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Multiplicity of Normalized Solutions for Schrödinger Equations

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Abstract

In this paper, we consider the following nonlinear Schrödinger equation with an L^2-constraint:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \quad u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $a, \mu > 0$, $2 < q < 2 + \frac{4}{N} < p < 2^*$, $2q + 2N - pN < 0$ and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier. We deal with the concave and convex cases of energy functional constraints on the L^2 sphere, and prove the existence of infinitely solutions with positive energy levels.

Keywords Nonlinear Schrödinger equation · Multiplicity · Normalized solution · Truncated functional · Variational methods

Mathematics Subject Classification 35A15 · 35B33 · 35J10 · 35Q30

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1 Introduction

In [21, 22], Soave considered the existence and properties of ground states for the nonlinear Schrödinger equation with combined power nonlinearities

$$-\Delta u = \lambda u + \mu |u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N, N \geq 1, \tag{$\mathcal{P}$}$$

having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2,$$

under different assumptions on $a > 0$, $\mu \in \mathbb{R}$ and

$$2 < q \leq 2 + \frac{4}{N} \leq p \leq 2^*, \quad q \neq p,$$

i.e. the two nonlinearities have different characters with respect to the L^2-critical exponent $\bar{p} := 2 + \frac{4}{N}$. The cases $p > \bar{p}$ and $p < \bar{p}$ are called mass L^2-supercritical and mass L^2-subcritical, respectively, which comes from the Gagliardo-Nirenberg inequality (see [23]). We recall that, for every $N \geq 1$ and $p \in (2, 2^*)$, there exists a constant $C_{N,p}$ depending on N and on p such that

$$\|u\|_p^p \leq C_{N,p}^p \|\nabla u\|_2^{p\delta_p} \|u\|_2^{p(1-\delta_p)} \quad \text{for all } u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $\delta_p := \frac{N(p-2)}{2p}$ and we denote by $C_{N,p}$ the best constant in the Gagliardo-Nirenberg inequality.

Here and in what follows, 2^* denotes the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ (that is, $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$).

We look for solutions of problem ($\mathcal{P}$) having a prescribed L^2-norm, which are often referred to as normalized solutions. More precisely, for given $a > 0$, we look for a couple of solution $(u_a, \lambda_a) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ to problem ($\mathcal{P}$) with

$$\int_{\mathbb{R}^N} |u_a|^2 \mathrm{d}x = a^2.$$

The solution u_a to the problem ($\mathcal{P}$) corresponds to a critical point of the following C^1 functional $\mathcal{J} : H_r^1(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{J}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{p}\|u\|_p^p - \frac{1}{q}\|u\|_q^q$$

restricted to the sphere in $L^2(\mathbb{R}^N)$:

$$\mathcal{S}(a) = \left\{ u \in H_r^1(\mathbb{R}^N) : \|u\|_2^2 = a^2 \right\}.$$

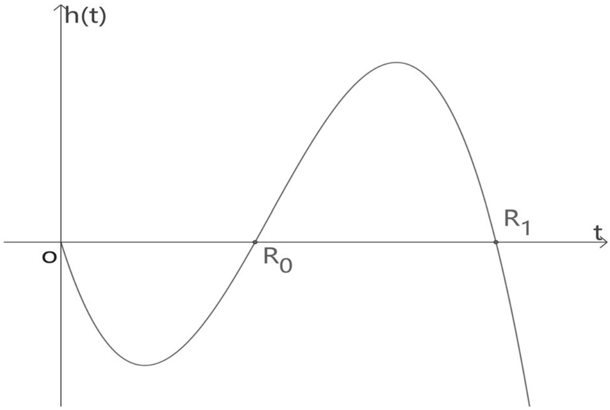

Fig. 1 The relationship between functional h and t.

It is clear that for each critical point $u_a \in \mathcal{S}(a)$ of $\mathcal{J}|_{\mathcal{S}(a)}$ corresponds to a Lagrange multiplier $\lambda_a \in \mathbb{R}$ such that (u_a, λ_a) solves problem $(\mathcal{P})$. Therefore, to obtain such a solution, it is necessary to find the critical point of $\mathcal{J}$ on the constraint $\mathcal{S}(a)$.

In recent decades, the question of finding solutions of nonlinear Schrödinger equations with prescribed L^2-norm has received a special attention. This approach seems particularly meaningful from the physical point of view, since, in addition to being a conserved quantity for the time dependent equation, the mass has often a clear physical meaning; For instance, it represents the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation, two main fields of application of the nonlinear Schrödinger equations. For more related results on normalized solutions of nonlinear Schrödinger equations, we refer to [1, 5–7, 14–17, 20, 25, 26] and the references therein.

Notice that, for the case of $2 < p < \bar{p} < q \leq 2^*$, Soave [21, 22] applied the Gagliardo-Nirenberg inequality (1.1) to create the corresponding energy functional $h \in C^2(\mathbb{R}^+, \mathbb{R})$

$$h(t) := \frac{1}{2}t^2 - \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q} t^{q\delta_q} - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} t^{p\delta_p}$$

such that

$$\mathcal{J}(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q} \|\nabla u\|_2^{q\delta_q} - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} \|\nabla u\|_2^{p\delta_p} = h(\|\nabla u\|_2).$$

Recalling that $2 < q < \bar{p} < p \leq 2^*$, so that $q\delta_q < 2$ and $2 < p\delta_p \leq 2^*$. Under certain conditions of $a > 0$ and $\mu > 0$, function h has a concave-convex structure (see Fig. 1).

