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On a Conjecture of Petrov and Tolev Related to Chen's Theorem

Guang-Liang Zhou[1] · Yingchun Cai[1]

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Abstract

For any real number y, let $[y]$ be the largest integer not exceeding y. Petrov and Tolev conjectured that there exists a constant $c_0 > 1$ such that if $1 < c < c_0$, then every sufficiently large natural number N can be represented as

$$N = [p^c] + [m^c],$$

where p is a prime and m is a natural number having at most 2 prime factors. And, they proved that when c is close to 1, specifically when $1 < c \leq 1485/1484 = 1.00067\ldots$, every sufficiently large natural number N can be represented as $N = [p^c] + [m^c]$ with m having at most 53 prime factors.
In this paper, we show that if $1 < c \leq 1.0198$, then every sufficiently large natural number N can be written as $N = [p^c] + [m^c]$, where p is a prime and m is a natural number having at most 10 prime factors. This improves the result of Petrov and Tolev.

Keywords Additive problem · Sieve · Fractional powers

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1 Introduction

The famous Goldbach conjecture states that any even number greater than 2 can be written as the sum of two prime numbers. Let P_r denote an almost-prime with at most

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✉ Yingchun Cai
yingchuncai@mail.tongji.edu.cn

Guang-Liang Zhou
guangliangzhou@126.com

[1] School of Mathematical Science, Tongji University, Shanghai 200092, People's Republic of China

r prime factors counted with multiplicity. In 1966, Chen [2] announced his remarkable theorem–Chen's theorem: every sufficiently large even integer N can written as

$$N = p + P_2,$$

where and in what follows p, with or without subscript, is a prime. And the detail was published in [3].

The ancient Waring problem says that for every natural number $k \geq 2$ there exists a positive integer $s = s(k)$ such that every natural number is a sum of at most s kth powers of natural numbers. In 1934, Segal [12] generalized the Waring problem to fractional exponents. And, he showed that for any fixed real number $c > 1$, there exists a positive integer $s = s(c)$ such that every sufficiently large natural number N can be written as

$$N = [x_1^c] + [x_2^c] + \cdots + [x_s^c],$$

where $x_1, x_2, \ldots, x_s$ are non-negative integers. On the other hand, some mathematicians consider that how large c can be for the fixed $s \geq 2$. In 1973, Deshouillers [4] proved that if $1 < c < 4/3$, then every sufficiently large natural number N can be represented as

$$N = [x_1^c] + [x_2^c],$$

where x_1 and x_2 are non-negative integers. Later, the domain of c was improved to $1 < c < 55/41$ and $1 < c < 3/2$, respectively by Gritsenko [7] and Konyagin [9]. In addition, in 2009, Kumchev [10] proved that if $1 < c < 16/15$, then every sufficiently large natural number N can be represented as

$$N = [p^c] + [m^c], \tag{1.1}$$

where m is a positive integer. Recently, the range of c obtained by Kumchev was improved by Yu [14] to $1 < c < 11/10$. Furthermore, Petrov and Tolev [11] proved that if $1 < c < 29/28$, then every sufficiently large natural number N can be represented as (1.1) with m is an almost prime with at most $[52/(29 - 28c)] + 1$ prime factors. Inspired by Chen's theorem, Petrov and Tolev [11] proposed the following interesting conjecture:

Conjecture 1.1 *There exists a constant $c_0 > 1$ such that if $1 < c < c_0$, then every sufficiently large natural number N can be represented as*

$$N = [p^c] + [P_2^c].$$

In the present paper, we improve the results of Petrov and Tolev when c close to 1. And, we state our theorem in the following.

Theorem 1.1 *Suppose that* $1 < c \leq 1.0198$. *Then every sufficiently large natural number* N *can be represented as*

$$N = [p^c] + [m^c],$$

where p *is a prime and* m *is an almost prime with at most* 10 *prime factors.*

Remark 1.1 While our method does not yield a general formula for the number of prime factors of m in terms of c, for every specific $c \in (1, 29/28)$, one can apply our method to get an improvement to Petrov and Tolev's result.

