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Characterizations of Minimal Elements in a Non-commutative L_p-Space

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Abstract

For $1 \leq p < \infty$, let $L_p(\mathcal{M}, \tau)$ be the non-commutative L_p-space associated with a von Neumann algebra $\mathcal{M}$, where $\mathcal{M}$ admits a normal semifinite faithful trace τ. Using the trace τ, Banach duality formula and Gâteaux derivative, this paper characterizes an element $a \in L_p(\mathcal{M}, \tau)$ such that

$$\|a\|_p = \inf\{\|a + b\|_p : b \in \mathcal{B}_p\},$$

where $\mathcal{B}_p$ is a closed linear subspace of $L_p(\mathcal{M}, \tau)$ and $\|\cdot\|_p$ is the norm on $L_p(\mathcal{M}, \tau)$. Such an a is called $\mathcal{B}_p$-minimal. In particular, minimal elements related to the finite-diagonal-block type closed linear subspaces

$$\mathcal{B}_p = \bigoplus_{i=1}^{\infty} e_i \mathcal{S} e_i$$

(converging with respect to $\|\cdot\|_p$) are considered, where $\{e_i\}_{i=1}^{\infty}$ is a sequence of mutually orthogonal and τ-finite projections in a σ-finite von Neumann algebra $\mathcal{M}$, and $\mathcal{S}$ is the set of elements in $\mathcal{M}$ with τ-finite supports.

Keywords Minimal elements · Trace · Banach duality formula · Gâteaux derivative

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1 Introduction

The non-commutative L_p-space theory was laid out in the 1950s by Segal [26] and Dixmier [12]. It is the intersection of operator theory and classical L^p-space theory, as well has been widely studied, extended and applied. In this paper, we explore the minimal elements in a non-commutative L_p-space, which are closely related to the orthogonality in metric geometry.

Suppose $\mathcal{H}$ is a Hilbert space and $\mathcal{K}$ is its closed linear subspace. For $h \in \mathcal{H}$, there exists a unique $k_0 \in \mathcal{K}$ such that

$$\|h - k_0\| = \operatorname{dist}(h, \mathcal{K}) = \inf\{\|h - k\| : k \in \mathcal{K}\},$$

where $\|\cdot\|$ is the norm induced by the inner product of $\mathcal{H}$. Replacing $h - k_0$ with h_0, one has

$$\|h_0\| = \inf\{\|h_0 + k\| : k \in \mathcal{K}\}.$$

Such an h_0 is called $\mathcal{K}$-minimal [25, Definition 5.2].

In the absence of inner product, Birkhoff [6] and James [15] study the orthogonality in a normed linear space, firstly. Suppose $\mathcal{X}$ is a normed linear space over $\mathbb{C}$ and $x, y \in \mathcal{X}$, then x is said to be Birkhoff–James orthogonal to y if

$$\|x\| \leq \|x + \lambda y\| \text{ for all } \lambda \in \mathbb{C}.$$

Thereafter, with the help of Hahn–Banach Theorem, Lumer [19] and Giles [14] carry over the notion of inner product on a Hilbert space to the semi-inner-product on a normed linear space, put forward that x and y in a continuous semi-inner-product space are Birkhoff–James orthogonal if and only if their semi-inner-product is 0 [14, Theorem 2]. Let $\mathcal{Y}$ be a closed linear subspace of $\mathcal{X}$. Then $x_0 \in \mathcal{X}$ is said to be $\mathcal{Y}$-minimal if it is Birkhoff–James orthogonal to each $y \in \mathcal{Y}$, or equivalently, if

$$\|x_0\| = \inf\{\|x_0 + y\| : y \in \mathcal{Y}\}.$$

The existence of minimal elements allows the description of minimal length curves (curves with minimal length joining fixed endpoints) of metric geometry in homogeneous spaces, and the characterization of minimal elements in various Banach spaces has attracted the attention of many scholars. For instance, [13] studies minimal elements and the corresponding minimal length curves of a homogeneous space $\mathcal{P}$ in a C*-algebra context. [3, 4, 18, 22, 32] are devoted to characterizing and constructing $D_n(\mathbb{R})$-minimal hermitian matrices in $M_n(\mathbb{C})$, in the sense of operator norm, where $M_n(\mathbb{C})$ is the algebra of complex $n \times n$ matrices and $D_n(\mathbb{R})$ is the algebra of real diagonal $n \times n$ matrices. For the study of minimal length curves in an infinite dimensional manifold, as well as the corresponding works on $D(K(H))$-minimal compact operators, one can refer to [2, 9, 10, 21, 31], where $\mathcal{H}$ is a complex separable Hilbert space with an orthonormal basis $\{\xi_i\}_{i=1}^{\infty}$, $K(\mathcal{H})$ is the algebra of compact operators on $\mathcal{H}$, and

$$D(K(\mathcal{H})) = \{D \in K(\mathcal{H}) : \langle D\xi_i, \xi_j \rangle = 0 \text{ when } i \neq j\}$$

is the set of diagonal compact operators. Moreover, [5, 28] study the best approximation and orthogonality in Hilbert C*-modules, which are closely related to minimal elements. Recently, with the help of semi-inner-product, minimal elements in p-Schatten ideals are explored in [7, 8]. With a view to the geometric property of orthogonality in a non-commutative L_p-space, illuminated by the idea of [7, 8], this paper devotes to characterizing an element $a \in L_p(\mathcal{M}, \tau)$ such that

$$\|a\|_p = \inf\{\|a + b\|_p : b \in \mathcal{B}_p\},$$

where $\|\cdot\|_p$ is the norm on $L_p(\mathcal{M}, \tau)$ and $\mathcal{B}_p$ is a closed linear subspace of $L_p(\mathcal{M}, \tau)$.

We briefly describe the contents of this paper. Section 2 lists some basic notions and prevalent results we will use throughout this paper. Section 3 provides the semi-inner-product on $L_p(\mathcal{M}, \tau)$ specifically and characterizes $\mathcal{B}_p$-minimal elements in terms of disjoint supports and the trace τ. Section 4 describes $\mathcal{B}_p$-minimal elements through the Gâteaux derivative of norm $\|\cdot\|_p$ and the Banach duality formula, respectively. In Sect. 5, minimal elements related to the finite-diagonal-block type closed linear subspaces

$$\mathcal{B}_p = \bigoplus_{i=1}^{\infty} e_i \mathcal{S} e_i$$

(converging with respect to $\|\cdot\|_p$) of $L_p(\mathcal{M}, \tau)$ are taken into account, where $\{e_i\}_{i=1}^{\infty}$ is a sequence of mutually orthogonal and τ-finite projections in a σ-finite von Neumann algebra $\mathcal{M}$, and $\mathcal{S}$ is the set of elements in $\mathcal{M}$ with τ-finite supports.

2 Preliminaries

In this section we give some basic concepts and prevalent results on non-commutative L_p-spaces. One can refer to [23, Chapter 34] and [29] for more details.