Naturally, it is known that $\mathcal{J}$ has local minima and mountain path geometric structures on the constraint $\mathcal{S}(a)$. Therefore, Soave proved the existence of mountain pass solutions and local minimum solutions. After that, Alves, Ji and Miyagaki [3] studies problem ($\mathcal{P}$) with $p \in (2 + \frac{4}{N}, 2^*)$, $q = 2^*$ and $N \geq 3$, there exists $\mu^* > 0$ such that problem ($\mathcal{P}$) admits a couple $(u_a, \lambda_a) \in H^1(\mathbb{R}^N) \times \mathbb{R}^-$ of weak solutions, where u_a is a radial, positive ground state solution of problem ($\mathcal{P}$) on $\mathcal{S}(a)$. In particular, replace $\mu|u|^{p-2}u + |u|^{q-2}u$ by $f(u)$ such that f satisfies some critical growth conditions with $N = 2$, then problem ($\mathcal{P}$) admits a couple $(u_a, \lambda_a) \in H^1(\mathbb{R}^2) \times \mathbb{R}^-$ of weak solutions. Subsequently, Alves, Ji and Miyagaki in [2] introduced a truncation function in $\mathcal{J}$. Precisely, they considered the truncated functional

$$\mathcal{J}_\chi(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{\chi(\|\nabla u\|_2)}{p}\|u\|_p^p,$$

where $\chi \in C_0^\infty(\mathbb{R}^+, [0, 1])$ is nonincreasing and satisfies

$$\chi(t) = \begin{cases} 1, & t \in [0, R_0], \\ 0, & t \in [R_1, \infty). \end{cases}$$

Here R_0 and R_1 are given as in Fig. 1. Also by applying the Gagliardo-Nirenberg inequality (1.1), one shows that

$$\mathcal{J}_\chi(u) \geq h_1(\|\nabla u\|_2),$$

where

$$h_1(t) := \frac{1}{2}t^2 - \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q} t^{q\delta_q} - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} \chi(t) t^{p\delta_p}.$$

Under certain conditions of $a > 0$ and $\mu > 0$, from Fig. 1 the image of h_1 is as Fig. 2, which implies that Alves, Ji and Miyagaki in [2] use a minimax theorem for a class of constrained even functionals that is proved in Jeanjean and Lu [16] to obtain the multiplicity of the solution of the energy functional $\mathcal{J}_\chi$ at the negative energy level. In fact, if $\mathcal{J}_\chi(u) \leq 0$ then $\|\nabla u\|_2 < R_0$, and $\mathcal{J}(v) = \mathcal{J}_\chi(v)$, for all v in a small neighborhood of u in $H^1\left(\mathbb{R}^N\right)$. Therefore, here the critical points of $\mathcal{J}_\chi$ are also are actually the critical points of $\mathcal{J}$.

In addition, this approach turns out to be useful also from the purely mathematical perspective, since it gives a better insight of the properties of the stationary solutions for ($\mathcal{P}$), such as stability or instability, can see [9, 10, 21, 22] and the references therein.

Moreover, we refer to [8], where Bartsch and Willem considered the model problem

$$\begin{cases} -\Delta u = \mu|u|^{q-2}u + \lambda|u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where Ω is a domain of $\mathbb{R}^N$ and $1 < q < 2 < p < 2^*$. The corresponding energy is defined on $H_0^1(\Omega)$ by

$$\varphi_{\lambda,\mu}(u) := \int_\Omega \left[\frac{|\nabla u|^2}{2} - \frac{\mu|u|^q}{q} - \frac{\lambda|u|^p}{p}\right] dx.$$

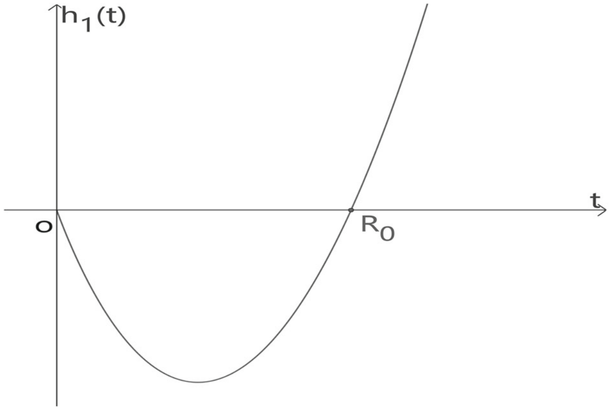

Fig. 2 The relationship between functional h_1 and t.

Then they showed that

- For every $\lambda > 0$, $\mu \in \mathbb{R}$, problem (1.2) has a sequence of solutions $\{u_k\}$ such that $\varphi_{\lambda,\mu}(u_k) \to \infty, k \to \infty$;
- For every $\mu > 0$, $\lambda \in \mathbb{R}$, problem (1.2) has a sequence of solutions $\{v_k\}$ such that $\varphi_{\lambda,\mu}(v_k) < 0$ and $\varphi_{\lambda,\mu}(v_k) \to 0, k \to \infty$.

Inspired by above results, a natural guess that problem ($\mathcal{P}$) also possesses an unbounded sequence of solutions $\{(u_k, \lambda_k)\} \subset H^1(\mathbb{R}^N) \times \mathbb{R}^-$ with $\|u_k\|_2^2 = a^2$ for each $k \in \mathbb{N}^+$, $\|\nabla u_k\|_2^2 \to +\infty$ and $\mathcal{J}(u_k) \to +\infty$ as $k \to +\infty$. So, in this article we attempted to provide a positive answer. The main results of this paper are stated as:

Theorem 1.1 *Assume that $2 < q < \bar{p} < p < 2^*$, $2q + 2N - pN < 0$ and $N \geq 3$. For $a > 0$ and $\mu > 0$ let us also suppose that*

$$\left(\mu a^{(1-\delta_q)q}\right)^{p\delta_p-2}\left(a^{(1-\delta_p)p}\right)^{2-q\delta_q} < \left(\frac{q(p\delta_p-2)}{2C_{N,q}^q(q\delta_q-p\delta_p)}\right)^{p\delta_p-2}\left(\frac{p(2-q\delta_q)}{2C_{N,p}^p(q\delta_q-p\delta_p)}\right)^{2-q\delta_q} \tag{A.1}$$

and

$$\left(\mu a^{(1-\delta_q)q}\right)^{p\delta_p-2}\left(a^{(1-\delta_p)p}\right)^{-q\delta_q} < \left(\frac{q}{\beta_{max}C_{N,q}^q}\right)^{p\delta_p-2}\left(\frac{p}{2C_{N,p}^p}\right)^{q\delta_q}, \tag{A.2}$$

where β_{max} defined in (2.1), then problem ($\mathcal{P}$) possesses an unbounded sequence of pairs of radial solutions $\{(u_k, \lambda_k)\} \subset H^1(\mathbb{R}^N) \times \mathbb{R}^-$ with $\|u_k\|_2^2 = a^2$ for each $k \in \mathbb{N}^+$, $\|\nabla u_k\|_2^2 \to +\infty$ and $\mathcal{J}(u_k) \to +\infty$ as $k \to +\infty$.

