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Extremal Graphs for the $K_{1,2}$-Isolation Number of Graphs

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Abstract

For any non-negative integer k and any graph G, a subset $S \subseteq V(G)$ is said to be a $K_{1,k+1}$-isolating set of G if $G - N[S]$ does not contain $K_{1,k+1}$ as a subgraph. The $K_{1,k+1}$-isolation number of G, denoted by $\iota_k(G)$, is the minimum cardinality of a $K_{1,k+1}$-isolating set of G. Recently, Zhang and Wu (2021) proved that if G is a connected n-vertex graph and $G \notin \{P_3, C_3, C_6\}$, then $\iota_1(G) \leq \frac{2}{7}n$. In this paper, we characterize all extremal graphs attaining this bound, which resolves a problem proposed by Zhang and Wu (Discrete Appl Math 304:365–374, 2021).

Keywords Partial domination · Isolating set · Isolation number · Extremal graphs

Mathematics Subject Classification 05C69 · 05C35

1 Introduction

In this paper we only consider finite graphs without loops or multiple edges. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For any $v \in V(G)$, the *open neighborhood* $N_G(v)$ of v is the set of neighbors of v in G, and the *closed neighborhood* $N_G[v]$ of v is the set $N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $d_G(v) = |N_G(v)|$, and the maximum degree of G is denoted by $\Delta(G)$. The *open neighborhood* of a subset $S \subseteq V(G)$ is the set $N_G(S) = \bigcup_{v \in S} N_G(v) \backslash S$,

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and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. For two distinct vertices $u, v \in V(G)$, let $d_G(u, v)$ denote the distance between u and v in G. For any $S \subseteq V(G)$, we denote by $G[S]$ and $G - S$ the subgraphs of G induced by S and $V(G) \backslash S$, respectively, and write $G - s$ instead of $G - \{s\}$ if $S = \{s\}$. For two disjoint subsets $S, T \subseteq V(G)$, let $E_G(S, T)$ denote the set of edges of G with one endvertex in S and the other endvertex in T. When the graph is clear from the context, we may omit the subscript G from the notation. We use P_n, C_n, K_n and $K_{1,n-1}$ to denote the path, the cycle, the complete graph and the star with n vertices, respectively. Let C_n^+ be the graph obtained by attaching a pendant edge to one vertex of C_n, and K_n^- the graph obtained from K_n by removing one edge. For any positive integer k, we denote by $[k]$ the set $\{1, 2, \dots, k\}$. We write $A := B$ to rename B as A.

Let G be a graph and $\mathcal{F}$ be a family of graphs. A subset $D \subseteq V(G)$ is said to be an $\mathcal{F}$-isolating set of G if $G - N[D]$ does not contain any member of $\mathcal{F}$ as a subgraph. The *$\mathcal{F}$-isolation number* of G, denoted by $\iota(G, \mathcal{F})$, is the minimum cardinality of an $\mathcal{F}$-isolating set of G. This concept was recently introduced by Caro and Hansberg [5] as a natural extension of the classical domination problem [7–10]. Recall that a subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) \setminus D$ has at least one neighbor in D. The *domination number* of G is the minimum cardinality of a dominating set of G. One can easily see that the $\{K_1\}$-isolation number is just the domination number. For the sake of brevity, for any non-negative integer k, a $\{K_{1,k+1}\}$-isolating set of G will be called a *$K_{1,k+1}$-isolating set* or a *k-isolating set* of G, and the $\{K_{1,k+1}\}$-isolation number of G will be called the *$K_{1,k+1}$-isolation number* or the k-isolation number of G. Moreover, we simply write $\iota_k(G)$ instead of $\iota(G, \{K_{1,k+1}\})$. Thus, a set $D \subseteq V(G)$ is a 0-isolating set of G if $G - N[D]$ consists of isolated vertices only (i.e., $V(G) \setminus N[D]$ is an independent set), and a set $D \subseteq V(G)$ is a 1-isolating set of G if $G - N[D]$ consists of isolated vertices and isolated edges only (i.e., every component of $G - N[D]$ contains at most two vertices).

There have been some interesting results about $\mathcal{F}$-isolation number of graphs. Caro and Hansberg [5] proved that $\iota_k(G) \le \frac{n}{k+2}$ for every n-vertex graph G, and moreover, $\iota_k(G) \le \frac{n}{k+3}$ when $G \ncong K_{1,k+1}$ is an n-vertex tree. It was also shown in [5] that if $G \ncong C_5$ is a connected graph with $n \ge 3$ vertices, then $\iota_0(G) \le \frac{n}{3}$. Borg, Fenech and Kaemawichanurat [2] proved that if G is a connected n-vertex graph, then $\iota(G, \{K_k\}) \le \frac{n}{k+1}$ unless $G \cong K_k$, or $k = 2$ and $G \cong C_5$. Borg [1] showed that if $G \ncong C_3$ is a connected n-vertex graph, then $\iota(G, \mathcal{C}) \le \frac{n}{4}$ where $\mathcal{C} = \{C_k : k \ge 3\}$. Yan [14] proved that $\iota(G, \{K_4^-\}) \le \frac{n}{5}$ for every connected graph G with $n \ge 10$ vertices. More results on $\mathcal{F}$-isolation number of graphs can be found in [3, 4, 6, 11–13, 15, 17].

In 2021, Zhang and Wu [16] investigated $\iota_1(G)$ for general connected graphs and derived the following result.

Theorem 1.1 (Zhang and Wu [16]) *If G is a connected n-vertex graph and $G \notin \{P_3, C_3, C_6\}$, then $\iota_1(G) \le \frac{2}{7}n$. Moreover, this bound is sharp.*

At the end of their paper, Zhang and Wu [16, Problem 3.2] asked for a complete characterization of all extremal graphs attaining the bound stated in Theorem 1.1. In this paper, we resolve this problem. For this purpose, we need to define several families of graphs.

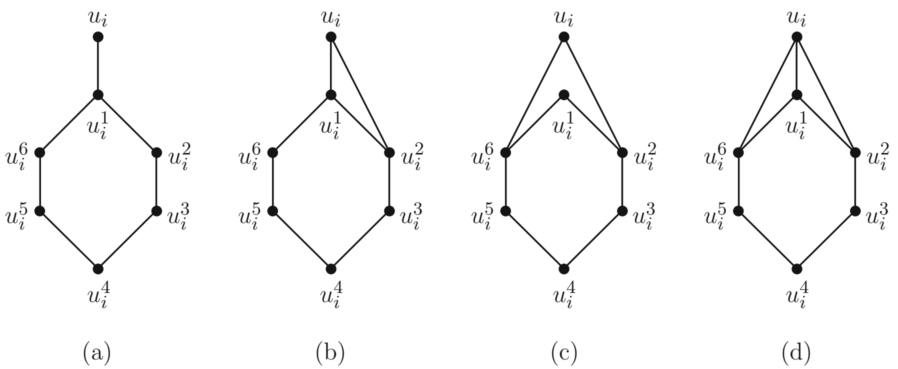

Fig. 1 The four ways of joining u_i to H_i

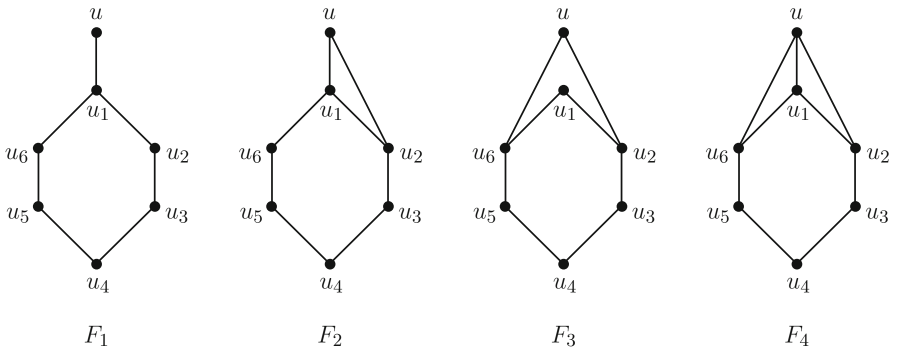

Fig. 2 The graphs in $\mathcal{F}$

For any positive integer t, let F be an arbitrary connected t-vertex graph with $V(F) = \{u_i : 1 \leq i \leq t\}$ and let $H_1, H_2, \ldots, H_t$ be t vertex-disjoint copies of C_6 such that $H_i := u_i^1 u_i^2 u_i^3 u_i^4 u_i^5 u_i^6 u_i^1$ for each $i \in [t]$. Let G be the graph obtained from F and $H_1, H_2, \ldots, H_t$, in which u_i is joined to H_i (for each $i \in [t]$) with one of the following four ways:

(i) u_i is only adjacent to u_i^1 (see Fig. 1a);
(ii) u_i is only adjacent to u_i^1 and u_i^2 (see Fig. 1b);
(iii) u_i is only adjacent to u_i^2 and u_i^6 (see Fig. 1c);
(iv) u_i is only adjacent to u_i^1, u_i^2 and u_i^6 (see Fig. 1d).

For each $i \in [t]$, we say that $G_i := G[V(H_i) \cup \{u_i\}]$ is a C_6-*constituent* of G and u_i is the C_6-*connection of* G_i in G. Let $\mathcal{F}$ be the set of graphs depicted in Fig. 2. Then, it is easy to observe that $G_i \in \mathcal{F}$ for each $i \in [t]$ under isomorphism.

Let $\mathcal{G}_1$ be the set of all graphs which can be constructed from the above ways. (It is clear that $\mathcal{F} \subseteq \mathcal{G}_1$.) Let $\mathcal{G}_2$ and $\mathcal{G}_3$ be the sets of graphs depicted in Figs. 3 and 4, respectively.

We can now state the main result of this paper.

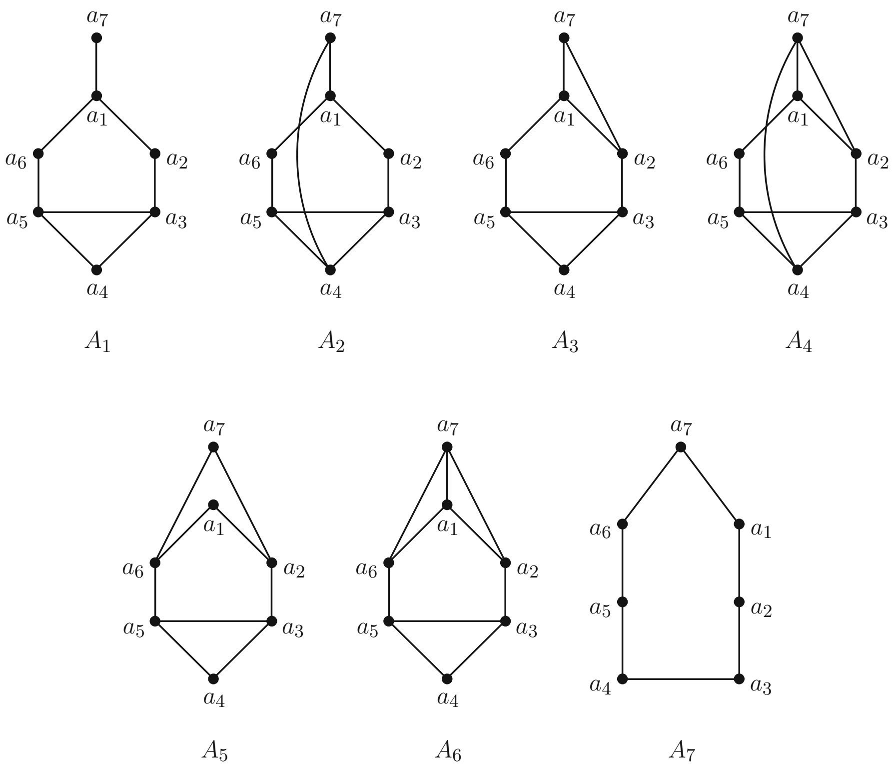

Fig. 3 The graphs in $\mathcal{G}_2$

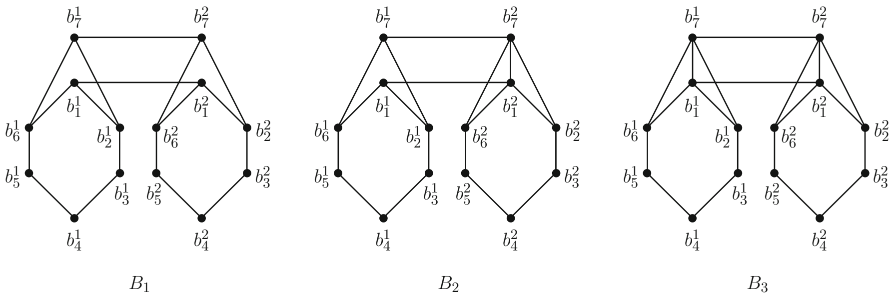

Fig. 4 The graphs in $\mathcal{G}_3$

Theorem 1.2 *For any connected n-vertex graph G, we have* $\iota_1(G) = \frac{2}{7}n$ *if and only if* $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

The rest of the paper is organized as follows. In Sect. 2, we state several known lemmas and introduce some structural properties of the graphs in $\mathcal{F}$, $\mathcal{G}_2$ and $\mathcal{G}_3$. In Sect. 3, we give the proof of Theorem 1.2.

2 Preliminaries

In this section, we state several lemmas and observations which will be used in the next section to prove the main result of this paper.

The following five lemmas were proved in [1, 5, 16].

Lemma 2.1 (Caro and Hansberg [5]) $\iota_1(P_n) = \lceil \frac{n-2}{5} \rceil$ *for every* $n \geq 2$.

