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Vertex Decomposability of the Stanley–Reisner Complex of a Path Ideal

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Abstract

The t-path ideal $I_t(G)$ of a graph G is the square-free monomial ideal generated by the monomials which correspond to the paths of length t in G. In this paper, we prove that the Stanley–Reisner complex of the 2-path ideal $I_2(G)$ of an (undirected) tree G is vertex decomposable. As a consequence, we show that the Alexander dual $I_2(G)^\vee$ of $I_2(G)$ has linear quotients. For each $t \geq 3$, we provide a counterexample of a tree for which the Stanley–Reisner complex of $I_t(G)$ is not vertex decomposable.

Keywords Path ideals · Vertex decomposable · Componentwise linear · Linear quotients

Mathematics Subject Classification Primary 05E40 · 13C14 · 13D02

1 Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring, where $\mathbb{K}$ is a field and n is a positive integer. In literature, there are several ways to associate a square-free monomial ideal in R to algebraic objects. In [11], Villarreal introduced the concept of edge ideal of a graph. Let G be a simple graph on the vertex set $[n]$. Then the **edge ideal** of G is the square-free monomial ideal given by

$$I(G) = \langle x_u x_w : \{u, w\} \in E(G)\rangle.$$

The concept of edge ideal is generalized by Conca and De Negri [4]. They introduced the concept of t-path ideal of a graph G. A path of length t in G is a sequence

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$w_1, \ldots, w_{t+1}$ of vertices in G such that $\{w_1, w_2\}, \ldots, \{w_t, w_{t+1}\}$ are distinct edges of G. The **t-path ideal** of G is the square-free monomial ideal given by

$$I_t(G) = \langle x_{w_1} \cdots x_{w_{t+1}} : w_1, \ldots, w_{t+1} \text{ is a path in } G \rangle.$$

Observe that $I_1(G) = I(G)$.

Billera and Provan [10] introduced the concept of vertex decomposability for pure simplicial complexes. Björner and Wachs [3] extended this concept for non-pure simplicial complexes (see Definition 2.1). Numerous researchers have investigated the vertex decomposability of the Stanley–Reisner complex Δ_I for different classes of square-free monomial ideals. Ajdani and Jahan [1] proved that $\Delta_{I_t(C_n)}$ is vertex decomposable if and only if $t = n - 1$, $t = n$, or n is odd and $t = (n - 1)/2$. Authors in [2] proved that the Stanley–Reisner complex of the facet ideal $I(\Delta)$ of a simplicial tree Δ is vertex decomposable (see [5] for the definition of facet ideal). In [8], He and Van Tuyl showed that if G is a rooted tree, then there exists a simplicial tree Δ such that $I_t(G) = I(\Delta)$. Combining these results, we obtain that if G is a rooted tree, then the complex $\Delta_{I_t(G)}$ is vertex decomposable. These results motivate us to explore the vertex decomposability of the complex $\Delta_{I_t(G)}$ for a broader class of graphs G. This paper explores the vertex decomposability of the Stanley–Reisner complex $\Delta_{I_t(G)}$ when G is an undirected tree. We prove that the complex $\Delta_{I_t(G)}$ is vertex decomposable when $t = 2$ (see Theorem 3.5). For each $t \geq 3$, we give counter-example of a tree G for which the complex $\Delta_{I_t(G)}$ is not vertex decomposable (see Example 3.8).

Let $I \subset R$ be a square-free monomial ideal. If Δ_I is a vertex decomposable simplicial complex, then the Alexander dual $I^\vee$ of I has linear quotients, and hence it is componentwise linear (see Definitions 2.5, 2.6, Lemma 2.7). Consequently, we obtain that the Alexander dual $I_2(G)^\vee$ of the path ideal $I_2(G)$ has linear quotients, and hence it is componentwise linear.

We now give a brief overview of this paper. In the upcoming section, we introduce basic notions of graph theory and commutative algebra. In Sect. 3, we prove the main result of this paper (Theorem 3.5, Corollary 3.6).