- Denote by $\mathcal{M}$ a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and by $\mathcal{M}_+$ its positive part. A **trace** on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying

  (1) $\tau(x + \lambda y) = \tau(x) + \lambda\tau(y)$, for $x, y \in \mathcal{M}_+$ and $\lambda \in \mathbb{R}_+$;
  (2) $\tau(x^*x) = \tau(xx^*)$, for $x \in \mathcal{M}$.

  Moreover, τ is said to be **normal** if $\sup_i \tau(x_i) = \tau(\sup_i x_i)$ for each bounded increasing net $\{x_i\}_{i\in\Lambda}$ in $\mathcal{M}_+$; to be **semifinite** if for any non-zero $x \in M_+$ there is a non-zero $y \in M_+$ such that $y \leq x$ and $\tau(y) < \infty$; and to be **faithful** if $x \in \mathcal{M}_+$ with $\tau(x) = 0$ implies that $x = 0$. In the rest of this paper, the von Neumann algebra $\mathcal{M}$ **always admits** a normal semifinite faithful trace τ. Denote by $P(\mathcal{M})$ the set of projections in $\mathcal{M}$, namely, $e \in P(\mathcal{M})$ if $e = e^2 = e^*$. There always exists an increasing net $\{e_i\}_{i\in\Lambda} \subset P(\mathcal{M})$ such that $\tau(e_i) < \infty$ for each $i \in \Lambda$ and $e_i \to I$ with respect to the strong operator topology, where I is the identity of $\mathcal{M}$.

- For $x \in \mathcal{M}$, let $x = u|x|$ be its polar decomposition, where u is a partial isometry from $(\ker x)^{\perp}$ onto $\overline{\operatorname{ran} x}$ and $|x| = (x^*x)^{\frac{1}{2}}$ is the absolute value of x. Denote by $l(x) = uu^*$ and $r(x) = u^*u$ the left and right **support** for x, respectively. If $x \in \mathcal{M}_+$, then $l(x) = r(x)$ and we write the support as $s(x)$. Set

$$\mathcal{S}_+ = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\},$$

and let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$, namely, the set of elements in $\mathcal{M}$ with τ-finite support. If $x \in \mathcal{S}$ and $0 < p < \infty$, then $|x|^p \in \mathcal{S}$. Moreover, define $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$, then $\|\cdot\|_p$ is a norm on $\mathcal{S}$ when $1 \le p < \infty$ and is a quasi-norm on $\mathcal{S}$ when $0 < p < 1$. The completion of $(\mathcal{S}, \|\cdot\|_p)$, denoted by $L_p(\mathcal{M}, \tau)$, is called the **non-commutative L_p-space** associated with $(\mathcal{M}, \tau)$. In this paper we focus on the case $1 \le p < \infty$, for which $L_p(\mathcal{M}, \tau)$ forms a Banach space. For the sake of convenience, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm.

- Let $1 \le p < \infty$ and take $x \in L_p(\mathcal{M}, \tau)$. Then x is a **closed densely defined** operator on $\mathcal{H}$. More specifically, its domain $D(x)$ is dense in $\mathcal{H}$ and its graph $G(x) = \{(\xi, x\xi) : \xi \in D(x)\}$ is closed in $\mathcal{H} \oplus \mathcal{H}$. The adjoint x^* of x is defined such that $\langle xf, g\rangle = \langle f, x^*g\rangle$ for all $f \in D(x)$ and $g \in D(x^*)$, where $D(x^*) = \{g \in \mathcal{H} : f \to \langle xf, g\rangle$ is continuous on $D(x)\}$. If $x = x^*$, then x is said to be self-adjoint. Similar to bounded linear operators, x has a unique polar decomposition $x = u|x|$, where u is a partial isometry from $(\ker x)^{\perp}$ onto $\overline{x(D(x))}$. In addition, the left and right supports for x can be defined. For more details on closed densely defined operators one can refer to [11, Chapter X] and [24, Chapter 13].

The following Lemma 2.1 is crucial to this paper.

Lemma 2.1 [23, 29] *The following statements hold:*

(1) *$\mathcal{S}$ is a strongly dense involutive ideal of $\mathcal{M}$. Moreover, for $x \in \mathcal{M}$, $x \in \mathcal{S}$ if and only if there is an $e \in P(\mathcal{M})$ with $\tau(e) < \infty$ such that $exe = x$.*
(2) *$|\tau(x)| \le \|x\|_1$ for $x \in \mathcal{S}$. Moreover, τ can be extended to a continuous linear functional on $L_1(\mathcal{M}, \tau)$.*
(3) *For $x \in L_p(\mathcal{M}, \tau)$ and $a, b \in \mathcal{M}$,*

$$\|x\|_p = \|x^*\|_p = \||x|\|_p, \ \ \|axb\|_p \le \|a\|\|x\|_p\|b\|. \tag{1}$$

(4) *(Hölder inequality) Suppose $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|\tau(xy)| \le \|x\|_p\|y\|_q$$

for $x \in L_p(\mathcal{M}, \tau)$ and $y \in L_q(\mathcal{M}, \tau)$.
(5) *Let $\{a_i\}_{i\in\Lambda}$ be a bounded net in $\mathcal{M}$ such that $a_i \to a$ with respect to the strong operator topology, then $xa_i \to xa$ in $L_p(\mathcal{M}, \tau)$ for any $x \in L_p(\mathcal{M}, \tau)$.*

3 Characterizations of $\mathcal{B}_p$-Minimal Elements

The aim of this section is to characterize $\mathcal{B}_p$-minimal elements in terms of disjoint supports and the normal semifinite faithful trace τ.

Definition 3.1 Let $1 \leq p < \infty$ and $\mathcal{B}_p$ be a closed linear subspace of $L_p(\mathcal{M}, \tau)$. We say that $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if

$$\|a\|_p = \inf\{\|a+b\|_p : b \in \mathcal{B}_p\}.$$

Remark 3.2 With $\mathcal{B}_p$ as above, suppose $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal.

(1) λa is $\mathcal{B}_p$-minimal for all $\lambda \in \mathbb{C}$, since

$$\begin{aligned}\|\lambda a\|_p = |\lambda|\|a\|_p &= |\lambda| \inf\{\|a+b\|_p : b \in \mathcal{B}_p\} \\ &= \inf\{\|\lambda a+b\|_p : b \in \mathcal{B}_p\}.\end{aligned}$$

(2) a^* is $\mathcal{B}_p$-minimal provided that $\mathcal{B}_p$ is $*$-closed. Indeed, since a is $\mathcal{B}_p$-minimal, $\|a\|_p = \|a^*\|_p$ and $\mathcal{B}_p = \mathcal{B}_p^*$, then

$$\begin{aligned}\|a^*\|_p = \|a\|_p &= \inf\{\|(a+b)\|_p : b \in \mathcal{B}_p\} \\ &= \inf\{\|(a+b)^*\|_p : b \in \mathcal{B}_p^*\} \\ &= \inf\{\|a^*+b^*\|_p : b^* \in \mathcal{B}_p\} \\ &= \inf\{\|a^*+b\|_p : b \in \mathcal{B}_p\}.\end{aligned}$$

(3) Suppose u and v are two unitary operators in $\mathcal{M}$ and $\mathcal{B}_p$ is (u, v)-invariant (namely, $u\mathcal{B}_p v = \mathcal{B}_p$), then uav is $\mathcal{B}_p$-minimal. Indeed, according to Lemma 2.1 (3), one has

$$\|x\|_p = \|u^*uxvv^*\|_p \leq \|uxv\|_p \leq \|x\|_p, \quad \forall x \in L_p(\mathcal{M}, \tau)$$

so the norm $\|\cdot\|_p$ is unitary invariant. Therefore, if $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal, then

$$\begin{aligned}\|uav\|_p = \|a\|_p &= \inf\{\|u(a+b)v\|_p : b \in \mathcal{B}_p\} \\ &= \inf\{\|uav+b\|_p : b \in \mathcal{B}_p\},\end{aligned}$$

which implies that uav is $\mathcal{B}_p$-minimal as well.