Lemma 2.2 (Caro and Hansberg [5]) $\iota_1(C_n) = \lceil \frac{n}{5} \rceil$ *for every* $n \geq 3$.

Lemma 2.3 (Zhang and Wu [16]) *Let* G *be a graph and* S *be a vertex subset of* G. *If* D *is a dominating set of* $G[S]$, *then* $\iota_1(G) \leq |D| + \iota_1(G - S)$.

Lemma 2.4 (Zhang and Wu [16]) *Let* G *be a graph and* S *be a vertex subset of* G. *If* $G[S]$ *has a* 1*-isolating set* D *such that* $E(S \backslash N[D], V(G) \backslash S) = \emptyset$, *then* $\iota_1(G) \leq |D| + \iota_1(G - S)$.

Lemma 2.5 (Borg [1], Zhang and Wu [16]) *If* $G_1, G_2, \ldots, G_s$ *are the distinct components of a graph* G, *then* $\iota_1(G) = \sum_{i=1}^{s} \iota_1(G_i)$.

The next three observations are easy to verify, hence we omit the proofs here.

Observation 2.6 *The graphs in* $\mathcal{F}$ *satisfy the following properties:*

- *(i)* $\{u_1, u_4\}$ *is a dominating set of* F_i *for each* $i \in \{1, 2, 4\}$, *and* $\{u_2, u_5\}$ *is a dominating set of* F_3;
- *(ii)* $\{u, u_4\}$ *is a* 1*-isolating set of* F_i *for each* $i \in [4]$;
- *(iii)* $\{u_j, u_{j+3}\}$ *is a* 1*-isolating set of* F_i *for each* $i \in [4]$ *and* $j \in [6]$, *where the subscript* $j + 3$ *is taken modulo* 6 *when* $j \in \{4, 5, 6\}$.

Observation 2.7 *The graphs in* $\mathcal{G}_2$ *satisfy the following properties:*

- *(i)* $A_i - a_j$ *is connected for each* $i, j \in [7]$, *unless* $i = j = 1$;
- *(ii)* $A_i - a_j \ncong C_6$ *for each* $i, j \in [7]$;
- *(iii)* $\{a_j, a_{j+3}\}$ *is a* 1*-isolating set of* A_i *for each* $i, j \in [7]$, *where the subscript* $j+3$ *is taken modulo* 6 *when* $j \in \{4, 5, 6, 7\}$.

Observation 2.8 *The graphs in* $\mathcal{G}_3$ *satisfy the following properties:*

- *(i)* $B_i - b_j^k$ *is connected for each* $i \in [3]$, $j \in [7]$ *and* $k \in [2]$;
- *(ii)* $\{b_j^k, b_{j+3}^k, b_4^{3-k}, b_7^{3-k}\}$ *is a* 1*-isolating set of* B_i *for each* $i \in [3]$, $j \in [7]$ *and* $k \in [2]$, *where the subscript* $j + 3$ *is taken modulo* 6 *when* $j \in \{4, 5, 6, 7\}$.

The final three lemmas in this section show that every graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ attains the bound stated in Theorem 1.1.

Lemma 2.9 *If* $G \in \mathcal{G}_1$, *then* $\iota_1(G) = \frac{2}{7}|V(G)|$.

Proof Let $G_1, G_2, \ldots, G_t$ be the C_6-constituents of G, and for each $i \in [t]$, let u_i be the C_6-connection of G_i and $H_i := u_i^1 u_i^2 u_i^3 u_i^4 u_i^5 u_i^6 u_i^1$ be the copy of C_6 in G_i. Then $|V(G)| = 7t$. It is easy to see that for any 1-isolating set D of G, we must have $|D \cap V(G_i)| \geq 2$ for each $i \in [t]$, which implies that $\iota_1(G) \geq 2t$. On the other hand, we notice that $\{u_i, u_i^4 : 1 \leq i \leq t\}$ is a 1-isolating set of G (by Observation 2.6(ii)), and thus $\iota_1(G) \leq 2t$. Hence, we conclude that $\iota_1(G) = 2t = \frac{2}{7}|V(G)|$. □

Lemma 2.10 *If $G \in \mathcal{G}_2$, then $\iota_1(G) = 2 = \frac{2}{7}|V(G)|$.*

Proof Since $G \in \mathcal{G}_2$, we have $G = A_i$ for some $i \in [7]$, and hence $|V(G)| = 7$. It is easy to verify that for any $j \in [7]$, $G - N[a_j]$ contains a component with at least three vertices, and thus contains a $K_{1,2}$. This shows that $\iota_1(G) \geq 2$. On the other hand, it follows from Observation 2.7(iii) that $\{a_1, a_4\}$ is a 1-isolating set of G, which implies that $\iota_1(G) \leq 2$. Therefore, we derive that $\iota_1(G) = 2 = \frac{2}{7}|V(G)|$. □

Lemma 2.11 *If $G \in \mathcal{G}_3$, then $\iota_1(G) = 4 = \frac{2}{7}|V(G)|$.*

Proof Since $G \in \mathcal{G}_3$, we see that $G = B_i$ for some $i \in [3]$, and thus $|V(G)| = 14$. Then by Observation 2.8(ii), we know that $\{b_1^1, b_4^1, b_4^2, b_7^2\}$ is a 1-isolating set of G, which means that $\iota_1(G) \leq 4$.

We next show that $\iota_1(G) \geq 4$. Suppose to the contrary that there exists a 1-isolating set D of G such that $|D| \leq 3$. Then $D \cap \{b_j^1 : 2 \leq j \leq 6\} \neq \emptyset$ and $D \cap \{b_j^2 : 2 \leq j \leq 6\} \neq \emptyset$; otherwise, either $\{b_3^1, b_4^1, b_5^1\}$ or $\{b_3^2, b_4^2, b_5^2\}$ induces a $K_{1,2}$ in $G - N[D]$, which contradicts the assumption that D is a 1-isolating set of G. Moreover, since $|D| \leq 3$, we deduce that $|D \cap \{b_j^k : 1 \leq j \leq 7\}| \leq 1$ for some $k \in [2]$. This implies that $D \cap \{b_1^k, b_7^k\} = \emptyset$, $|D \cap \{b_j^k : 2 \leq j \leq 6\}| = 1$, $|D \cap \{b_j^{3-k} : 2 \leq j \leq 6\}| \geq 1$ and $|D \cap \{b_1^{3-k}, b_7^{3-k}\}| \leq 1$. By symmetry between b_2^k and b_6^k and by symmetry between b_3^k and b_5^k, we may assume that $D \cap \{b_j^k : 2 \leq j \leq 6\} = \{b_p^k\}$ for some $p \in \{2, 3, 4\}$. If $p = 2$, then $\{b_4^k, b_5^k, b_6^k\}$ induces a $K_{1,2}$ in $G - N[D]$, a contradiction. If $p = 3$, then it follows from $|D \cap \{b_1^{3-k}, b_7^{3-k}\}| \leq 1$ that either $\{b_5^k, b_6^k, b_1^k\}$ (if $b_1^{3-k} \notin D$) or $\{b_5^k, b_6^k, b_7^k\}$ (if $b_7^{3-k} \notin D$) induces a $K_{1,2}$ in $G - N[D]$, again a contradiction. Hence, we have $p = 4$. But then, we can derive that either $\{b_2^k, b_6^k, b_1^k\}$ (if $b_1^{3-k} \notin D$) or $\{b_2^k, b_6^k, b_7^k\}$ (if $b_7^{3-k} \notin D$) induces a $K_{1,2}$ in $G - N[D]$, giving a contradiction. This shows that $\iota_1(G) \geq 4$.

Therefore, we conclude that $\iota_1(G) = 4 = \frac{2}{7}|V(G)|$. □

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Our proof is inductive and follows the line of that in [16], but the arguments are much more complicated.

First, we prove the following lemma which verifies the necessity of Theorem 1.2 when $n = 7$.

Lemma 3.1 *Let G be a connected graph with 7 vertices. If $\iota_1(G) = 2$, then $G \in \mathcal{F} \cup \mathcal{G}_2$.*

Proof Since G is connected and $|V(G)| = 7$, we see that $\Delta(G) \geq 2$. If there exists a vertex $v \in V(G)$ such that $d(v) \geq 4$, then it follows from $|V(G)| = 7$ that $|V(G - N[v])| \leq 2$ and thus $\{v\}$ is a 1-isolating set of G, contradicting the assumption that $\iota_1(G) = 2$. Therefore, we have $2 \leq \Delta(G) \leq 3$.

First, suppose $\Delta(G) = 2$. Then $G \in \{P_7, C_7\}$. Since $\iota_1(P_7) = 1$ and $\iota_1(C_7) = 2 = \iota_1(G)$, we know that $G = C_7 \cong A_7 \in \mathcal{G}_2$.

Next, suppose $\Delta(G) = 3$. Let $V(G) = \{v_i : 1 \le i \le 7\}$ such that $N(v_1) = \{v_2, v_3, v_4\}$. Since $\Delta(G) = 3$ and G is connected, we have $G[N[v_1]] \in \{K_{1,3}, C_3^+, K_4^-\}$. Furthermore, we can conclude that $G - N[v_1] \in \{P_3, C_3\}$; otherwise, $\{v_1\}$ is a 1-isolating set of G, which contradicts the assumption that $\iota_1(G) = 2$. We consider different cases according to the structures of $G[N[v_1]]$ and $G - N[v_1]$.

Case 1. $G[N[v_1]] \cong K_{1,3}$.

Subcase 1.1. $G - N[v_1] \cong C_3$.

In this subcase, $v_5v_6, v_5v_7, v_6v_7 \in E(G)$. Since $\Delta(G) = 3$, we observe that each vertex in $\{v_5, v_6, v_7\}$ has at most one neighbor in $\{v_2, v_3, v_4\}$. On the other hand, we also claim that each vertex in $\{v_2, v_3, v_4\}$ has at most one neighbor in $\{v_5, v_6, v_7\}$; otherwise, suppose v_i has two neighbors in $\{v_5, v_6, v_7\}$ for some $i \in \{2, 3, 4\}$, then $\{v_i\}$ is a 1-isolating set of G, giving a contradiction. Since G is connected, we may assume by symmetry that $v_2v_5 \in E(G)$ and $v_2v_6, v_2v_7 \notin E(G)$. Note that $|E(\{v_3, v_4\}, \{v_6, v_7\})| \le 2$. If $E(\{v_3, v_4\}, \{v_6, v_7\}) = \emptyset$, then $\{v_2\}$ is a 1-isolating set of G, a contradiction. If $|E(\{v_3, v_4\}, \{v_6, v_7\})| = 1$, say $v_3v_6 \in E(G)$, then we have $G \cong A_1 \in \mathcal{G}_2$. If $|E(\{v_3, v_4\}, \{v_6, v_7\})| = 2$, say $v_3v_6, v_4v_7 \in E(G)$, then we can deduce that $G \cong A_2 \in \mathcal{G}_2$.

Subcase 2.1. $G - N[v_1] \cong P_3$.

Without loss of generality, suppose $v_5v_6, v_6v_7 \in E(G)$ and $v_5v_7 \notin E(G)$. Since $\Delta(G) = 3$, we notice that v_6 has at most one neighbor in $\{v_2, v_3, v_4\}$.

Suppose first that v_5 or v_7 has two neighbors in $\{v_2, v_3, v_4\}$. By symmetry, we may assume that $v_5v_2, v_5v_3 \in E(G)$. Then we see that $v_4v_7 \in E(G)$; otherwise, $\{v_5\}$ is a 1-isolating set of G, giving a contradiction. If $E(\{v_2, v_3, v_4\}, \{v_6, v_7\}) = \{v_4v_7\}$, then we have $G \cong F_3 \in \mathcal{F}$. Assume now that $|E(\{v_2, v_3, v_4\}, \{v_6, v_7\})| \ge 2$. If $v_iv_7 \in E(G)$ for some $i \in \{2, 3\}$, then $\{v_i\}$ is a 1-isolating set of G, a contradiction. Hence, $v_2v_7, v_3v_7 \notin E(G)$. Moreover, we know that $v_2v_6, v_3v_6 \notin E(G)$; otherwise, suppose $v_iv_6 \in E(G)$ for some $i \in \{2, 3\}$, then $\{v_i\}$ is a 1-isolating set of G, again a contradiction. Since $|E(\{v_2, v_3, v_4\}, \{v_6, v_7\})| \ge 2$, we can derive that $E(\{v_2, v_3, v_4\}, \{v_6, v_7\}) = \{v_4v_6, v_4v_7\}$. Then, it is easy to check that $G \cong A_5 \in \mathcal{G}_2$.

Now, suppose each vertex in $\{v_5, v_7\}$ has at most one neighbor in $\{v_2, v_3, v_4\}$. This implies that each vertex in $\{v_2, v_3, v_4\}$ also has at most one neighbor in $\{v_5, v_6, v_7\}$; otherwise, suppose v_i has two neighbors in $\{v_5, v_6, v_7\}$ for some $i \in \{2, 3, 4\}$, then $\{v_i\}$ is a 1-isolating set of G, a contradiction. We further claim that v_6 has no neighbor in $\{v_2, v_3, v_4\}$; otherwise, suppose $v_6v_i \in E(G)$ for some $i \in \{2, 3, 4\}$, then $\{v_i\}$ is a 1-isolating set of G, again a contradiction. Since G is connected, we may assume by symmetry that $v_2v_5 \in E(G)$ and $v_2v_6, v_2v_7 \notin E(G)$. Note that $|E(\{v_3, v_4\}, \{v_7\})| \le 1$. If $E(\{v_3, v_4\}, \{v_7\}) = \emptyset$, then $\{v_2\}$ is a 1-isolating set of G, giving a contradiction. If $|E(\{v_3, v_4\}, \{v_7\})| = 1$, say $v_3v_7 \in E(G)$, then we have $G \cong F_1 \in \mathcal{F}$.