It is reasonable to assume ($A.1$) and ($A.2$) in the Theorem 1.1, because when we have $(p\delta_p - 2)(1-\delta_q)q - q\delta_q(1-p)\delta_p > 0$ at $2q + 2N - pN < 0$, then there is $a > 0$ satisfying both assume ($A.1$) and ($A.2$).

Compared with [8], which works in a bounded region and has some compactness, while we work in the whole space and lack compactness, so additional restrictions are needed to ensure compactness. In addition, our method of proving is different, we are not a direct generalization of [8]. Compared with [2], we both adopted the truncation method, but the parts we truncated were different. On the one hand, the energy functional has an extra local term by truncating, which brings some difficulties to our subsequent estimation and compactility proof, while in [2], we can clearly see that when $\|\nabla u\|_2^2$ is bounded, $\mathcal{J}_\chi = \mathcal{J}$ is satisfied, that is, the local term has no effect on the proof. On the other hand, $\|\nabla u\|_2^2$ has at least k solutions at the negative energy level, and we have multiple solutions at the positive energy level.

In the proof of Theorem 1.1 we shall work on the space $H_r^1(\mathbb{R}^N)$, because it has a compact embedding. Moreover, by Palais' principle of symmetric criticality, see [18], we know that the critical points of $\mathcal{J}$ in $H_r^1(\mathbb{R}^N)$ are in fact critical points in whole $H^1(\mathbb{R}^N)$. To prove the Theorem 1.1 we shall adapt for our case a truncation function found in Peral Alonso [19, Chapter 2, Theorem 2.4.6].

2 Preliminaries

We recall the functional h:

$$h(t) := \frac{1}{2}t^2 - \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q} t^{q\delta_q} - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} t^{p\delta_p}.$$

Since $a > 0$, $\mu > 0$ and $q\delta_q < 2 < p\delta_p$, we have that $h(0^+) = 0^-$ and $h(+\infty) = -\infty$. The following proposition states the role of assumption ($A.1$).

Proposition 2.1 ([21, See Lemma 5.1.]) *Under assumption ($A.1$), the function h has a local strict minimum at negative level and a global strict maximum at positive level. Moreover, there exist $0 < R_0 < R_1 < \infty$, depending on $a > 0$ and $\mu > 0$, such that $h(R_0) = 0 = h(R_1)$ and $h(t) > 0$ iff $t \in (R_0, R_1)$, (see Fig. 1).*

Under assumptions to ($A.1$), the function $\hat{h}$ has a global strict maximum at positive level, and there exist $0 < R_1 < R_2 < \infty$, depending on $a > 0$, such that $\hat{h}(R_2) = 0$, where

$$\hat{h}(t) = \frac{1}{2}t^2 - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} t^{p\delta_p},$$

and

$$R_2 = \left(\frac{p}{2C_{N,p}^p}\right)^{\frac{1}{p\delta_p - 2}} a^{-\frac{(1-\delta_p)p}{p\delta_p - 2}}.$$

For $0 < R_0 < R_1 < \infty$, fix $\tau : \mathbb{R}^+ \to [0, 1]$ as being a C^∞ function that satisfies

$$\tau(x) = \begin{cases} 0 \text{ if } & x \le R_0, \\ 1 \text{ if } & x \ge R_1. \end{cases}$$

From proposition 2.1 we know that for the function h, we have R_0 and R_1 dependent of $a > 0$ and $\mu > 0$ such that $h_1(R_0) = 0 = h_1(R_1)$ and $h(t) > 0$ iff $t \in (R_0, R_1)$. For any fix $\mu > 0$, we define the following functional H, denoted by

$$H(R, a) = \frac{1}{2}R^2 - \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q} R^{q\delta_q} - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} R^{p\delta_p} = h(R).$$

For any $a_1, a_2 > 0$ that satisfies $a_1 > a_2$, there is obviously

$$H(R_0(a_2), a_1) > H(R_0(a_2), a_2) = 0 = H(R_1(a_2), a_2) < H(R_1(a_2), a_1).$$

According to the structure of functional h, we can obtain

$$R_0(a_2) > R_0(a_1) \text{ and } R_1(a_2) < R_1(a_1),$$

therefore, $a \mapsto \mathcal{R}(a) := R_1(a) - R_0(a)$ is non-increasing and under the assumption $(A.1)$ $\mathcal{R}(a)$ has a lower bound $\alpha > 0$. Now, under assumption $(A.1)$, for any $a > 0$ we can take τ such that τ' has a uniform upper bound, and we remember that the uniform upper bound is β_1, where we have $\tau'(x) \in [0, \beta_1)$ when $x \in [0, \infty)$ (Rule out that if $a > 0$ is small enough it may not be possible to find τ such that τ' has no uniform upper bound).

By the same token, we have a similar conclusion for any fixed $a > 0$, for any $\mu > 0$ we can take τ such that τ' has a uniform upper bound β_2. Therefore, for any $a > 0$ and $\mu > 0$ we can take τ such that τ' has a uniform upper bound under the assumption $(A.1)$, which we remember

$$\beta_{max} := \max\{\beta_1, \beta_2\}. \tag{2.1}$$

Thus we have $\tau'(x) \in [0, \beta_{max})$ when $x \in [0, \infty)$.

