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Weighted Second Order Adams Inequality in the Whole Space $\mathbb{R}^4$

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Abstract

The main result of this paper is to establish a weighted second-order Adams-type inequality on the whole set of $\mathbb{R}^4$. As an application of this result, we prove the existence of a solution for a Kirchhoff-type equation involving non-linearity with subcritical or critical exponential growth. In the critical case, the associated energy loses its compactness. To avoid this problem, we add an asymptotic condition to the nonlinearity

Keywords Second order weighted Adams' inequality · Moser–Trudinger's inequality · Nonlinearity of exponential growth · Mountain pass method · Compactness level

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1 Introduction and Main Results

We first give an outline of Trudinger-Moser inequalities in classical Sobolev spaces of the first order. We also discuss Adams inequalities in higher-order Sobolev spaces.

In the literature, the notion of critical exponential growth is linked to Trudinger-Moser inequalities. For bounded domains $\Omega \subset \mathbb{R}^{\mathbb{N}}$, and in the Sobolev space $W_0^{1,N}(\Omega)$, these inequalities [32, 36] are given by

$$\sup_{\int_\Omega |\nabla u|^N \leq 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} \, dx < +\infty \text{ if and only if } \alpha \leq \alpha_N,$$

where $\alpha_N = \omega_{N-1}^{\frac{1}{N-1}}$ with ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. Later, the Trudinger-Moser inequality was improved to weighted inequalities [10, 12]. When the weight is of logarithmic type, Calanchi and Ruf [11] extend the Trudinger-Moser inequality and proved the following results in the weighted Sobolev space, $W_{0,rad}^{1,N}(B,\rho) = \text{closure}\{u \in C_{0,rad}^\infty(B) \mid \int_B |\nabla u|^N \rho(x) dx < \infty\}$, where B denote the unit ball of $\mathbb{R}^N$, $N \geq 2$.

Theorem 1.1 *[11]*

(i) *Let* $\beta \in [0, 1)$ *and let* ρ *given by* $\rho(x) = \big(\log \frac{1}{|x|}\big)^{\beta(N-1)}$, *then*

$$\int_B e^{|u|^\gamma} dx < +\infty, \quad \forall \ u \in W_{0,rad}^{1,N}(B,\rho), \quad \textit{if and only if } \gamma \leq \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

and

$$\sup_{\substack{u \in W_{0,rad}^{1,N}(B,\rho) \\ \int_B |\nabla u|^N w(x) dx \leq 1}} \int_B e^{\alpha |u|^{\gamma_{N,\beta}}} dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_{N,\beta} = N[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}}$$

where ω_{N-1} *is the area of the unit sphere* S^{N-1} *in* $\mathbb{R}^N$ *and* N' *is the Hölder conjugate of* N.

(ii) *Let* ρ *given by* $\rho(x) = \big(\log \frac{e}{|x|}\big)^{N-1}$, *then*

$$\int_B exp\{e^{|u|^{\frac{N}{N-1}}}\} dx < +\infty, \quad \forall \ u \in W_{0,rad}^{1,N}(B,\rho)$$

and

$$\sup_{\substack{u \in W_{0,rad}^{1,N}(B,\rho) \\ \|u\|_\rho \leq 1}} \int_B exp\{\beta e^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{\frac{N}{N-1}}}\} dx < +\infty \quad \Leftrightarrow \quad \beta \leq N,$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ and N' is the Hölder conjugate of N.

These types of results are mainly derived from calculations involving integrals and series. These types of calculations are currently at the heart of the mathematical news (see [21, 22, 33, 34]). Also, we like to recall, for instance, the study made by [8], [9], [14] and reference therein.

The Theorem 1.1 has allowed the exploration of second-order weighted elliptic problems in dimensions where $N \geq 2$. As a result, Calanchi et al. [13] established the existence of a non-trivial radial solution for an elliptic problem defined on the unit ball in $\mathbb{R}^2$, where the non-linearities exhibit double exponential growth at infinity. Following this, Deng et al. studied the following problem

$$\begin{cases} -\text{div}(\sigma(x)|\nabla u(x)|^{N-2}\nabla u(x)) = f(x,u) & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 2$ and the nonlinearity $f(x,u)$ is continuous in $B \times \mathbb{R}$ and has critical growth in the sense of Theorem 1.1. The authors have proved that there is a non-trivial solution to this problem, using the mountain pass Theorem. Similar results are proven by Chetouane and Jaidane [15, 24] and Zhang [38]. Furthermore, problem (1.1), involving a potential, has been studied by Baraket and Jaidane [7]. Also, we point out that recently, Abid et al. and Jaidane [1, 25] have proved the existence of a nontrivial solution for the following logarithmic weighted Kirchhoff problem

$$\begin{cases} -g\Big(\int_B \tau(x)|\nabla u|^N + \overline{V}(x)|u|^N dx\Big)\text{div}(\tau(x)|\nabla u|^{N-2}\nabla u + \overline{V}(x)|u|^{N-2}u) = f(x,u) & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 2$, the weight $\tau(x) = \Big(\log \frac{e}{|x|}\Big)^{\beta(N-1)}$, with $\beta = 1$ or $\beta \in [0,1)$, the reaction term $f(x,u)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\big(e^{\alpha t^{\frac{N}{(N-1)}}}\big)$ or $e^{\alpha t^{\frac{N}{(N-1)(1-\beta)}}}$, as $t \to +\infty$, for some $\alpha > 0$ and the potential $\overline{V}$ is a positive and continuous function on $\overline{B}$. The authors proved that there is a non-trivial solution to this problem using Nehari method and weighted Trudinger-Moser inequality [11].

These Kirchhoff-type equations are inspired by the following well-known Kirchhoff problem [26]

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = f(x,u), \tag{1.2}$$

where ρ, P_0, h, E, L represent physical quantities. This model extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. We call (1.2) a nonlocal problem since the equation contains an integral over $[0, L]$ which makes the study of it interesting. Later, Lions in his pioneering work [29] presented an abstract functional analysis framework to (1.2). We mention that non-local problems also arise in other areas, for instance, biological systems where the function u describes a process that depends on the average of itself (for example, population density), see for instance [3, 4] and its references.

In recent years, Aouaoui and Jlel [6] have extended the work of Calanchi and Ruf to the whole $\mathbb{R}^2$ space, by considering the following weight

$$\upsilon_\beta(x) = \begin{cases} \left(\log\left(\frac{e}{|x|}\right)\right)^\beta & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geq 1, \end{cases} \tag{1.3}$$

where, $0 < \beta \leq 1$ and $\chi : [1, +\infty[\rightarrow]0, +\infty[$ is a continuous function such that $\chi(1) = 1$ and $\inf_{t \in [1,+\infty[} \chi(t) > 0$. The authors consider the space E_β as the space of all radial functions of the completion of $C_0^\infty\left(\mathbb{R}^2\right)$ with respect to the norm

$$\|u\|_{E_\beta}^2 = \int_{\mathbb{R}^2} |\nabla u|^2 \upsilon_\beta(x) dx + \int_{\mathbb{R}^2} u^2 dx = |\nabla u|^2_{L^2(\mathbb{R}^2, \upsilon_\beta)} + |u|^2_{L^2(\mathbb{R}^2)}.$$

The authors proved the following result:

Theorem 1.2 *Let $\beta \in (0, 1)$ and ω_β be defined by (1.3). For all $u \in E_\beta$, we have*

$$\int_{\mathbb{R}^2} \left(e^{|u|^{\frac{2}{1-\beta}}} - 1\right) dx < +\infty.$$

Moreover, if $\alpha < \tau_\beta$, then

$$\sup_{u \in E_\beta, \|u\|_{E_\beta} \leq 1} \int_{\mathbb{R}^2} \left(e^{\alpha |u|^{\frac{2}{1-\beta}}} - 1\right) dx < +\infty \tag{1.4}$$

where $\tau_\beta = 2\big[2\pi(1-\beta)\big]^{\frac{1}{1-\beta}}$. If $\alpha > \tau_\beta$, then

$$\sup_{u \in E_\beta, \|u\|_{E_\beta} \leq 1} \int_{\mathbb{R}^2} \left(e^{\alpha |u|^{\frac{2}{1-\beta}}} - 1\right) dx = +\infty.$$

We now give an historic of second order Adams inequalities. For bounded domains $\Omega \subset \mathbb{R}^4$, in [2, 35] the authors extended the Trudinger Moser inequality to the higher order space $W_0^{2,2}(\Omega)$ and obtained

$$\sup_{u \in S} \int_\Omega (e^{\alpha u^2)} - 1) dx < +\infty \quad \Leftrightarrow \quad \alpha \leq 32\pi^2$$

where

$$S = \{u \in W_0^{2,2}(\Omega) \ | \ \Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{1}{2}} \leq 1\}.$$

When Ω is replaced by the whole space $\mathbb{R}^4$, Ruff and Sani [35] established the corresponding Adams type inequality as follows:

$$\sup_{\|u\|_{W^{2,2}} \leq 1} \int_\Omega (e^{\alpha u^2)} - 1) dx < +\infty \quad \Leftrightarrow \quad \alpha \leq 32\pi^2 \tag{1.5}$$

where $\|u\|^2_{W^{2,2}(\mathbb{R}^4)} = \int_{\mathbb{R}^4} |\Delta u|^2 dx + 2\int_{\mathbb{R}^4} |\nabla u|^2 dx + \int_{\mathbb{R}^4} u^2 dx.$

Recently, Adams-type inequalities on the logarithmic weighted Sobolev space

$$W^{2,2}_{0,rad}(B, w) = closure\{u \in C^\infty_{0,rad}(B) \ | \ \int_B \big(\log(\frac{e}{|x|})^\beta |\Delta u|^2 dx < \infty\}$$

of radial function in the unit ball B of $\mathbb{R}^4$ has been established. More precisely, in [37] the authors proved the following result:

Theorem 1.3 *[37] Let $\beta \in (0, 1)$ and let $w = (\log(\frac{e}{|x|}))^\beta$, then*

$$\sup_{\substack{u \in W^{2,2}_{0,rad}(B,w) \\ \int_B w(x)|\Delta u|^2 dx \leq 1}} \int_B e^{\alpha |u|^{\frac{2}{1-\beta}}} dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_\beta = 4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}}.$$

This last result permitted the authors in [18, 23] to investigate the following weighted problem

$$\begin{cases} g\big(\int_B (w(x)|\Delta u|^2 + |\nabla u|^2 + V(x)u^2) dx\big)\big[\Delta(w(x)\Delta u) - \Delta u + V(x)u)\big] = \ f(x, u) \ \text{in} \quad B \\ u = \dfrac{\partial u}{\partial n} = 0 \qquad \text{on} \ \ \partial B, \end{cases}$$

when $g = 1$ or g is not constant and verifying some mild conditions and where $B = B(0, 1)$ is the unit open ball in $\mathbb{R}^4$, $f(x, t)$ is a radial function with respect to x and the weight $w(x)$ is given by

$$w(x) = \left(\log \frac{e}{|x|}\right)^\beta, \ \ \beta \in (0, 1)\cdot$$

The Kirchhoff function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous positive function and the potential V is a positive continuous function on $\overline{B}$ and bounded away from zero in B. The authors proved that this problem has a positive ground state solution. The existence

result was proved by combining minimax techniques and weighted Trudinger-Moser inequality.
It should be noticed that several works involving weighted elliptic equations of Kirchhof type with critical non-linearities in the sense of Theorem 1.1 or Theorem 1.3 have been investigated (see [1, 23, 25]).
Recently, Meng et al. [30], studied the following fourth order equation of Kirchhoff type namely:

$$\Delta^2 u - (a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \; u \in H^2(\mathbb{R}^N),$$

with concave-convexe nonlinearities. The authors prove that there are at least two positive solutions. They used the Nehari manifold, Ekeland variational principle and the theory of Lagrange multipliers.