Case 2. $G[N[v_1]] \cong C_3^+$.

Without loss of generality, suppose $v_2v_3 \in E(G)$ and $v_2v_4, v_3v_4 \notin E(G)$. Since $\Delta(G) = 3$, we observe that each vertex in $\{v_2, v_3\}$ has at most one neighbor in $\{v_5, v_6, v_7\}$.

Subcase 2.1. $G - N[v_1] \cong C_3$.

In this subcase, $v_5v_6, v_5v_7, v_6v_7 \in E(G)$ and it follows from $\Delta(G) = 3$ that each vertex in $\{v_5, v_6, v_7\}$ has at most one neighbor in $\{v_2, v_3, v_4\}$.

First, suppose v_4 has two neighbors in $\{v_5, v_6, v_7\}$. By symmetry, we may assume that $v_4v_6, v_4v_7 \in E(G)$. Note that $|E(\{v_2, v_3\}, \{v_5\})| \leq 1$. If $E(\{v_2, v_3\}, \{v_5\}) = \emptyset$, then $\{v_4\}$ is a 1-isolating set of G, a contradiction. If $|E(\{v_2, v_3\}, \{v_5\})| = 1$, say $v_2v_5 \in E(G)$, then it is straightforward to verify that $G \cong A_6 \in \mathcal{G}_2$.

Next, suppose v_4 has one neighbor in $\{v_5, v_6, v_7\}$, say $v_4v_7 \in E(G)$ and $v_4v_5, v_4v_6 \notin E(G)$. Notice that $|E(\{v_2, v_3\}, \{v_5, v_6\})| \leq 2$. If $E(\{v_2, v_3\}, \{v_5, v_6\}) = \emptyset$, then $\{v_4\}$ is a 1-isolating set of G, a contradiction. If $|E(\{v_2, v_3\}, \{v_5, v_6\})| = 1$, say $v_2v_5 \in E(G)$, then we deduce that $G \cong A_3 \in \mathcal{G}_2$. If $|E(\{v_2, v_3\}, \{v_5, v_6\})| = 2$, say $v_2v_5, v_3v_6 \in E(G)$, then we can conclude that $G \cong A_4 \in \mathcal{G}_2$.

Finally, suppose v_4 has no neighbor in $\{v_5, v_6, v_7\}$. Since G is connected, we may assume by symmetry that $v_2v_5 \in E(G)$. But then, we see that $\{v_2\}$ is a 1-isolating set of G, giving a contradiction.

Subcase 2.2. $G - N[v_1] \cong P_3$.

Without loss of generality, suppose $v_5v_6, v_6v_7 \in E(G)$ and $v_5v_7 \notin E(G)$. Since $\Delta(G) = 3$, we know that v_6 has at most one neighbor in $\{v_2, v_3, v_4\}$. If $v_6v_i \in E(G)$ for some $i \in \{2, 3\}$, then we have $G[N[v_6]] \cong K_{1,3}$ and $G - N[v_6] \cong P_3$, and we are back to Subcase 1.2 by relabeling v_6 as v_1 (and relabeling other vertices of G appropriately). If $v_6v_4 \in E(G)$, then we can derive that $G[N[v_6]] \in \{K_{1,3}, C_3^+\}$ and $G - N[v_6] \cong C_3$, and we are back to Subcase 1.1 or Subcase 2.1 by relabeling v_6 as v_1 (and relabeling other vertices of G appropriately). Therefore, we may assume that v_6 has no neighbor in $\{v_2, v_3, v_4\}$. This implies that v_4 has at most one neighbor in $\{v_5, v_7\}$; otherwise, we conclude that $v_4v_5, v_4v_7 \in E(G)$ and $\{v_4\}$ is a 1-isolating set of G, a contradiction.

First, suppose v_4 has one neighbor in $\{v_5, v_7\}$, say $v_4v_7 \in E(G)$ and $v_4v_5 \notin E(G)$. Then we deduce that $E(\{v_2, v_3\}, \{v_5\}) \neq \emptyset$; otherwise, $\{v_4\}$ is a 1-isolating set of G, giving a contradiction. By symmetry, we may assume that $v_2v_5 \in E(G)$. Note that $|E(\{v_3\}, \{v_5, v_7\})| \leq 1$. If $E(\{v_3\}, \{v_5, v_7\}) = \emptyset$, then we have $G \cong F_2 \in \mathcal{F}$. If $E(\{v_3\}, \{v_5, v_7\}) = \{v_3v_5\}$, then we see that $G \cong F_4 \in \mathcal{F}$. If $E(\{v_3\}, \{v_5, v_7\}) = \{v_3v_7\}$, then $\{v_3\}$ is a 1-isolating set of G, a contradiction.

Next, suppose v_4 has no neighbor in $\{v_5, v_7\}$. Since G is connected, we may assume by symmetry that $v_2v_5 \in E(G)$. But now, it is easy to check that $\{v_2\}$ is a 1-isolating set of G, a contradiction.

Case 3. $G[N[v_1]] \cong K_4^-$.

Without loss of generality, suppose $v_2v_3, v_3v_4 \in E(G)$ and $v_2v_4 \notin E(G)$. Since $\Delta(G) = 3$, we notice that each vertex in $\{v_2, v_4\}$ has at most one neighbor in $\{v_5, v_6, v_7\}$ and v_3 has no neighbor in $\{v_5, v_6, v_7\}$.

Subcase 3.1. $G - N[v_1] \cong C_3$.

In this subcase, $v_5v_6, v_5v_7, v_6v_7 \in E(G)$. Since G is connected, we may assume by symmetry that $v_2v_5 \in E(G)$. Then we have $G[N[v_5]] \cong C_3^+$ and $G - N[v_5] \cong C_3$, and we are back to Subcase 2.1 by relabeling v_5 as v_1 (and relabeling other vertices of G appropriately).

Subcase 3.2. $G - N[v_1] \cong P_3$.

Without loss of generality, suppose $v_5v_6, v_6v_7 \in E(G)$ and $v_5v_7 \notin E(G)$. If $v_6v_i \in E(G)$ for some $i \in \{2, 4\}$, then we know that $G[N[v_6]] \cong K_{1,3}$ and $G - N[v_6] \cong C_3$, and we are back to Subcase 1.1 by relabeling v_6 as v_1 (and relabeling other vertices of G appropriately). Hence, we may assume that v_6 has no neighbor in $\{v_2, v_4\}$. Since G is connected, we may further assume by symmetry that $v_2v_5 \in E(G)$. If $v_4v_7 \notin E(G)$, then $\{v_2\}$ is a 1-isolating set of G, a contradiction. If $v_4v_7 \in E(G)$, then we can derive that $G \cong F_4 \in \mathcal{F}$.

This completes the proof of Lemma 3.1. □

We are now ready to prove Theorem 1.2, which we restate below for convenience.

Theorem 1.2 For any connected n-vertex graph G, we have $\iota_1(G) = \frac{2}{7}n$ if and only if $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Proof If $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, then by Lemmas 2.9, 2.10 and 2.11, we conclude that $\iota_1(G) = \frac{2}{7}n$. This proves the sufficiency of Theorem 1.2.

In the following, we will prove the necessity by induction on n. Since $\iota_1(G) = \frac{2}{7}n$, we see that $n = 7p$ for some positive integer p. If $n = 7$, then it follows from Lemma 3.1 that $G \in \mathcal{F} \cup \mathcal{G}_2 \subseteq \mathcal{G}_1 \cup \mathcal{G}_2$. So we may assume that $n \geq 14$.

If $\Delta(G) = 2$, then $G \in \{P_n, C_n\}$. Note that $\iota_1(P_n) = \lceil \frac{n-2}{5} \rceil$ and $\iota_1(C_n) = \lceil \frac{n}{5} \rceil$ by Lemmas 2.1 and 2.2. Since $\lceil \frac{n-2}{5} \rceil \leq \lceil \frac{n}{5} \rceil < \frac{2}{7}n$ when $n \geq 14$, we derive a contradiction to the assumption that $\iota_1(G) = \frac{2}{7}n$.

Therefore, we may assume that $\Delta(G) \geq 3$. Let v be a vertex in G such that $d(v) = \Delta(G)$. Then $V(G) \neq N[v]$; otherwise, we deduce that $\{v\}$ is a 1-isolating set of G and thus $\iota_1(G) = 1 < \frac{2}{7}n$ (since $n \geq 14$), a contradiction. Define $G' := G - N[v]$. Let $\mathcal{H}$ be the set of components of G'. For any $x \in N(v)$ and any $H \in \mathcal{H}$, we say that x is *linked to* H or H is *linked to* x if x has at least one neighbor in H. Let $\mathcal{H}_b := \{H \in \mathcal{H} : H \in \{P_3, C_3, C_6\}\}$ and $\mathcal{H}_g := \mathcal{H} \setminus \mathcal{H}_b$. Then by Theorem 1.1 and the induction hypothesis, we conclude that for each $H \in \mathcal{H}_g$, $\iota_1(H) \leq \frac{2}{7}|V(H)|$ with equality if and only if $H \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Note that $\{v\}$ is a dominating set of $G[N[v]]$. If $\mathcal{H}_b = \emptyset$, then by Lemmas 2.3, 2.5 and Theorem 1.1, we have

$$\begin{aligned}\iota_1(G) &\leq |\{v\}| + \iota_1(G') = 1 + \sum_{H \in \mathcal{H}_g} \iota_1(H) \leq 1 + \sum_{H \in \mathcal{H}_g} \frac{2}{7}|V(H)| \\ &= 1 + \frac{2}{7}(n - 1 - \Delta(G)) \leq 1 + \frac{2}{7}(n-4) = \frac{2}{7}n - \frac{1}{7} < \frac{2}{7}n,\end{aligned}$$

a contradiction. Hence, we may assume that $\mathcal{H}_b \neq \emptyset$.

For any given $x \in N(v)$, let $\mathcal{H}_{b,x} := \{H \in \mathcal{H}_b : H \text{ is linked to } x \text{ only}\}$ and $\mathcal{H}_{g,x} := \{H \in \mathcal{H}_g : H \text{ is linked to } x \text{ only}\}$. We have two cases to consider.

Case 1. There exists a vertex $x \in N(v)$ such that $\mathcal{H}_{b,x} \neq \emptyset$.
Define $\mathcal{H}_{b,x}^1 := \{H \in \mathcal{H}_{b,x} : H \in \{P_3, C_3\}\}$ and $\mathcal{H}_{b,x}^2 := \{H \in \mathcal{H}_{b,x} : H \cong C_6\}$. Let $b_1 = |\mathcal{H}_{b,x}^1|$ and $b_2 = |\mathcal{H}_{b,x}^2|$. Since $\mathcal{H}_{b,x} \neq \emptyset$, we know that $b_1 + b_2 = |\mathcal{H}_{b,x}| \geq 1$. For each $H \in \mathcal{H}_{b,x}^1$, let y_H be one neighbor of x in H. For each $H \in \mathcal{H}_{b,x}^2$, let y_H be one neighbor of x in H and z_H the unique vertex in H with $d_H(y_H, z_H) = 3$.

Let $X := \{x\} \cup \left(\bigcup_{H \in \mathcal{H}_{b,x}} V(H)\right)$ and let G_v be the component of $G - X$ containing v. Then the components of $G - X$ are G_v and the members of $\mathcal{H}_{g,x}$. We consider two subcases according to whether $G_v \in \{P_3, C_3, C_6\}$ or not.

Subcase 1.1. $G_v \notin \{P_3, C_3, C_6\}$.

Let $D := \{x\} \cup \{z_H : H \in \mathcal{H}^2_{b,x}\}$. Then D is a 1-isolating set of $G[X]$ and $E(X \backslash N[D], V(G) \backslash X) = \emptyset$. By Lemmas 2.4, 2.5 and Theorem 1.1, we derive that

$$\begin{aligned} \iota_1(G) &\le |D| + \iota_1(G - X) = 1 + b_2 + \sum_{H \in \mathcal{H}_{g,x} \cup \{G_v\}} \iota_1(H) \le 1 + b_2 + \sum_{H \in \mathcal{H}_{g,x} \cup \{G_v\}} \frac{2}{7}|V(H)| \\ &= 1 + b_2 + \frac{2}{7}(n - 1 - 3b_1 - 6b_2) = \frac{2}{7}n - \frac{1}{7}(6b_1 + 5b_2 - 5) \le \frac{2}{7}n. \qquad (1) \end{aligned}$$

Since $\iota_1(G) = \frac{2}{7}n$, we conclude that all inequalities in (1) should be equalities, which implies that $b_1 = 0$, $b_2 = 1$ and $\iota_1(H) = \frac{2}{7}|V(H)|$ for each $H \in \mathcal{H}_{g,x} \cup \{G_v\}$. It follows from $b_1 = 0$ and $b_2 = 1$ that $\mathcal{H}_{b,x}$ contains exactly one member (which is isomorphic to C_6), and hence $|X| = 7$ and $|D| = 2$. Moreover, by the induction hypothesis, we have $H \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ for each $H \in \mathcal{H}_{g,x} \cup \{G_v\}$. We now prove that $G \in \mathcal{G}_1$.