2 Notation and Preliminaries

From now on, let N be a sufficiently large natural number and

$$1 < c \leq 1.0198, \quad \theta = \frac{1}{c}$$

be the positive real numbers. Put

$$P = \delta N^{\theta}, \quad \delta = 10^{-9}.$$

Put $e(y) = e^{2\pi i y}$. As usual, $\mu(n)$ denotes the Möbius function. Let $\rho(t) = \frac{1}{2} - \{t\}$, where $\{t\}$ is the fractional part of t. Define (ξ_d^+) and (ξ_d^-) the upper bound and lower bound beta-sieves of level D respectively (see Chapter 11 of [5]), for which we have

$$\sum_{d|n} \xi^-(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \xi^+(d). \tag{2.1}$$

For $z \geq 2$, define

$$P(z) = \prod_{p<z} p \quad \text{and} \quad V(z) = \prod_{p<z} \left(1 - \frac{1}{p}\right).$$

Then, by Theorem 11.12 of Friedlander and Iwaniec [5], we have

$$\sum_{d|P(z)} \frac{\xi^+(d)}{d} \leq V(z)\,(F(s) + o(1)) \tag{2.2}$$

and

$$\sum_{d|P(z)} \frac{\xi^-(d)}{d} \geq V(z)\,(f(s) + o(1)), \tag{2.3}$$

where $F(s)$ and $f(s)$ are the standard upper and lower bound functions of the linear sieve, and

$$s = \frac{\log D}{\log z}.$$

Lemma 2.1 *For $F(s)$ and $f(s)$, We have*

$$
\begin{aligned}
&F(s)=\frac{2e^{\gamma}}{s},\quad 0<s\le 3;\\
&F(s)=\frac{2e^{\gamma}}{s}\left(1+\int_2^{s-1}\frac{\log(t-1)}{t}\right),\quad 3\le s\le 5;\\
&F(s)=\frac{2e^{\gamma}}{s}\left(1+\int_2^{s-1}\frac{\log(t-1)}{t}+\int_2^{s-3}\frac{\log(t-1)}{t}dt\right.\\
&\qquad\left.\int_{t+2}^{s-1}\frac{1}{u}\log\frac{u-1}{t+1}\right),\quad 5\le s\le 7;\\
&f(s)=0,\quad 0<s\le 2;\\
&f(s)=\frac{2e^{\gamma}\log(s-1)}{s},\quad 2\le s\le 4;\\
&f(s)=\frac{2e^{\gamma}}{s}\left(\log(s-1)+\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}du\right),\quad 4\le s\le 6;\\
&f(s)=\frac{2e^{\gamma}}{s}\left(\log(s-1)+\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}du\right.\\
&\qquad\left.+\int_2^{s-4}\frac{\log(t-1)}{t}dt\int_{t+2}^{s-2}\frac{1}{u}\log\frac{u-1}{t+1}\log\frac{s}{u+2}du\right),\quad 6\le s\le 8.
\end{aligned}
$$

Proof See [8, (3.11) and (3.12)]. □

We denote

$$D=N^{\eta},$$

where $\eta=\eta(c)>0$ is a constant. Let

$$\Sigma_j=\sum_{d|P(z)}\xi_d^{-}\sum_{P<p\le 2P}(\log p)\rho\Big(-\frac{1}{d}(N+j-[p^c])^{\theta}\Big),\quad j=0,1. \tag{2.4}$$

Lemma 2.2 *Let*

$$\frac{28}{29}<\theta<1,\quad \eta<\frac{29\theta-28}{26}.$$

Then we have

$$\Sigma_0,\Sigma_1\ll\frac{N^{2\theta-1}}{(\log N)^2},\quad j=0,1.$$

Proof See (23), (24), (71), and (73) of [11]. □

Lemma 2.3 (Vaaler's theorem) *For each $H \ge 2$ there are numbers c_h, $1 \le h \le H$, and d_h, $0 \le h \le H$, such that*

$$\rho(t) = \sum_{1 \le |h| \le H} c_h e(ht) + \Delta_H(t),$$

where

$$|\Delta_H(t)| \le \sum_{0 \le |h| \le H} d_h e(ht)$$

and

$$|c_h| \ll \frac{1}{|h|}, \quad |d_h| \ll \frac{1}{H}.$$

Proof See [13].

Lemma 2.4 (Van der Corput's Theorem) *Suppose that ϑ is a real valued function with two continuous derivatives on interval I. Suppose also that there is some $\lambda > 0$ such that*

$$|\vartheta''| \asymp \lambda$$

on I. Then

$$\sum_{n \in I} e(\vartheta(n)) \ll |I|\lambda^{1/2} + \lambda^{-1/2}.$$

Proof See [6, Theorem 2.2]. □

3 A Key Mean Estimation

In this section, we prove a mean estimation similar to Lemma 2.2, which plays a crucial role in the proof of Theorem 1.1.