Now, we denote by E the space of all radial functions of the completion of $C_0^\infty(\mathbb{R}^4)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^4} |\Delta u|^2 w_\beta(x)dx + \int_{\mathbb{R}^4} |\nabla u|^2 dx + \int_{\mathbb{R}^4} u^2 dx.$$

where the weight $w_\beta(x)$ is given by

$$w_\beta(x) = \begin{cases} \left(\log\left(\frac{e}{|x|}\right)\right)^\beta & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geq 1, \end{cases} \tag{1.6}$$

with $\frac{1}{4} < \beta \leq 1$, $\chi : [1, +\infty[\to [1, +\infty[$ is a continuous function such that $\chi(1) = 1$ and $\inf_{t\in[1,+\infty[} \chi(t) \geq 1$. Also, we suppose that there exists a positive constant $M > 0$ such that

$$\frac{1}{r^8}\left(\int_1^r t^3\chi(t)dt\right)\left(\int_1^r \frac{t^3}{\chi(t)}dt\right) \leq M, \forall r \geq 1, \tag{1.7}$$

$$\frac{1}{r^8}\left(\int_1^r t^3\chi(t)dt\right) \leq M, \forall r \geq 1, \tag{1.8}$$

and

$$\frac{\max_{r\leq t\leq 4r} \chi(t)}{\min_{r\leq t\leq 4r} \chi(t)} \leq M, \forall r \geq 1. \tag{1.9}$$

We give some examples of functions $\chi : [1, +\infty[\to [1, +\infty[$ satisfying the conditions (1.7), (1.8) and (1.9):

• Any continuous function χ such that $\chi(1) = 1$ and $1 \leq \inf_{t\geq 1} \chi(t) \leq \sup_{t\geq 1} \chi(t) < +\infty$.

• $\chi(t) = t^{\delta}$, $-4 < \delta < 4$.
• $\chi(t) = 1 + \log t$.

Since the weight w_β belongs to the Muckenhoupt's class A_2, then $C_0^\infty\left(\mathbb{R}^4\right)$ is dense in the space E (see Lemma 1). It follows that the space E can be seen as

$$E = \left\{u \in L^2_{rad}\left(\mathbb{R}^4\right), \int_{\mathbb{R}^4}\left(|\Delta u|^2 w_\beta(x) + |\nabla u|^2\right)dx < +\infty\right\},$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^4} |\Delta u|^2 w_\beta(x)dx + \int_{\mathbb{R}^4} |\nabla u|^2 dx + \int_{\mathbb{R}^4} u^2 dx.$$

We note that this norm is issued from the Euclidean inner product scalar

$$\langle u, v\rangle = \int_B \left(w_\beta(x)\Delta u \Delta v + \nabla u.\nabla v + uv\right) dx.$$

We first prove a weighted second order Adams inequality which is similar to (1.5) in the set of $\mathbb{R}^4$ that is:

Theorem 1.4 *Let $\beta \in (\frac{1}{4}, 1)$ and let w_β given by (1.6). Then*

(i)

$$\int_{\mathbb{R}^4}\left(e^{|u|^{\frac{2}{1-\beta}}} - 1\right)dx < +\infty, \quad \forall u \in E. \tag{1.10}$$

(ii)

$$\sup_{\substack{u\in E\\ \|u\|\le 1}} \int_{\mathbb{R}^4}\left(e^{\alpha|u|^{\frac{2}{1-\beta}}} - 1\right)dx < +\infty \quad \Leftrightarrow \quad \alpha \le \alpha_\beta = 4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}}. \tag{1.11}$$

As an application of this last result, we study the non local following weighted problem

$$g\Big(\int_{\mathbb{R}^4}(w_\beta(x)|\Delta u|^2 + |\nabla u|^2 + u^2)dx\Big)\big[\Delta(w_\beta(x)\Delta u) - \Delta u + u)\big] = f(u) \ \text{ in } \ \mathbb{R}^4, \tag{1.12}$$

where the weight is given by (1.6).The non linearity $f(t)$ is continuous in $\mathbb{R}$ and behaves like $\exp\{\alpha t^{\frac{2}{1-\beta}}\}$ as $|t| \to +\infty$, for some $\alpha > 0$. The Kirchhoff function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous positive function which will be specified later.

In this paper, we set

$$\gamma := \frac{2}{1-\beta}\cdot$$

We now give some definitions of the notion of the exponential growth for the non-linearity f. In view of inequality (1.11), we say that f has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$,

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^\gamma}} = 0, \quad \forall\, \alpha \text{ such that } \alpha > \alpha_0 \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^\gamma}} = +\infty, \ \forall\ \alpha < \alpha_0. \tag{1.13}$$

According to inequality (1.10), we say that f has subcritical growth at $+\infty$ if

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^\gamma}} = 0, \quad \forall\, \alpha > 0.$$

Let us now present our results. In this paper, we always assume that the nonlinearities $f(t)$ have critical growth with $\alpha_0 > 0$ or that $f(t)$ has subcritical growth and fulfils these conditions:

(H_1) The non-linearity $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exist $t_0 > 0$ and $M_0 > 0$, such that $0 < F(t) = \displaystyle\int_0^t f(s)ds \leq M_0\big|f(t)\big|$ for $t \geq t_0$.

(H_3) There exists $\theta > 4$, such that $0 < \theta F(t) = \theta \displaystyle\int_0^t f(s)ds \leq tf(t), \quad \forall t \in \mathbb{R}\setminus\{0\}$.

(H_4) $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 0$.

(H_5) There exist p , $p > 4$ and $A > 0$ such that

$$F(t) \geq A\frac{|t|^p}{p} \quad \text{for all} \ \ t \in \mathbb{R}.$$

We give an example of such non linearity f. Let $f(t) = At^{p-1} + A\alpha_0(\gamma-1)\frac{t^{\gamma-1}}{p}e^{\alpha_0 t^\gamma}$ with $f(t) = 0$ for $t \leq 0$. It is clear that $F(t) = A\frac{t^p}{p} + \frac{A}{p}e^{\alpha_0|t|^\gamma}$. Since, $\displaystyle\lim_{t\to+\infty} \frac{F(t)}{f(t)} = 0$, then ($H_2$) is satisfied. Also, the conditions (H_4) and (H_5) are verified. It is clear that for $t > (\frac{1}{\alpha_0(\gamma-1)})^{\frac{1}{\gamma}}$ the condition (H_3) is satisfied.

Now, we define the Kirchhoff function g and set out the conditions for it. The function g is continuous on $\mathbb{R}^+$ and fulfils the conditions :

(G_1) There exists $g_0 > 0$ sucht that $g(t) \geq g_0$ for all $t \geq 0$ and

$$G(t+s) \geq G(t) + G(s) \ \ \forall\ \ s, t \geq 0;$$

where

$$G(t)=\int_0^t g(s)ds,$$

(G_2) $\dfrac{g(t)}{t}$ is nonincreasing for $t>0$.

The assmption (G_2) implies that $\dfrac{g(t)}{t}\leq g(1)$ for all $t\geq 1$. Also, as a consequence of (G_2), a simple calculation shows that

$$\frac{1}{2}G(t)-\frac{1}{4}g(t)t \text{ is nondecreasing for } t\geq 0. \tag{1.14}$$

Consequently, one has

$$\frac{1}{2}G(t)-\frac{1}{4}g(t)t\geq 0,\quad \forall t\geq 0. \tag{1.15}$$

A typical example of a function g fulfilling the conditions
(G_1) and (G_2) is given by
- $g(t)=g_0+at,\quad g_0,a>0.$
- $g(t)=1+\ln(1+t).$

We say that u is a solution to the problem (1.12), if u is a weak solution in the following sense.

Definition 1.1 A function u is called a solution to (1.12) if $u\in E$ and

$$g(\|u\|^2)\langle u,\varphi\rangle=\int_{\mathbb{R}^4} f(u)\,\varphi\,dx,\qquad \text{for all}\quad \varphi\in E. \tag{1.16}$$

It is straightforward to see that finding weak solutions to the problem (1.12) is equivalent to finding non-zero critical points of the following functional on E :

$$\mathcal{J}(u)=\frac{1}{2}G(\|u\|^2)-\int_{\mathbb{R}^4}F(x,u)dx, \tag{1.17}$$

where $F(u)=\displaystyle\int_0^u f(t)dt.$
We prove the following results:
In the critical case, we prove

Theorem 1.5 *Assume that the function f has a critical growth at $+\infty$ and satisfies the conditions (H_1), (H_2), (H_3) and (H_4). In addition, suppose that (G_1) and (G_2) are satisfied. Then, there exists $A_0>0$ such that problem (1.12) has a nontrivial weak solution for all $A>A_0$.*

In the subcritical case, we have :

Theorem 1.6 *Assume that the function f has a subcritical growth at $+\infty$ and satisfies the conditions (H_1), (H_2) and (H_3) . In addition, suppose that (G_1) and (G_2) are satisfied. Then, problem (1.12) has a nontrivial weak solution.*

In general, the treatment of fourth-order partial differential equations is an interesting subject. An interest in the investigation of these equations has been stimulated by their applications in the following fields: in micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, see [16, 19, 31]. However many applications are generated by elliptic problems, such as the study of traveling waves in suspension bridges, radar imaging (see, for example [5, 28]).