In recent works, Li et al. [20] and Bottazzi et al. [8] point out that operators x and y in a p-Schatten ideal have disjoint supports if and only if $\|x+y\|_p^p = \|x\|_p^p + \|y\|_p^p$ $(0 < p < \infty)$. Following this idea, we characterize $\mathcal{B}_p$-minimal elements in $L_p(\mathcal{M}, \tau)$ through disjoint supports. Let $L_p^+(\mathcal{M}, \tau)$ be the set of positive elements in $L_p(\mathcal{M}, \tau)$. For $x, y \in L_p^+(\mathcal{M}, \tau)$, $x \geq y$ means that $x - y \in L_p^+(\mathcal{M}, \tau)$.

Definition 3.3 [29] For $a \in L_p(\mathcal{M}, \tau)$, let $a = u|a|$ be its polar decomposition, where u is a partial isometry from $(\ker a)^\perp$ onto $\overline{aD(a)}$, $D(a)$ is the domain of a, and

$|a| = (a^*a)^{\frac{1}{2}}$ is the absolute value of a. We say that $l(a) = uu^*$ is the left support for a and $r(a) = u^*u$ is the right support for a.

Definition 3.4 Let $1 \le p < \infty$ and $\mathcal{B}_p$ be a closed linear subspace of $L_p(\mathcal{M}, \tau)$. We say that $a \in L_p(\mathcal{M}, \tau)$ and $\mathcal{B}_p$ have disjoint left supports if

$$l(a)l(b) = 0 \text{ for all } b \in \mathcal{B}_p,$$

and have disjoint right supports if

$$r(a)r(b) = 0 \text{ for all } b \in \mathcal{B}_p,$$

where $l(\cdot)$ and $r(\cdot)$ mean the left and right supports, respectively.

Theorem 3.5 *Let $1 \le p < \infty$ and $\mathcal{B}_p$ be a closed linear subspace of $L_p(\mathcal{M}, \tau)$. If $a \in L_p(\mathcal{M}, \tau)$ and $\mathcal{B}_p$ have disjoint left (or right) supports, then a is $\mathcal{B}_p$-minimal.*

Proof Recall that for $x \in L_p(\mathcal{M}, \tau)$, $l(x)$ is the projection from H onto $\overline{\operatorname{ran} x}$ and $r(x)$ is the projection from H onto $(\ker x)^\perp$. Since each x in $L_p(\mathcal{M}, \tau)$ is closed and densely defined, then $(\operatorname{ran} x)^\perp = \ker x^*$, $(\operatorname{ran} x^*)^\perp = \ker x$ and $x = x^{**}$ [11, Proposition X.1.6 and X.1.13].

Suppose that a and $\mathcal{B}_p$ have disjoint left supports first. For $b \in \mathcal{B}_p$, $l(a)l(b) = 0$ implies that

$$\operatorname{ran} a \subset (\operatorname{ran} b)^\perp = \ker b^*, \quad \operatorname{ran} b \subset (\operatorname{ran} a)^\perp = \ker a^*.$$

Thus, $b^*a = a^*b = 0$ and

$$|a+b|^2 = a^*a + a^*b + b^*a + b^*b = |a|^2 + |b|^2 \ge |a|^2.$$

According to [16, Lemma 3.2], one has

$$\|a+b\|_p^p = \tau(|a+b|^{2\cdot\frac{p}{2}}) \ge \tau(|a|^{2\cdot\frac{p}{2}}) = \|a\|_p^p,$$

so a is $\mathcal{B}_p$-minimal.

Using similar techniques, if a and $\mathcal{B}_p$ have disjoint right supports, then

$$\overline{\operatorname{ran} a^*} = (\ker a)^\perp \subset (\ker b)^{\perp\perp} = \ker b, \quad \overline{\operatorname{ran} b^*} \subset \ker a$$

and so $ba^* = ab^* = 0$ for each $b \in \mathcal{B}_p$. Hence

$$|a^* + b^*|^2 = |a^*|^2 + |b^*|^2 \ge |a^*|^2,$$
$$\|a+b\|_p = \|a^* + b^*\|_p \ge \|a^*\|_p = \|a\|_p.$$

The desired result follows. □

In 1961, Lumer [19] carried over the concept of inner product on Hilbert spaces to semi-inner-product on normed linear spaces, excepting the conjugate linear property. Later, G.R. Giles pointed out that every normed linear space can be represented as a semi-inner-product space with the homogeneity property (see [14, Theorem 1]). Before moving forward, let us recall revalent notions.

Definition 3.6 [14, Page 437], [19, Definition 1]

(1) Let $\mathcal{X}$ be a normed linear space. A mapping $[\cdot,\cdot]:\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ satisfying

(a) $[\alpha x+\beta z, y]=\alpha[x,y]+\beta[z,y]$;
(b) $\|x\|=[x,x]^{\frac{1}{2}}$;
(c) $|[x,y]|^2\le[x,x][y,y]$;

for all $x,y,z\in\mathcal{X}$ and $\alpha,\beta\in\mathbb{C}$ is called a *semi-inner-product* on $\mathcal{X}$, and then $(\mathcal{X},[\cdot,\cdot])$ is called a semi-inner-product space.

(2) A semi-inner-product space $(\mathcal{X},[\cdot,\cdot])$ is said to have the *homogeneity property* if $[\cdot,\cdot]$ also satisfies

(d) $[x,\alpha y]=\overline{\alpha}[x,y]$

for all $x,y\in\mathcal{X}$ and $\alpha\in\mathbb{C}$.

(3) A semi-inner-product space $(\mathcal{X},[\cdot,\cdot])$ is said to be *continuous* if

(e) $\mathrm{Re}([x,y+\lambda x])\to\mathrm{Re}([x,y])$ for real $\lambda\to 0$,

for every x,y in the unit sphere $S(\mathcal{X})=\{x\in\mathcal{X}:\|x\|=1\}$, where $\mathrm{Re}([x,y])$ is the real part of $[x,y]$.

Draw on the experience of [8, 27], we show the semi-inner-product on $L_p(\mathcal{M},\tau)$ specifically, where $\mathcal{M}$ admits a normal semifinite faithful trace τ and $1<p<\infty$.