From now on, we take $z = N^{\frac{1}{200}}$. Let

$$R_j^{(k)} = \sum_{d | P(z)} \xi_d^+ \sum_{m \in \mathcal{M}_k} \rho\Big(-\frac{1}{d}(N + j - [m^c])^\theta\Big), \quad j = 0, 1, \tag{3.1}$$

where

$$\begin{aligned}\mathcal{M}_k = \{m = p_1 \cdots p_k : (1 - (2\delta)^c)^\theta N^\theta - 1 \le p_1 \cdots p_k \\ < (1 - \delta^c)^\theta N^\theta + 1, z \le p_1 \le \cdots \le p_k\}.\end{aligned} \tag{3.2}$$

By the prime number theorem, we have

$$|\mathcal{M}_k| \le (1+o(1)) \sum_{z \le p_1 \le \cdots \le p_{k-1} \le \left(\frac{(1-\delta^c)^\theta N^\theta+1}{p_1\cdots p_{k-2}}\right)^{\frac{1}{2}}} \left(\frac{(1-\delta^c)^\theta N^\theta+1}{p_1\dots p_{k-1}\log\frac{(1-\delta^c)^\theta N^\theta+1}{p_1\cdots p_{k-1}}} - \frac{(1-(2\delta)^c)^\theta N^\theta-1}{p_1\dots p_{k-1}\log\frac{(1-(2\delta)^c)^\theta N^\theta-1}{p_1\cdots p_{k-1}}}\right)$$

Taking $x=(1-\delta^c)^\theta N^\theta+1$ in [1, (4.29)] and by some routine arguments we get that

$$|\mathcal{M}_k| \le ((1-\delta^c)^\theta-(1-(2\delta)^c)^\theta+o(1))c_k\frac{N^\theta}{\log N^\theta}, \tag{3.3}$$

where

$$c_k=\int_{k-1}^{199}\frac{dt_1}{t_1}\int_{k-2}^{t_1-1}\frac{dt_2}{t_2}\cdots\int_3^{t_{k-4}-1}\frac{dt_{k-3}}{t_{k-3}}\int_2^{t_{k-3}-1}\frac{\log(t_{k-2}-1)dt_{k-2}}{t_{k-2}}.$$

To compute the bound c_k we used the *Mathematica* technical computing software. For example, we use the following code to calculate c_{11}.
NIntegrate [(Log[t9 − 1]/t9)*(1/t8)*(1/t7)*(1/t6)*(1/t5)*(1/t4)*(1/t3)*(1/t2)*(1/t1), {t1, 10, 199}, {t2, 9, t1 − 1}, {t3, 8, t2 − 1}, {t4, 7, t3 − 1}, {t5, 6, t4 − 1}, {t6, 5, t5 − 1}, {t7, 4, t6 − 1}, {t8, 3, t7 − 1}, {t9, 2, t8 − 1}]
In fact, the estimate of c_k for $k \ge 15$ has already been given in [1, (4.30)]. Whatever, we have

$$\begin{aligned}&c_{11}<0.580195,\quad c_{12}<0.185152,\quad c_{13}<0.052602,\quad c_{14}<0.018655,\\&c_{15}<0.003088,\quad c_{16}<0.000646,\quad c_{17}<0.000124,\quad c_{18}<0.000011,\\&c_k<0.000001\ \text{ for }\ 19\le k\le 199.\end{aligned}$$

Hence, we have

$$\sum_{k=11}^{199}|\mathcal{M}_k| \le (0.840654+o(1))((1-\delta^c)^\theta-(1-(2\delta)^c)^\theta)\frac{N^\theta}{\log N^\theta}. \tag{3.4}$$

Proposition 3.1 *Let*

$$\frac{28}{29}<\theta<1,\quad \eta<\frac{29\theta-28}{26}.$$

If $k \ge 2$, then we have

$$R_0^{(k)}+R_1^{(k)} \ll \frac{N^{2\theta-1}}{(\log N)^3}.$$

Proof Let $Z \geq 2$ be any fixed integer. From page S43 and S44 of [11], we know that there exists a series of periodic functions $g_s(t)$, $s = 0, 1, \ldots, 2Z-1$ with a period of 1 and has the following properties:

$$0 < g_s(t) \leq 1 \ \text{ for } \ \left| t - \frac{s}{2Z} \right| < \frac{1}{2Z}, \tag{3.5}$$

$$g_s(t) = 0 \ \text{ for } \ \frac{1}{2Z} < \left| t - \frac{s}{2Z} \right| < \frac{1}{2}, \tag{3.6}$$

and

$$\sum_{s=0}^{2Z-1} g_s(t) = 1 \ \text{ for all } \ t \in \mathbb{R}. \tag{3.7}$$