2 Preliminaries

In this section, we discuss some fundamental notation and terminology used in the paper. The vertex set and the edge set of a finite simple graph G, are denoted throughout by $V(G)$ and $E(G)$, respectively. For $A \subset V(G)$, the **induced subgraph** of G on A, denoted by $G[A]$, is the graph with vertex set A and edge set $\{\{u, w\} \in E(G) : u, w \in A\}$. For simplicity, we use notation $G \backslash A$ for the induced subgraph of G on $V(G) \backslash A$. For a vertex $v \in V(G)$, the set

$$N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$$

is called the **neighborhood** of v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the **closed neighborhood** of v in G. The **degree** of vertex v in G is defined by $\deg_G(v) = |N_G(v)|$.

By a **walk** in G, we mean a sequence $w_1, \ldots, w_{t+1}$ of vertices in G such that $\{w_i, w_{i+1}\} \in E(G)$ for all $1 \le i \le t$. If a walk $w_1, w_2, \ldots, w_{t+1}$ has all distinct vertices, except possibly $w_1 = w_{t+1}$, then it is called a **path** of length t. A path $w_1, w_2, \ldots, w_{t+1}$ with $w_1 = w_{t+1}$ is called a **cycle**. A connected graph without cycles is called a **tree**. Let G be a tree on the vertex set $[n]$. Then G is called a **star graph** if there exists $v \in [n]$ such that $\{v, w\} \in E(G)$ for all $w \in [n] \backslash \{v\}$.

Notation: Let G be a simple graph. Then G° denotes the graph obtained by removing isolated vertices of G.

Let Δ be a simplicial complex on the vertex set $[n]$. A subset $\sigma \subset [n]$ is called a **face** of Δ if $\sigma \in \Delta$. Otherwise, σ is called a **nonface** of Δ. A maximal face of Δ with respect to inclusion is called a **facet** of Δ. The set of all minimal (with respect to inclusion) nonfaces of Δ is denoted by $\mathcal{N}(\Delta)$.

Let σ be a face of a simplicial complex Δ. The simplicial complex defined by the formula

$$\mathrm{del}_\Delta(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset\}$$

is called the **deletion** of σ, while the simplicial complex defined by the formula

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$$

is called the **link** of σ. Let $v \in [n]$ be a vertex of Δ. Then we write $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ for $\mathrm{link}_\Delta(\{v\})$ and $\mathrm{del}_\Delta(\{v\})$, respectively.

Definition 2.1 (*see* [3]) Let Δ be a simplicial complex. A vertex v of Δ is called a **shedding vertex** of Δ if every facet of $\mathrm{del}_\Delta(v)$ is a facet of Δ. Further, Δ is said to be **vertex decomposable** if either it has only one facet, or it has a shedding vertex v such that both $\mathrm{del}_\Delta(v)$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable.

Observe that v is a shedding vertex of Δ if and only if no face of $\mathrm{link}_\Delta(v)$ is a facet of $\mathrm{del}_\Delta(v)$. By using inductive argument for the vertex decomposability, we make the following simple observation.

Lemma 2.2 *Let Δ be a simplicial complex on the vertex set $[n]$ and $\{w_1, \ldots, w_r\} \subset [n]$. Further, let $\Omega_0 = \Delta$, $\Omega_i = \mathrm{del}_{\Omega_{i-1}}(w_i)$ and $\Phi_i = \mathrm{link}_{\Omega_{i-1}}(w_i)$ for all $1 \le i \le r$. If*

(i) *Ω_r is a vertex decomposable simplicial complex,*
(ii) *Φ_i is a vertex decomposable simplicial complex, for all $1 \le i \le r$, and*
(iii) *w_i is a shedding vertex of Ω_{i-1}, for all $1 \le i \le r$,*

then Δ is a vertex decomposable simplicial complex.

Let $\mathbb{K}$ be a field. Throughout, we write R for the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. For a subset σ of $[n]$, x^σ denotes the monomial $\prod_{i \in \sigma} x_i$ in R. Let Δ be a simplicial complex on the vertex set $[n]$. Then the **Stanley–Reisner ideal** of Δ, denoted by I_Δ, is the square-free monomial ideal of R given by

$$I_\Delta = \langle x^\sigma : \sigma \in \mathcal{N}(\Delta) \rangle.$$

Further, let I be a square-free monomial ideal of R. Then the **Stanley–Reisner complex** of I is the simplicial complex given by $\Delta_I = \{\sigma \subset [n] : x^\sigma \notin I\}$.