Proposition 3.7 *Suppose* $1<p<\infty$. *For* $x,y\in L_p(\mathcal{M},\tau)$, *define*

$$[x,y]=\|y\|_p^{2-p}\tau(|y|^{p-1}u^*x), \tag{2}$$

where $y=u|y|$ *is the polar decomposition of* y. *Then*

$$[\cdot,\cdot]:L_p(\mathcal{M},\tau)\times L_p(\mathcal{M},\tau)\to\mathbb{C}$$

is a semi-inner-product on $L_p(\mathcal{M},\tau)$ *having the homogeneity property.*

Proof Suppose $\frac{1}{p}+\frac{1}{q}=1$. Take $x,y,z\in L_p(\mathcal{M},\tau)$ and $\alpha,\beta\in\mathbb{C}$. Let us check that the mapping $[\cdot,\cdot]$ defined in (2) satisfies (a–d) in Definition 3.6.

(a) For $y=u|y|\in L_p(\mathcal{M},\tau)$ one has $|y|^{p-1}\in L_q(\mathcal{M},\tau)$, since

$$\tau\Big((|y|^{p-1})^q\Big)=\tau(|y|^p)<\infty.$$

Moreover, using the Hölder inequality,

$$\left\||y|^{p-1}u^*x\right\|_1\le\left\||y|^{p-1}\right\|_q\|u^*x\|_p\le\left\||y|^{p-1}\right\|_q\|u^*\|\|x\|_p<\infty,$$

$|y|^{p-1}u^*x$ is in $L_1(\mathcal{M},\tau)$. Recall that τ is linear on $L_1(\mathcal{M},\tau)$, one has

$$\begin{aligned}[\alpha x+\beta z, y] &= \|y\|_p^{2-p}\tau(|y|^{p-1}u^*(\alpha x+\beta z)) \\ &= \alpha\|y\|_p^{2-p}\tau(|y|^{p-1}u^*x)+\beta\|y\|_p^{2-p}\tau(|y|^{p-1}u^*z) \\ &= \alpha[x,y]+\beta[z,y].\end{aligned}$$

(b) First we claim that $u^*u|y| = |y|$. Indeed, suppose $\xi \in \operatorname{ran} y^*$, then $\xi = y^*\eta$ for some $\eta \in (\ker y^*)^\perp \cap D(y^*) = \overline{\operatorname{ran} y} \cap D(y^*)$ and so

$$\overline{\operatorname{ran} y^*y} = \overline{\operatorname{ran} y^*} = (\ker y)^\perp. \tag{3}$$

Applying (3) to $|y|$, one has $\overline{\operatorname{ran}|y|} = \overline{\operatorname{ran}|y|^2} = \overline{\operatorname{ran} y^*y}$ and thus $\operatorname{ran}|y|$ is dense in $(\ker y)^\perp$. Recall that u^*u is the projection onto $(\ker y)^\perp$, $u^*u|y| = |y|$ as asserted. Therefore,

$$\begin{aligned}[y,y] &= \|y\|_p^{2-p}\tau(|y|^{p-1}u^*y) = \|y\|_p^{2-p}\tau(|y|^{p-1}u^*u|y|) \\ &= \|y\|_p^{2-p}\tau(|y|^p) = \|y\|_p^2.\end{aligned}$$

(c) Since $|y|^{p-1} \in L_q(\mathcal{M},\tau)$, it follows from the Hölder inequality that

$$\begin{aligned}|[x,y]|^2 &= \|y\|_p^{4-2p}\left|\tau(|y|^{p-1}u^*x)\right|^2 \\ &\le \|y\|_p^{4-2p}\left(\left\||y|^{p-1}u^*\right\|_q^2\right)\|x\|_p^2,\end{aligned}$$

meanwhile, by Lemma 2.1 (2),

$$\begin{aligned}\left\||y|^{p-1}u^*\right\|_q &\le \||y|^{p-1}\|_q\|u^*\| \\ &= \left\||y|^{p-1}\right\|_q = \tau\left((|y|^{p-1})^q\right)^{\frac{1}{q}} \\ &= \tau(|y|^p)^{\frac{1}{q}} = \left(\|y\|_p\right)^{\frac{p}{q}} = \|y\|_p^{p-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}|[x,y]|^2 &\le \|y\|_p^{4-2p}\|y\|_p^{2p-2}\|x\|_p^2 \\ &= \|y\|_p^2\|x\|_p^2 = [x,x][y,y].\end{aligned}$$

(d) Observe that $\alpha y = (\frac{\alpha}{|\alpha|}u)(|\alpha y|)$ is the polar decomposition of αy,

$$\begin{aligned}[x,\alpha y] &= \|\alpha y\|_p^{2-p}\tau(|\alpha y|^{p-1}(\tfrac{\alpha}{|\alpha|}u)^*x) \\ &= (|\alpha|^{(2-p)+(p-1)-1})(\overline{\alpha})\|y\|_p^{2-p}\tau(|y|^{p-1}u^*x) \\ &= \overline{\alpha}[x,y].\end{aligned}$$

The proof is completed. □

To characterize $\mathcal{B}_p$-minimal elements, it is necessary to review some basic definitions and known results on geometric theory of Banach space.

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. We call $x_0 \in S(\mathcal{X})$ a **smooth** point of the unit ball $B(\mathcal{X}) = \{x \in \mathcal{X} : \|x\| \leq 1\}$, if there is a unique $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x_0) = 1$; and call the norm $\|\cdot\|$ is **Gâteaux differentiable** at $x_0 \in S(\mathcal{X})$, if for any $y \in S(\mathcal{X})$ and $\lambda \in \mathbb{R}$

$$D_{x_0}(y) = \lim_{\lambda \to 0} \frac{\|x_0 + \lambda y\| - \|x_0\|}{\lambda}$$

exists. Accordingly, the Banach space $\mathcal{X}$ is said to be smooth if each $x \in S(\mathcal{X})$ is a smooth point of $B(\mathcal{X})$, and is said to be Gâteaux differentiable if $\|\cdot\|$ is Gâteaux differentiable at each $x \in S(\mathcal{X})$. It is well known that $\mathcal{X}$ is smooth if and only if $\mathcal{X}$ is Gâteaux differentiable [1, Theorem 2.1].

Lemma 3.8 [23, Corollary 5.2] *For $1 < p < \infty$, $L_p(\mathcal{M}, \tau)$ is uniformly convex and smooth.*

Let $1 < p < \infty$ and $\mathcal{B}_p$ be a closed linear subspace of $L_p(\mathcal{M}, \tau)$. Since the Banach space $L_p(\mathcal{M}, \tau)$ is smooth, in other words, it is Gâteaux differentiable, then the semi-inner product defined in (2) is continuous [14, Theorem 3]. Moreover, by [14, Theorem 2], $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if and only if

$$[b, a] = 0 \text{ for all } b \in \mathcal{B}_p,$$

equivalently,

$$\tau(|a|^{p-1}u^*b) = 0 \text{ for all } b \in \mathcal{B}_p,$$

where $a = u|a|$ is the polar decomposition of a. We obtain the following theorem.