Furthermore, we have

$$g_s(t) = \sum_{|n| \leq Z(\log N)^4} \beta_n^{(s)} e(nt) + O(N^{-\log\log N}), \quad s = 0, 1, \ldots, 2Z-1, \tag{3.8}$$

where

$$\beta_n^{(s)} \leq \frac{1}{2Z}. \tag{3.9}$$

By Lemma 2.3, we can write

$$R_j^{(k)} = R_{j1}^{(k)} + R_{j2}^{(k)}, \tag{3.10}$$

where

$$R_{j1}^{(k)} = \sum_{d \leq D} \xi_d^+ \sum_{m \in \mathcal{M}_k} \sum_{1 \leq |h| \leq H} c_h e\Big(-\frac{h}{d}(N + j - [m^c])^\theta \Big)$$

and

$$R_{j2}^{(k)} = \sum_{d \leq D} \xi_d^+ \sum_{m \in \mathcal{M}_k} \Delta_H \Big(-\frac{h}{d}(N + j - [m^c])^\theta \Big).$$

Let

$$W_j(v) = \sum_{m \in \mathcal{M}_k} e(v(N + j - [m^c])^\theta).$$

Changing the order of summation together with Lemma 2.3, we obtain

$$R_{j1}^{(k)} \ll \sum_{d \leq D} \sum_{1 \leq h \leq H} \frac{1}{h} \left| W_j\Big(\frac{h}{d}\Big) \right|, \tag{3.11}$$

and

$$R_{j2}^{(k)} \ll \sum_{d\le D} \frac{W(0)}{H} + \sum_{d\le D} \sum_{1\le h\le H} \frac{1}{H} \left| W_j\left(\frac{h}{d}\right) \right|. \tag{3.12}$$

Hence, by (3.3), we obtain

$$R_{j2}^{(k)} \ll \frac{1}{\log N} \sum_{d\le D} \frac{N^\theta}{H} + \sum_{d\le D} \sum_{1\le h\le H} \frac{1}{H} \left| W_j\left(\frac{h}{d}\right) \right|.$$

We choose

$$H = dN^{1-\theta}(\log N)^3.$$

Combining (3.10), (3.11) and (3.12), we obtain

$$R_j^{(k)} \ll \frac{N^{2\theta-1}}{(\log N)^3} + \sum_{d\le D} \sum_{1\le h\le H} \frac{1}{h} \left| W_j\left(\frac{h}{d}\right) \right|, \quad j = 0, 1. \tag{3.13}$$

Now, we consider the sum $W_j(v)$. From (3.7) it follows that

$$W_j(v) \ll \sum_{m\in\mathcal{M}_k} e(v(N + j - [m^c])^\theta) \sum_{s=0}^{2Z-1} g_s(m^c) = \sum_{s=0}^{2Z-1} W_j^{(s)}(v),$$

where

$$W_j^{(s)}(v) = \sum_{m\in\mathcal{M}_k} g_s(m^c) e(v(N + j - [m^c])^\theta).$$

By (3.2), (3.8) and (3.9). we have

$$\begin{aligned} W_j^{(0)}(v) &\ll \sum_{m\in\mathcal{M}_k} g_0(m^c) \le \sum_{AN^\theta-1\le m<BN^\theta+1} g_0(m^c) \\ &\ll \frac{N^\theta}{Z} + \left| \sum_{AN^\theta-1\le m<BN^\theta+1} \sum_{1\le|n|\le Z(\log N)^4} \beta_n e(nm^c) \right| + 1 \\ &\ll \frac{N^\theta}{Z} + \frac{1}{Z} \sum_{1\le|n|\le Z(\log N)^4} |H_n| + 1, \end{aligned} \tag{3.14}$$

where $A = (1 - (2\delta)^c)^\theta$, $B = (1 - \delta^c)^\theta$ and

$$H_n = \sum_{AN^\theta-1\le m<BN^\theta+1} e(nm^c).$$

Let $\vartheta_n(x) = nx^c$. It is easy to verify that $\vartheta_n''(x) \asymp nN^{1-2\theta}$ uniformly for $AN^\theta - 1 \leq x < BN^\theta + 1$. Hence, by Lemma 2.4, we get

$$H_n \ll N^\theta (nN^{1-2\theta})^{1/2} + (nN^{1-2\theta})^{-1/2} \ll (nN)^{1/2}. \tag{3.15}$$