Suppose first that there exists a member $H' \in \mathcal{H}_{g,x} \cup \{G_v\}$ such that $H' \in \mathcal{G}_2 \cup \mathcal{G}_3$, where the vertices of H' are labeled as shown in Figs. 3 or 4. Let w be one neighbor of x in H'. Define $Y := X \cup \{w\}$. Then, one can easily see that D is a 1-isolating set of $G[Y]$ and $E(Y \setminus N[D], V(G) \setminus Y) = \emptyset$. Note that $H' - w \notin \{P_3, C_3, C_6\}$ (since $|V(H' - w)| \in \{6, 13\}$ and by Observation 2.7(ii)). If $H' - w$ is connected, then by Lemmas 2.4, 2.5 and Theorem 1.1, we deduce that

$$\begin{aligned} \iota_1(G) &\le |D| + \iota_1(G - Y) = 2 + \iota_1(H' - w) + \sum_{H \in (\mathcal{H}_{g,x} \cup \{G_v\}) \setminus \{H'\}} \iota_1(H) \\ &\le 2 + \frac{2}{7}|V(H' - w)| + \sum_{H \in (\mathcal{H}_{g,x} \cup \{G_v\}) \setminus \{H'\}} \frac{2}{7}|V(H)| = 2 + \frac{2}{7}(n - 8) = \frac{2}{7}n - \frac{2}{7} < \frac{2}{7}n, \end{aligned}$$

a contradiction. So we may assume that $H' - w$ is not connected. Then by Observation 2.7(i) and Observation 2.8(i), we see that $H' \cong A_1$ and $w = a_1$. It is easy to observe that $\{a_4\}$ is a 1-isolating set of $H' - w$, and thus $\iota_1(H' - w) = 1$. By Lemmas 2.4, 2.5 and Theorem 1.1, we have

$$\begin{aligned} \iota_1(G) &\le |D| + \iota_1(G - Y) = 2 + \iota_1(H' - w) + \sum_{H \in (\mathcal{H}_{g,x} \cup \{G_v\}) \setminus \{H'\}} \iota_1(H) \\ &\le 2 + 1 + \sum_{H \in (\mathcal{H}_{g,x} \cup \{G_v\}) \setminus \{H'\}} \frac{2}{7}|V(H)| = 3 + \frac{2}{7}(n - 14) = \frac{2}{7}n - 1 < \frac{2}{7}n, \end{aligned}$$

again a contradiction.

Now, suppose $H \in \mathcal{G}_1$ for each $H \in \mathcal{H}_{g,x} \cup \{G_v\}$. Let $H_1, H_2, \ldots, H_s$ be the members of $\mathcal{H}_{g,x} \cup \{G_v\}$. For each $i \in [s]$, let $G_{i,1}, G_{i,2}, \ldots, G_{i,t_i}$ be the C_6-constituents of H_i, and for each $j \in [t_i]$, let $u_{i,j}$ be the C_6-connection of $G_{i,j}$ and $H_{i,j} := u^1_{i,j}u^2_{i,j}u^3_{i,j}u^4_{i,j}u^5_{i,j}u^6_{i,j}u^1_{i,j}$ be the copy of C_6 in $G_{i,j}$. It is clear that $G_{i,j} \in \mathcal{F}$ for each $i \in [s]$ and $j \in [t_i]$.

For each $i \in [s]$ and $j \in [t_i]$, let

$$D_{i,j} := \begin{cases} \{u^1_{i,j}, u^4_{i,j}\}, & \text{if } G_{i,j} \in \{F_1, F_2, F_4\}, \\ \{u^2_{i,j}, u^5_{i,j}\}, & \text{if } G_{i,j} \cong F_3. \end{cases}$$

Define $D^* := \bigcup_{1 \le i \le s,\ 1 \le j \le t_i} D_{i,j}$. Then, it follows from Observation 2.6(i) that D^* is a dominating set of $G - X$ and $|D^*| = \frac{2}{7}(n-7) = \frac{2}{7}n - 2$. By Lemma 2.3 and Theorem 1.1, we know that

$$\iota_1(G) \le |D^*| + \iota_1(G[X]) \le \left(\frac{2}{7}n - 2\right) + 2 = \frac{2}{7}n. \tag{2}$$

Since $\iota_1(G) = \frac{2}{7}n$, we derive that all inequalities in (2) should be equalities, which shows that $\iota_1(G[X]) = 2$. Then by Lemma 3.1, we have $G[X] \in \mathcal{F} \cup \mathcal{G}_2$. Notice that $G[X]$ contains an induced C_6 (since $G[X]$ contains the unique member of $\mathcal{H}_{b,x}$ which is isomorphic to C_6) and no graph in $\mathcal{G}_2$ contains an induced C_6, we further conclude that $G[X] \in \mathcal{F}$.

Let $D' := D \cup \{u_{i,j}, u^4_{i,j} : 1 \le i \le s \text{ and } 1 \le j \le t_i\}$ and $W := \{x\} \cup \{u_{i,j} : 1 \le i \le s \text{ and } 1 \le j \le t_i\}$. Then, it is easy to verify that D' is a 1-isolating set of G (by Observation 2.6(ii)) and $|D'| = 2 + \frac{2}{7}(n-7) = \frac{2}{7}n = \iota_1(G)$. To prove that $G \in \mathcal{G}_1$, it remains to show that $N(x) \cap V(H_i) \subseteq W$ for each $i \in [s]$. If there exists some $i \in [s]$ such that $t_i = 1$, $H_i \in \{F_3, F_4\}$ and $u^1_{i,1}$ is the unique neighbor of x in H_i, then we exchange the labels of $u_{i,1}$ and $u^1_{i,1}$ (note that in this case, the two vertices $u_{i,1}$ and $u^1_{i,1}$ are symmetric in H_i). After this modification, let w_i be one neighbor of x in H_i for each $i \in [s]$. If, for some $i \in [s]$, w_i can be chosen such that $w_i = u^k_{i,j}$ for some $j \in [t_i]$ and $k \in [6]$ (and we further choose $w_i = u^k_{i,j}$ such that $k \ne 1$ if possible), then $D'' := (D' \setminus \{u_{i,j}, u^4_{i,j}\}) \cup \{u^{k+3}_{i,j}\}$ (where the superscript $k+3$ is taken modulo 6 when $k \in \{4, 5, 6\}$) is a 1-isolating set of G of size $\iota_1(G) - 1$, a contradiction. Hence, we derive that for each $i \in [s]$, $w_i = u_{i,j}$ for some $j \in [t_i]$ (i.e., $N(x) \cap V(H_i) \subseteq W$). This implies that $G[W]$ is connected, and thus $G \in \mathcal{G}_1$ whose C_6-constituents are $G[X], G_{1,1}, G_{1,2}, \ldots, G_{1,t_1}, \ldots, G_{s,1}, G_{s,2}, \ldots, G_{s,t_s}$.

Subcase 1.2. $G_v \in \{P_3, C_3, C_6\}$.

Let $Y := X \cup V(G_v)$. Then the components of $G - Y$ are the members of $\mathcal{H}_{g,x}$.

First, suppose $G_v \in \{P_3, C_3\}$. Then, it is straightforward to check that $D := \{x\} \cup \{z_H : H \in \mathcal{H}^2_{b,x}\}$ is a 1-isolating set of $G[Y]$ and $E(Y \setminus N[D], V(G) \setminus Y) = \emptyset$.

By Lemmas 2.4, 2.5 and Theorem 1.1, we deduce that

$$\begin{aligned}\iota_1(G) &\le |D| + \iota_1(G-Y) = 1 + b_2 + \sum_{H\in\mathcal{H}_{g,x}} \iota_1(H) \le 1 + b_2 + \sum_{H\in\mathcal{H}_{g,x}} \frac{2}{7}|V(H)|\\ &= 1 + b_2 + \frac{2}{7}(n-4-3b_1-6b_2) = \frac{2}{7}n - \frac{1}{7}(6b_1+5b_2+1) < \frac{2}{7}n,\end{aligned}$$

a contradiction.

Next, suppose $G_v \cong C_6$. Define $D := \{x, w\} \cup \{z_H : H \in \mathcal{H}^2_{b,x}\}$, where w is the unique vertex in G_v with $d_{G_v}(v, w) = 3$. Then, one can easily see that D is a 1-isolating set of $G[Y]$ and $E(Y\backslash N[D], V(G)\backslash Y) = \emptyset$. By Lemmas 2.4, 2.5 and Theorem 1.1, we have

$$\begin{aligned}\iota_1(G) &\le |D| + \iota_1(G-Y) = 2 + b_2 + \sum_{H\in\mathcal{H}_{g,x}} \iota_1(H) \le 2 + b_2 + \sum_{H\in\mathcal{H}_{g,x}} \frac{2}{7}|V(H)|\\ &= 2 + b_2 + \frac{2}{7}(n-7-3b_1-6b_2) = \frac{2}{7}n - \frac{1}{7}(6b_1+5b_2) < \frac{2}{7}n,\end{aligned}$$

again a contradiction.

Case 2. $\mathcal{H}_{b,x} = \emptyset$ for every $x \in N(v)$.

By the definition of $\mathcal{H}_{b,x}$, we see that each member of $\mathcal{H}_b$ is linked to at least two vertices in $N(v)$. We deal with two subcases according to whether $\mathcal{H}_b$ contains some member that is isomorphic to C_3 or C_6.

Subcase 2.1. There exists a member $H^* \in \mathcal{H}_b$ such that $H^* \in \{C_3, C_6\}$.

We choose a vertex $x \in N(v)$ such that $x \in N(V(H^*))$. Let y be one neighbor of x in H^*. Since H^* is linked to at least two vertices in $N(v)$, we observe that $E(V(H^*), N(v) \setminus \{x\}) \ne \emptyset$. Let $x'y'$ be an edge in $E(V(H^*), N(v) \setminus \{x\})$ with $x' \in N(v) \setminus \{x\}$ and $y' \in V(H^*)$ (note that it is possible that $y' = y$).

Let $X := \{x\} \cup V(H^*)$ and let G_v be the component of $G - X$ containing v. Then the components of $G - X$ are G_v and the members of $\mathcal{H}_{g,x}$.

Claim 1 *If there exists a member $H' \in \mathcal{H}_{g,x}$ satisfying $H' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, then H' contains a 1-isolating set $D_{H'}$ such that $|D_{H'}| = \frac{2}{7}|V(H')|$ and $x \in N[D_{H'}]$.*

Proof Let w be a neighbor of x in H'. We define a subset $D_{H'} \subseteq V(H')$ as follows.

First, suppose $H' \in \mathcal{G}_1$. Let $G_1, G_2, \ldots, G_t$ be the C_6-constituents of H', and for each $i \in [t]$, let u_i be the C_6-connection of G_i and $H_i := u_i^1u_i^2u_i^3u_i^4u_i^5u_i^6u_i^1$ be the copy of C_6 in G_i. Then $w \in V(G_i)$ for some $i \in [t]$. It is clear that either $w = u_i$ or $w = u_i^j$ for some $j \in [6]$. Let

$$D_i := \begin{cases} \{w, u_i^4\}, & \text{if } w = u_i,\\ \{w, u_i^{j+3}\}, & \text{if } w = u_i^j \text{ for some } j \in [6],\end{cases}$$

where the superscript $j+3$ is taken modulo 6 when $j \in \{4, 5, 6\}$. For each $k \in [t]$ and $k \neq i$, let $D_k := \{u_k, u_k^4\}$. Define $D_{H'} := \bigcup_{1 \le k \le t} D_k$. Then by Observation 2.6(ii) and (iii), we conclude that $D_{H'}$ is a 1-isolating set of H'.

Next, suppose $H' \in \mathcal{G}_2$. Then $H' \cong A_i$ for some $i \in [7]$ (where the vertices of H' are labeled as shown in Fig. 3), and $w = a_j$ for some $j \in [7]$. Define $D_{H'} := \{w, a_{j+3}\}$, where the subscript $j+3$ is taken modulo 6 when $j \in \{4, 5, 6, 7\}$. By Observation 2.7(iii), we know that $D_{H'}$ is a 1-isolating set of H'.

Finally, suppose $H' \in \mathcal{G}_3$. Then $H' \cong B_i$ for some $i \in [3]$ (where the vertices of H' are labeled as shown in Fig. 4), and $w = b_j^k$ for some $j \in [7]$ and $k \in [2]$. Define $D_{H'} := \{w, b_{j+3}^k, b_4^{3-k}, b_7^{3-k}\}$, where the subscript $j+3$ is taken modulo 6 when $j \in \{4, 5, 6, 7\}$. Then, it follows from Observation 2.8(ii) that $D_{H'}$ is a 1-isolating set of H'.

In all three situations, it is straightforward to verify that $|D_{H'}| = \frac{2}{7}|V(H')|$ and $x \in N[D_{H'}]$ (since $w \in D_{H'}$ and $x \in N(w)$). □

Claim 2 *If $\mathcal{H}_{g,x} \neq \emptyset$, then $\iota_1(H) < \frac{2}{7}|V(H)|$ for each $H \in \mathcal{H}_{g,x}$.*

Proof Suppose to the contrary (and by Theorem 1.1) that there exists a member $H' \in \mathcal{H}_{g,x}$ such that $\iota_1(H') = \frac{2}{7}|V(H')|$. Then by the induction hypothesis, we have $H' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Let $Y := V(H^*) \cup V(G_v)$ and $Z := V(H') \cup \{x\}$. Since $x'y' \in E(G)$ with $x' \in V(G_v)$ and $y' \in V(H^*)$, we notice that $G[Y]$ is connected. This implies that the components of $G - Z$ are $G[Y]$ and the members of $\mathcal{H}_{g,x} \setminus \{H'\}$. Moreover, since $H^* \in \{C_3, C_6\}$ and $x'y' \in E(G)$, we derive that y' has degree at least 3 in $G[Y]$, and thus $G[Y] \notin \{P_3, C_3, C_6\}$.