In the sequel, let us consider the truncated functional

$$\mathcal{J}_T(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu\tau(\|\nabla u\|_2)}{p}\|u\|_q^q - \frac{1}{p}\|u\|_p^p.$$

Thus

$$\mathcal{J}_T(u) \ge h_2(\|\nabla u\|_2),$$

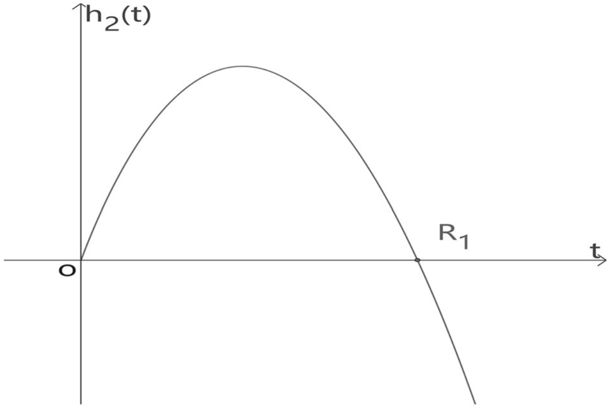

Fig. 3 The relationship between functional h_2 and t.

where

$$h_2(t) = \frac{1}{2}t^2 - \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q} t^{q\delta_q} \tau(t) - \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p} t^{p\delta_p}, \quad \text{(see Fig. 3)}.$$

The truncated functional $\mathcal{J}_T$ and the relationship between $\mathcal{J}_T$ and $\mathcal{J}$ are noteworthy. By reference [2, See Lemma 3.1] we get the following lemma.

Lemma 2.1 *Assume that $N \geq 3, 2 < q < \bar{p} < p < 2^*$ and (A.1) holds, then the functional $\mathcal{J}_T$ has some important properties:*

(i) $\mathcal{J}_T \in C^1(H_r^1(\mathbb{R}^N), \mathbb{R})$.

(ii) *If $\mathcal{J}_T \leq 0$ then $\|\nabla u\|_2^2 \geq R_1$, and $\mathcal{J}(v) = \mathcal{J}_T(v)$, for all v in a small neighborhood of u in $H_r^1(\mathbb{R}^N)$.*

In order to recover some compacity, we will work in $E = H_r^1(\mathbb{R}^N)$, provided with the standard scalar product and norm: $\|u\|_H^2 = \|\nabla u\|_2^2 + \|u\|_2^2$. Here and in the sequel we write $\|u\|_p^p$ to denote the L^p-norm. For convenience, $C_1, C_2, \cdots$ denote various positive constants.

3 Proof of Theorem 1.1

To prove our conclusion, we use the proof technique in [5], but here our nonlinear term does not satisfy the conditions in [5]. The main theorem's proof will follow from several lemmas. We fix a strictly increasing sequence of finite-dimensional linear subspaces $V_n \subset E$ such that $\bigcup_n V_n$ is dense in E.

Lemma 3.1 *([5, See Lemma 2.1.]) For $2 < r < 2^*$ there holds:*

$$\mu_n(r) = \inf_{u \in V_{n-1}^{\perp}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx}{(\int_{\mathbb{R}^N} |u|^r dx)^{2/r}} = \inf_{u \in V_{n-1}^{\perp}} \frac{\|u\|_H^2}{\|u\|_r^2} \to \infty \; as \; n \to \infty.$$

Here we give the following definition

$$\int_{\mathbb{R}^N} F(u)dx = \frac{\mu\tau(\|\nabla u\|_2)}{q}\|u\|_q^q + \frac{1}{p}\|u\|_p^p. \tag{3.1}$$

We introduce now the constant

$$K = \max_{u\in H_r^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} F(u)dx}{\|u\|_p^p + \|u\|_q^q},$$

which is well defined when combined with (3.1). For $n \in \mathbb{N}$ we define

$$\rho_n = \frac{M_n^{p/(p-2)}}{L^{2/(p-2)}},$$

where

$$M_n = [\mu_n(q)^{-q/2} + \mu_n(p)^{-p/2}]^{-2/p} \quad \text{and} \quad L = 3K \max_{\theta>0} \frac{(a+\theta^2)^{p/2}}{a+\theta^p}.$$

By Lemma 3.1 we have $\rho_n \to \infty$ as $n \to \infty$. We also define

$$B_n := \left\{u \in V_{n-1}^{\perp} \cap \mathcal{S}(a) : \|\nabla u\|_2^2 = \rho_n\right\},$$

here $V_{n-1}^{\perp}$ is the orthogonal complement of V_{n-1}. Then we have:

Lemma 3.2 $b_n = \inf_{u\in B_n} \mathcal{J}_T(u) \to \infty$ *as* $n \to \infty$.

Proof For any $a > 0$ and $u \in B_n$, because of $\rho_n \to \infty$ as $n \to \infty$, we have $\|\nabla u\|_2^2 + a^2 > 1$ when n is large enough. Since $2 < q < \bar{p} < p < 2^*$ we deduce using the preceding lemma with $r = p$ and $r = q$,

$$\begin{aligned}
\mathcal{J}_T(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu\tau(\|\nabla u\|_2)}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - K\|u\|_q^q - K\|u\|_p^p \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{K}{\mu_n(q)^{q/2}}\left(\|\nabla u\|_2^2 + a^2\right)^{q/2} - \frac{K}{\mu_n(p)^{p/2}}\left(\|\nabla u\|_2^2 + a^2\right)^{p/2} \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - K\left[\mu_n(q)^{-q/2} + \mu_n(p)^{-p/2}\right]\left(\|\nabla u\|_2^2 + a^2\right)^{p/2} \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{K}{M_n^{p/2}}\left(\|\nabla u\|_2^2 + a^2\right)^{p/2} \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{L}{3M_n^{p/2}}\left(\|\nabla u\|_2^p + a^2\right) \\
&\geq \frac{1}{2}\rho_n - \frac{L}{3M_n^{p/2}}\rho_n^{p/2}
\end{aligned}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) \rho_n \to \infty.$$

The proof of the lemma is complete. □

Lemma 3.3 *Assume that* $N \geq 3$ *and* $2 < q < \bar{p} < p < 2^*$. *Then there exists* $0 < \rho_0 < R_0^2$ *such that*

$$0 < \sup_{u \in M_1} \mathcal{J}_T(u) < b_0 := \inf_{u \in M_2} \mathcal{J}_T(u), \tag{3.2}$$

where

$$M_1 := \left\{u \in \mathcal{S}(a), \|\nabla u\|_2^2 \leq \rho_0/2\right\}, \quad M_2 := \left\{u \in \mathcal{S}(a), \|\nabla u\|_2^2 = \rho_0\right\}.$$