This paper is organized as follows. In Sect. 2, we present some necessary preliminary knowledge about functional space. In Sect. 3, we prove some preliminary results that will be useful in our proofs. Section 4 is devoted for the proof of Theorem 1.3. Section 5 concerned the variational framework of problem (1.12). In Sect. 6, we give the proof of Therems 1.5 and 1.6.
Through this paper, the constants C or c may change from line to another and we sometimes index the constants in order to show how they change.

2 Weighted Lebesgue and Sobolev Spaces Setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, bounded or unbounded, possibly even equal to the whole $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$, $W^{m,p}(\Omega, w)$, $W_0^{m,p}(\Omega, w)$ and some of their properties that will be used later. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on Ω and two measurable functions are considered as the same element if they are equal almost everywhere. Following Drabek et al. and Kufner in [17, 27], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \{u : \Omega \to \mathbb{R} \text{ measurable}; \quad \int_\Omega w(x)|u|^p\, dx < \infty\}$$

for any real number $1 \leq p < \infty$.
This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \Big(\int_\Omega w(x)|u|^p\, dx\Big)^{\frac{1}{p}}.$$

For $m \geq 2$, let w be a given family of weight functions w_τ, $|\tau| \leq m$, $w = \{w_\tau(x)\ x \in \Omega,\ \ |\tau| \leq m\}$.

In [17], the corresponding weighted Sobolev space was defined as

$$\begin{aligned} W^{m,p}(\Omega, w) &= \{u \in L^p(\Omega), D^\tau u \in L^p(\Omega, w)\ \ \forall\ \ 1 \leq |\tau| \leq m-1, D^\tau u \\ &\in L^p(\Omega, w)\ \ \forall\ \ |\tau| = m\} \end{aligned}$$

endowed with the following norm:

$$\|u\|_{W^{m,p}(\Omega,w)} = \left(\sum_{|\tau|\le m-1} \int_\Omega |D^\tau u|^p dx + \sum_{|\tau|=m} \int_\Omega w(x)|D^\tau u|^p dx \right)^{\frac{1}{p}},$$

where $w_\tau = 1$ for all $|\tau| < k$, $w_\tau = w$ for all $|\tau| = k$.
If we suppose also that $w(x) \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{m,p}(\Omega, w)$ and we can introduce the space

$$W_0^{m,p}(\Omega, w)$$

as the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega, w)$. Moroever, the injection

$$W^{m,p}(\Omega, w) \hookrightarrow W^{m-1,p}(\Omega) \text{ is compact.}$$

Also, $(L^p(\Omega, w), \|\cdot\|_{p,w})$ and $(W^{m,p}(\Omega, w), \|\cdot\|_{W^{m,p}(\Omega,w)})$ are separable, reflexive Banach spaces provided that $w(x)^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$. Then the space

$$E = \{u \in L^2_{rad}(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} \big(w_\beta(x)|\Delta u|^2 + |\nabla u|^2\big)dx < +\infty\}$$

is a Banach and reflexive space.
We have the following result

Lemma 1 *$C^\infty_{0,rad}\left(\mathbb{R}^4\right)$ is dense in the space*

$$\left\{u \in L^2_{rad}\left(\mathbb{R}^4\right), \int_{\mathbb{R}^4} \left(|\Delta u|^2 w_\beta(x) + |\nabla u|^2\right)dx < +\infty\right\}.$$

Proof it suffice to see that ω_β belongs to the Muckenhoupt's class A_2 (we also say that ω_β is an A_2-weight), that is

$$\sup \left(\frac{1}{|B|} \int_B w_\beta(x)dx \right) \left(\frac{1}{|B|} \int_B \left(w_\beta(x)\right)^{-1} dx \right) < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^4$.
Let $r > 0$ and $x_0 \in \mathbb{R}^4$. Denote by $B\,(x_0, r)$ (resp. $B(0, r)$) the open ball of $\mathbb{R}^4$ of center x_0 and radius r (resp. of center 0 and radius r).

- First case: Suppose that $B\,(x_0, r) \cap B(0, r) \neq \emptyset$. Thus, $B\,(x_0, r) \subset B(0, 3r)$ which implies that

$$\frac{1}{|B(x_0,r)|^2}\left(\int_{B(x_0,r)} w_\beta(x)dx\right)\left(\int_{B(x_0,r)}\frac{dx}{w_\beta(x)}\right)$$
$$\leq \frac{c}{r^8}\left(\int_0^{3r} w_\beta(t)t^3dt\right)\left(\int_0^{3r}\frac{t^3}{w_\beta(t)}dt\right). \tag{2.1}$$

If $3r < 1$, then

$$\frac{c}{r^8}\left(\int_0^{3r} w_\beta(t)t^3dt\right)\left(\int_0^{3r}\frac{t}{w_\beta(t)}dt\right)$$
$$= \frac{c}{r^8}\left(\int_0^{3r} t^3(1-\log t)^\beta dt\right)\left(\int_0^{3r}\frac{t^3}{(1-\log t)^\beta}dt\right).$$

But, a simple computation gives

$$\limsup_{r\to 0^+}\frac{c}{r^8}\left(\int_0^{3r} t^3(1-\log t)^\beta dt\right)\left(\int_0^{3r}\frac{t^3}{(1-\log t)^\beta}dt\right) < +\infty. \tag{2.2}$$

If $3r \geq 1$, then

$$\frac{c}{r^8}\left(\int_0^{3r}\omega_\beta(t)t^3dt\right)\left(\int_0^{3r}\frac{t^3}{w_\beta(t)}dt\right)$$

$$= \frac{c}{r^8}\left(\int_0^{1} t^3(1-\log t)^\beta dt + \int_1^{3r} t^3\chi(t)dt\right)$$
$$\left(\int_0^{1}\frac{t^3}{(1-\log t)^\beta}dt + \int_1^{3r}\frac{t^3}{\chi(t)}dt\right). \tag{2.3}$$

Since $\inf_{t\geq 1}\chi(t)\geq 1$, then

$$\limsup_{r\to+\infty}\frac{1}{r^8}\int_1^{3r}\frac{t^3}{\chi(t)}dt = 0 < +\infty\cdot \tag{2.4}$$

On the other hand, by (1.8), we infer

$$\limsup_{r\to+\infty}\frac{1}{r^8}\int_1^{3r} t^3\chi(t)dt < +\infty\cdot \tag{2.5}$$

Hence, in view of (2.4), (2.5) and (2.3), it remains to show that

$$\limsup_{r\to+\infty}\frac{1}{r^8}\left(\int_1^{3r}t^3\chi(t)dt\right)\left(\int_1^{3r}\frac{t^3}{\chi(t)}dt\right)<+\infty.$$

But this fact can immediately be deduced from (1.8). Combining (2.2) and (2.3), we deduce from (2.1) that there exists a constant $D_0>0$ independent of x_0 and r such that

$$\frac{1}{|B(x_0,r)|^2}\left(\int_{B(x_0,r)}\omega_\beta(x)dx\right)\left(\int_{B(x_0,r)}\frac{dx}{\omega_\beta(x)}\right)\le D_0. \tag{2.6}$$

- Second case: Suppose that $B(x_0,r)\cap B(0,r)=\emptyset$. In this case, we have

$$\frac{|x_0|}{2}\le|x|\le 2|x_0|,\ \forall x\in B(x_0,r).$$

Hence,

$$\begin{aligned}&\frac{1}{|B(x_0,r)|^2}\left(\int_{B(x_0,r)}w_\beta(x)dx\right)\left(\int_{B(x_0,r)}\frac{dx}{w_\beta(x)}\right)\\ &\le\left(\frac{\sup_{\frac{|x_0|}{2}\le|x|\le2|x_0|}w_\beta(t)}{\inf_{\frac{|x_0|}{2}\le|x|\le2|x_0|}w_\beta(t)}\right)\le\sup_{\tau>0}\left(\frac{\sup_{\tau\le t\le4\tau}w_\beta(t)}{\inf_{\tau\le t\le4\tau}w_\beta(t)}\right).\end{aligned} \tag{2.7}$$

If $4\tau<1$, then

$$\frac{\sup_{\tau\le t\le4\tau}w_\beta(t)}{\inf_{\tau\le t\le4\tau}w_\beta(t)}=\frac{(1-\log\tau)^\beta}{(1-\log(4\tau))^\beta}.$$

In view of the fact that

$$\sup_{0<\tau<\frac14}\left(\frac{1-\log\tau}{1-\log(4\tau)}\right)^\beta<+\infty,$$

it follows that

$$\sup_{0<\tau<\frac14}\left(\frac{\sup_{\tau\le t\le4\tau}w_\beta(t)}{\inf_{\tau\le t\le4\tau}w_\beta(t)}\right)<+\infty. \tag{2.8}$$

If $\frac{1}{4} \leq \tau < 1$, then

$$\frac{\sup_{\tau \leq t \leq 4\tau} w_\beta(t)}{\inf_{\tau \leq t \leq 4\tau} w_\beta(t)} \leq \frac{\sup_{\frac{1}{4} \leq t \leq 4} w_\beta(t)}{\inf_{\frac{1}{4} \leq t \leq 4} w_\beta(t)} < +\infty,$$

and consequently,

$$\sup_{\frac{1}{4} \leq \tau < 1} \left(\frac{\sup_{\tau \leq t \leq 4\tau} w_\beta(t)}{\inf_{\tau \leq t \leq 4\tau} w_\beta(t)} \right) < +\infty\cdot \tag{2.9}$$

If $\tau \geq 1$, then it follows

$$\frac{\sup_{\alpha \leq t \leq 4\alpha} \omega_\beta(t)}{\inf_{\tau \leq t \leq 4\tau} \omega_\beta(t)} = \frac{\sup_{\tau \leq t \leq 4\tau} \chi(t)}{\inf_{\tau \leq t \leq 4\tau} \chi(t)} \leq M,$$

and consequently,

$$\sup_{\tau \geq 1} \left(\frac{\sup_{\tau \leq t \leq 4\tau} \omega_\beta(t)}{\inf_{\tau \leq t \leq 4\tau} \omega_\beta(t)} \right) < +\infty. \tag{2.10}$$

Combining (2.8) and (2.9), we deduce from that there exists a constant $C_2 > 0$ independent of x_0 and r such that

$$\frac{1}{|B\,(x_0, r)|^2} \left(\int_{B(x_0,r)} w_\beta(x)dx \right) \left(\int_{B(x_0,r)} \frac{dx}{w_\beta(x)} \right) \leq C_2. \tag{2.11}$$

This completes the proof.

3 Some Useful Preliminary Results

In this section, we will derive several technical lemmas for our use later. First we begin by the radial lemma.