By Claim 1, let $D_{H'}$ be a 1-isolating set of H' such that $|D_{H'}| = \frac{2}{7}|V(H')|$ and $x \in N[D_{H'}]$. Then, it is easy to check that $D_{H'}$ is also a 1-isolating set of $G[Z]$ and $E(Z \setminus N[D_{H'}], V(G) \setminus Z) = \emptyset$. By Lemmas 2.4, 2.5 and Theorem 1.1, we have

$$
\begin{aligned}
\iota_1(G) &\le |D_{H'}| + \iota_1(G - Z) = \frac{2}{7}|V(H')| + \iota_1(G[Y]) + \sum_{H \in \mathcal{H}_{g,x} \setminus \{H'\}} \iota_1(H) \\
&\le \frac{2}{7}|V(H')| + \frac{2}{7}|Y| + \sum_{H \in \mathcal{H}_{g,x} \setminus \{H'\}} \frac{2}{7}|V(H)| = \frac{2}{7}|V(H')| + \frac{2}{7}(n - 1 - |V(H')|) \\
&= \frac{2}{7}n - \frac{2}{7} < \frac{2}{7}n,
\end{aligned}
$$

a contradiction. □

Claim 3 $H^* \cong C_6$.

Proof Suppose to the contrary that $H^* \cong C_3$. Note that $\{y\}$ is a dominating set of $G[X]$. If $G_v \notin \{P_3, C_3, C_6\}$, then by Lemmas 2.3, 2.5 and Theorem 1.1, we conclude that

$$
\iota_1(G) \le |\{y\}| + \iota_1(G - X) = 1 + \iota_1(G_v) + \sum_{H \in \mathcal{H}_{g,x}} \iota_1(H)
$$

$$\leq 1 + \frac{2}{7}|V(G_v)| + \sum_{H \in \mathcal{H}_{g,x}} \frac{2}{7}|V(H)| = 1 + \frac{2}{7}(n-4) = \frac{2}{7}n - \frac{1}{7} < \frac{2}{7}n,$$

giving a contradiction. So we may assume that $G_v \in \{P_3, C_3, C_6\}$. Define $Y := X \cup V(G_v)$. Then the components of $G - Y$ are the members of $\mathcal{H}_{g,x}$. Since $n \geq 14$ and $|Y| \in \{7, 10\}$, we deduce that $\mathcal{H}_{g,x} \neq \emptyset$.

First, suppose $G_v \in \{P_3, C_3\}$. Then, one can easily see that $\{v, y\}$ is a dominating set of $G[Y]$. By Lemmas 2.3, 2.5 and Claim 2, we have

$$\begin{aligned} \iota_1(G) &\leq |\{v, y\}| + \iota_1(G - Y) \\ &= 2 + \sum_{H \in \mathcal{H}_{g,x}} \iota_1(H) < 2 + \sum_{H \in \mathcal{H}_{g,x}} \frac{2}{7}|V(H)| = 2 + \frac{2}{7}(n-7) = \frac{2}{7}n, \end{aligned}$$

a contradiction.

Next, suppose $G_v \cong C_6$. Recall that $x'y'$ is an edge in $E(V(H^*), N(v) \setminus \{x\})$ with $x' \in N(v) \setminus \{x\}$ and $y' \in V(H^*)$ (see the beginning of Subcase 2.1). Then, it is easy to check that $\{v, x'\}$ is a 1-isolating set of $G[Y]$ and $E(Y \setminus N[\{v, x'\}], V(G) \setminus Y) = \emptyset$. By Lemmas 2.4, 2.5 and Claim 2, we see that

$$\begin{aligned} \iota_1(G) &\leq |\{v, x'\}| + \iota_1(G - Y) = 2 + \sum_{H \in \mathcal{H}_{g,x}} \iota_1(H) < 2 + \sum_{H \in \mathcal{H}_{g,x}} \frac{2}{7}|V(H)| \\ &= 2 + \frac{2}{7}(n-10) = \frac{2}{7}n - \frac{6}{7} < \frac{2}{7}n, \end{aligned}$$

again a contradiction. □

By Claim 3, let $H^* := yy_1z_1zz_2y_2y$. We then consider two subcases according to whether $G_v \in \{P_3, C_3, C_6\}$ or not.

Subcase 2.1.1. $G_v \notin \{P_3, C_3, C_6\}$.

Notice that $\{y, z\}$ is a dominating set of $G[X]$. If $\mathcal{H}_{g,x} \neq \emptyset$, then by Lemmas 2.3, 2.5, Theorem 1.1 and Claim 2, we know that

$$\begin{aligned} \iota_1(G) &\leq |\{y, z\}| + \iota_1(G - X) = 2 + \iota_1(G_v) + \sum_{H \in \mathcal{H}_{g,x}} \iota_1(H) \\ &< 2 + \frac{2}{7}|V(G_v)| + \sum_{H \in \mathcal{H}_{g,x}} \frac{2}{7}|V(H)| = 2 + \frac{2}{7}(n-7) = \frac{2}{7}n, \end{aligned}$$

a contradiction. Therefore, we may assume that $\mathcal{H}_{g,x} = \emptyset$. This shows that $V(G) = X \cup V(G_v)$. By Lemma 2.3 and Theorem 1.1, we have

$$\iota_1(G) \leq |\{y, z\}| + \iota_1(G - X) = 2 + \iota_1(G_v) \leq 2 + \frac{2}{7}|V(G_v)| = 2 + \frac{2}{7}(n-7) = \frac{2}{7}n. \quad (3)$$

Since $\iota_1(G) = \frac{2}{7}n$, we derive that all inequalities in (3) should be equalities, which means that $\iota_1(G_v) = \frac{2}{7}|V(G_v)|$. By the induction hypothesis, we conclude that $G_v \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Claim 4 $E(\{z, z_1, z_2\}, N(v) \setminus \{x\}) = \emptyset$.

Proof Suppose to the contrary that $E(\{z, z_1, z_2\}, N(v) \setminus \{x\}) \neq \emptyset$. Define $Y := \{x, y, y_1, y_2\}$. It is clear that $\{y\}$ is a dominating set of $G[Y]$. Since $V(G - Y) = \{z, z_1, z_2\} \cup V(G_v)$ and $E(\{z, z_1, z_2\}, N(v) \setminus \{x\}) \neq \emptyset$, we deduce that $G - Y$ is connected. Moreover, we have $G - Y \notin \{P_3, C_3, C_6\}$ (since $|V(G - Y)| = n - 4 \geq 10$). By Lemma 2.3 and Theorem 1.1, we see that

$$\iota_1(G) \leq |\{y\}| + \iota_1(G - Y) \leq 1 + \frac{2}{7}|V(G - Y)| = 1 + \frac{2}{7}(n - 4) = \frac{2}{7}n - \frac{1}{7} < \frac{2}{7}n,$$

a contradiction. □

Define $X' := \{y, y_1, y_2\}$. Recall that $x'y' \in E(V(H^*), N(v)\setminus\{x\})$ with $x' \in N(v) \setminus \{x\}$ and $y' \in V(H^*)$ (see the beginning of Subcase 2.1). By Claim 4, we know that $E(V(H^*), N(v)\setminus\{x\}) = E(X', N(v)\setminus\{x\})$ and $y' \in X'$. Let

$$D := \begin{cases} \{y, z\}, & \text{if } y' = y, \\ \{y, y'\}, & \text{if } y' \in \{y_1, y_2\}. \end{cases}$$

Claim 5 $G_v \in \mathcal{G}_1$.

Proof Suppose to the contrary that $G_v \in \mathcal{G}_2 \cup \mathcal{G}_3$, where the vertices of G_v are labeled as shown in Figs. 3 or 4. Let $Y := X \cup \{x'\}$. Then D is a 1-isolating set of $G[Y]$ with $E(Y\setminus N[D], V(G)\setminus Y) = \emptyset$ and $G - Y = G_v - x'$. Note that $G_v - x' \notin \{P_3, C_3, C_6\}$ (since $|V(G_v - x')| \in \{6, 13\}$ and by Observation 2.7(ii)). If $G_v - x'$ is connected, then by Lemma 2.4 and Theorem 1.1, we have

$$\begin{aligned} \iota_1(G) &\leq |D| + \iota_1(G - Y) = 2 + \iota_1(G_v - x') \leq 2 + \frac{2}{7}|V(G_v - x')| \\ &= 2 + \frac{2}{7}(n - 8) = \frac{2}{7}n - \frac{2}{7} < \frac{2}{7}n, \end{aligned}$$

a contradiction. So we may assume that $G_v - x'$ is not connected. Then by Observation 2.7(i) and Observation 2.8(i), we derive that $G_v \cong A_1$ and $x' = a_1$ (and hence $n = |X| + |V(G_v)| = 14$). But then, it is straightforward to verify that $D \cup \{a_4\}$ is a 1-isolating set of G of size 3, contradicting the assumption that $\iota_1(G) = \frac{2}{7}n = 4$. □

Claim 6 $G_v \in \mathcal{F}$.

Proof Suppose to the contrary that $G_v \notin \mathcal{F}$. Then by Claim 5, we conclude that $|V(G_v)| \geq 14$. Let $G_1, G_2, \ldots, G_t$ (with $t \geq 2$) be the C_6-constituents of G_v, and for each $i \in [t]$, let u_i be the C_6-connection of G_i and $H_i := u_i^1 u_i^2 u_i^3 u_i^4 u_i^5 u_i^6 u_i^1$ be the copy of C_6 in G_i. Define $W := \{u_k : 1 \leq k \leq t\}$.

Suppose $x' = u_i^j$ for some $i \in [t]$ and $j \in [6]$. Let $D' := D \cup \{u_i^{j+3}\} \cup \{u_k, u_k^4 : 1 \leq k \leq t \text{ and } k \neq i\}$, where the superscript $j+3$ is taken modulo 6 when $j \in \{4, 5, 6\}$. Then by Observation 2.6(ii), one can easily check that D' is a 1-isolating set of G. Since $|D'| = 3 + \frac{2}{7}(n-14) = \frac{2}{7}n - 1 < \frac{2}{7}n$, we obtain a contradiction to the assumption that $\iota_1(G) = \frac{2}{7}n$.

Hence, we have $x' \in W$. Moreover, since the preceding reasoning applies to every edge $x'y' \in E(X', N(v)\backslash\{x\})$ with $x' \in N(v)\backslash\{x\}$ and $y' \in X'$, we can further deduce that $N(X') \cap (N(v) \setminus \{x\}) \subseteq W$. We consider two possibilities depending on whether $v \in W$ or not.

- Suppose $v \in W$. Define $D^* := \{z\} \cup \{u_k, u_k^4 : 1 \leq k \leq t\}$. Since $x \in N(v)$, $N(X') \cap (N(v)\backslash\{x\}) \subseteq W$ and by Observation 2.6(ii), we see that D^* is a 1-isolating set of G. Notice that $|D^*| = 1 + \frac{2}{7}(n-7) = \frac{2}{7}n - 1 < \frac{2}{7}n$, we derive a contradiction.
- Suppose $v \notin W$. Since $N(X') \cap (N(v)\backslash\{x\}) \subseteq W$, we know that $N(X') \cap (N(v)\backslash\{x\}) = \{u_i\}$ and $v = u_i^j$ for some $i \in [t]$ and $j \in \{1, 2, 6\}$. Define $D^* := \{x, z, u_i^{j+3}\} \cup \{u_k, u_k^4 : 1 \leq k \leq t \text{ and } k \neq i\}$, where the superscript $j+3$ is taken modulo 6 when $j = 6$. Then by Observation 2.6(ii), it is easy to see that D^* is a 1-isolating set of G. Since $|D^*| = 3 + \frac{2}{7}(n-14) = \frac{2}{7}n - 1 < \frac{2}{7}n$, we also derive a contradiction.

□

It follows from Claim 6 that $n = |X| + |V(G_v)| = 14$, and thus $\iota_1(G) = \frac{2}{7}n = 4$. Suppose the vertices of G_v are labeled as shown in Fig. 2. Observe that the two vertices u and u_1 are symmetric in F_3 and F_4. Therefore, we may assume that $N(X') \cap V(G_v) \neq \{u_1\}$ when $G_v \in \{F_3, F_4\}$; otherwise, we can exchange the labels of u and u_1.

Claim 7 $N(X') \cap V(G_v) = \{u\}$.

Proof Suppose to the contrary that $N(X') \cap V(G_v) \neq \{u\}$. Note that $x' \in N(X') \cap V(G_v)$. Since $x'y'$ can be arbitrarily chosen in $E(X', N(v) \setminus \{x\})$ (with $x' \in N(v)\backslash\{x\}$ and $y' \in X'$), we may assume that $x' \neq u$. If x' can be chosen such that $x' = u_i$ for some $i \in \{2, 3, 4, 5, 6\}$, then it is easy to verify that $D \cup \{u_{i+3}\}$ (where the subscript $i+3$ is taken modulo 6 when $i \in \{4, 5, 6\}$) is a 1-isolating set of G of size 3, giving a contradiction.

Hence, we have $x' = u_1$, which shows that $N(X') \cap V(G_v) \subseteq \{u, u_1\}$. If $G_v \in \{F_1, F_2\}$, then $D \cup \{u_4\}$ is a 1-isolating set of G of size 3, a contradiction. Therefore, we derive that $G_v \in \{F_3, F_4\}$. Since $N(X') \cap V(G_v) \neq \{u\}$ (by the assumption) and $N(X') \cap V(G_v) \neq \{u_1\}$ (by the argument before Claim 7), we can further conclude that $N(X') \cap V(G_v) = \{u, u_1\}$. Then, it follows from $N(X') \cap V(G_v) \subseteq N(v)\backslash\{x\}$ that $v \in \{u_2, u_6\}$. By symmetry, we may assume that $v = u_2$. But then, we deduce that $\{x, z, u_6\}$ is a 1-isolating set of G, again a contradiction. □

Now, we complete the proof of Subcase 2.1.1 by considering all four possibilities of G_v.