Proof Now, using equation (3.1) and Gagliardo-Nirenberg inequality (1.1) and taking into account that $\|u\|_2^2 = a^2$

$$\begin{aligned}\int_{\mathbb{R}^N} F(u)dx &= \frac{\mu\tau(\|\nabla u\|_2)}{q}\|u\|_q^q + \frac{1}{p}\|u\|_p^p \\ &\leq \frac{\mu C_{N,q}^q a^{(1-\delta_q)q}}{q}\|\nabla u\|_2^{q\delta_q}\tau(\|\nabla u\|_2) + \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p}\|\nabla u\|_2^{p\delta_p}.\end{aligned}$$

Then we have for $\|\nabla u\|_2 < R_0$ small enough,

$$\int_{\mathbb{R}^N} F(u)dx \leq \frac{C_{N,p}^p a^{(1-\delta_p)p}}{p}\|\nabla u\|_2^{p\delta_p}. \tag{3.3}$$

Next, let $0 < \rho < R_0$ be arbitrary but fixed and suppose $u, v \in \mathcal{S}(a)$ are such that $\|\nabla u\|_2^2 \leq \rho/2$ and $\|\nabla v\|_2^2 = \rho$. Then, for $\rho > 0$ small enough

$$\begin{aligned}\mathcal{J}_T(v) - \mathcal{J}_T(u) &= \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} F(v)dx + \int_{\mathbb{R}^N} F(u)dx \\ &\geq \frac{\rho}{4} - \int_{\mathbb{R}^N} F(v)dx \\ &\geq \frac{\rho}{4} - C\rho^{p\delta_p} \\ &\geq \frac{\rho}{8},\end{aligned}$$

using (3.3) and $p\delta_p = \frac{N(q-2)}{2} > 2$. The proof of the lemma is complete. □

In order to set up a min-max scheme, let

$$\varphi : \mathbb{R} \times E \to E, \ \varphi(s, u) = s * u,$$

be the action of the group $\mathbb{R}$ on E defined by

$$(s * u)(x) = e^{sN/2}u(e^s x) \text{ for } s \in \mathbb{R}, u \in E, x \in \mathbb{R}^N.$$

Observe that $s * u \in \mathcal{S}(a)$ if $u \in \mathcal{S}(a)$, and that for $u \in \mathcal{S}(a)$

$$\|\nabla(s * u)\|_2^2 \to 0, \ \mathcal{J}_T(s * u) \to 0 \quad \text{as} \quad s \to -\infty. \tag{3.4}$$

Moreover, $\int_{\mathbb{R}^N} F(u)dx \geq \frac{1}{p}\|u\|_p^p$ for all $u \in H^1(\mathbb{R}^N)$, and therefore

$$\begin{aligned}
\mathcal{J}_T(s * u) &= \frac{1}{2}\|\nabla(s * u)\|_2^2 - \int_{\mathbb{R}^N} F(s * u)dx \\
&\leq \frac{1}{2}\|\nabla(s * u)\|_2^2 - \frac{1}{p}\|s * u\|_p^p \\
&= \frac{e^{2s}}{2}\|\nabla u\|_2^2 - \frac{e^{-Ns}e^{(Nsp)/2}}{p}\|u\|_p^p \to -\infty \quad \text{as} \quad s \to \infty,
\end{aligned}$$

because $-Ns + (Nsp)/2 > 2s$. As a consequence we obtain for $u \in \mathcal{S}(a)$ that

$$\|\nabla(s * u)\|_2^2 \to \infty, \ \mathcal{J}_T(s * u) \to -\infty \quad \text{as} \quad s \to \infty. \tag{3.5}$$

Due to (3.4) and (3.5), there exists $s_n > 0$ such that

$$\tilde{\gamma}_n : [0, 1] \times (\mathcal{S}(a) \cap V_n) \to \mathcal{S}(a), \ \tilde{\gamma}_n(t, u) = (2s_n t - s_n) * u,$$

satisfies (with ρ_0, b_0 from Lemma 3.3, b_n from Lemma 3.2):

$$\|\nabla\tilde{\gamma}_n(0, u)\|_2^2 < \rho_0 < \rho_n, \ \|\nabla\tilde{\gamma}_n(1, u)\|_2^2 > \rho_n,$$

and

$$0 < \mathcal{J}_T(\tilde{\gamma}_n(0, u)) < \max\{b_0, b_n\}, \ \mathcal{J}_T(\tilde{\gamma}_n(1, u)) < b_n, \tag{3.6}$$

uniformly for $u \in \mathcal{S}(a) \cap V_n$. Now we define

$$\Gamma_n := \left\{ \gamma : [0, 1] \times (\mathcal{S}(a) \cap V_n) \to \mathcal{S}(a) \,\middle|\, \begin{array}{l} \gamma \text{ is continuous, odd in } u, \\ \gamma(0, u) = \tilde{\gamma}_n(0, u), \ \gamma(1, u) = \tilde{\gamma}_n(1, u) \end{array} \right\}.$$

Clearly we have $\tilde{\gamma}_n \in \Gamma_n$. Here we define the mountain pass value

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{\substack{t \in [0,1] \\ u \in \mathcal{S}(a) \cap V_n}} \mathcal{J}_T(\gamma(t, u)).$$

For the sake of subsequent lemmas, in the following we recall some properties of the cohomological index for spaces with an action of the group $G = \{-1, 1\}$. This index goes back to [11] and has been used in a variational setting in [12]. It associates to a G-space X an element $i(X) \in \mathbb{N}_0 \cup \{\infty\}$. We only need the following properties.

(I_1) If G acts on $\mathbb{S}^{n-1}$ via multiplication, then $i(\mathbb{S}^{n-1}) = n$.

(I_2) If there exists an equivariant map $X \to Y$, then $i(X) \leq i(Y)$.