We assume that

$$Z \ll N^{(2\theta-1)/3} (\log N)^{-4}. \tag{3.16}$$

Then by (3.14), (3.15) and (3.16), we obtain

$$W_j^{(0)}(v) \ll \frac{N^\theta}{Z} + N^{1/2} Z^{1/2} \log^6 N \ll \frac{N^\theta}{Z}. \tag{3.17}$$

Now, we consider the sums $W_j^{(s)}(v)$ for $1 \leq s \leq 2Z - 1$. From (3.5) we know that $g_s(m^c)$ vanishes unless $\{m^c\} \in [(s-1)/(2Z), (s+1)/(2Z)]$. Hence, the only summands in the sums $W_s(v)$ are those for which

$$\{m^c\} = \frac{s}{2Z} + O\Big(\frac{1}{Z}\Big).$$

And in this case we have

$$v(N + j - [m^c])^\theta = v\Big(N + j - m^c + \frac{s}{2Z}\Big)^\theta + O\Big(\frac{vN^{\theta-1}}{Z}\Big)$$

and so

$$e(v(N + j - [m^c])^\theta) = e\Big(v\Big(N + j - m^c + \frac{s}{2Z}\Big)^\theta\Big) + O\Big(\frac{vN^{\theta-1}}{Z}\Big).$$

Hence, we have

$$W_j^{(s)}(v) = V_j^{(s)}(v) + O\Big(\frac{vN^{\theta-1}}{Z} \sum_{m \in \mathcal{M}_k} g_s(m^c)\Big),$$

where

$$V_j^{(s)}(v) = \sum_{m \in \mathcal{M}_k} g_s(m^c) e\Big(v\Big(N + j - m^c + \frac{s}{2Z}\Big)^\theta\Big). \tag{3.18}$$

Thus, by (3.17), we get

$$W_j(v) = \sum_{s=1}^{2Z-1} W_j^{(s)}(v) + W_j^{(0)}(v) = \sum_{s=1}^{2Z-1} V_j^{(s)}(v) + O(\Xi) + O\Big(\frac{N^\theta}{Z}\Big),$$

where

$$\Xi = \frac{vN^{\theta-1}}{Z} \sum_{m \in \mathcal{M}_k} \sum_{s=1}^{2Z-1} g_s(m^c).$$

By (3.3), (3.5) and (3.6), we have

$$\Xi \ll \frac{vN^{2\theta-1}}{Z \log N}.$$

Now, we have

$$W_j(v) = \sum_{s=1}^{2Z-1} V_j^{(s)}(v) + O\Big(\frac{vN^{2\theta-1}}{Z \log N} + \frac{N^\theta}{Z}\Big). \tag{3.19}$$

Take

$$v = \frac{h}{d}, \quad \text{where } 1 \le d \le D, \quad 1 \le h \le H = dN^{1-\theta}(\log N)^3.$$

Obviously, we have $vN^{2\theta-1} \ll N^\theta (\log N)^3$. So we can rewrite (3.19) as

$$W_j(v) = \sum_{s=1}^{2Z-1} V_j^{(s)}(v) + O\Big(\frac{N^\theta}{Z}(\log N)^2\Big). \tag{3.20}$$

Combining (3.13) and (3.20), we obtain

$$R_j^{(k)} \ll \frac{N^{2\theta-1}}{(\log N)^3} + \sum_{d \le D} \sum_{1 \le h \le H} \frac{1}{h} \sum_{s=1}^{2Z-1} \Big| V_j^{(s)}\Big(\frac{h}{d}\Big)\Big| + \sum_{d \le D} \sum_{1 \le h \le H} \frac{1}{h} \frac{N^\theta}{Z} (\log N)^2.$$

We choose Z such that

$$Z \asymp dN^{1-\theta}(\log N)^7.$$

Hence, we have

$$R_j^{(k)} \ll \frac{N^{2\theta-1}}{(\log N)^3} + \sum_{d \le D} \sum_{1 \le h \le H} \frac{1}{h} \sum_{s=1}^{2Z-1} \Big| V_j^{(s)}\Big(\frac{h}{d}\Big)\Big|. \tag{3.21}$$