- Suppose $G_v \cong F_1$. Since $N(X') \cap V(G_v) \subseteq N(v)\backslash\{x\}$ and by Claim 7, we see that $E(\{u\}, X') \neq \emptyset$ and $v = u_1$. This implies that $\Delta(G) = d(v) = 4$ (since v has degree 3 in G_v and $vx \in E(G)$).

First, suppose $uy \in E(G)$. Then we notice that $uy_1, uy_2 \in E(G)$; otherwise, $\{x, z, u_4\}$ is a 1-isolating set of G, contradicting the assumption that $\iota_1(G) = 4$. If x has at most one neighbor in $\{u_2, u_6\}$, then $\{z, u, u_4\}$ is a 1-isolating set of G, a contradiction. So we may assume that $xu_2, xu_6 \in E(G)$. Then, it is straightforward to check that $G \cong B_3 \in \mathcal{G}_3$.

Next, suppose by symmetry that $uy \notin E(G)$ and $uy_1 \in E(G)$. Then we have $uy_2 \in E(G)$; otherwise, $\{x, z, u_4\}$ is a 1-isolating set of G, giving a contradiction. If $E(\{x\}, \{u_2, u_6\}) = \emptyset$, then $\{z, u, u_4\}$ is a 1-isolating set of G, a contradiction. If $|E(\{x\}, \{u_2, u_6\})| = 1$, say $xu_2 \in E(G)$, then either $\{z, u, x\}$ (if $xu_5 \in E(G)$) or $\{z, u, u_3\}$ (if $xu_5 \notin E(G)$) is a 1-isolating set of G, again a contradiction. Hence, we may assume that $|E(\{x\}, \{u_2, u_6\})| = 2$ (i.e., $xu_2, xu_6 \in E(G)$). Then, one can easily see that $G \cong B_2 \in \mathcal{G}_3$.

- Suppose $G_v \cong F_2$. Since $N(X') \cap V(G_v) \subseteq N(v)\backslash\{x\}$ and by Claim 7, we know that $E(\{u\}, X') \neq \emptyset$ and $v \in \{u_1, u_2\}$. This means that $\Delta(G) = d(v) = 4$ (since v has degree 3 in G_v and $vx \in E(G)$). By symmetry, we may assume that $v = u_1$.
  First, suppose $uy \in E(G)$. Since $\Delta(G) = 4$, we conclude that $|E(\{u\}, \{y_1, y_2\})| \leq 1$. If $E(\{u\}, \{y_1, y_2\}) = \emptyset$, then $\{x, z, u_4\}$ is a 1-isolating set of G, a contradiction. Therefore, we derive that $|E(\{u\}, \{y_1, y_2\})| = 1$. By symmetry, we may assume that $uy_1 \in E(G)$ and $uy_2 \notin E(G)$. Then we have $xy_2, xu_6 \in E(G)$; otherwise, $\{z, u, u_4\}$ is a 1-isolating set of G, a contradiction. But now, we can deduce that $\{x, z, u_2\}$ is a 1-isolating set of G, again a contradiction.
  Next, suppose by symmetry that $uy \notin E(G)$ and $uy_1 \in E(G)$. Then we see that $E(\{x\}, \{z, z_1\}) \neq \emptyset$; otherwise, $\{y_2, u, u_4\}$ is a 1-isolating set of G, giving a contradiction. If $xz \in E(G)$, then $\{x, y_1, u_3\}$ is a 1-isolating set of G, a contradiction. Hence, we have $xz \notin E(G)$, and thus $E(\{x\}, \{z, z_1\}) = \{xz_1\}$. Then, it is easy to observe that $\{x, z_2, u_3\}$ is a 1-isolating set of G, again a contradiction.
- Suppose $G_v \cong F_3$. Since $N(X') \cap V(G_v) \subseteq N(v)\backslash\{x\}$ and by Claim 7, we notice that $v \in \{u_2, u_6\}$. By symmetry, we may assume that $v = u_2$. But then, we derive that $\{x, z, u_6\}$ is a 1-isolating set of G, a contradiction.
- Suppose $G_v \cong F_4$. Since $N(X') \cap V(G_v) \subseteq N(v)\backslash\{x\}$ and by Claim 7, we conclude that $E(\{u\}, X') \neq \emptyset$ and $v \in \{u_1, u_2, u_6\}$. This shows that $\Delta(G) = d(v) = 4$ (since v has degree 3 in G_v and $vx \in E(G)$), and thus $|E(\{u\}, X')| = 1$. If $v \in \{u_2, u_6\}$, suppose by symmetry that $v = u_2$, then $\{x, z, u_6\}$ is a 1-isolating set of G, giving a contradiction. Therefore, we know that $v = u_1$.
  First, suppose $uy \in E(G)$. Then we deduce that $xy_1, xy_2 \in E(G)$; otherwise, $\{z, u, u_4\}$ is a 1-isolating set of G, a contradiction. This implies that $G \cong B_3 \in \mathcal{G}_3$.
  Next, suppose by symmetry that $uy \notin E(G)$ and $uy_1 \in E(G)$. Then we have $xz_1 \in E(G)$; otherwise, $\{z_2, u, u_4\}$ is a 1-isolating set of G, a contradiction. If $d(x) = 3$, then we see that $G \cong B_2 \in \mathcal{G}_3$. So we may assume that $d(x) = \Delta(G) = 4$. If $xu_i \in E(G)$ for some $i \in \{2, 3, 4, 5, 6\}$, then $\{x, z_2, u_{i+3}\}$ (where the subscript $i + 3$ is taken modulo 6 when $i \in \{4, 5, 6\}$) is a 1-isolating set of G, a contradiction. Hence, we can derive that x has no neighbor in $\{u_i : 2 \leq i \leq 6\}$. Moreover, we claim that x also has no neighbor in $\{z, z_2, y_2\}$; otherwise, $\{x, u, u_4\}$ is a 1-isolating set of G, again a contradiction. Since $d(x) = 4$, we conclude that $xy_1 \in E(G)$. Then, it is easy to verify that $G \cong B_3 \in \mathcal{G}_3$.

Subcase 2.1.2. $G_v \in \{P_3, C_3, C_6\}$.

In this subcase, we have $\Delta(G) = d(v) = 3$ (since v has degree 2 in G_v and $vx \in E(G)$). Since $H^* \cong C_6$ and $xy \in E(G)$, we know that y has no neighbor in $N(v)\setminus\{x\}$ and each vertex in $\{y_1, y_2, z, z_1, z_2\}$ has at most one neighbor in $N(v)\setminus\{x\}$.

Let $Y := X \cup V(G_v)$. Then the components of $G - Y$ are the members of $\mathcal{H}_{g,x}$. Since $n \geq 14$ and $|Y| \in \{10, 13\}$, we deduce that $\mathcal{H}_{g,x} \neq \emptyset$.

First, suppose $G_v \cong P_3$. If each vertex in $N(v) \setminus \{x\}$ has at most one neighbor in $\{y_1, y_2\}$, then we define $D := \{x, z\}$; otherwise, if some vertex in $N(v) \setminus \{x\}$ is adjacent to both y_1 and y_2, then we let x'' be the other vertex in $N(v) \setminus \{x\}$ (and thus $x''y_1, x''y_2 \notin E(G)$) and define

$$D := \begin{cases} \{y, y_2\}, & \text{if } E(\{x''\}, \{z, z_1\}) = \emptyset, \\ \{y, x''\}, & \text{if } E(\{x''\}, \{z, z_1\}) \neq \emptyset. \end{cases}$$

In all possibilities, one can easily check that D is a 1-isolating set of $G[Y]$ and $E(Y\setminus N[D], V(G)\setminus Y) = \emptyset$. Then by Lemmas 2.4, 2.5 and Claim 2, we have

$$\begin{aligned} \iota_1(G) &\leq |D| + \iota_1(G - Y) = 2 + \sum_{H\in\mathcal{H}_{g,x}} \iota_1(H) < 2 + \sum_{H\in\mathcal{H}_{g,x}} \frac{2}{7}|V(H)| \\ &= 2 + \frac{2}{7}(n - 10) = \frac{2}{7}n - \frac{6}{7} < \frac{2}{7}n, \end{aligned}$$

a contradiction.

Next, suppose $G_v \cong C_3$. Recall that $x'y' \in E(V(H^*), N(v)\setminus\{x\})$ with $x' \in N(v)\setminus\{x\}$ and $y' \in V(H^*)$ (see the beginning of Subcase 2.1). If $E(\{z, z_1, z_2\}, N(v)\setminus\{x\}) = \emptyset$, then $y' \in \{y_1, y_2\}$ and we define $D := \{y, y'\}$; otherwise, we may assume $x'y' \in E(V(H^*), N(v) \setminus \{x\})$ is chosen such that $x' \in N(v)\setminus\{x\}$ and $y' \in \{z, z_1, z_2\}$, and then define $D := \{y, x'\}$. In both possibilities, we see that D is a 1-isolating set of $G[Y]$ and $E(Y\setminus N[D], V(G)\setminus Y) = \emptyset$. By Lemmas 2.4, 2.5 and Claim 2, we derive that

$$\begin{aligned} \iota_1(G) &\leq |D| + \iota_1(G - Y) = 2 + \sum_{H\in\mathcal{H}_{g,x}} \iota_1(H) < 2 + \sum_{H\in\mathcal{H}_{g,x}} \frac{2}{7}|V(H)| \\ &= 2 + \frac{2}{7}(n - 10) = \frac{2}{7}n - \frac{6}{7} < \frac{2}{7}n, \end{aligned}$$

again a contradiction.

Finally, suppose $G_v \cong C_6$. Define $D := \{x, z, w\}$, where w is the unique vertex in G_v with $d_{G_v}(v, w) = 3$. Then, it is easy to see that D is a 1-isolating set of $G[Y]$ and $E(Y\setminus N[D], V(G)\setminus Y) = \emptyset$. By Lemmas 2.4, 2.5 and Claim 2, we conclude that

$$\begin{aligned} \iota_1(G) &\leq |D| + \iota_1(G - Y) = 3 + \sum_{H\in\mathcal{H}_{g,x}} \iota_1(H) < 3 + \sum_{H\in\mathcal{H}_{g,x}} \frac{2}{7}|V(H)| \\ &= 3 + \frac{2}{7}(n - 13) = \frac{2}{7}n - \frac{5}{7} < \frac{2}{7}n, \end{aligned}$$

giving a contradiction.

Subcase 2.2. $H \cong P_3$ for every $H \in \mathcal{H}_b$.

Let $X := N[v] \cup \Big(\bigcup_{H\in\mathcal{H}_b} V(H)\Big)$. Then the components of $G-X$ are the members of $\mathcal{H}_g$. For the sake of brevity, let $\Delta = \Delta(G)$ and $b = |\mathcal{H}_b|$. Then $\Delta \geq 3$ and $b \geq 1$ (since $\mathcal{H}_b \neq \emptyset$). For each $H \in \mathcal{H}_b$, let y_H be the unique vertex of degree 2 in H.

Define $S := \{v\} \cup \{y_H : H \in \mathcal{H}_b\}$ and $T := N(v)$. It is clear that S is a dominating set of $G[X]$ and T is a 1-isolating set of $G[X]$ with $E(X \setminus N[T], V(G)\setminus X) = \emptyset$. By Lemmas 2.3, 2.4, 2.5 and Theorem 1.1, we know that

$$\begin{aligned}\iota_1(G) &\leq |S| + \iota_1(G-X) = 1 + b + \sum_{H\in\mathcal{H}_g} \iota_1(H) \leq 1 + b + \sum_{H\in\mathcal{H}_g} \frac{2}{7}|V(H)| \\ &= 1 + b + \frac{2}{7}(n - 1 - \Delta - 3b) = \frac{2}{7}n - \frac{1}{7}(2\Delta - b - 5) \end{aligned} \tag{4}$$

and

$$\begin{aligned}\iota_1(G) &\leq |T| + \iota_1(G-X) = \Delta + \sum_{H\in\mathcal{H}_g} \iota_1(H) \leq \Delta + \sum_{H\in\mathcal{H}_g} \frac{2}{7}|V(H)| \\ &= \Delta + \frac{2}{7}(n - 1 - \Delta - 3b) = \frac{2}{7}n - \frac{1}{7}(6b + 2 - 5\Delta). \end{aligned} \tag{5}$$

If $b < 2\Delta - 5$, then by (4), we have

$$\iota_1(G) \leq \frac{2}{7}n - \frac{1}{7}(2\Delta - b - 5) < \frac{2}{7}n,$$

contradicting the assumption that $\iota_1(G) = \frac{2}{7}n$. On the other hand, if $b > \frac{1}{6}(5\Delta - 2)$, then it follows from (5) that

$$\iota_1(G) \leq \frac{2}{7}n - \frac{1}{7}(6b + 2 - 5\Delta) < \frac{2}{7}n,$$

again a contradiction. Therefore, we can deduce that $2\Delta - 5 \leq b \leq \frac{1}{6}(5\Delta - 2)$. Then a simple calculation shows that either $\Delta = 4$ and $b = 3$, or $\Delta = 3$ and $b \in \{1, 2\}$. We consider three subcases according to the values of Δ and b.