(I_3) Let $X = X_0 \cup X_1$ be metrisable and $X_0, X_1 \subset X$ be closed G-invariant subspaces. Let Y be a G-space, and consider a continuous map $\phi : [0, 1] \times Y \to X$ such that each $\phi_t = \phi(t, \cdot) : Y \to X$ is equivariant. If $\phi_0(Y) \subset X_0$ and $\phi_1(Y) \subset X_1$, then

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \geq i(Y).$$

Properties (I_1) and (I_2) are standard and hold also for the Krasnoselskii genus. Property (I_3) has been proven in [4, Corollary 4.11, Remark 4.12].

We now need the following linking property, and it is proved by the above properties.

Lemma 3.4 *For every $\gamma \in \Gamma_n$, there exists $(t, u) \in [0, 1] \times (\mathcal{S}(a) \cap V_n)$ such that $\gamma(t, u) \in B_n$.*

Proof Let $\mathcal{T}_{n-1} : E \to V_{n-1}$ be the orthogonal projection, and set

$$h_n : \mathcal{S}(a) \to V_{n-1} \times \mathbb{R}^+, \quad u \mapsto (\mathcal{T}_{n-1}u, \|\nabla u\|_2^2).$$

Then clearly $B_n = h_n^{-1}(0, \rho_n)$. We fix $\gamma \in \Gamma_n$ and consider the map

$$\phi = h_n \circ \gamma : [0, 1] \times (\mathcal{S}(a) \cap V_n) \to V_{n-1} \times \mathbb{R}^+ =: X.$$

Since

$$\phi_0(\mathcal{S}(a) \cap V_n) \subset V_{n-1} \times (0, \rho_n] =: X_0$$

and

$$\phi_1(\mathcal{S}(a) \cap V_n) \subset V_{n-1} \times (\rho_n, \infty] =: X_1,$$

it follows from (I_1) to (I_3) that

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \geq i(\mathcal{S}(a) \cap V_n) = \dim V_n.$$

If there would not exist $(t, u) \in [0, 1] \times (\mathcal{S}(a) \cap V_n)$ with $\gamma(t, u) \in B_n$, then

$$\mathrm{Im}(\phi) \cap X_0 \cap X_1 \subset (V_{n-1} \setminus 0) \times \{\rho_n\}.$$

Now (I_1) and (I_2) imply that

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \leq i((V_{n-1} \setminus 0) \times \{\rho_n\}) = \dim V_{n-1},$$

contradicting $\dim V_{n-1} < \dim V_n$. □

It follows from Lemma 3.3 that

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{\substack{t \in [0,1] \\ u \in \mathcal{S}(a) \cap V_n}} \mathcal{J}_T(\gamma(t,u)) \geq b_n = \inf_{u \in B_n} \mathcal{J}_T(u) \to \infty. \tag{3.7}$$

Clearly by (3.2) and (3.6) there also holds

$$c_n \geq b_0 > 0. \tag{3.8}$$

We recall the stretched functional from [15], see also [13]:

$$\bar{\mathcal{J}}_T : \mathbb{R} \times E \to \mathbb{R}, \quad (s, u) \mapsto \mathcal{J}_T(s * u).$$

Now we define

$$\bar{\Gamma}_n := \left\{ \bar{\gamma} : [0,1] \times (\mathcal{S}(a) \cap V_n) \to \mathbb{R} \times \mathcal{S}(a) \,\middle|\, \begin{array}{l} \bar{\gamma} \text{ is continuous, odd in } u, \\ \varphi \circ \bar{\gamma} \in \Gamma_n \end{array} \right\},$$

where $\varphi(s, u) = s * u$ and

$$\bar{c}_n = \inf_{\bar{\gamma} \in \bar{\Gamma}_n} \max_{\substack{t \in [0,1] \\ u \in \mathcal{S}(a) \cap V_n}} \bar{\mathcal{J}}_T(\bar{\gamma}(t,u)).$$

Reference [5, Lemma 2.5], we also have conclusions $c_n = \bar{c}_n$ for c_n and $\bar{c}_n$.

Next, we will show that c_n is a critical value of $\mathcal{J}_T$, which is an important part of the proof of Theorem 1.1. We fix n from now on.

Lemma 3.5 *There exists a Palais-Smale sequence $\{u_k^n\}$ for $\mathcal{J}_T$ at the level c_n satisfying $P(u_k^n) \to 0$ as $k \to \infty$, where*

$$P(u) = \|\nabla u\|_2^2 - \delta_q \mu \tau(\|\nabla u\|_2) \|u\|_q^q - \frac{\mu \tau'(\|\nabla u\|_2)}{q} \|\nabla u\|_2 \|u\|_q^q - \delta_p \|u\|_p^p. \tag{3.9}$$

Proof For $\gamma \in \Gamma_n$ there holds by (3.6), (3.7), (3.8), and the definition of Γ_n:

$$\begin{aligned} c_n \geq \max\{b_0, b_n\} &> \max\left\{ \max_{u \in \mathcal{S}(a) \cap V_n} \mathcal{J}_T(\tilde{\gamma}_n(0,u)), \max_{u \in \mathcal{S}(a) \cap V_n} \mathcal{J}_T(\tilde{\gamma}_n(1,u)) \right\} \\ &= \max\left\{ \max_{u \in \mathcal{S}(a) \cap V_n} \mathcal{J}_T(\gamma(0,u)), \max_{u \in \mathcal{S}(a) \cap V_n} \mathcal{J}_T(\gamma(1,u)) \right\}. \end{aligned}$$

Using the fact $c_n = \bar{c}_n$ we obtain a sequence $\{\gamma_k^n\}$ in Γ_n such that

$$\max_{[0,1] \times (\mathcal{S}(a) \cap V_n)} \bar{\mathcal{J}}_T \left(0, \gamma_k^n\right) \to c_n.$$