Now, we consider the sums $V_j^{(s)}(h/d)$. By (3.8) and (3.18), we have

$$V_j^{(s)}\Big(\frac{h}{d}\Big) = \sum_{m\in\mathcal{M}_k}\Big(\sum_{|n|\le Z(\log N)^4}\beta_n^{(s)}e(nm^c)\Big)e\Big(v\Big(N+j-m^c+\frac{s}{2Z}\Big)^\theta\Big)+O(N^{-10})$$
$$= \sum_{|n|\le Z(\log N)^4}\beta_n^{(s)}U\Big(N+j+\frac{s}{2Z},n,\frac{h}{d}\Big)+O(N^{-10})$$
$$\ll \frac{1}{Z}\sum_{|n|\le R}\sup_{T\in[N,N+2]}\left|U\left(T,n,\frac{h}{d}\right)\right|, \tag{3.22}$$

where

$$U=U(T,n,v)=\sum_{m\in\mathcal{M}_k}e(nm^c+v(T-m^c)^\theta)\ \text{ and }\ R=dN^{1-\theta}(\log N)^{12}.$$

Inserting (3.22) into (3.21), we get

$$R_0^{(k)}+R_1^{(k)} \ll \frac{N^{2\theta-1}}{(\log N)^3}+\sum_{d\le D}\sum_{1\le h\le H}\frac{1}{h}\sum_{|n|\le R}\sup_{T\in[N,N+2]}\left|U\Big(T,n,\frac{h}{d}\Big)\right|. \tag{3.23}$$

Recall the definition of $\mathcal{M}_k$ in (3.2). Let $k\ge 2$ and $n=p_1\cdots p_k\in\mathcal{M}_k$. By some routine arguments, we can rewrite n as $n=rs$ with

$$N^{\frac{\theta}{200}}<r\le N^{\frac{1}{2}}<s<N^{\frac{199\theta}{200}}.$$

In fact, it is easy to see that $U\Big(T,n,\frac{h}{d}\Big)$ is a summation similar to (121) in [11]. Through the same argument as Sect. 3.6 of [11], there is almost no need for adjustment, and we can get that if

$$\frac{28}{29}<\theta<1,\quad \delta<\frac{29\theta-28}{26},$$

then

$$\sum_{d\le D}\sum_{1\le h\le H}\frac{1}{h}\sum_{|n|\le R}\sup_{T\in[N,N+2]}\left|U\Big(T,n,\frac{h}{d}\Big)\right|\ll\frac{N^{2\theta-1}}{(\log N)^3}.$$

So, we have

$$R_0^{(k)}+R_1^{(k)}\ll\frac{N^{2\theta-1}}{(\log N)^3}.$$

This completes the proof. □

4 The Proof of Theorem 1.1

To prove the theorem, we consider the lower bound of the sum

$$\Gamma = \sum_{\substack{P<p\le 2P, m\in\mathbb{N}\\ [p^c]+[m^c]=N\\ m=P_{10}}} (\log p).$$

By the trivial inequality

$$\begin{aligned}\Gamma \ge & \sum_{\substack{P<p\le 2P,\ m\in\mathbb{N}\\ [p^c]+[m^c]=N\\ (m,P(z))=1}} (\log p) - \sum_{\substack{P<p\le 2P,\ m\in\mathbb{N}\\ [p^c]+[(p_1\dots p_{10}m)^c]=N\\ z\le p_1\le\dots\le p_{10},(m,P(p_{10})=1)}} (\log p)\\ \ge & \sum_{\substack{P<p\le 2P,\ m\in\mathbb{N}\\ [p^c]+[m^c]=N\\ (m,P(z))=1}} (\log p) - \sum_{\substack{\ell,m\in\mathbb{N},\ P<\ell\le 2P\\ [\ell^c]+[(p_1\dots p_{10}m)^c]=N\\ z\le p_1\le\dots\le p_{10},(m,P(p_{10})=1)\\ (\ell,P(z))=1}} (\log \ell)\\ =&\Gamma_1-\Gamma_2. \end{aligned} \tag{4.1}$$

From (6), (17), (18) and (19) in [11], we know that

$$\Gamma_1 \ge \Sigma + \Sigma_0 - \Sigma_1,$$

where Σ_j, $j = 0, 1$ are defined by (2.4) and

$$\Sigma \ge A(N)V(z)(f(s)+o(1))$$

with

$$A(N) = \theta \sum_{P<p\le 2P} (\log p)((N-[p^c])^{\theta-1} + O(N^{\theta-2}))$$

and

$$s = \frac{\log D}{\log z}.$$

By Lemma 2.2 and Proposition 3.1, we can take

$$\eta = \frac{29\theta-28}{26} - \varepsilon.$$

So we have

$$s = \frac{200(29\theta-28)}{26} + o(1). \tag{4.2}$$

From the definition of P and the prime number theorem, we obtain

$$A(N) \geq (\delta\theta(1-(2\delta)^c)^{\theta-1} + o(1))N^{2\theta-1}.$$

Hence, by Lemma 2.2 and the fact

$$V(z) = \prod_{p<z}\left(1-\frac{1}{p}\right) \asymp \frac{1}{\log z} \asymp \frac{1}{\log N}, \tag{4.3}$$

we get

$$\Gamma_1 \geq (\theta(\delta(1-(2\delta)^c)^{\theta-1}) + o(1))N^{2\theta-1}V(z)(f(s)+o(1)). \tag{4.4}$$