Theorem 3.9 *Let $1 < p < \infty$ and $\mathcal{B}_p$ be a closed linear subspace of $L_p(\mathcal{M}, \tau)$. Then $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if and only if*

$$\tau(|a|^{p-1}u^*b) = 0 \text{ for all } b \in \mathcal{B}_p,$$

where $a = u|a|$ is the polar decomposition of a.

One can simplify Theorem 3.9 when the $\mathcal{B}_p$-minimal element is self-adjoint. Notice that $a = u|a| = |a|u^*$ when a is self-adjoint, moreover, $|a|^{p-2} = a^{p-2}$ when p is an even integer, one has the following corollary.

Corollary 3.10 *Let $2 \leq p < \infty$ and $\mathcal{B}_p$ be a closed linear subspace of $L_p(\mathcal{M}, \tau)$.*
(1) *A self-adjoint element $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if and only if*

$$\tau(|a|^{p-2}ab) = 0 \text{ for all } b \in \mathcal{B}_p.$$

In particular, when p is an even integer, a self-adjoint element $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if and only if

$$\tau(a^{p-1}b) = 0 \text{ for all } b \in \mathcal{B}_p.$$

(2) *A positive element $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if and only if*

$$\tau(a^{p-1}b) = 0 \text{ for all } b \in \mathcal{B}_p.$$

Example 3.11 Let $M_n(\mathbb{C})$ be the algebra of complex $n \times n$ matrices. For $A \in M_n(\mathbb{C})$, denote by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$ the set of eigenvalues of A, in counting multiplicity. Then $\|A\|_p = \left(\sum_{i=1}^n |\lambda_i(A)|^p\right)^{\frac{1}{p}}$. A class of positive minimal matrices in $M_2(\mathbb{C})$ will be provided below.

Let $p = 3$, denote $\mathcal{B}_3 = \mathbb{C} \oplus 0$ and take $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}_3$. The positive $\mathcal{B}_3$-minimal matrix in $M_2(\mathbb{C})$ must have the form $\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$, where $a_{22} \geq 0$. Indeed, suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & a_{22} \end{pmatrix}$ is a positive $\mathcal{B}_3$-minimal matrix, where $a_{11}, a_{22} \geq 0$. By Corollary 3.10 (2) one has

$$\mathrm{tr}(A^2E) = \mathrm{tr}(A^2E^2) = \mathrm{tr}(EA^2E) = \mathrm{tr}\begin{pmatrix} a_{11}^2 + |a_{12}|^2 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

and then $a_{11} = a_{12} = 0$. Moreover, since that

$$\left\| \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\|_3 = \sqrt[3]{|x|^3 + a_{22}^3} \geq a_{22} = \left\| \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} \right\|_3$$

for all $x \in \mathbb{C}$, $A = \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$ is $\mathcal{B}_3$-minimal.

Remark 3.12 (1) The $\mathcal{B}_p$-minimal element must exists (considering 0).
(2) From Example 3.11 one can see the $\mathcal{B}_p$-minimal element may not be unique.

4 Banach Duality Formula and Minimal Elements

As an application of the Hahn–Banach Theorem, [9, Proposition 4] and [21, Lemma 4] put forward the Banach duality formula between sets of compact operators and trace class operators. Then [7, Proposition 3.3] generalizes this result to a p-Schatten ideal and its dual, i.e. q-Schatten ideal, where $\frac{1}{p} + \frac{1}{q} = 1$. The Banach duality formula connects a Banach space $\mathcal{X}$ and its dual $\mathcal{X}^*$, which is also a tool to characterize minimal elements. In this section, we characterize $\mathcal{B}_p$-minimal elements by the Banach duality formula between $L_p(\mathcal{M}, \tau)$ and its dual given below.

Lemma 4.1 (Banach duality formula for non-commutative L_p-space) *Let* $1 \leq p < \infty$ *and* $\mathcal{B}_p$ *be a closed linear subspace of* $L_p(\mathcal{M}, \tau)$. *Denote*

$$\mathcal{B}_p^{\perp(\tau)} = \{y \in L_q(\mathcal{M}, \tau) : \tau(by) = 0 \text{ for all } b \in \mathcal{B}_p\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $a \in L_p(\mathcal{M}, \tau)$,

$$\inf\{\|a + b\|_p : b \in \mathcal{B}_p\} = \sup\{|\tau(ay)| : y \in \mathcal{B}_p^{\perp(\tau)}, \|y\|_q = 1\}. \tag{4}$$

Proof Take $y \in \mathcal{B}_p^{\perp(\tau)}$ with $\|y\|_q = 1$. With the help of Hölder inequality,

$$|\tau(ay)| = |\tau(ay + by)| \leq \|a + b\|_p \|y\|_q = \|a + b\|_p$$

for all $b \in \mathcal{B}_p$, so

$$\sup\{|\tau(ay)| : y \in \mathcal{B}_p^{\perp(\tau)}, \|y\|_q = 1\} \leq \inf\{\|a + b\|_p : b \in \mathcal{B}_p\}.$$

Without loss of generality, suppose $a \notin \mathcal{B}_p$, then $\inf\{\|a + b\|_p : B \in \mathcal{B}_p\} > 0$ as $\mathcal{B}_p$ is closed. According to the Hahn–Banach Theorem [11, Corollary III.6.8], there is a linear functional $f \in (L_p(\mathcal{M}, \tau))^*$ such that $\|f\| = 1$, $f|_{\mathcal{B}_p} = 0$ and

$$f(a) = \text{dist}(a, \mathcal{B}_p) = \inf\{\|a + b\|_p : b \in \mathcal{B}_p\}.$$

Since $(L_p(\mathcal{M}, \tau))^* = L_q(\mathcal{M}, \tau)$ [23, Page 1464], there exists a unique y_0 in $L_q(\mathcal{M}, \tau)$ such that

$$\|y_0\|_q = \|f\| = 1 \text{ and } f(\cdot) = \tau(\cdot y_0).$$

In addition, $f|_{\mathcal{B}_p} = 0$ implies that $y_0 \in \mathcal{B}_p^{\perp(\tau)}$. One gets

$$\inf\{\|a + b\|_p : b \in \mathcal{B}_p\} = \tau(ay_0) \leq \sup\{|\tau(ay)| : y \in \mathcal{B}_p^{\perp(\tau)}, \|y\|_q = 1\}.$$

The equation (4) holds. □

Theorem 4.2 *Let* $1 < p < \infty$, $\mathcal{B}_p$ *be a closed linear subspace of* $L_p(\mathcal{M}, \tau)$ *and* $a(\neq 0) \in L_p(\mathcal{M}, \tau)$. *The following two statements hold:*

(1) *a is $\mathcal{B}_p$-minimal if and only if*

$$\|a\|_p = \sup\{|\tau(ay)| : y \in \mathcal{B}_p^{\perp(\tau)}, \|y\|_q = 1\}.$$

Moreover, the supremum of the right side can be obtained at $y_a = \frac{|a|^{p-1}u^*}{\|a\|_p^{p-1}}$, *where* $a = u|a|$ *is the polar decomposition of* a;

(2) *If a is $\mathcal{B}_p$-minimal, then* $D_a(b) = 0$ *for all* $b \in \mathcal{B}_p$, *where* $D_a(b)$ *is the Gâteaux derivative of* $\|\cdot\|_p$ *at* a *in the* b *direction.*

Proof (1) It is obvious from Lemma 4.1 that $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal if and only if

$$\|a\|_p = \sup\{|\tau(ay)| : y \in \mathcal{B}_p^{\perp(\tau)}, \|y\|_q = 1\}. \tag{5}$$

Let us prove that the supremum of right side of (5) can be obtained at

$$y_a = \frac{|a|^{p-1}u^*}{\|a\|_p^{p-1}}$$

when a is $\mathcal{B}_p$-minimal.