Lemma 2 *Let $u \in E$. Then*

$$|u(x)| \leq \frac{1}{\pi} \frac{1}{|x|^{\frac{3}{2}}} \|u\|^{\frac{1}{2}} \quad \textit{for} \;\; |x| \geq 1,$$

Proof We prove the lemma for all $u \in C^{\infty}_{0,rad}(\mathbb{R}^4)$ and use density arguments to conclude. Let $\phi(s) = u(x), |x| = s$. For $r \geq 1$, using the Hölder inequality, Young inequality one has

$$
\begin{aligned}
(\phi(r))^2 &= 2\int_r^{+\infty} \phi'(s)\phi(s)ds \leq 2\int_r^{+\infty} |\phi'(s)\phi(s)s^{\frac{3}{2}}s^{\frac{-3}{2}}|ds \\
&\leq 2\Big(\int_r^{+\infty} |\phi'(s)|^2 s^3 ds\Big)^{\frac{1}{2}}\Big(\int_r^{+\infty} \frac{|\phi(s)|^2}{s^6} s^3 ds\Big)^{\frac{1}{2}} \\
&= \frac{1}{\pi^2}\Big(2\pi^2\int_r^{+\infty} |\phi'(s)|^2 s^3 ds\Big)^{\frac{1}{2}}\Big(2\pi^2\int_r^{+\infty} \frac{|\phi(s)|^2}{s^6} s^3 ds\Big)^{\frac{1}{2}} \\
&\leq \frac{1}{\pi^2}\frac{1}{r^3}\Big(\int_{|x|>r} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{|x|>r} u^2 dx\Big)^{\frac{1}{2}} \\
&\leq \frac{1}{r^3}\frac{1}{\pi^2}\|u\|.
\end{aligned}
$$

We recall that

$$
W^{2,2}_{0,rad}(B, w) = closure\{u \in C^{\infty}_{0,rad}(B) \ \mid \ \int_B \big(\log(\frac{e}{|x|})\big)^{\beta}|\Delta u|^2 dx < \infty\}.
$$

□

We have the following results.

Lemma 3 *Let u be a radially symmetric function in $C^2_0(B)$. Then, we have*

(i) [37] For all $|x| < 1$,

$$
\begin{aligned}
|u(x)| \ &\leq \frac{1}{2\sqrt{2}\pi}\frac{||\log(\frac{e}{|x|})|^{1-\beta} - 1|^{\frac{1}{2}}}{\sqrt{1-\beta}}\int_B w_\beta(x)|\Delta u|^2 dx \\
&\leq \frac{1}{2\sqrt{2}\pi}\frac{||\log(\frac{e}{|x|})|^{1-\beta} - 1|^{\frac{1}{2}}}{\sqrt{1-\beta}}\|u\|^2.
\end{aligned}
$$

(ii) $\displaystyle\int_{|x|<1} e^{|u|^{\gamma}} dx < +\infty, \ \ \forall u \in W^{2,2}_{0,rad}(B, w).$

(iii) The following embedding is continuous

$$
E \hookrightarrow L^q(\mathbb{R}^4) \ \ for\ all\ \ q \geq 2.
$$

(vi) E is compactly embedded in $L^q(\mathbb{R}^4)$ for all $q \geq 2$.

Proof (i) see [37]

(ii) From (i) and using the identity $\log(\frac{e}{|x|}) - |\log(|x|)| = 1 \ \ \forall x \in B$ and the fact that $\sqrt{t-1} \leq \sqrt{t}, \forall t \geq 1$, we get

$$
|u(x)|^{\gamma} \leq \frac{1}{\alpha_\beta}\Big|| \log(\frac{e}{|x|})|^{1-\beta} - 1\Big|^{\frac{1}{1-\beta}}\|u\|^{\gamma} \ \leq \frac{1}{\alpha_\beta}\big(1 + \big|\log(|x|)\big|\big) \ \|u\|^{\gamma}.
$$

Hence, using the fact that the function $r \mapsto r^3 e^{\frac{\|u\|^{\gamma}(1+|\log r|)}{\alpha\beta}}$ is increasing, we get

$$\int_{|x|<1} e^{|u|^{\gamma}} dx \leq 2\pi^2 \int_0^1 r^3 e^{\frac{\|u\|^{\gamma}(1+|\log r|)}{\alpha\beta}} dr \leq \frac{\pi^2}{2} e^{\frac{\|u\|^{\gamma}}{\alpha\beta}} < +\infty, \quad \forall \; u \in W^{2,2}_{0,rad}(B, w).$$

Then (ii) follows by density.

(iii) Since $w_{\beta}(x) \geq 1$, then by Sobolev theorem, the following embedding are continuous

$$E \hookrightarrow W^{2,\frac{N}{2}}_{rad}(\mathbb{R}^4, w_{\beta}) \hookrightarrow W^{2,\frac{N}{2}}_{rad}(\mathbb{R}^4) \hookrightarrow L^q(\mathbb{R}^4) \;\; \forall q \geq 2.$$

The embedding $E \to L^q(\mathbb{R}^4)$ is compact. In fact, set $Q(s) = |s|^q$ and $P(s) = |s|^{q+\epsilon_0} + |s|^{q-\epsilon_0}$, where $0 < \epsilon_0 < q-2$. Clearly, $\dfrac{Q(s)}{P(s)} \to 0$ as $|s| \to +\infty$, and $\dfrac{Q(s)}{P(s)} \to 0$ as $|s| \to 0$. Let $(u_n)_n \in E$ be such that $u_n \hookrightarrow 0$ weakly in E and $u_n(x) \to 0$ $a.e. x \in \mathbb{R}^4$. By the continuity of the embedding $E \hookrightarrow L^{q+\varepsilon_0}(\mathbb{R}^4)$ and $E \hookrightarrow L^{q-\varepsilon_0}(\mathbb{R}^4)$, we obtain that

$$\sup_n \int_{\mathbb{R}^4} |P(u_n)| < +\infty.$$

On the other hand, by Lemma 2, $u_n(x) \to 0$ as $|x| \to +\infty$, uniformly in $n \in \mathbb{N}$. Therefore, we can apply the compactness Strauss Lemma, to deduce that $Q(u_n) \to 0$ strongly in $L^1(\mathbb{R}^4)$.

This concludes the lemma. □

Lemma 4 *[20] Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_{\Omega} |f(x, u_n)u_n| dx \leq C,$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \;\; in \;\; L^1(\Omega).$$

4 Proof of Theorem 1.4

We first show the first point of the theorem 1.4. We have for all $u \in E$,

$$\int_{\mathbb{R}^4} (e^{|u|^{\frac{2}{1-\beta}}} - 1) dx = \int_{|x|\geq 1} (e^{|u|^{\frac{2}{1-\beta}}} - 1) dx + \int_{|x|<1} (e^{|u|^{\frac{2}{1-\beta}}} - 1) dx. \quad (4.1)$$

On one side,

$$\int_{|x|\geq 1}(e^{|u|^{\frac{2}{1-\beta}}}-1)dx=\sum_{k=1}^{+\infty}\frac{1}{k!}\int_{|x|\geq 1}|u|^{\frac{2k}{1-\beta}}dx. \tag{4.2}$$

From Lemma 2, we get

$$\begin{aligned}\int_{|x|\geq 1}|u|^{\frac{2k}{1-\beta}}dx \leq \frac{1}{\pi}\|u\|^{\frac{k}{1-\beta}}\int_1^{+\infty}\frac{1}{r^{\frac{3k}{1-\beta}-3}}dr &= \frac{1}{\pi}\|u\|^{\frac{k}{1-\beta}}\frac{1}{\frac{3k}{1-\beta}-4}\\ &\leq \frac{1}{\pi}\|u\|^{\frac{k}{1-\beta}}\frac{1-\beta}{4\beta-1}, \quad \text{for all } k\geq 1;\\ &\frac{1}{4}<\beta<1.\end{aligned} \tag{4.3}$$

Associating (4.2) and (4.3) gives us

$$\int_{|x|\geq 1}(e^{|u|^{\frac{2}{1-\beta}}}-1)dx \leq \frac{1}{\pi}\frac{1-\beta}{4\beta-1}\sum_{k=1}^{+\infty}\frac{\|u\|^{\frac{k}{1-\beta}}}{k!}=\frac{1}{\pi}\frac{1-\beta}{4\beta-1}e^{\|u\|^{\frac{1}{1-\beta}}}<+\infty \tag{4.4}$$

We will now approximate the second integral in (4.1). Set

$$v(x)=\begin{cases}u(x)-u(e_1), & 0\leq |x|<1,\\ 0, & |x|\geq 1,\end{cases} \tag{4.5}$$

where $e_1=(1,0,0,0)\in\mathbb{R}^4$. Clearly $v\in W^{2,2}_{0,rad}(B,w_\beta)$.
For all $\varepsilon>0$, we have

$$|u|^{\frac{2}{1-\beta}}=|v+u(e_1)|^{\frac{2}{1-\beta}}\leq (1+\varepsilon)|v|^{\frac{2}{1-\beta}}+\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}|u(e_1)|^{\frac{2}{1-\beta}}.$$

Then, from Lemma 3 (ii), we have

$$\begin{aligned}\int_{|x|<1}e^{|u|^{\frac{2}{1-\beta}}}dx &\leq \int_{|x|<1}e^{(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}|u(e_1)|^{\frac{2}{1-\beta}}}dx\\ &\leq e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}|u(e_1)|^{\frac{2}{1-\beta}}}\int_{|x|<1}e^{(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}dx<+\infty\cdot\end{aligned} \tag{4.6}$$

Using (4.1), (4.4), (4.6) and Lemma 3 (ii), we conclude that

$$\int_{\mathbb{R}^4}(e^{|u|^{\frac{2}{1-\beta}}}-1)dx<+\infty, \quad \text{for all } u\in E.$$

This ends the proof of the first point .
By (4.4) we have

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|\ge 1}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx \le \sup_{u\in E,\|u\|\le 1}\frac{1}{\pi}\frac{1-\beta}{4\beta-1}e^{\|u\|^{\frac{1}{1-\beta}}} \le \frac{1}{\pi}\frac{1-\beta}{4\beta-1}e. \tag{4.7}$$

Furthermore, by using (4.6) and the radial lemma 3(ii), we obtain

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|\ge 1}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx \le \sup_{u\in E,\|u\|\le 1}C\frac{1-\beta}{4\beta-1}e^{\|u\|^{\frac{1}{1-\beta}}} \le C\frac{1-\beta}{4\beta-1}e. \tag{4.8}$$