Now Ekeland's variational principle implies the existence of a Palais-Smale sequence $\{(s_k^n, u_k^n)\}$ for $\bar{\mathcal{J}}_T|_{\mathbb{R}\times\mathcal{S}(a)}$ at the level c_n such that $s_k^n \to 0$. From $\bar{\mathcal{J}}_T(s,u) = \bar{\mathcal{J}}_T(0, s*u)$ and for every $\psi \in H^1(\mathbb{R}^N)$ we deduce

$$\left(\partial_s \bar{\mathcal{J}}_T\right)(s,u) = \left(\partial_s \bar{\mathcal{J}}_T\right)(0, s*u) \quad \text{and} \quad \left(\partial_u \bar{\mathcal{J}}_T\right)(s,u)[\psi] = \left(\partial_u \bar{\mathcal{J}}_T\right)(0, s*u)[s*\psi]$$

so that $\left\{\left(0, s_k^n * u_k^n\right)\right\}$ is also a Palais-Smale sequence for $\bar{\mathcal{J}}_T|_{\mathbb{R}\times S}$ at the level c_n. Thus we may assume that $s_k^n = 0$. This implies, firstly, that $\left\{u_k^n\right\}$ is a PalaisSmale sequence for $\mathcal{J}_T$ at the level c_n, and secondly, using $\partial_s \bar{\mathcal{J}}_T\left(0, u_k^n\right) \to 0$ as $k \to \infty$, that is $P(u_k^n) \to 0$ as $k \to \infty$. □

Lemma 3.6 *Assume that $N \geq 3$ and $2 < q < \bar{p} < p < 2^*$. (A.2) is satisfied for any $a > 0$ and $\mu > 0$, if the sequence $\{u_k\}$ in $\mathcal{S}(a)$ satisfies $\mathcal{J}_T'(u_k) \to 0$, $\mathcal{J}_T(u_k) \to c > 0$, and $P(u_k) \to 0$ as $k \to \infty$, then it is bounded in E and has a convergent subsequence.*

Proof **Claim 1:** The sequence $\{u_k\}$ is bounded in E.

Suppose $\{u_k\}$ is unbounded, that is, $\|\nabla u_k\|_2^2 \to \infty$ as $k \to \infty$. As $P(u_k) \to 0$ as $k \to \infty$, we observe that

$$\|u_k\|_p^p = \frac{1}{\delta_p}\|\nabla u_k\|_2^2 - \frac{\mu\delta_q \tau(\|\nabla u_k\|_2)}{\delta_p}\|u_k\|_q^q - \frac{\mu\tau'(\|\nabla u_k\|_2)}{q\delta_p}\|\nabla u_k\|_2\|u_k\|_q^q + o(1).$$

Whence

$$\begin{aligned}\mathcal{J}_T(u_k) = &\left(\frac{1}{2} - \frac{1}{p\delta_p}\right)\|\nabla u_k\|_2^2 - \frac{\mu}{q}\left(1 - \frac{q\delta_q}{p\delta_p}\right)\tau(\|\nabla u_k\|_2)\|u_k\|_q^q \\ &+ \frac{\mu\tau'(\|\nabla u_k\|_2)}{qp\delta_p}\|\nabla u_k\|_2\|u_k\|_q^q + o(1),\end{aligned}$$

by the Gagliardo-Nirenberg inequality (1.1) and $\tau'(x) \in [0, \beta_{max})$ we have that

$$c + 1 \geq \mathcal{J}_T(u_k) \geq \left(\frac{1}{2} - \frac{1}{p\delta_p}\right)\|\nabla u_k\|_2^2 - \frac{\mu}{q}\left(1 - \frac{q\delta_q}{p\delta_p}\right)\tau(\|\nabla u_k\|_2) C_{N,q}^q a^{(1-\delta_q)q}\|\nabla u_k\|_2^{q\delta_q},$$

this implies that

$$\|\nabla u_k\|_2^2 \leq C a^{(1-\delta_q)q}\tau(\|\nabla u_k\|_2^2)\|\nabla u_k\|_2^{q\delta_q} + C,$$

since $q\delta_q < 2$, the boundedness of $\{u_k\}$ follows also in this case.

As $\{u_k\}$ is bounded in $H_r^1(\mathbb{R}^N)$, and $H_r^1(\mathbb{R}^N) \hookrightarrow L^l(\mathbb{R}^N)$ compactly for $l \in (2, 2^*)$, there exists $u \in H_r^1(\mathbb{R}^N)$ such that up to a subsequence

$$u_k \rightharpoonup u \text{ in } H_r^1(\mathbb{R}^N),\ u_k \to u \text{ in } L^l(\mathbb{R}^N) \text{ and } u_k \to u \ a.e. \text{ in } \mathbb{R}^N.$$

Claim 2: The weak limit u is nontrivial, that is, $u \not\equiv 0$.

Since $\mathcal{J}_T(u_k) \to c \neq 0$, using the fact that $P(u_k) \to 0$ and $\tau'(x) \in [0, \beta_{max})$ for all $x \in \mathbb{R}$, we had $u_k \to 0$ we would find by strong $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ convergence that

$$\begin{aligned}\mathcal{J}_T(u_k) &= \frac{\mu}{q}\left(\frac{q\delta_q}{2}-1\right)\tau(\|\nabla u_k\|_2)\|u_k\|_q^q + \frac{\mu\tau'(\|\nabla u_k\|_2)}{2q}\|\nabla u_k\|_2\|u_k\|_q^q \\ &\quad - \frac{1}{p}\left(\frac{p\delta_q}{2}-1\right)\|u_k\|_p^p + o(1) \to 0,\end{aligned}$$

that is a contradiction.

Claim 3: $\lambda_k \to \lambda < 0$.