Subcase 2.2.1. $\Delta = 4$ and $b = 3$.

Let $\mathcal{H}_b = \{H_1, H_2, H_3\}$. Since H_i is linked to at least two vertices in $N(v)$ for each $i \in [3]$ and $|N(v)| = \Delta = 4$, we see that there must exist a vertex $x \in N(v)$ such that x is linked to at least two members of $\mathcal{H}_b$. Without loss of generality, we may assume that x is linked to both H_1 and H_2. Let y be any vertex in H_3. Then, it is easy to observe that $D := \{v, x, y\}$ is a 1-isolating set of $G[X]$ and $E(X \setminus N[D], V(G) \setminus X) = \emptyset$. By Lemmas 2.4, 2.5 and Theorem 1.1, we have

$$\iota_1(G) \leq |D| + \iota_1(G-X) = 3 + \sum_{H\in\mathcal{H}_g} \iota_1(H) \leq 3 + \sum_{H\in\mathcal{H}_g} \frac{2}{7}|V(H)|$$

$$= 3 + \frac{2}{7}(n - 14) = \frac{2}{7}n - 1 < \frac{2}{7}n,$$

a contradiction.

Subcase 2.2.2. $\Delta = 3$ and $b = 2$.

Let $\mathcal{H}_b = \{H_1, H_2\}$. Since H_i is linked to at least two vertices in $N(v)$ for each $i \in [2]$ and $|N(v)| = \Delta = 3$, we derive that there must exist a vertex $x \in N(v)$ such that x is linked to both H_1 and H_2. Then, one can easily see that $D := \{v, x\}$ is a 1-isolating set of $G[X]$ and $E(X \backslash N[D], V(G) \backslash X) = \emptyset$. By Lemmas 2.4, 2.5 and Theorem 1.1, we conclude that

$$\begin{aligned} \iota_1(G) &\leq |D| + \iota_1(G - X) = 2 + \sum_{H \in \mathcal{H}_g} \iota_1(H) \leq 2 + \sum_{H \in \mathcal{H}_g} \frac{2}{7}|V(H)| \\ &= 2 + \frac{2}{7}(n - 10) = \frac{2}{7}n - \frac{6}{7} < \frac{2}{7}n, \end{aligned}$$

a contradiction.

Subcase 2.2.3. $\Delta = 3$ and $b = 1$.

Let $N(v) = \{x_1, x_2, x_3\}$, and let H^* be the unique member of $\mathcal{H}_b$ with $V(H^*) = \{y_1, y_2, y_3\}$ and $y_1y_2, y_2y_3 \in E(H^*)$. Since $\Delta = 3$, we know that y_2 has at most one neighbor in $N(v)$ and each vertex in $\{y_1, y_3\}$ has at most two neighbors in $N(v)$.

Recall that $X = N[v] \cup V(H^*)$ and the components of $G - X$ are the members of $\mathcal{H}_g$. Then, it follows from $n \geq 14$ and $|X| = 7$ that $\mathcal{H}_g \neq \emptyset$. Since $\iota_1(G) = \frac{2}{7}n$, we deduce that all inequalities in (4) (by letting $\Delta = 3$ and $b = 1$) should be equalities, which implies that $\iota_1(H) = \frac{2}{7}|V(H)|$ for each $H \in \mathcal{H}_g$. By the induction hypothesis, we have $H \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ for each $H \in \mathcal{H}_g$. For each $i \in [3]$, let $\mathcal{H}'_{g,x_i} := \{H \in \mathcal{H}_g : H \text{ is linked to } x_i\}$. It is clear that $\mathcal{H}_{g,x_i} \subseteq \mathcal{H}'_{g,x_i}$ for each $i \in [3]$.

Claim 8 *For any $i \in [3]$ and any member $H' \in \mathcal{H}'_{g,x_i}$, there exists a 1-isolating set $D_{H'}$ of H' such that $|D_{H'}| = \frac{2}{7}|V(H')|$ and $x_i \in N[D_{H'}]$.*

Proof Since $H' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, the proof is the same as that of Claim 1 and is omitted here. □

Claim 9 *For any $i \in [3]$ and any member $H' \in \mathcal{H}'_{g,x_i}$, if there exists a 1-isolating set D of $G[X]$ such that $|D| = 2$, $x_i \in D$ and $E(X \backslash N[D], V(G) \backslash X) = \emptyset$, then $H' \in \mathcal{G}_1$.*

Proof Suppose to the contrary that $H' \in \mathcal{G}_2 \cup \mathcal{G}_3$, where the vertices of H' are labeled as shown in Figs. 3 or 4. Let w be one neighbor of x_i in H'. Define $Y := X \cup \{w\}$. Since D is a 1-isolating set of $G[X]$ containing x_i and $E(X \backslash N[D], V(G) \backslash X) = \emptyset$, we see that D is also a 1-isolating set of $G[Y]$ and $E(Y \setminus N[D], V(G) \setminus Y) = \emptyset$. Notice that $w \neq a_1$ when $H' \cong A_1$ (since $\Delta = 3$). Then, it follows from Observation 2.7(i) and Observation 2.8(i) that $H' - w$ is connected, which means that the components of $G - Y$ are $H' - w$ and the members of $\mathcal{H}_g \setminus \{H'\}$. Moreover, since $|V(H' - w)| \in \{6, 13\}$ and by Observation 2.7(ii), we derive that $H' - w \notin \{P_3, C_3, C_6\}$. By Lemmas 2.4, 2.5 and Theorem 1.1, we have

$$\iota_1(G) \le |D| + \iota_1(G - Y) = 2 + \iota_1(H' - w) + \sum_{H \in \mathcal{H}_g \setminus \{H'\}} \iota_1(H)$$
$$\le 2 + \frac{2}{7}|V(H' - w)| + \sum_{H \in \mathcal{H}_g \setminus \{H'\}} \frac{2}{7}|V(H)| = 2 + \frac{2}{7}(n - 8) = \frac{2}{7}n - \frac{2}{7} < \frac{2}{7}n,$$

a contradiction. □

Claim 10 *If $\mathcal{H}_{g,x_i} \neq \emptyset$ for some $i \in [3]$, then $G \in \mathcal{G}_1$.*

Proof Without loss of generality, suppose that $\mathcal{H}_{g,x_1} \neq \emptyset$ and let H' be a member of $\mathcal{H}_{g,x_1}$. Note that $|\mathcal{H}_{g,x_1}| \le 2$ (since $\Delta = 3$).

Let $Y := V(H') \cup \{x_1\}$ and let G_v be the component of $G - Y$ containing v. Since H^* is linked to at least two vertices in $N(v)$, we observe that $H^* \subseteq G_v$. This shows that the components of $G - Y$ are G_v and the members of $\mathcal{H}_{g,x_1} \setminus \{H'\}$. By Claim 8, let $D_{H'}$ be a 1-isolating set of H' such that $|D_{H'}| = \frac{2}{7}|V(H')|$ and $x_1 \in N[D_{H'}]$. Then, it is straightforward to verify that $D_{H'}$ is also a 1-isolating set of $G[Y]$ and $E(Y \backslash N[D_{H'}], V(G) \backslash Y) = \emptyset$. Notice that $G_v \notin \{P_3, C_3\}$ (since $|V(G_v)| \ge 6$). If $G_v \ncong C_6$, then by Lemmas 2.4, 2.5 and Theorem 1.1, we conclude that

$$\iota_1(G) \le |D_{H'}| + \iota_1(G - Y) = \frac{2}{7}|V(H')| + \iota_1(G_v) + \sum_{H \in \mathcal{H}_{g,x_1} \setminus \{H'\}} \iota_1(H)$$
$$\le \frac{2}{7}|V(H')| + \frac{2}{7}|V(G_v)| + \sum_{H \in \mathcal{H}_{g,x_1} \setminus \{H'\}} \frac{2}{7}|V(H)| = \frac{2}{7}|V(H')| + \frac{2}{7}(n - 1 - |V(H')|)$$
$$= \frac{2}{7}n - \frac{2}{7} < \frac{2}{7}n,$$

giving a contradiction. So we may assume that $G_v \cong C_6$. This implies that $V(G_v) = X \setminus \{x_1\}$, and thus $\mathcal{H}_g = \mathcal{H}_{g,x_1}$ (i.e., $\mathcal{H}'_{g,x_2} = \mathcal{H}'_{g,x_3} = \emptyset$).

Without loss of generality, suppose $G_v := vx_2y_1y_2y_3x_3v$. Since $\Delta = 3$, we know that x_1 has at most one neighbor in $\{x_2, x_3, y_1, y_2, y_3\}$. If $x_1y_2 \in E(G)$, then we deduce that H' is the unique member of $\mathcal{H}_g$ (since $\Delta = 3$) and $D_{H'} \cup \{x_1\}$ is a 1-isolating set of G, which means that

$$\iota_1(G) \le |D_{H'}| + 1 = \frac{2}{7}|V(H')| + 1 = \frac{2}{7}(n - 7) + 1 = \frac{2}{7}n - 1 < \frac{2}{7}n,$$

a contradiction. Hence, we may further assume that $x_1y_2 \notin E(G)$. This shows that $G[X] \in \{F_1, F_2, F_3\} \subseteq \mathcal{F}$.

Let $D := \{x_1, y_2\}$. Then D is a 1-isolating set of $G[X]$ containing x_1 and $E(X \backslash N[D], V(G) \backslash X) = \emptyset$ (since $\mathcal{H}_g = \mathcal{H}_{g,x_1}$). By Claim 9, we see that $H' \in \mathcal{G}_1$. Since H' can be arbitrarily chosen in $\mathcal{H}_g$, we can derive that each member of $\mathcal{H}_g$ belongs to $\mathcal{G}_1$. Let $H_1, \ldots, H_s$ (with $s \le 2$) be the members of $\mathcal{H}_g$. For each $i \in [s]$, let $G_{i,1}, G_{i,2}, \ldots, G_{i,t_i}$ be the C_6-constituents of H_i, and for each $j \in [t_i]$, let $u_{i,j}$ be the C_6-connection of $G_{i,j}$ and $H_{i,j} := u^1_{i,j}u^2_{i,j}u^3_{i,j}u^4_{i,j}u^5_{i,j}u^6_{i,j}u^1_{i,j}$ be the copy of C_6 in $G_{i,j}$.

Define $D' := D \cup \{u_{i,j}, u^4_{i,j} : 1 \le i \le s \text{ and } 1 \le j \le t_i\}$ and $W := \{x_1\} \cup \{u_{i,j} : 1 \le i \le s \text{ and } 1 \le j \le t_i\}$. Then, one can easily check that D' is a 1-isolating set of G (by Observation 2.6(ii)) and $|D'| = 2 + \frac{2}{7}(n-7) = \frac{2}{7}n = \iota_1(G)$. To show that $G \in \mathcal{G}_1$, it remains to prove that $N(x_1) \cap V(H_i) \subseteq W$ for each $i \in [s]$. If there exists some $i \in [s]$ such that $t_i = 1$, $H_i \cong F_3$ and $u^1_{i,1}$ is the unique neighbor of x_1 in H_i, then we exchange the labels of $u_{i,1}$ and $u^1_{i,1}$ (note that in this case, the two vertices $u_{i,1}$ and $u^1_{i,1}$ are symmetric in H_i). After this modification, let w_i be one neighbor of x_1 in H_i for each $i \in [s]$. If, for some $i \in [s]$, w_i can be chosen such that $w_i = u^k_{i,j}$ for some $j \in [t_i]$ and $k \in [6]$ (and we further choose $w_i = u^k_{i,j}$ such that $k \ne 1$ if possible), then it is easy to observe that $D'' := (D' \setminus \{u_{i,j}, u^4_{i,j}\}) \cup \{u^{k+3}_{i,j}\}$ (where the superscript $k+3$ is taken modulo 6 when $k \in \{4, 5, 6\}$) is a 1-isolating set of G of size $\iota_1(G) - 1$, a contradiction. Therefore, we deduce that for each $i \in [s]$, $w_i = u_{i,j}$ for some $j \in [t_i]$ (i.e., $N(x_1) \cap V(H_i) \subseteq W$). This implies that $G[W]$ is connected, and thus $G \in \mathcal{G}_1$ whose C_6-constituents are $G[X], G_{1,1}, G_{1,2}, \ldots, G_{1,t_1}, \ldots, G_{s,1}, G_{s,2}, \ldots, G_{s,t_s}$. □

By Claim 10, we may assume that $\mathcal{H}_{g,x_i} = \emptyset$ for each $i \in [3]$ in the remainder of the proof. This means that each member of $\mathcal{H}_g$ is linked to at least two vertices in $N(v)$.

Claim 11 *y_2 has no neighbor in $N(v)$.*

Proof Suppose this is false. Since $\Delta = 3$, we may assume by symmetry that $y_2x_2 \in E(G)$ and $y_2x_1, y_2x_3 \notin E(G)$. Define $Y := \{x_2, y_1, y_2, y_3\}$. Then $\{y_2\}$ is a dominating set of $G[Y]$. Since each member of $\mathcal{H}_g$ is linked to at least two vertices in $N(v)$, we conclude that $G - Y$ is connected. Moreover, we have $G - Y \notin \{P_3, C_3, C_6\}$ (since $|V(G - Y)| = n - 4 \ge 10$). By Lemma 2.3 and Theorem 1.1, we know that

$$\iota_1(G) \le |\{y_2\}| + \iota_1(G - Y) \le 1 + \frac{2}{7}|V(G - Y)| = 1 + \frac{2}{7}(n-4) = \frac{2}{7}n - \frac{1}{7} < \frac{2}{7}n,$$

giving a contradiction. □

Claim 12 *If y_1 or y_3 has two neighbors in $N(v)$, then $G \cong B_1 \in \mathcal{G}_3$.*

Proof Without loss of generality, we may assume that $y_1x_1, y_1x_2 \in E(G)$.