On the other hand, by (4.6) and using the radial lemma 3(i), we get

$$\begin{aligned}
&\sup_{u\in E,\|u\|\le 1}\int_{|x|\le 1}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx\\
&\le \sup_{u\in E,\|u\|\le 1}\int_{|x|<1} e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}|u(e_1)|^{\frac{2}{1-\beta}}} e^{(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}dx\\
&\le \sup_{u\in E,\|u\|\le 1}\int_{|x|<1} e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}(c\|u\|)^{\frac{2}{1-\beta}}} e^{(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}dx\\
&\le e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}(c)^{\frac{2}{1-\beta}}} \sup_{u\in E,\|u\|\le 1}\int_{|x|<1} e^{(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}dx.
\end{aligned} \tag{4.9}$$

Let $\alpha < \alpha_\beta$. It is evident that there exists $\varepsilon > 0$ such that $\alpha(1+\varepsilon) < \alpha_\beta$.
Having in mind that for all $u \in E \;\; u \neq 0$ with $\|u\| \le 1$, we have

$$\begin{aligned}
\|v\|^2_{W^{2,2}_{0,rad}(B)} &= \int_B |\Delta v|^2\left(\log\left(\frac{e}{|x|}\right)\right)^\beta dx\\
&= \int_B |\Delta u|^2 w_\beta(x)dx \le \|u\|^2 \le 1,
\end{aligned} \tag{4.10}$$

then,

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}dx \le \sup\Big\{\int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^{\frac{2}{1-\beta}}}dx,\quad v\in W^{2,2}_{0,rad}(w,B),\\ \|v\|_{W^{2,2}_{0,rad}(B,w)} \le 1\Big\}.$$

So by (4.9) and Lemma 3 (ii), there exists a positive constant $C(\beta)$ depending only on β such that

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|<1} e^{\alpha|u|^{\frac{2}{1-\beta}}}dx \le e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}(c))^{\frac{2}{1-\beta}}} C(\beta). \tag{4.11}$$

Combining (4.9) and (4.10), we get

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|<1}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx < +\infty.$$

Furthermore

$$\int_{|x|\ge 1}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx = \sum_{k=1}^{+\infty}\frac{\alpha^k}{k!}\int_{|x|\ge 1}|u|^{\frac{2k}{1-\beta}}dx. \tag{4.12}$$

Combining (4.3) and (4.12), we infer

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|\ge 1}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx < +\infty. \tag{4.13}$$

It follows from (4.9) and (4.13) that

$$\sup_{u\in E,\|u\|\le 1}\int_{\mathbb{R}^4}(e^{\alpha|u|^{\frac{2}{1-\beta}}}-1)dx < +\infty, \quad \text{for all}\, \alpha < \alpha_\beta.$$

Let's consider the case $\alpha = \alpha_\beta$. It is clear that (4.8) holds for $\alpha = \alpha_\beta$. So, we get

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|\ge 1}(e^{\alpha_\beta|u|^{\frac{2}{1-\beta}}}-1)dx < +\infty. \tag{4.14}$$

We shall show that

$$\sup_{u\in E,\|u\|\le 1}\int_{|x|<1}(e^{\alpha_\beta|u|^{\frac{2}{1-\beta}}}-1)dx < +\infty. \tag{4.15}$$

For this, we consider $u \in E$, $u \ne 0$ such that $\|u\| \le 1$ and $\varepsilon > 0$ such that

$$(1+\varepsilon)^{1-\beta} = \frac{1}{\left(\int_{|x|<1}|\Delta u|^2 w_\beta(x)dx + \int_{|x|<1}|\nabla u|^2 dx + \int_{|x|<1}|v|^2 dx\right)}.$$

Moreover, we have a similar inequality to (4.6) that is

$$\begin{aligned}\int_{|x|<1} e^{\alpha_\beta |u|^{\frac{2}{1-\beta}}} dx &\le \int_{|x|<1} e^{\alpha_\beta (1+\varepsilon)|v|^{\frac{2}{1-\beta}}} e^{\alpha_\beta \left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}} |u(e_1)|^{\frac{2}{1-\beta}}} dx \\ &\le e^{\alpha_\beta \left(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{\beta+1}{1-\beta}} |u(e_1)|^{\frac{2}{1-\beta}}} \int_{|x|<1} e^{(1+\varepsilon)\alpha_\beta |v|^{\frac{2}{1-\beta}}} dx,\end{aligned} \tag{4.16}$$

where v is given by (4.5). On the other hand, we have from the proof of the radial Lemma 2,

$$\begin{aligned}|u(e_1)|^{\frac{2}{1-\beta}} &\le C_4 \left(\int_{|x|\ge 1} \left(w_\beta |\Delta u|^2 + |\nabla u|^2 + |u + u(e_1) - u(e_1)|^2\right) dx \right)^{\frac{1}{1-\beta}} \\ &\le C_4 \left(\int_{\mathbb{R}^4} \left(w_\beta |\Delta u|^2 + |\nabla u|^2 + |u|^2\right) dx \right. \\ &\left. - \int_{|x|<1} \left(w_\beta |\Delta u|^2 + |\nabla u|^2 + |u - u(e_1) + u(e_1)|^2\right) dx \right)^{\frac{1}{1-\beta}} \\ &\le C_4 \left(1 - \int_{|x|<1} \left(w_\beta |\Delta u|^2 + |\nabla u|^2 + |u - u(e_1)|^2 + |u(e_1)|^2\right) dx \right)^{\frac{1}{1-\beta}} \\ &\le C_4 \left(1 - C_5 - \int_{|x|<1} \left(w_\beta |\Delta u|^2 + |\nabla u|^2 + |u - u(e_1)|^2\right) dx \right)^{\frac{1}{1-\beta}} \\ &\le C_4 \left(1 - \int_{|x|<1} \left(w_\beta |\Delta u|^2 + |\nabla u|^2 + |u - u(e_1)|^2\right) dx \right)^{\frac{1}{1-\beta}} \\ &\le C_4 \left(1 - \frac{1}{(1+\varepsilon)^{1-\beta}}\right)^{\frac{1}{1-\beta}}.\end{aligned} \tag{4.17}$$

Also,

$$\begin{aligned}&\int_{|x|<1} |(1+\varepsilon)^{\frac{2}{1-\beta}} \Delta v|^2 w_\beta(x) dx + \int_{|x|<1} |(1+\varepsilon)^{\frac{2}{1-\beta}} \nabla v|^2 dx \\ &+ \int_{|x|<1} |(1+\varepsilon)^{\frac{2}{1-\beta}} v|^2 dx = 1.\end{aligned}$$

Then, by Theorem 1.3, there exists $C > 0$ such that

$$\int_{|x|<1} e^{(1+\varepsilon)\alpha_\beta |v|^{\frac{2}{1-\beta}}} dx < C. \tag{4.18}$$

Using (4.17), we get,

$$\int_{|x|<1} e^{\alpha_\beta |u|^{\frac{2}{1-\beta}}}\,dx \le C\exp\left(\alpha_\beta\Big(1-\frac{1}{(1+\varepsilon)^{\frac{1-\beta}{1+\beta}}}\Big)^{-\frac{1+\beta}{1-\beta}}\big(1-\frac{1}{(1+\varepsilon)^{1-\beta}}\big)^{\frac{1}{1-\beta}}\right).$$

But the function $\Upsilon : t \to \left(1-\frac{1}{t^{\frac{1-\beta}{1+\beta}}}\right)^{-\frac{1+\beta}{1-\beta}}\left(1-\frac{1}{t^{1-\beta}}\right)^{\frac{1}{1-\beta}}$ defined on $(1+\varepsilon_0, +\infty)$, $\varepsilon_0 > 0$ is decreasing and verifies $\lim_{t\to+\infty}\Upsilon(t) = 1$. It follows that Υ is bounded and consequently we get

$$\int_{|x|<1} e^{\alpha_\beta |u|^{\frac{2}{1-\beta}}}\,dx < +\infty. \tag{4.19}$$

Consequently, (4.15) is valid.

In the next step , we show that if $\alpha > \alpha_\beta$, then the supremum is infinite. Now, we will use particular functions [37], namely the Adams' functions. We consider the sequence defined for all $n \ge 3$ by

$$w_n(x) = \begin{cases} \left(\dfrac{4\log(e\sqrt[4]{n})}{\alpha_\beta}\right)^{\frac{1}{\gamma}} - \dfrac{|x|^{2(1-\beta)}}{2\big(\frac{\alpha_\beta}{4n}\big)^{\frac{1}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{\gamma-1}{\gamma}}} + \dfrac{1}{2\big(\frac{\alpha_\beta}{4}\big)^{\frac{1}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{\gamma-1}{\gamma}}} & \text{if } 0 \le |x| \le \frac{1}{\sqrt[4]{n}} \\ \dfrac{\Big(\log(\frac{e}{|x|})\Big)^{1-\beta}}{\Big(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\Big)^{\frac{1}{\gamma}}} & \text{if } \frac{1}{\sqrt[4]{n}} \le |x| \le \frac{1}{2} \\ \zeta_n & \text{if } |x| \ge \frac{1}{2} \end{cases} \tag{4.20}$$

where $\zeta_n \in C^\infty_{0,rad}(B)$ is such that

$$\zeta_n(\frac{1}{2}) = \frac{1}{\big(\frac{\alpha_\beta}{16}\log(e^4 n)\big)^{\frac{1}{\gamma}}}\big(\log 2e\big)^{1-\beta},\ \frac{\partial\zeta_n}{\partial r}(\frac{1}{2}) = \frac{-2(1-\beta)}{\Big(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\Big)^{\frac{1}{\gamma}}}\big(\log(2e)\big)^{-\beta}$$

$\zeta_n(1) = \dfrac{\partial\zeta_n}{\partial r}(1) = 0$ and $\xi_n, \nabla\xi_n, \Delta\xi_n$ are all $o\left(\dfrac{1}{[\log(e\sqrt[4]{n})]^{\frac{1}{\gamma}}}\right)$. Here, $\dfrac{\partial\zeta_n}{\partial r}$ denotes the first derivative of ζ_n in the radial variable $r = |x|$.

Let $v_n(x) = \dfrac{w_n}{\|w_n\|}$. We have, $v_n \in E$, $\|v_n\|^2 = 1$.