According to Theorem 3.9, if a is $\mathcal{B}_p$-minimal, then

$$\tau(by_a) = \tau(y_a b) = \frac{1}{\|a\|_p^{p-1}}\tau(|a|^{p-1}u^*b) = 0$$

for all $b \in \mathcal{B}_p$, hence $y_a \in \mathcal{B}_p^{\perp(\tau)}$. In addition, by the proof of Proposition 3.7 (b) we know $u^*u|a| = |a|$, then

$$y_a y_a^* = \frac{|a|^{p-1}u^*u|a|^{p-1}}{\|a\|_p^{2p-2}} = \frac{|a|^{2p-2}}{\|a\|_p^{2p-2}},$$

$|y_a^*| = \frac{|a|^{p-1}}{\|a\|_p^{p-1}}$ and

$$\|y_a\|_q = \|y_a^*\|_q = \left(\frac{\tau(|a|^{(p-1)q})}{\|a\|_p^{(p-1)q}}\right)^{\frac{1}{q}} = \left(\frac{\tau(|a|^p)}{\|a\|_p^p}\right)^{\frac{1}{q}} = 1.$$

Moreover,

$$\tau(ay_a) = \frac{\tau(u|a|^p u^*)}{\|a\|_p^{p-1}} = \frac{\tau(|a|^{p-1}u^*u|a|)}{\|a\|_p^{p-1}} = \frac{\tau(|a|^p)}{\|a\|_p^{p-1}} = \|a\|_p.$$

The desired result is obtained.

(2) By Lemma 3.8 and its previous statements, the norm $\|\cdot\|_p$ is Gâteaux differentiable at each $a \in L_p(\mathcal{M}, \tau)$. Moreover, according to [1, Theorem 1.1] and [17, Proposition 1.3],

$$D_a(x) = \lim_{\lambda \to 0} \frac{\|a + \lambda x\|_p - \|a\|_p}{\lambda} = \operatorname{Re} f_a(x)$$

for $x \in L_p(\mathcal{M}, \tau)$, where $f_a(\cdot)$ is the unique linear functional on $L_p(\mathcal{M}, \tau)$ such that $\|f_a\| = 1$ and $f_a(a) = \|a\|_p$. Combined with the first part proof, one gets

$f_a(\cdot) = \tau(\cdot y_a)$ and

$$D_a(x) = \text{Re } \tau(xy_a) = \frac{1}{\|a\|_p^{p-1}} \text{Re } [x, a].$$

Thus, if $a \in L_p(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal, then $D_a(b) = 0$ for all $b \in \mathcal{B}_p$. □

Remark 4.3 Let $\mathcal{A}$ be a Banach space, $\mathcal{B}$ be a proper closed linear subspace of $\mathcal{A}$ and $a(\neq 0) \in \mathcal{A}$ be $\mathcal{B}$-minimal. A **witness** to the $\mathcal{B}$-minimality of a is a linear functional f on $\mathcal{A}$ such that $\|f\| = 1$, $f|_{\mathcal{B}} = 0$ and $f(a) = \|a\|$ [25, Page 2267–2268]. By the proof of Theorem 4.2, we know that in the non-commutative L_p-space context $(1 < p < \infty)$, $\tau(\cdot y_a)$ is a witness to the $\mathcal{B}_p$-minimality of a, where $y_a = \frac{|a|^{p-1}u^*}{\|a\|_p^{p-1}}$ and $a = u|a|$ is the polar decomposition of a.

5 Minimal Elements Related to Finite-Diagonal-Block Type $\mathcal{B}_p$

Let $\mathcal{H}$ be a complex separable Hilbert space with an orthonormal basis $\{\xi_i\}_{i=1}^{\infty}$. Make a partition of $\mathbb{Z}^+$ by $\{1, 2, \ldots, \lambda_1\}$, $\{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\}$, $\{\lambda_1 + \lambda_2 + 1, \lambda_1 + \lambda_2 + 2, \ldots, \lambda_1 + \lambda_2 + \lambda_3\}, \ldots$, such that each set is finite. Set $\Lambda_0 = 0$, $\Lambda_k = \sum_{i=1}^{k} \lambda_i$ and

$$\mathcal{H}_{\Lambda_k} = \text{span}\{\xi_i : \Lambda_{k-1} + 1 \leq i \leq \Lambda_k\}.$$

Denote by P_{Λ_k} the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\Lambda_k}$, and let

$$W(\mathcal{H}) = \sum_{k=1}^{\infty} P_{\Lambda_k} F(\mathcal{H}) P_{\Lambda_k} \tag{6}$$

(converging with respect to the operator norm), where $F(\mathcal{H})$ is the algebra of finite rank operators on $\mathcal{H}$. The elements of $W(\mathcal{H})$ are compact operators with finite square matrices of a fixed type in their diagonals, and $W(\mathcal{H})$ is a C*-subalgebra of $K(\mathcal{H})$ [31]. Such a $W(\mathcal{H})$ is said to be **finite-diagonal-block type**. As a continuation and improvement of previous works on $D(K(\mathcal{H}))$-minimal compact operators [2, 9, 10, 21, 31], we characterize minimal elements related to a finite-diagonal-block type C*-subalgebra of $K(\mathcal{H})$ in [31].

Similar to the $W(\mathcal{H})$ in (6), we can construct a closed linear subspace $\mathcal{B}_p$ of $L_p(\mathcal{M}, \tau)$ with finite-diagonal-block type. Some interesting results about minimal elements related to such a type $\mathcal{B}_p$ can be drawn. Two projections e_1 and e_2 in $P(\mathcal{M})$ are said to be orthogonal if $e_1 e_2 = 0$. It is easy to check that if e_1 and e_2 are two orthogonal projections in $P(\mathcal{M})$, then $e_1 \mathcal{S} e_1 \cap e_2 \mathcal{S} e_2 = \{0\}$.