Suppose $y_3x_3 \notin E(G)$. Let $Y := \{x_1, x_2, y_1, y_2, y_3\}$ and let G_v be the component of $G - Y$ containing v. Then the components of $G - Y$ are G_v and the members of $\mathcal{H}_g \setminus \mathcal{H}'_{g,x_3}$. It is clear that $\{y_1\}$ is a 1-isolating set of $G[Y]$ and $E(Y \setminus N[y_1], V(G) \setminus Y) = \emptyset$. Since either $|V(G_v)| = 2$ (if $\mathcal{H}'_{g,x_3} = \emptyset$) or $|V(G_v)| \ge 9$ (if $\mathcal{H}'_{g,x_3} \ne \emptyset$), we deduce that $G_v \notin \{P_3, C_3, C_6\}$. By Lemmas 2.4, 2.5 and Theorem 1.1, we have

$$\iota_1(G) \le |\{y_1\}| + \iota_1(G - Y) = 1 + \iota_1(G_v) + \sum_{H \in \mathcal{H}_g \setminus \mathcal{H}'_{g,x_3}} \iota_1(H)$$

$$\leq 1 + \frac{2}{7}|V(G_v)| + \sum_{H \in \mathcal{H}_g \setminus \mathcal{H}'_{g,x_3}} \frac{2}{7}|V(H)| = 1 + \frac{2}{7}(n-5) = \frac{2}{7}n - \frac{3}{7} < \frac{2}{7}n,$$

a contradiction.

Hence, we may assume that $y_3x_3 \in E(G)$. Let $Z := \{x_1, x_2, y_1, y_2\}$ and let G'_v be the component of $G - Z$ containing v. Then $\{y_1\}$ is a dominating set of $G[Z]$ and the components of $G - Z$ are G'_v and the members of $\mathcal{H}_g \setminus \mathcal{H}'_{g,x_3}$. Note that either $|V(G'_v)| = 3$ (if $\mathcal{H}'_{g,x_3} = \emptyset$) or $|V(G'_v)| \geq 10$ (if $\mathcal{H}'_{g,x_3} \neq \emptyset$). This shows that $G'_v \ncong C_6$. Moreover, we see that $G'_v \ncong C_3$ (since $vy_3 \notin E(G)$). If $G'_v \ncong P_3$, then by Lemmas 2.3, 2.5 and Theorem 1.1, we derive that

$$\iota_1(G) \leq |\{y_1\}| + \iota_1(G - Z) = 1 + \iota_1(G'_v) + \sum_{H \in \mathcal{H}_g \setminus \mathcal{H}'_{g,x_3}} \iota_1(H)$$
$$\leq 1 + \frac{2}{7}|V(G'_v)| + \sum_{H \in \mathcal{H}_g \setminus \mathcal{H}'_{g,x_3}} \frac{2}{7}|V(H)| = 1 + \frac{2}{7}(n-4) = \frac{2}{7}n - \frac{1}{7} < \frac{2}{7}n,$$

a contradiction. So we may assume that $G'_v \cong P_3$. This implies that $V(G'_v) = \{v, x_3, y_3\}$ and $\mathcal{H}'_{g,x_3} = \emptyset$. Since each member of $\mathcal{H}_g$ is linked to at least two vertices in $N(v)$, we conclude that each member of $\mathcal{H}_g$ is linked to both x_1 and x_2. Then, it follows from $\Delta = 3$ and Claim 11 that $|\mathcal{H}_g| = 1$ and $G[X] \cong F_3$.

Let H be the unique member of $\mathcal{H}_g$. For each $i \in [2]$, let w_i be the unique neighbor of x_i in H (since $\Delta = 3$). Define $D := \{v, x_1\}$. Then D is a 1-isolating set of $G[X]$ containing x_1 and $E(X \setminus N[D], V(G) \setminus X) = \emptyset$. By Claim 9, we have $H \in \mathcal{G}_1$. We distinguish two possibilities depending on whether $H \cong F_3$ or not.

- Suppose $H \cong F_3$ (where the vertices of H are labeled as shown in Fig. 2). Then $n = |X| + |V(H)| = 14$, and thus $\iota_1(G) = \frac{2}{7}n = 4$. If $\{w_1, w_2\} = \{u, u_1\}$, then it is easy to see that $G \cong B_1 \in \mathcal{G}_3$. Therefore, we may assume by symmetry that $w_1 \notin \{u, u_1\}$. Then $w_1 = u_i$ for some $i \in \{3, 4, 5\}$ (since $\Delta = 3$). But now, we know that $D \cup \{u_{i+3}\}$ (where the subscript $i+3$ is taken modulo 6 when $i \in \{4, 5\}$) is a 1-isolating set of G of size 3, a contradiction.
- Suppose $H \ncong F_3$. Let $G_1, G_2, \ldots, G_t$ be the C_6-constituents of H, and for each $i \in [t]$, let u_i be the C_6-connection of G_i and $H_i := u_i^1u_i^2u_i^3u_i^4u_i^5u_i^6u_i^1$ be the copy of C_6 in G_i. Define $W := \{u_k : 1 \leq k \leq t\}$.
  Suppose $w_1 = u_i^j$ for some $i \in [t]$ and $j \in [6]$. Let $D' := D \cup \{u_i^{j+3}\} \cup \{u_k, u_k^4 : 1 \leq k \leq t \text{ and } k \neq i\}$, where the superscript $j + 3$ is taken modulo 6 when $j \in \{4, 5, 6\}$. Then by Observation 2.6(ii), we can deduce that D' is a 1-isolating set of G. Since $|D'| = 3 + \frac{2}{7}(n - 14) = \frac{2}{7}n - 1 < \frac{2}{7}n$, we obtain a contradiction to the assumption that $\iota_1(G) = \frac{2}{7}n$.
  Hence, we have $w_1 \in W$. By the same argument as that for w_1 (with $D := \{v, x_1\}$ being replaced by $D^* := \{v, x_2\}$), we can also show that $w_2 \in W$. Then, it is straightforward to verify that $D'' := \{x_3\} \cup \{u_k, u_k^4 : 1 \leq k \leq t\}$ is a 1-isolating set of G (by Observation 2.6(ii)). Since $|D''| = 1 + \frac{2}{7}(n - 7) = \frac{2}{7}n - 1 < \frac{2}{7}n$, we also derive a contradiction.

□

By Claim 12, we may further assume that each vertex in $\{y_1, y_3\}$ has at most one neighbor in $N(v)$ in the following proof. This means that $|E(\{y_1, y_3\}, N(v))| \leq 2$. On the other hand, since H^* is linked to at least two vertices in $N(v)$ and by Claim 11, we see that $|E(\{y_1, y_3\}, N(v))| \geq 2$. Therefore, we conclude that $|E(\{y_1, y_3\}, N(v))| = 2$. Without loss of generality, we may assume that $E(\{y_1, y_3\}, N(v)) = \{y_1x_1, y_3x_2\}$.

Claim 13 $G[X] \cong F_1$.

Proof Suppose to the contrary that $G[X] \ncong F_1$. Then $x_ix_j \in E(G)$ for some $i, j \in [3]$. If $x_1x_2 \in E(G)$, then it follows from $\Delta = 3$ that each member of $\mathcal{H}_g$ is linked to x_3 only (i.e., $\mathcal{H}_{g,x_3} \neq \emptyset$), giving a contradiction. So we may assume by symmetry that $x_1x_3 \in E(G)$. Since $\Delta = 3$ and each member of $\mathcal{H}_g$ is linked to at least two vertices in $N(v)$, we know that $\mathcal{H}_g$ contains exactly one member, say H, which is linked to both x_2 and x_3. By Claim 8, let D_H be a 1-isolating set of H such that $|D_H| = \frac{2}{7}|V(H)|$ and $x_2 \in N[D_H]$. Then, it is easy to check that $D := D_H \cup \{x_1\}$ is a 1-isolating set of G. Notice that

$$|D| = |D_H| + 1 = \frac{2}{7}|V(H)| + 1 = \frac{2}{7}(n-7) + 1 = \frac{2}{7}n - 1 < \frac{2}{7}n,$$

we obtain a contradiction to the assumption that $\iota_1(G) = \frac{2}{7}n$. □

Claim 14 $|\mathcal{H}_g| = 1$.

Proof Suppose to the contrary that $|\mathcal{H}_g| \geq 2$. Since $\Delta = 3$ and each member of $\mathcal{H}_g$ is linked to at least two vertices in $N(v)$, we deduce that $\mathcal{H}_g$ contains exactly two members, say H_1 and H_2, such that H_1 is linked to both x_1 and x_3 and H_2 is linked to both x_2 and x_3. By Claim 8, let D_{H_1} be a 1-isolating set of H_1 such that $|D_{H_1}| = \frac{2}{7}|V(H_1)|$ and $x_1 \in N[D_{H_1}]$, and let D_{H_2} be a 1-isolating set of H_2 such that $|D_{H_2}| = \frac{2}{7}|V(H_2)|$ and $x_3 \in N[D_{H_2}]$. Define $D := D_{H_1} \cup D_{H_2} \cup \{x_2\}$. Then, one can easily see that D is a 1-isolating set of G. Since

$$|D| = |D_{H_1}| + |D_{H_2}| + 1 = \frac{2}{7}|V(H_1)| + \frac{2}{7}|V(H_2)| + 1 = \frac{2}{7}(n-7) + 1 = \frac{2}{7}n - 1 < \frac{2}{7}n,$$

we derive a contradiction. □

By Claim 14, let H be the unique member of $\mathcal{H}_g$. Since H is linked to at least two vertices in $N(v)$ and by symmetry between x_1 and x_2, we may assume that H is linked to x_1 and at least one vertex in $\{x_2, x_3\}$. Define $D := \{v, x_1\}$. Then D is a 1-isolating set of $G[X]$ containing x_1 and $E(X \backslash N[D], V(G) \backslash X) = \emptyset$. By Claim 9, we have $H \in \mathcal{G}_1$. Let $G_1, G_2, \ldots, G_t$ be the C_6-constituents of H, and for each $i \in [t]$, let u_i be the C_6-connection of G_i and $H_i := u_i^1u_i^2u_i^3u_i^4u_i^5u_i^6u_i^1$ be the copy of C_6 in G_i. Let w be the unique neighbor of x_1 in H (since $\Delta = 3$). Note that when $t = 1$ and $H \cong F_3$, the two vertices u_1 and u_1^1 are symmetric in H. Hence, we may assume that

$w \neq u_1^1$ when $t = 1$ and $H \cong F_3$; otherwise, we can exchange the labels of u_1 and u_1^1.

Suppose $w = u_i^j$ for some $i \in [t]$ and $j \in [6]$. Define $D' := D \cup \{u_i^{j+3}\} \cup \{u_k, u_k^4 : 1 \leq k \leq t \text{ and } k \neq i\}$, where the superscript $j+3$ is taken modulo 6 when $j \in \{4, 5, 6\}$. Then, it follows from Observation 2.6(ii) that D' is a 1-isolating set of G. Since $|D'| = 3 + \frac{2}{7}(n-14) = \frac{2}{7}n - 1 < \frac{2}{7}n$, we obtain a contradiction to the assumption that $\iota_1(G) = \frac{2}{7}n$.

Therefore, we may assume that $w = u_i$ for some $i \in [t]$. This (together with $\Delta = 3$) shows that $G_k \ncong F_4$ for any $k \in [t]$. Let $Y := X \cup V(G_i)$. For each $k \in [t]$ and $k \neq i$, let

$$D_k := \begin{cases} \{u_k^1, u_k^4\}, & \text{if } G_k \in \{F_1, F_2\}, \\ \{u_k^2, u_k^5\}, & \text{if } G_k \cong F_3. \end{cases}$$

Define $D^* := \bigcup_{1 \leq k \leq t,\, k \neq i} D_k$. Then by Observation 2.6(i), we see that D^* is a dominating set of $G - Y$ and $|D^*| = \frac{2}{7}(n-14) = \frac{2}{7}n - 4$. If $G_i \ncong F_1$ or $|E(\{x_3\}, \{u_i^2, u_i^6\})| \leq 1$, then $\{x_2, u_i, u_i^4\}$ is a 1-isolating set of $G[Y]$ (and thus $\iota_1(G[Y]) \leq 3$) and it follows from Lemma 2.3 that

$$\iota_1(G) \leq |D^*| + \iota_1(G[Y]) \leq \left(\frac{2}{7}n - 4\right) + 3 = \frac{2}{7}n - 1 < \frac{2}{7}n,$$

a contradiction. So we may assume that $G_i \cong F_1$ and $|E(\{x_3\}, \{u_i^2, u_i^6\})| = 2$ (i.e., $x_3u_i^2, x_3u_i^6 \in E(G)$). Moreover, we have $x_2u_i \in E(G)$; otherwise, $\{x_3, y_1, u_i^4\}$ is a 1-isolating set of $G[Y]$ (and hence $\iota_1(G[Y]) \leq 3$) and we can conclude from Lemma 2.3 that

$$\iota_1(G) \leq |D^*| + \iota_1(G[Y]) \leq \left(\frac{2}{7}n - 4\right) + 3 = \frac{2}{7}n - 1 < \frac{2}{7}n,$$

again a contradiction. Since $\Delta = 3$, we derive that $t = 1$ and $H \cong F_1$. Now, it is straightforward to verify that $G \cong B_1 \in \mathcal{G}_3$.

This completes the proof of Theorem 1.2. □

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