We compute $\Delta w_n(x)$, we get

$$\Delta w_n(x) = \begin{cases} \dfrac{-(1-\beta)(4-2\beta)|x|^{-2\beta}}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{1}{\gamma}}\left(\log(e\sqrt[4]{n})\right)^{\frac{\gamma-1}{\gamma}}} & \text{if } 0 \le |x| \le \frac{1}{\sqrt[4]{n}} \\ \dfrac{-(1-\beta)}{|x|^2}\dfrac{\left(\log(\frac{e}{|x|})\right)^{-\beta}\left(2+\beta\left(\log\frac{e}{|x|}\right)^{-1}\right)}{\left(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\right)^{\frac{1}{\gamma}}} & \text{if } \frac{1}{\sqrt[4]{n}} \le |x| \le \frac{1}{2} \\ \Delta\zeta_n & \text{if } |x| \ge \frac{1}{2} \end{cases}$$

So,

$$\|\Delta w_n\|_{2,w}^2 = \underbrace{2\pi^2\int_0^{\frac{1}{\sqrt[4]{n}}} r^3|\Delta w_n(x)|^2\left(\log\frac{e}{r}\right)^\beta dr}_{I_1} + \underbrace{2\pi^2\int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} r^3|\Delta w_n(x)|^2\left(\log\frac{e}{r}\right)^\beta dr}_{I_2}$$
$$+\underbrace{2\pi^2\int_{\frac{1}{2}}^{1} r^3|\Delta\zeta_n(x)|^2\left(\log\frac{e}{r}\right)^\beta dr + 2\pi^2\int_1^{+\infty}|\Delta\zeta_n|^2\chi(r)r^3dr}_{I_3}$$

By using integration by parts, we obtain,

$$\begin{aligned} I_1 &= 2\pi^2\frac{(1-\beta)^2(4-2\beta)^2}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{2}{\gamma}}\left(\log(e\sqrt[4]{n})\right)^{\frac{2(\gamma-1)}{\gamma}}}\int_0^{\frac{1}{\sqrt[4]{n}}} r^{3-4\beta}\left(\log\frac{e}{r}\right)^\beta dr \\ &= 2\pi^2\frac{(1-\beta)^2(4-2\beta)^2}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{2}{\gamma}}\left(\log(e\sqrt[4]{n})\right)^{\frac{2(\gamma-1)}{\gamma}}}\left[\frac{r^{4-4\beta}}{4-4\beta}(\log\frac{e}{r})^\beta\right]_0^{\frac{1}{\sqrt[4]{n}}} \\ &+ 2\pi^2\frac{\beta(1-\beta)^2(4-2\beta)^2}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{2}{\gamma}}\left(\log(e\sqrt[4]{n})\right)^{\frac{2(\gamma-1)}{\gamma}}}\int_0^{\frac{1}{\sqrt[4]{n}}}\frac{r^{4-4\beta}}{4-4\beta}\left(\log\frac{e}{r}\right)^{\beta-1}dr \\ &= o\left(\frac{1}{\log e\sqrt[4]{n}}\right). \end{aligned}$$

Also,

$$\begin{aligned} I_2 &= 2\pi^2\frac{(1-\beta)^2}{\left(\frac{\alpha_\beta}{4}\right)^{\frac{2}{\gamma}}\left(\log(e\sqrt[4]{n})\right)^{\frac{2}{\gamma}}}\int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}}\frac{1}{r}\left(\log\frac{e}{r}\right)^{-\beta}\left(2+\beta\left(\log\frac{e}{r}\right)^{-1}\right)^2dr \\ &= -2\pi^2\frac{(1-\beta)^2}{\left(\frac{\alpha_\beta}{4}\right)^{\frac{2}{\gamma}}\left(\log(e\sqrt[4]{n})\right)^{\frac{2}{\gamma}}}\left[\frac{\beta^2}{-1-\beta}\left(\log\frac{e}{r}\right)^{-\beta-1}+4\left(\log\frac{e}{r}\right)^{-\beta}\right. \\ &\left.+\frac{4}{1-\beta}\left(\log\frac{e}{r}\right)^{1-\beta}\right]_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} \end{aligned}$$

$$= 1 + o\Big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\Big)\cdot$$

and $I_3 = o\Big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\Big)$. Then $\|\Delta w_n\|_{2,w}^2 = 1 + o\Big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\Big)$.

Lemma 5 *The Adams' function given by (4.20) verifies* $\lim_{n\to+\infty} \|w_n\|^2 = 1$.

Proof We have

$$\begin{aligned}\|w_n\|^2 &= \int_{\mathbb{R}^4} w(x)|\Delta w_n|^2 dx + \int_{\mathbb{R}^4} |\nabla w_n|^2 dx + \int_{\mathbb{R}^4} w_n^2 dx\\ &= 1 + o\Big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\Big) + \int_{0\le|x|\le\frac{1}{\sqrt[4]{n}}} w_n^2 dx + \int_{\sqrt[4]{n}\le|x|\le\frac{1}{2}} w_n^2 dx + \int_{|x|\ge\frac{1}{2}} \zeta_n^2 dx\\ &+ \underbrace{\int_{0\le|x|\le\frac{1}{\sqrt[4]{n}}} |\nabla w_n|^2 dx}_{I_1'} + \underbrace{\int_{\sqrt[4]{n}\le|x|\le\frac{1}{2}} |\nabla w_n|^2 dx}_{I_2'} + \underbrace{\int_{|x|\ge\frac{1}{2}} |\nabla \zeta_n|^2 dx}_{I_3'}\cdot\end{aligned}$$

We have,

$$\begin{aligned}I_1' &= 2\pi^2 \frac{(1-\beta)^2}{\big(\frac{\alpha_\beta}{4n}\big)^{\frac{2}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{2(\gamma-1)}{\gamma}}} \int_0^{\frac{1}{\sqrt[4]{n}}} r^{4-2\beta} dr\\ &= 2\pi^2 \frac{(1-\beta)^2}{\big(\frac{\alpha_\beta}{4n}\big)^{\frac{2}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{2(\gamma-1)}{\gamma}}} \left[\frac{r^{5-2\beta}}{5-2\beta}\right]_0^{\frac{1}{\sqrt[4]{n}}}\\ &= 2\pi^2 \frac{(1-\beta)^2}{(5-2\beta)n^{\frac{1}{4}+\frac{\beta}{2}}\big(\frac{\alpha_\beta}{4}\big)^{\frac{2}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{2(\gamma-1)}{\gamma}}}\\ &= o\Big(\frac{1}{n^{\frac{1}{4}+\frac{\beta}{2}}\log e\sqrt[4]{n}}\Big)\cdot\end{aligned}$$

Also, using the fact that the function $r \mapsto r\big(\log\frac{e}{r}\big)^{-2\beta}$ is increasing on [0, 1], we get

$$\begin{aligned}I_2' &= 2\pi^2 \frac{(1-\beta)^2}{\big(\frac{\alpha_\beta}{4}\big)^{\frac{2}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{2}{\gamma}}} \int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} r\Big(\log\frac{e}{r}\Big)^{-2\beta} dr\\ &\le \frac{\pi^2}{2} \frac{(1-\beta)^2}{\big(\frac{\alpha_\beta}{4}\big)^{\frac{2}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{2}{\gamma}}}\big(\log 2e\big)^{-2\beta}\\ &= o\bigg(\frac{1}{[\log(e\sqrt[4]{n})]^{\frac{1}{\gamma}}}\bigg)\end{aligned}$$

and $I_3' = o\Big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\Big)$. For $|x| \le \frac{1}{\sqrt[4]{n}}$,

$$w_n^2 \le \left(\left(4\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)^{\frac{1}{\gamma}} + \frac{1}{2(\frac{\alpha_\beta}{4})^{\frac{1}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{\gamma-1}{\gamma}}}\right)^2.$$

Then,

$$\int_{0\le|x|\le\frac{1}{\sqrt[4]{n}}} w_n^2dx \;\le 2\pi^2\left(\left(4\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)^{\frac{1}{\gamma}} + \frac{1}{2(\frac{\alpha_\beta}{4})^{\frac{1}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{\gamma-1}{\gamma}}}\right)^2 \int_0^{\frac{1}{\sqrt[4]{n}}} r^3dr = o_n(1).$$

Also,

$$\int_{\frac{1}{\sqrt[4]{n}}\le|x|\le\frac{1}{2}} w_n^2dx = 2\pi^2\frac{1}{\left(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\right)^{\frac{1}{\gamma}}}\int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} r^3\big(\log(\frac{e}{r})\big)^2dr.$$

Using the fact that $\big(\log(\frac{e}{r})\big)^2 \le (\frac{e}{r})^2$, we obtain

$$\int_{\frac{1}{\sqrt[4]{n}}\le|x|\le\frac{1}{2}} w_n^2dx \le 2\pi^2\frac{1}{\left(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\right)^{\frac{1}{\gamma}}}\int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} e^2rdr = \pi^2\frac{e^2}{2\left(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\right)^{\frac{1}{\gamma}}}$$
$$= o_n(1).$$

Finaly,

$$\int_{|x|\ge\frac{1}{2}} w_n^2dx = \int_{|x|\ge\frac{1}{2}} \zeta_n^2dx = o_n(1).$$

Then, $\| w_n\|^2 = 1 + o\Big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\Big)$.

From the definition of w_n, it is easy to see that

$$-\frac{|x|^{2(1-\beta)}}{2\big(\frac{\alpha_\beta}{4n}\big)^{\frac{1}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{\gamma-1}{\gamma}}} + \frac{1}{2(\frac{\alpha_\beta}{4})^{\frac{1}{\gamma}}\big(\log(e\sqrt[4]{n})\big)^{\frac{\gamma-1}{\gamma}}} \ge 0 \;\text{ for all }\; 0\le|x|\le\frac{1}{\sqrt[4]{n}}.$$

Then, for all $0 \leq |x| \leq \frac{1}{\sqrt[4]{n}}$, $w_n^2 \geq \left(4\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)^{\frac{2}{\gamma}}$ · So, we get

$$\int_{0\leq|x|\leq\frac{1}{\sqrt[4]{n}}} w_n^2 dx \geq 2\pi^2\left(4\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)^{\frac{2}{\gamma}} \int_0^{\frac{1}{\sqrt[4]{n}}} r^3 dr = o_n(1)$$

and using the fact that the function $r \mapsto r^3\big(\log(\frac{e}{r})\big)^2$ is increasing on $[0, \frac{1}{2}]$, we get

$$\int_{\frac{1}{\sqrt[4]{n}}\leq|x|\leq\frac{1}{2}} w_n^2 dx \geq 2\pi^2\frac{1}{\left(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\right)^{\frac{2}{\gamma}}} \int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} r^3\big(\log(\frac{e}{r})\big)^2 dr$$

$$\geq \pi^2\frac{1}{2n\left(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\right)^{\frac{2}{\gamma}}} = o_n(1)$$

Consequently, $1 + o_1\big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\big) \leq \|w_n\|^2 \leq 1 + o_2\big(\frac{1}{(\log e\sqrt[4]{n})^{\frac{2}{\gamma}}}\big)$. The Lemma is proved.