Remark 2.3 (*see* [7]) Let Δ be a simplicial complex on the vertex set $[n]$ and v be a vertex in Δ. Then, we have $I_{\mathrm{del}_\Delta(v)} = I_\Delta + \langle x_v \rangle$ and $I_{\mathrm{link}_\Delta(v)} = (I_\Delta : \langle x_v \rangle) + \langle x_v \rangle$.

The t-path ideal of a graph G is the main object of study in this paper.

Definition 2.4 Let G be a graph on the vertex set $[n]$ and $t \geq 1$ be an integer. Then the t**-path ideal** of G, denoted by $I_t(G)$, is the square-free monomial ideal of R defined by

$$I_t(G) = \langle x_{w_1} \cdots x_{w_{t+1}} : w_1, \ldots, w_{t+1} \text{ is a path in } G \rangle.$$

Specifically, the ideal $I_1(G)$, which is denoted by $I(G)$, is known as the **edge ideal** of G.

Let P be a finitely generated $\mathbb{Z}$-graded R-module and $\beta^R_{i,j}(P)$ denote $(i, j)^{th}$ graded Betti number of P. We say that P has a **linear resolution** if there exists a integer d such that $\beta^R_{i,i+b}(P) = 0$ for all i and for all $b \neq d$.

Definition 2.5 Let I be a homogeneous ideal of R. For $j \in \mathbb{N}$, let $I_{<j>}$ denote the ideal generated by all homogeneous elements of I of degree j. We say that I is **componentwise linear** if $I_{<j>}$ has a linear resolution for all j.

Definition 2.6 Let $I \subset R$ be a monomial ideal. We say that I has **linear quotients** if there exists an ordering $M_1, \ldots, M_r$ of minimal generators of I such that the ideal $\langle M_1, \ldots, M_{i-1} \rangle : \langle M_i \rangle$ is generated by a subset of $\{x_1, \ldots, x_n\}$ for all $2 \leq i \leq r$.

The following lemma is an important tool to determine componentwise linearity of a square-free monomial ideal.

Lemma 2.7 *Let $I \subset R$ be a square-free monomial ideal. If Δ_I is vertex decomposable, then the Alexander dual $I^\vee$ of I has linear quotients, and hence it is componentwise linear.*

Proof The conclusion follows directly from [9, Theorem 8.2.5].

3 The Stanley–Reisner Complex of *t*-Path Ideals

The main aim of this section is to prove that the Stanley–Reisner complex $\Delta_{I_2(G)}$ is vertex decomposable when G is a tree. We start by making the following observation.

Lemma 3.1 *Let $w \in [n]$ and $I \subset R$ be a square-free monomial ideal. If there exists $u \in [n]$ with $u \neq w$ such that $x_u x_w \in I$ and $x_w | M$ for all monomials M in I with $x_u | M$, then w is a shedding vertex of Δ_I.*

Proof Let σ be a facet of $\mathrm{del}_{\Delta_I}(w)$. Then there exists a facet σ' of Δ_I such that $\sigma \subset \sigma'$. It is sufficient to prove that $\sigma = \sigma'$. For a contradiction, suppose $\sigma \neq \sigma'$ and $z \in \sigma' \backslash \sigma$. If $z \neq w$, then $\sigma \cup \{z\} \in \mathrm{del}_{\Delta_I}(w)$ which contradicts the fact that σ is a facet of $\mathrm{del}_{\Delta_I}(w)$. Thus we must have $z = w$ or, equivalently, $\sigma' = \sigma \cup \{w\}$. Since $x_u x_w \in I$, it follows that $\{u, w\} \notin \Delta_I$, and hence $u \notin \sigma$. This implies that $\sigma \cup \{u\} \notin \mathrm{del}_{\Delta_I}(w)$. Now the fact $u \neq w$ implies that $\sigma \cup \{u\} \notin \Delta_I$ which means $x^{\sigma \cup \{u\}} \in I$. By given hypothesis, $x_w | x^{\sigma \cup \{u\}}$, i.e. $w \in \sigma$, a contradiction. Therefore, $\sigma = \sigma'$.