Obviously, we have

$$\Gamma_2 \leq \sum_{k=11}^{199} \sum_{\substack{\ell\in\mathbb{N} \\ [\ell^c]+[m^c]=N \\ m\in\mathcal{M}_k,\ (\ell,P(z))=1}} (\log \ell) = (1+o(1)) \sum_{k=11}^{199} (\log P)\Sigma_2^k, \tag{4.5}$$

where

$$\Sigma_2^k = \sum_{\substack{\ell\in\mathbb{N} \\ [\ell^c]+[m^c]=N \\ m\in\mathcal{M}_k,\ (\ell,P(z))=1}} 1.$$

From (2.1), we find

$$\Sigma_2^k = \sum_{\substack{\ell\in\mathbb{N} \\ [\ell^c]+[m^c]=N \\ m\in\mathcal{M}_k}} \sum_{d|(\ell,P(z))} \mu(d) \leq \sum_{\substack{\ell\in\mathbb{N} \\ [\ell^c]+[m^c]=N \\ m\in\mathcal{M}_k}} \sum_{d|(\ell,P(z))} \xi^+(d).$$

By exchanging the order of summation, we obtain

$$\Sigma_2^k \leq \sum_{d|P(z)} \xi^+(d)G_{d,k}, \quad \text{where } G_{d,k} = \sum_{\substack{\ell\in\mathbb{N} \\ [\ell^c]+[m^c]=N \\ m\in\mathcal{M}_k,\ \ell\equiv 0 \pmod d}} 1. \tag{4.6}$$

By the identity

$$\sum_{a\leq m<b} 1 = [-a]-[-b] = b-a-\rho(-b)+\rho(-a),$$

we have

$$G_{d,k}=\sum_{m\in\mathcal{M}_k}\sum_{\substack{\ell\in\mathbb{N}\\ [\ell^c]+[m^c]=N\\ \ell\equiv 0\ (\mathrm{mod}\ d)}}1=\sum_{m\in\mathcal{M}_k}\sum_{(1/d)(N-[m^c])^\theta\le\ell<(1/d)(N+1-[m^c])^\theta}1$$
$$=\sum_{m\in\mathcal{M}_k}\frac{(N+1-[m^c])^\theta-(N-[m^c])^\theta}{d}$$
$$+\sum_{m\in\mathcal{M}_k}\rho\Big(-\frac{1}{d}(N-[m^c])^\theta\Big)-\sum_{m\in\mathcal{M}_k}\rho\Big(-\frac{1}{d}(N+1-[m^c])^\theta\Big). \quad (4.7)$$

Combining (4.6), (4.7) and the identity

$$(N+1-[m^c])^\theta=(N-[m^c])^\theta+\theta(N-[m^c])^{\theta-1}+O(N^{\theta-2}),$$

we obtain

$$\Sigma_2^k\le R^{(k)}+R_0^{(k)}-R_1^{(k)}, \quad (4.8)$$

where $R_j^{(k)}$, $j=0,1$ are defined by (3.1) and

$$R^{(k)}=\theta\sum_{d|P(z)}\frac{\xi^+(d)}{d}\sum_{m\in\mathcal{M}_k}((N-[m^c])^{\theta-1}+O(N^{\theta-2})).$$

By (2.2) and (3.2), we have

$$R^{(k)}\le\theta(2\delta)^{1-c}N^{\theta-1}\Big(\sum_{m\in\mathcal{M}_k}1\Big)V(z)(F(s)+o(1)), \quad (4.9)$$

where

$$s=\frac{200(29\theta-28)}{26}+o(1).$$

By (3.4), (4.3), (4.5), (4.8), (4.9) and Proposition 3.1, we have

$$\Gamma_2\le(0.840654+o(1))\theta(2\delta)^{1-c}((1-\delta^c)^\theta-(1-(2\delta)^c)^\theta)F(s)V(z)N^{2\theta-1}. \quad (4.10)$$

From (4.1), (4.4) and (4.10), as long as

$$L=(1-(2\delta)^c)^{\theta-1}f(s)-0.840654\delta^{-1}(2\delta)^{1-c}((1-\delta^c)^\theta$$
$$-(1-(2\delta)^c)^\theta)F(s)>0,$$

we can deduce that

$$\Gamma \gg \frac{N^{2\theta-1}}{\log N},$$

which leads to the theorem. Recall that $\delta = 10^{-9}$. By (4.2) and Lemma 2.1, one can use the software *Mathematica* to run the following code, which shows that $L[1.0198] > 0.0017884$.