Let $v_n(x) = \dfrac{w_n}{\|w_n\|}$ and $\overline{\alpha} = \frac{\alpha}{\alpha_\beta}$. We have for all $0 \leq |x| \leq \frac{1}{\sqrt[4]{n}}$, $v_n^\gamma \geq \left(4\frac{\log(e\sqrt[4]{n})}{\|w_n\|^\gamma\alpha_\beta}\right)$. So when $\alpha > \alpha_\beta$, for any $u \in E$, $\|u\| \leq 1$, we have

$$\begin{aligned}\sup_{u\in E,\|u\|\leq 1}\int_{\mathbb{R}^4}(e^{\alpha|u|^{\frac{2}{1-\beta}}} - 1)dx &\geq \lim_{n\longrightarrow+\infty}\int_{|x|\leq\frac{1}{\sqrt[4]{n}}}(e^{\alpha|v_n^\gamma|^{\frac{2}{1-\beta}}} - 1)dx\\ &\geq \lim_{n\longrightarrow+\infty}\int_0^{\frac{1}{\sqrt[4]{n}}}\left(r^3 e^{4\overline{\alpha}\log\left(e\sqrt[4]{n}\right)} - r^4\right)dr\\ &\geq \lim_{n\longrightarrow\infty}\frac{\pi^2}{2}\left(\frac{e^{4n}}{n} - \frac{1}{5n^{\frac{5}{4}}}\right) = +\infty.\end{aligned}$$

Then,

$$\sup_{u\in E,\|u\|\leq 1}\int_{\mathbb{R}^4}(e^{\alpha|u|^{\frac{2}{1-\beta}}} - 1)dx = +\infty \ \forall\ \alpha > \alpha_\beta.$$

□

5 The Variational Formulation for the Problem (1.12)

Note that, by the hypothesis (H_4), for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$|f(t)| \leq \varepsilon|t|, \ \ \forall\ 0 < |t| \leq \delta_0. \tag{5.1}$$

Moreover, since f is critical at infinity, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\forall t \geq C_\varepsilon \quad |f(t)| \leq \varepsilon \exp(\, a|t|^\gamma - 1) \ \text{ with } \ a > \alpha_0. \tag{5.2}$$

In particular, we obtain for $q \geq 2$,

$$|f(t)| \leq \frac{\varepsilon}{C_\varepsilon^{q-1}} |t|^{q-1} \exp(a\, |t|^\gamma - 1) \ \text{ with } \ a > \alpha_0. \tag{5.3}$$

Hence, using (5.1), (5.2), (5.3) and the continuity of f, for every $\varepsilon > 0$, for every $q > 2$, there exists a positive constant C such that

$$|f(t)| \leq \varepsilon|t| + C|t|^{q-1}\big(e^{a\ |t|^\gamma} - 1\big), \qquad \forall\, t \in \mathbb{R}, \ \ \forall\, a > \alpha_0. \tag{5.4}$$

It follows from (5.4) and (H_3), that for all $\varepsilon > 0$, there exists $C > 0$ such that

$$F(t) \leq \varepsilon|t|^2 + C|t|^q\big(e^{a\ |t|^\gamma} - 1\big), \qquad \text{for all } \ t, \forall\, a > \alpha_0 \tag{5.5}$$

So, by (1.11) and (5.5) the functional $\mathcal{J}$ given by (1.17), is well defined. Moreover, by standard arguments, $\mathcal{J} \in C^1(E, \mathbb{R})$.

5.1 The Mountain Pass Geometry of the Energy

In the sequel, we prove that the functional $\mathcal{J}$ has a mountain pass geometry.

Lemma 6 *Assume that the hypothesis* (H_1), (H_2), (H_3) *and* (H_4) *hold. In addition, assume that* (G_1) *and* (G_2) *are satisfied, then,*
there exist $\rho,\ \beta_0 > 0$ *such that* $\mathcal{J}(u) \geq \beta_0$ *for all* $u \in E$ *with* $\|u\| = \rho$.

Proof From (5.4), for all $\varepsilon > 0$, there exists $C > 0$ such that

$$F(t) \leq \varepsilon|t|^2 + C|t|^q\big(e^{a\ t^\gamma - 1} - 1\big), \qquad \text{for all } \ t \in \mathbb{R}.$$

Then, using the last inequality, we get

$$\mathcal{J}(u) \geq \frac{1}{2} G(\|u\|^2) - \varepsilon \int_{\mathbb{R}^4} |u|^2 dx - C \int_{\mathbb{R}^4} |u|^q \big(e^{a\ u^\gamma} - 1\big)\, dx.$$

From the Hölder inequality and using the following inequality

$$\big(e^s - 1\big)^\nu \leq e^{\nu s} - 1, \ \ \forall \ \ s \geq 0 \ \ \forall \nu \geq 1,$$

and the condition (G_1), we obtain

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon \int_{\mathbb{R}^4} |u|^2 dx - C\Big(\int_{\mathbb{R}^4} \big(e^{2a\ |u|^\gamma} - 1\big) dx\Big)^{\frac{1}{2}} \|u\|_{2q}^q. \tag{5.6}$$

From the Theorem 1.1, if we choose $u \in E$ such that

$$2a\|u\|^{\gamma} \leq \alpha_{\beta}, \tag{5.7}$$

we get

$$\int_{\mathbb{R}^4} \big(e^{2a\ |u|^{\gamma}} - 1\big)dx = \int_{\mathbb{R}^4} \big(e^{2a\ \|u\|^{\gamma}(\frac{|u|}{\|u\|})^{\gamma}}) - 1\big)dx < +\infty.$$

On the other hand from Sobolev embedding Lemma 3, there exist constants $C_1 > 0$ and $C_2 > 0$ such that $\|u\|_{2q} \leq C_1\|u\|$ and $\|u\|_2^2 \leq C_1\|u\|^2$. So,

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon C_1\|u\|^2 - C\|u\|^q = \|u\|^2\big(\frac{g_0}{2} - \varepsilon C_1 - C\|u\|^{q-2}\big),$$

for all $u \in E$ satisfying (5.8). Since $q > 2$, we can choose $\rho = \|u\| \leq (\frac{\alpha_{\beta}}{2a})^{\frac{1}{\gamma}}$ and for ε such that $\frac{g_0}{2C_1} > \varepsilon$, there exists $\beta_0 = \rho^2\big(\frac{g_0}{2} - \varepsilon C_1 - C\rho^{q-2}\big) > 0$ with $\mathcal{J}(u) \geq \beta_0 > 0$. □

Lemma 7 *Suppose that* (H_1), (H_3), (H_4), (G_1) *and* (G_2) *hold. Then there exists* $e \in E$ *with*
$\mathcal{J}(e) < 0$ *and* $\|e\| > \rho$.

Proof Let $\overline{u} \in E\backslash\{0\}$, $\|\overline{u}\| = 1$. From the condition (G_2), for all $t \geq 1$, we have that

$$G(t) \leq \frac{g(1)}{2}t^2. \tag{5.8}$$

It follows from the condition (H_3) and (H_4) that there exist two positive constants C_1 and C_2 such that

$$F(t) \geq C_1|t|^{\theta} - C_2t^2, \ \ \forall t \in \mathbb{R}.$$

Therefore

$$\mathcal{J}(t\bar{u}) \leq \frac{g(1)}{4}t^4 - C_1\|\bar{u}\|_p^{\theta}\ \ t^{\theta} + C_2t^2\|u\|_2^2.$$

Since, $\theta > 4$, we get that

$$\lim_{t\to+\infty} \mathcal{J}(t\bar{u}) = -\infty.$$

We take $e = \bar{t}\bar{u}$, for some $\bar{t} > 0$ large enough. So, Lemma 7 is proved. □

5.2 Palais–Smale Sequences

Consider a $(PS)_c$ sequence (u_n) in E, for some $c \in \mathbb{R}$, that is

$$\mathcal{J}(u_n) = \frac{1}{2}G(\|u_n\|^2) - \int_{\mathbb{R}^4} F(u_n)dx \to c, \quad n \to +\infty \tag{5.9}$$

and

$$|\langle \mathcal{J}'(u_n), \varphi\rangle| = \left| g(\|u_n\|^2)\langle u_n, \varphi\rangle - \int_B f(x, u_n)\varphi dx \right| \leq \varepsilon_n \|\varphi\|, \tag{5.10}$$

for all $\varphi \in E$, where $\varepsilon_n \to 0$, as $n \to +\infty$.
It follows from (H_3) , (5.9) , (5.10) with $\varphi = u_n$ and (1.15) , that

$$g_0\Big(\frac{1}{4} - \frac{1}{\theta}\Big)\|u_n\|^2 \leq \mathcal{J}(u_n) - \frac{1}{\theta}\langle \mathcal{J}'(u_n), u_n\rangle \leq c + o_n(1)\|u_n\|, \quad \forall n \in \mathbb{N}.$$

So the sequence $\|u_n\|$ is bounded in $\mathbb{R}$ and

$$\limsup_{n\to+\infty} \|u_n\| \leq \Big(\frac{4\theta c}{g_0(\theta - 4)}\Big)^{\frac{1}{2}}. \tag{5.11}$$

By the mountain pass theorem of Ambrosetti and Rabinowitz, we know that

$$c = \inf_{\gamma\in\Lambda} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)) \geq \rho > 0$$

where

$$\Lambda := \{\gamma \in C([0, 1], E) \ \text{ such that } \ \gamma(0) = 0 \ \text{ and } \ \mathcal{J}(\gamma(1)) < 0\}.$$

We now look at the behaviour of level c as a function of parameter A, which is given by the hypothesis (H_4).

