **40**, 146–160 (2016)