F[x_] := Piecewise [{{(2E^EulerGamma)/x, 0 < x <= 3}, {((2E^EulerGamma)/x) * (1 + NIntegrate [Log [t−1]/t, {t, 2, x−1}]), 3 <= x < 5}, {((2E^EulerGamma)/x) * (1 + NIntegrate [Log[t−1]/t, {t, 2, x−1}] + NIntegrate [(Log [t−1]/(t * u)) * Log [(u−1)/(t+1)], {t, 2, x−3}, {u, t+2, x−1}]), 5 <= x<7}}]

f[x_] := Piecewise [{{((2E^EulerGamma)/x) * Log [x−1], 2 <= x <= 4}, {((2E^EulerGamma) /x) * (Log [x−1] + NIntegrate [Log [u−1]/(t * u), {t, 3, x−1}, {u, 2, t−1}]), 4 <= x < 6}, {((2E^Eu lerGamma)/x) * (Log [x−1] + NIntegrate [Log [u−1]/(t * u), {t, 3, x−1}, {u, 2, t−1}] + NIntegrate [(Log [t−1]/(t * u)) * Log [(u−1)/(t + 1)] * Log[x/(u + 2)], {t, 2, x−4}, {u, t + 2, x−2}]), 6 <= x <= 8}}]

L[x_] := ((1 − (2 * 10^(−9))^x)^ (1/x −1)) * f [(200 * (29–28x))/(26x)] − (10^9) * (2 * 10^(−9))^ (1 −x) ((1–10^(−9x))^ (1/x) − (1 − (2 * 10^(−9))^x)^ (1/x)) F [(200 * (29–28x))/(26x)] * 0.840654

From [5, Chapter 11], we know that $f(s)$ is an increasing function and $F(s)$ is a decreasing function. By a trivial argument, we can conclude that L is a decreasing function about c, which deduces that if $1 < c < 1.0198$, then $L > 0.0017884$. This completes the proof of the theorem.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Cai, Y.: Almost prime triples and Chen's theorem. Acta Arith. **179**, 233–250 (2017)
2. Chen, J.R.: On the representation of a large even integer as the sum of a prime and the product of at most two primes. Kexue Tongbao **17**, 385–386 (1966)
3. Chen, J.R.: On the representation of a larger even integer as the sum of a prime and the product of at most two primes. Sci. Sin. **16**, 157–176 (1973)
4. Deshouillers, J.-M.: Un problème binaire en théorie additive. Acta Arith. **25**, 393–403 (1973/1974)
5. Friedlander, J., Iwaniec, H.: Opera de Cribro. American Mathematical Society: Colloquium publications, vol. 57. American Mathematical Society, Providence (2010)
6. Graham, S.W., Kolesnik, G.: Van Der Corput's Method of Exponential Sums. Cambridge University Press, Cambridge (1991)
7. Gritsenko, S.A.: Three additive problems. Izv. Ross. Akad. Nauk. **41**, 447–464 (1992)

8. Halberstam, H., Heath-Brown, D.R., Richert, H.E.: Almost-primes in short intervals. In: Recent Progress in Analytic Number Theory I, pp. 69–103. Academic Press (1981)
9. Konyagin, S.V.: An additive problem with fractional powers. Mat. Zametki **73**, 633–636 (2003)
10. Kumchev, A.: A binary additive equation involving fractional powers. Int. J. Number Theory **5**, 281–292 (2009)
11. Petrov, Z.H., Tolev, D.I.: On an equation involving fractional powers with one prime and one almost prime variables. Proc. Steklov Inst. Math. **298**, S38–S56 (2017)
12. Segal, B.I.: Der Waringsche Satz für die Potenzen mit gebrochenem und mit irrationalem Exponenten. Trav. Inst. Phys. Math. Stekloff **5**, 73–86 (1934)
13. Vaaler, J.D.: Some extremal problems in Fourier analysis. Bull. Am. Math. Soc. (N.S.) **12**, 183–216 (1985)
14. Yu, G.: On a binary additive problem involving fractional powers. J. Number Theory **208**, 101–119 (2020)