Lemma 8 *For all $\epsilon > 0$, there exists $A_\epsilon > 0$ such that $c < \epsilon$, $\forall A > A_\epsilon$.*

Proof Let $\varphi \in E \setminus \{0\}$ be such that $\varphi \geq 0$ and $t > 0$. Based on the fact that

$$g(t) \leq g(1) + g(1)t, \quad \forall t \geq 0,$$

we get,

$$\begin{aligned}\mathcal{J}(t\varphi) &\leq \frac{1}{2}G(t^2\|\varphi\|^2) - At^p|\varphi|_p^p \leq \frac{1}{2}g(1)t^2\|\varphi\|^2 + \frac{1}{4}g(1)t^4\|\varphi\|^2 - At^p|\varphi|_p^p \\ &:= \psi(t).\end{aligned}$$

Let,

$$\psi_1(t) = \frac{1}{2}g(1)t^2\|\varphi\|^2 + \frac{1}{4}g(1)t^2\|\varphi\|^2 - At^p|\varphi|_p^p \ \text{ for } \ t \in [0, 1]$$

and

$$\psi_2(t) = \frac{1}{2}g(1)t^4\|\varphi\|^2 + \frac{1}{4}g(1)t^4\|\varphi\|^2 - At^p|\varphi|_p^p \ \text{ for } \ t \geq 1.$$

We have,

$$\sup_{t\geq 0} \psi(t) \leq \sup_{t\in[0,1]} \psi_1(t) \ \text{ or } \ \sup_{t\geq 0} \psi(t) \leq \sup_{t\geq 1} \psi_2(t)\cdot$$

The function ψ_1 achieves its maximum at the point $T_0 = \left(\frac{\frac{3}{2}g(1)\|\varphi\|^2}{p\ A|\varphi|_p^p}\right)^{\frac{1}{p-2}}$ for $t \in [0, 1]$ and ψ_2 at the point $T_1 = \left(\frac{3g(1)\|\varphi\|^2}{p\ A|\varphi|_p^p}\right)^{\frac{1}{p-4}}$. On the other hand, we have

$$\begin{aligned} c &\leq \sup_{t\geq 0} \mathcal{J}(t\varphi) \leq \sup_{t\geq 0} \psi_1(t) = \frac{3}{4}g(1)T_0^2\|\varphi\|^2 - AT_0^p|\varphi|_p^p \\ &\leq \frac{3}{4}g(1)\left(\frac{\frac{3}{2}g(1)\|\varphi\|^2}{p\ A|\varphi|_p^p}\right)^{\frac{2}{p-2}} \|\varphi\|^2 \to 0 \ \text{ as } \ A \to +\infty, \end{aligned}$$

or

$$\begin{aligned} c &\leq \sup_{t\geq 0} \mathcal{J}(t\varphi) \leq \sup_{t\geq 0} \psi_2(t) = \frac{3}{4}g(1)T_1^4\|\varphi\|^2 - AT_1^p|\varphi|_p^p \\ &\leq \frac{3}{4}g(1)\left(\frac{3g(1)\|\varphi\|^2}{p\ A|\varphi|_p^p}\right)^{\frac{2}{p-4}} \|\varphi\|^2 \to 0 \ \text{ as } \ A \to +\infty\cdot \end{aligned}$$

The lemma follows. □

6 Proof of Theorems 1.5 and 1.6

Proof of Theorem 1.6.

Since $\mathcal{J}$ has mountain pass geometry, then there exists a Palais-Smale sequence $(u_n) \subset E$ at the level c. For n large enough, there exists a constant $C > 0$ such that

$$\frac{1}{2}G(\|u_n\|^2) \leq C + \int_{\mathbb{R}^4} F(x, u_n)dx\cdot$$

From (H_4), it follows that

$$\int_{\mathbb{R}^4} F(x, u_n)dx \leq \frac{1}{\theta} \int_{\mathbb{R}^4} f(x, u_n)u_n dx$$

Using (5.10) with $\varphi = u_n$, we obtain

$$\int_{\mathbb{R}^4} f(u_n)u_n dx \leq \varepsilon_n \|u_n\| + g(\|u_n\|^2)\|u_n\|^2.$$

Therefore,

$$\frac{1}{2}G(\|u_n\|^2) \leq C + \frac{\varepsilon_n}{\theta}\|u_n\| + \frac{1}{\theta}g(\|u_n\|^2)\|u_n\|^2.$$

It follows from (1.16) that

$$\frac{1}{4}g(\|u_n\|^2)\|u_n\|^2 \leq \frac{1}{2}G(\|u_n\|^2) \leq \frac{\varepsilon_n}{\theta}\|u_n\| + \frac{1}{\theta}g(\|u_n\|^2)\|u_n\|^2.$$

Using the condition (G_1) and since $\theta > 4$, we get

$$0 \leq g_0(\frac{1}{4} - \frac{1}{\theta})\|u_n |^2 \leq C + \frac{\varepsilon_n}{\theta}\|u_n\|.$$

We deduce that the sequence (u_n) is bounded in E. As consequence, there exists $u \in E$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^q(\mathbb{R}^4)$, for all $q \geq 2$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^4$.
Our goal is to prove that $u_n \to u$ strongly in E. It is sufficient to prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^4} f(u_n)(u_n - u)dx = 0. \tag{6.1}$$

Let $R > 0$, we have

$$\int_{\mathbb{R}^4} \big|f(u_n)(u_n - u)\big|dx = \int_{|x|\geq R} \big|f(u_n)(u_n - u)\big|dx + \int_{|x|<R} \big|f(u_n)(u_n - u)\big|dx. \tag{6.2}$$

Using the Hölder inequality, we get

$$\int_{|x|\geq R} \big|f(u_n)(u_n - u)\big|dx \leq \left(\int_{|x|\geq R} (f(u_n))^2 dx\right)^{\frac{1}{2}} \left(\int_{|x|\geq R} (u_n - u)^2 dx\right)^{\frac{1}{2}}. \tag{6.3}$$

By (5.3), we have

$$\int_{|x|\geq R}(f(u_n))^2dx \leq \varepsilon\int_{|x|\geq R}u_n^2dx + C(\varepsilon)\int_{|x|\geq R}(e^{2a|u_n|^\gamma}-1)dx$$
$$+ C'(\varepsilon)\int_{|x|\geq R}|u_n|^q(e^{a|u_n|^\gamma}-1)dx \qquad (6.4)$$

Once again, using the Hölder inequality, we obtain

$$\int_{|x|\geq R}|u_n|^q(e^{a|u_n|^\gamma}-1)dx \leq \left(\int_{|x|\geq R}|u_n|^{2q}dx\right)^{\frac{1}{2}}\left(\int_{|x|\geq R}(e^{2a|u_n|^\gamma}-1)dx\right)^{\frac{1}{2}}$$

Now, by Lemma 8, there exists $A_1 > 0$ such that

$$\left(\frac{4\theta c}{g_0(\theta-4)}\right)^{\frac{1}{2}} \leq \left(\frac{\alpha_\beta}{2a}\right)^{\frac{1}{\gamma}}, \quad \forall A > A_1.$$

By (5.11), we have

$$\forall A > A_1, \quad \limsup_{n\to+\infty}\|u_n\| \leq \left(\frac{\alpha_\beta}{2a}\right)^{\frac{1}{\gamma}}.$$

It follows that

$$\int_{\mathbb{R}^4}(e^{2a|u_n|^\gamma}-1)dx < +\infty.$$

Hence, using (6.4) and the last result, we get

$$\int_{|x|\geq R}\left|f(u_n)(u_n-u)\right|dx \leq C\varepsilon + C'\varepsilon\int_{|x|\geq R}|u_n|^{2q}dx\cdot \qquad (6.5)$$

Using the radial lemma we get

$$\int_{|x|\geq R}|u_n|^{2q}dx \leq c\int_{|x|\geq R}\frac{1}{|x|^{3q}}dx \leq CR^{3-3q} \to 0 \ \ as \ \ R\to+\infty\cdot$$

Then, for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{|x|\geq R_\varepsilon}|u_n|^{2q}dx \leq \varepsilon.$$

It follows from (6.5) that there exists a positive constant C such that

$$\int_{|x|\geq R_\varepsilon}\left|f(u_n)(u_n-u)\right|dx \leq C\varepsilon, \quad \forall n\in\mathbb{N}. \qquad (6.6)$$

Furthermore, we have

$$\int_{|x|<R_\varepsilon} \big|f(u_n)(u_n-u)\big|dx \le \left(\int_{|x|<R_\varepsilon} (f(u_n))^2 dx\right)^{\frac{1}{2}} \left(\int_{|x|<R_\varepsilon} (u_n-u)^2 dx\right)^{\frac{1}{2}}.$$

Since B_{R_ε} is bounded and using the compact embedding $E \hookrightarrow\hookrightarrow L^2(B_{R_\varepsilon})$, we get

$$|u_n-u|_{L^2(B_{R_\varepsilon})} \to 0 \text{ as } n \to +\infty.$$

Also, do not forgot that

$$\sup_n \int_{|x|<R_\varepsilon} \big(f(u_n)\big)^2 dx < +\infty.$$

It follows that

$$\int_{|x|<R_\varepsilon} \big|f(u_n)(u_n-u)\big|dx \to 0 \text{ as } n \to +\infty. \tag{6.7}$$

Choosing $R = R_\varepsilon$ in (6.2) and combining (6.6) and (6.7) we get

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^4} \big|f(u_n)(u_n-u)\big|dx \le C\varepsilon.$$

Since ε is arbitrarily chosen, we deduce that (6.1) holds.
As a direct result, we can state that the point u is a critical point of $\mathcal{J}$ at level $c > \rho$. Consequently, problem (1.12) has a non-trivial weak solution.
Proof of Theorem 1.5. The energy $\mathcal{J}$ has mountain pass geometry, then there exists a Palais-Smale sequence $(u_n) \subset E$ at the level c. Then, as in the critical case, there exists $u \in E$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^q(\mathbb{R}^4)$, for all $q \ge 2$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^4$.
In the subcritical case (5.2) and (5.3) hold for all $a > 0$. By taking $a \le \frac{\alpha\beta}{2}\left(\frac{g_0(\theta-4)}{4\theta c}\right)^{\frac{1}{2}}$, its easy to deduce by (1.10) that

$$\sup_n \int_{\mathbb{R}^4} \big(f(u_n)\big)^2 dx < +\infty$$

and using the compact embedding $E \hookrightarrow\hookrightarrow L^2(\mathbb{R}^4)$, we get

$$|u_n-u|_{L^2(\mathbb{R}^4)} \to 0 \text{ as } n \to +\infty.$$

So,

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^4} \big|f(u_n)(u_n-u)\big|dx = 0.$$

Remark 6.1 In the subcritical case, the energy $\mathcal{J}$ satisfies the Palais-Smale condition at all level $c \in \mathbb{R}$. But in the critical case, the energy functional loses its compactness for all levels c such that $c \geq \dfrac{g_0(\theta - 4)}{4\theta}\big(\dfrac{\alpha_\beta}{2\alpha_0}\big)^2$.

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Declarations

Competing interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Consent of Publication We declare that this manuscript is original, has not been published before and is not currently being considered for publication elsewhere. We confirm that the manuscript has been read and approved and that there are no other persons who satisfied the criteria for authorship but are not listed.

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