By Willem [24, Proposition 5.12], there exists $\{\lambda_k\} \subset \mathbb{R}$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla u_k \cdot \nabla\psi dx - \mu\tau(\|\nabla u_k\|_2)\int_{\mathbb{R}^N} |u_k|^{q-2}u_k\psi dx - \int_{\mathbb{R}^N} |u_k|^{p-2}u_k\psi dx \\ &- \|u_k\|_p^p \frac{\mu\tau'(\|\nabla u_k\|_2)}{p}\left(\int_{\mathbb{R}^N} \nabla u_k \cdot \nabla\psi dx\right)^{\frac{1}{2}} = \int_{\mathbb{R}^N} \lambda_k u_k\psi dx + o(1)\|\psi\|_H,\end{aligned} \tag{3.10}$$

for every $\psi \in H^1(\mathbb{R}^N)$, where $o(1) \to 0$ as $n \to \infty$. The choice $\psi = u_k$ provides

$$\|\nabla u_k\|_2^2 - \mu\tau(\|\nabla u_k\|_2)\|u_k\|_q^q - \frac{\mu\tau'(\|\nabla u_k\|_2)}{q}\|\nabla u_k\|_2\|u_k\|_q^q - \|u_k\|_p^p = \lambda_k a^2 + o(1).$$

Recalling that $P(u_k) \to 0$, we have

$$\lambda_k a^2 = \mu(\delta_q - 1)\tau(\|\nabla u_k\|_2)\|u_k\|_q^q + (\delta_p - 1)\|u_k\|_p^p + o(1), \tag{3.11}$$

since $0 < \delta_q, \delta_p < 1$, we deduce that $\{\lambda_k\}$ is bounded and $\lambda_k \leq 0$. We now claim that

$$\lim_{k\to\infty} \|\nabla u_k\|_2 = A > 0.$$

If not, from Gagliardo-Nirenberg inequality (1.1) we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |u_k|^l dx \to 0 \ \text{ for } \ l \in (2, 2^*),$$

then

$$0 \neq c = \lim_{n\to\infty} \mathcal{J}_T(u_k) = \lim_{n\to\infty}\left[\frac{1}{2}\|\nabla u_k\|_2^2 - \frac{\mu\tau(\|\nabla u_k\|_2)}{q}\|u_k\|_q^q - \frac{1}{p}\|u_k\|_p^p\right] = 0.$$

Next, we proved that up to a subsequence $\lambda_k \to \lambda < 0$. Using the strong $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ convergence of $\{u_k\}$, by (3.11) we have that

$$\lambda a^2 = (\delta_q - 1)\tau(A)\|u\|_q^q + (\delta_p - 1)\|u\|_p^p,$$

since $0 < \delta_q, \delta_p < 1$ and $\tau(A) \geq 0$, we must have $\lambda < 0$.
Claim 4: $u_k \to u$ in $H_r^1(\mathbb{R}^N)$.

Up to a subsequence, let $\lim_{n\to\infty} \|\nabla u_k\|_2^2 = A^2 > 0$. Then, u satisfies

$$\|\nabla u\|_2^2 - \mu\tau(A)\|u\|_q^q - \frac{\mu\tau'(A)}{q}\|\nabla u\|_2\|u\|_q^q - \|u\|_p^p = \lambda\|u\|_2^2. \tag{3.12}$$

By (3.10) and (3.12), we obtain

$$\begin{aligned}
&\|\nabla u\|_2^2 - \mu\tau(A)\|u\|_q^q - \frac{\mu\tau'(A)}{p}\|\nabla u\|_2\|u\|_q^q - \lambda\|u\|_2^2 \\
&= \|u\|_p^p = \lim_{k\to\infty} \|u_k\|_p^p \\
&= \lim_{k\to\infty}\left[\|\nabla u_k\|_2^2 - \mu\tau(\|\nabla u_k\|_2)\|u_k\|_q^q - \frac{\mu\tau'(\|\nabla u_k\|_2)}{q}\|\nabla u_k\|_2\|u_k\|_q^q - \lambda\|u_k\|_2^2\right] \\
&\geq A^2 - \mu\tau(A)\|u\|_q^q - \frac{\mu\tau'(A)}{p}A\|u\|_q^q - \lambda\|u\|_2^2.
\end{aligned}$$

We claim that $1 - \|u\|_q^q \frac{\mu\tau'(A)}{q} > 0$. If not, then $q \leq \mu\tau'(A)\|u\|_q^q$. From the properties of function τ, we have the following two cases.

Case 1: If $A \in (0, R_0] \cup [R_1, +\infty)$ then $\tau'(A) = 0$, we get a contradiction

$$0 < q \leq \mu\tau'(A)\|u\|_q^q = 0.$$

Case 2: If $A \in [R_0, R_1]$ then $\tau'(x) \in [0, \beta_{max})$, by Proposition 2.1 and Gagliardo-Nirenberg inequality (1.1) we have

$$\begin{aligned}
q \leq \mu\tau'(A)\|u\|_q^q &\leq \mu\beta_{max} C_{N,q}^q a^{(1-\delta_q)q}\|\nabla u\|_2^{q\delta_q} \\
&\leq \mu\beta_{max} C_{N,q}^q a^{(1-\delta_q)q} R_2^{q\delta_q} \\
&\leq \mu\beta_{max} C_{N,q}^q \left(\frac{p}{2C_{N,p}^p}\right)^{\frac{q\delta_q}{p\delta_p-2}} a^{-q\delta_q\frac{(1-\delta_p)p}{p\delta_p-2}} a^{(1-\delta_q)q},
\end{aligned}$$

this contradicts condition (*A*.2).

Then we can deduce that $A = \|\nabla u\|_2$ and $\|u\|_2^2 = a^2$. Up to a subsequence, $u_n \to u$ strongly in $H_r^1(\mathbb{R}^N)$. □

Remark 3.1 When the formulas (*A*.1)) and (*A*.2) are satisfied, according to (3.7), Lemma 3.5 and 3.6 we know that the functional $\mathcal{J}_T$ has an unbounded sequence of pairs of radial solutions $\{(u_k, \lambda_k)\} \subset H^1(\mathbb{R}^N) \times \mathbb{R}^-$.

Proof of Theorem 1.1 From Remark 3.1, we know the functional $\mathcal{J}_T$ has an unbounded sequence of pairs of radial solutions $\{(u_k, \lambda_k)\} \subset H^1(\mathbb{R}^N)\times\mathbb{R}^-$, where $\|\nabla u_k\|_2^2 \to \infty$ as $k \to \infty$. By Lemma 2.1 we can see that $\mathcal{J}_T(u) = \mathcal{J}_T(u)$ when $\|\nabla u\|_2^2 \geq R_1$. Thus, we can obtain an unbounded subsequence, still denoted as $\{(u_k, \lambda_k)\}$, which is an unbounded sequence of pairs of radial solutionsan of the problem ($\mathcal{P}$). □

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