Theorem 5.1 *Let $\mathcal{M}$ be a σ-finite von Neumann algebra. Take $\{e_i\}_{i=1}^{\infty} \subset P(\mathcal{M})$ such that*

(a) $\tau(e_i) < \infty$;

(b) $e_ie_j = 0$ *when* $i \neq j$;
(c) $\sum_{i=1}^{\infty} e_i = I$, *with respect to the strong operator topology.*

Set

$$\mathcal{B}_p = \bigoplus_{i=1}^{\infty} e_i \mathcal{S} e_i \tag{7}$$

(converging with respect to $\|\cdot\|_p$*). The following statements hold:*

(1) *For* $1 < p < \infty$, $a \in L_p(\mathcal{M}, \tau)$ *is* $\mathcal{B}_p$*-minimal if and only if*

$$\tau(|a|^{p-1}u^*e_ixe_i) = 0 \text{ for all } x \in \mathcal{S} \text{ and } i \in \mathbb{Z}_+,$$

where $a = u|a|$ *is the polar decomposition of* a.

(2) *For* $2 \leq p < \infty$, *if* $a \in L_p^+(\mathcal{M}, \tau)$ *is* $\mathcal{B}_p$*-minimal, then* $e_ia^{p-1}e_i = 0$ *for all* $i \in \mathbb{Z}_+$.

(3) *For* $2 \leq p < \infty$, $a \in \mathcal{S}_+$ *is* $\mathcal{B}_p$*-minimal implies that* $a = 0$.

We say $\mathcal{B}_p$ *with the form* (7) *a* **finite-diagonal-block type** *closed linear subspace of* $L_p(\mathcal{M}, \tau)$.

Proof It is well known that for a σ-finite von Neumann algebra $\mathcal{M}$, the sequence of projections $\{e_i\}_{i=1}^{\infty}$ satisfying (a-c) must exist.

(1) Suppose that $\tau(|a|^{p-1}u^*e_ixe_i) = 0$ for all $x \in \mathcal{S}$ and $i \in \mathbb{Z}_+$. Take any $b = \sum_{i=1}^{\infty} e_ix_ie_i \in \mathcal{B}_p$, where $x_i \in \mathcal{S}$. Since $|a|^{p-1}u^* \in L_q(\mathcal{M}, \tau)$, one has

$$\begin{aligned}&\left\| |a|^{p-1}u^* \sum_{i=1}^{n} e_ix_ie_i - |a|^{p-1}u^*b \right\|_1 \\ &\quad \leq \left\| |a|^{p-1}u^* \right\|_q \left\| \sum_{i=1}^{n} e_ix_ie_i - b \right\|_p \to 0\end{aligned}$$

when $n \to \infty$. Moreover, as τ is continuous on $L_1(\mathcal{M}, \tau)$,

$$\begin{aligned}\tau(|a|^{p-1}u^*b) &= \lim_{n\to\infty} \tau(|a|^{p-1}u^* \sum_{i=1}^{n} e_ix_ie_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \tau(|a|^{p-1}u^*e_ix_ie_i) \\ &= 0.\end{aligned}$$

Using Theorem 3.9 and by the arbitrariness of b, a is $\mathcal{B}_p$-minimal. The necessity is obvious.

(2) Suppose $a \in L_p^+(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal. From Corollary 3.10 we know $\tau(a^{p-1}x) = 0$ for all $x \in \mathcal{B}_p$. Take $x = e_i$, then

$$\tau(a^{p-1}e_i) = \tau(a^{p-1}e_i^2) = \tau(e_i a^{p-1} e_i) = 0.$$

One can get that $e_i a^{p-1} e_i = 0$, since τ is faithful and $e_i a^{p-1} e_i$ is positive.

(3) Suppose $a \in \mathcal{S}_+$ is $\mathcal{B}_p$-minimal, then $\tau\Big(a^{p-1}(\sum_{i=1}^n e_i)\Big) = 0$ for each $n \in \mathbb{Z}_+$ (Corollary 3.10). Note that a^{p-1} is in $\mathcal{S}_+$ and then in $L_1(\mathcal{M}, \tau)$, according to Lemma 2.1 (5) one has $\lim_{n\to\infty} a^{p-1}\sum_{i=1}^n e_i = a^{p-1}$ in $L_1(\mathcal{M}, \tau)$. Since τ is continuous and faithful on $L_1(\mathcal{M}, \tau)$,

$$\tau(a^{p-1}) = \lim_{n\to\infty} \tau\left(a^{p-1}\left(\sum_{i=1}^n e_i\right)\right) = 0,$$

further, $a^{p-1} = a = 0$. □

Corollary 5.2 *Let $\mathcal{M}$ be a finite von Neumann algebra, $\{e_i\}_{i=1}^n$ be mutually orthogonal projections in $P(\mathcal{M})$ such that $\tau(e_i) < \infty$ and $\sum_{i=1}^n e_i = I$. Set*

$$\mathcal{B}_p = \bigoplus_{i=1}^n e_i \mathcal{S} e_i, \tag{8}$$

where $2 \le p < \infty$. Then $\mathcal{B}_p$ is a closed linear subspace of $L_p(\mathcal{M}, \tau)$, and $a \in L_p^+(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal implies that $a = 0$.

Proof Obviously, $\mathcal{B}_p$ is a linear subspace of $L_p(\mathcal{M}, \tau)$. Recall that a finite direct sum of closed sets is also closed, it is enough to prove that $e\mathcal{S}e$ is closed for any $e \in P(\mathcal{M})$ with $\tau(e) < \infty$.

Take z from the closure of $e\mathcal{S}e$ and suppose $ex_i e \to z$ in $L_p(\mathcal{M}, \tau)$ when $i \to \infty$, where $x_i \in S$. Note that

$$\|ex_i e - eze\|_p = \|e(ex_i e)e - eze\|_p \le \|e\| \|ex_i e - z\|_p \|e\|,$$

so $ex_i e \to eze$ in $L_p(\mathcal{M}, \tau)$ when $i \to \infty$. By the uniqueness of the limit, one has $z = eze = e(eze)e$. On the other hand, since $\tau(e) < \infty$, it follows from Lemma 2.1 (1) that $eze \in \mathcal{S}$, then $z \in e\mathcal{S}e$ and $e\mathcal{S}e$ is closed in $L_p(\mathcal{M}, \tau)$. Obviously, the $\mathcal{B}_p$ in (8) contains identity I. Thus if $a \in L_p^+(\mathcal{M}, \tau)$ is $\mathcal{B}_p$-minimal, then $\tau(a^{p-1}) = 0$ and $a^{p-1} = a = 0$. □

Example 5.3 Let $2 \le p < \infty$. Consider $M_n(\mathbb{C})$ and its closed linear subspace

$$\mathcal{B}_p = M_{\mu_1}(\mathbb{C}) \oplus M_{\mu_2}(\mathbb{C}) \oplus \cdots \oplus M_{\mu_k}(\mathbb{C}),$$

where $k \ge 1$, $1 \le \mu_i \le n$ and $\sum_{i=1}^k \mu_i = n$. Different from the $\mathcal{B}_3$ in Example 3.11, such a $\mathcal{B}_p$ contains the identity matrix I_n. By Corollary 5.2, the only positive $\mathcal{B}_p$-minimal matrix in $M_n(\mathbb{C})$ is the zero matrix.

Example 5.4 This example is obtained from [7]. Let $\mathcal{H}$ be a complex separable Hilbert space with an orthonormal basis $\{\xi_i\}_{i=1}^{\infty}$ and $S_p(\mathcal{H})$ be the set of p-Schatten class operators on $\mathcal{H}$, namely,

$$S_p(\mathcal{H}) = \{x \in K(\mathcal{H}) : \|x\|_p < \infty\},$$

where $\|x\|_p^p = \mathrm{tr}(|x|^p) = \sum_{i=1}^{\infty} \langle |x|^p \xi_i, \xi_i \rangle$. Set

$$\mathcal{B}_p = \bigoplus_{i=1}^{\infty} e_i F(\mathcal{H}) e_i,$$

(converging with respect to $\|\cdot\|_p$), where e_i is the orthogonal projection from $\mathcal{H}$ onto span$\{\xi_i\}$. That is, $\mathcal{B}_p$ consists of all diagonal p-Schatten operators. From Theorem 5.1 (1) we know if $a \in S_p^+(\mathcal{H})$ is $\mathcal{B}_p$−minimal ($p \geq 2$), then $e_i a^{p-1} e_i = 0$ for each $i \in \mathbb{Z}_+$. Further,

$$\begin{aligned} \mathrm{tr}(a^{p-1}) &= \sum_{i=1}^{\infty} \langle a^{p-1} \xi_i, \xi_i \rangle = \sum_{i=1}^{\infty} \langle a^{p-1} e_i \xi_i, e_i \xi_i \rangle \\ &= \sum_{i=1}^{\infty} \langle e_i a^{p-1} e_i \xi_i, \xi_i \rangle = 0, \end{aligned}$$

and a must be 0.

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References

1. Abatzoglu, T.J.: Norm derivatives on spaces of operators. Math. Ann. **239**, 129–135 (1979)
2. Andruchow, E., Larotonda, G.: The rectifiable distance in the unitary Fredholm group. Studia Math. **196**(2), 151–178 (2010)
3. Andruchow, E., Larotonda, G., Recht, L., Varela, A.: A characterization of minimal Hermitian matrices. Linear Algebra Appl. **436**, 2366–2374 (2012)
4. Andruchow, E., Mata-Lorenzo, L.E., Mendoza, A., Recht, L., Varela, A.: Minimal matrices and the corresponding minimal curves on flag manifolds in low dimension. Linear Algebra Appl. **430**(8), 1906–1928 (2009)
5. Arambašić, L., Rajić, R.: Operator version of the best approximation problem in Hilbert C*-modules. J. Math. Anal. Appl. **413**, 311–320 (2014)
6. Birkhoff, G.: Orthogonality in linear metric spaces. Duke Math. J. **1**, 169–172 (1935)
7. Bottazzi, T.: Best approximation by diagonal operators in Schatten ideals. Linear Algebra Appl. **620**, 1–26 (2021)
8. Bottazzi, T., Conde, C., Moslehian, M.S., Wójcik, P., Zamani, A.: Orthogonality and parallelism of operators on various Banach spaces. J. Aust. Math. Soc. **106**(2), 160–183 (2019)
9. Bottazzi, T., Varela, A.: Best approximation by diagonal compact operators. Linear Algebra Appl. **439**, 3044–3056 (2013)
10. Bottazzi, T., Varela, A.: Minimal length curves in unitary orbits of a Hermitian compact operator. Differ. Geom. Appl. **45**, 1–22 (2016)
11. Conway, J.B.: A Course in Functional Analysis, 2nd edn. Springer, New York (2003)

12. Dixmier, J.: Formes linéaires sur un anneau d'opérateurs. Bull. Soc. Math. France **81**, 9–39 (1953)
13. Durán, C.E., Mata-Lorenzo, L.E., Recht, L.: Metric geometry in homogeneous spaces of the unitary group of a C*-algebra: part I-minimal curves. Adv. Math. **184**(2), 342–366 (2004)
14. Giles, J.R.: Classes of semi-inner-product spaces. Trans. Am. Math. Soc. **129**, 436–446 (1967)
15. James, R.C.: Orthogonality in normed linear spaces. Duke Math. J. **12**, 291–301 (1945)
16. Kosaki, H.: Applications of uniform convexity of noncommutative L^p-spaces. Trans. Am. Math. Soc. **283**, 265–282 (1984)
17. Kečkić, D.J.: Gâteaux derivative of $B(H)$ norm. Proc. Am. Math. Soc. **133**(7), 2061–2067 (2005)
18. Klobouk, A., Varela, A.: Concrete minimal 3×3 Hermitian matrices and some general cases. Demonstr. Math. **50**(1), 330–350 (2017)
19. Lumer, G.: Semi-inner-product spaces. Trans. Am. Math. Soc. **100**, 29–43 (1961)
20. Li, Y., Li, Y.E.: Some characterizations of the trace norm triangle equality. Linear Algebra Appl. **484**, 396–408 (2015)
21. Li, Y., Wang, S.J., Cai, X.M.: Some characterizations of minimal compact normal operators. Linear Algebra Appl. **549**, 233–245 (2018)
22. Mendoza, A., Recht, L., Varela, A.: Supports for minimal Hermitian matrices. Linear Algebra Appl. **584**, 458–482 (2020)
23. Pisier, G., Xu, Q.H.: Non-commutative L_p-spaces. In: Johnson, W.B., Lindenstrauss, J. (ed.) Handbook of the Geometry of Banach Spaces, Vol 2. Elsevier Science B.V., Amsterdam (2003)
24. Rudin, W.: Functional Analysis, 2nd edn. McGraw-Hill Companies, New York (1991)
25. Rieffel, M.A.: Leibniz seminorms and best approximation from C*-subalgebras. Sci. China Math. **54**(11), 2259–2274 (2011)
26. Segal, I.: A non-commutative extension of abstract integration. Ann. Math. **57**, 401–457 (1953)
27. Sherman, D.: Noncommutative L^p structure encodes exacly Jordan structure. https://doi.org/10.48550/arXiv.math/0309365 (2004)
28. Wójcik, P., Zamani, A.: From norm derivatives to orthogonalities in Hilbert C*-modules. Linear Multilinear Algebra **71**(6), 875–888 (2023)
29. Xu, Q.H., Turdebek, N., Chen, Z.Q.: Operator Theory and Non-commutative L_p-Spaces. Science Press, Beijing (2010). (**in Chinese**)
30. Yeadon, F.J.: Isometries of non-commutative L_p-spaces. Math. Proc. Camb. Philos. Soc. **90**, 41–50 (1981)
31. Zhang, Y., Jiang, L.N.: Minimal Hermitian compact operators related to a C*-subalgebra of $K(H)$. J. Math. Anal. Appl. **506**(2), 125649 (2022)
32. Zhang, Y., Jiang, L.N., Han, Y.H.: Constructions of minimal Hermitian matrices related to a C*-subalgebra of $M_n(\mathbb{C})$. Proc. Am. Math. Soc. **151**, 73–84 (2023)

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