Now we introduce some notation that will be used repeatedly in the remaining part of this article. Let G be a graph on the vertex set $[n]$. For a vertex $v \in [n]$, we set

$$\mathcal{O}_G(v) = \{w \in N_G(v) : \deg_G(w) = 1\}$$

and

$$\mathcal{T}_G(v) = \{w \in N_G(v) : \deg_G(w) = 2, \deg_G(u) = 1 \text{ for all } u \in N_G(w) \backslash \{v\}\}.$$

Further, we set

$$\mathcal{U}_G(v) = \{u \in V(G) : u \in N_G(w) \backslash \{v\} \text{ for some } w \in \mathcal{T}_G(v)\}.$$

It is important to note that $\mathcal{U}_G(v) = \left(\bigcup_{w \in \mathcal{T}_G(v)} N_G(w) \right) \backslash \{v\}$. Thus, we have $|\mathcal{T}_G(v)| = |\mathcal{U}_G(v)|$.

The following lemma works as a tool for the proof of Theorem 3.5.

Lemma 3.2 *Let G be a tree on the vertex set $[n]$ with $n \geq 4$. Then there exists a vertex $v \in V(G)$ satisfying at least one of the following conditions:*

(i) $\deg_G(v) \geq 3$, $|\mathcal{O}_G(v)| \geq 2$ *and* $|N_G(v) \backslash \mathcal{O}_G(v)| \leq 1$.
(ii) $\deg_G(v) = 2$ *and* $\mathcal{T}_G(v) \neq \emptyset$.
(iii) $\deg_G(v) \geq 3$, $\mathcal{T}_G(v) \neq \emptyset$ *and* $|N_G(v) \backslash (\mathcal{O}_G(v) \cup \mathcal{T}_G(v))| \leq 1$.

Proof We prove the result by using induction on $n = |V(G)|$. If $n = 4$, then either G is a path graph, or G is a star graph. Hence the result holds.

Now suppose that $n > 4$. Let $\xi \in V(G)$ be such that $\deg_G(\xi) = 1$ and $G' = G \backslash \{\xi\}$. Further, let $N_G(\xi) = \{\zeta\}$. By induction hypothesis, there exists a vertex $v \in V(G')$ which satisfies at least one of the conditions (i), (ii), or (iii) as stated in the lemma. We consider the following three cases.

Case 1. When $\deg_{G'}(v) \geq 3$, $|\mathcal{O}_{G'}(v)| \geq 2$ and $|N_{G'}(v) \backslash \mathcal{O}_{G'}(v)| \leq 1$. In this case, $\deg_G(v) \geq 3$. If $\zeta \notin \mathcal{O}_{G'}(v)$, then $\mathcal{O}_{G'}(v) \subset \mathcal{O}_G(v)$ and $N_G(v) \backslash \mathcal{O}_G(v) = N_{G'}(v) \backslash \mathcal{O}_{G'}(v)$. This implies that v satisfies the condition (i). On the other hand, if $\zeta \in \mathcal{O}_{G'}(v)$, then $\{\zeta\} \subset \mathcal{T}_G(v)$ and $N_G(v) \setminus (\mathcal{O}_G(v) \cup \mathcal{T}_G(v)) \subset N_{G'}(v) \setminus \mathcal{O}_{G'}(v)$. Thus, v satisfies the condition (iii).

Case 2. When $\deg_{G'}(v) = 2$ and $\mathcal{T}_{G'}(v) \neq \emptyset$. Then $|\mathcal{U}_{G'}(v)| = 1$. Let $\mathcal{T}_{G'}(v) = \{w\}$ and $\mathcal{U}_{G'}(v) = \{u\}$. If $\zeta \notin \{u, v, w\}$, then $\deg_G(v) = 2$ and $\mathcal{T}_G(v) \neq \emptyset$. Thus, v satisfies the condition (ii). Now suppose that $\zeta \in \{u, v, w\}$. If

- $u = \zeta$, then $\deg_G(w) = 2$ and $\{u\} \subset \mathcal{T}_G(w)$ which means w satisfies the condition (ii).
- $w = \zeta$, then $\mathcal{O}_G(w) = \{u, \xi\}$ and $N_G(w) \setminus \mathcal{O}_G(w) = \{v\}$ which means w satisfies the condition (i).
- $v = \zeta$, then $\deg_G(v) = 3$, $\mathcal{T}_G(v) = \{w\}$ and $\mathcal{O}_G(v) = \{\xi\}$, means v satisfies the condition (iii).

Case 3. When $\deg_{G'}(v) \geq 3$, $\mathcal{T}_{G'}(v) \neq \emptyset$ and $|N_{G'}(v) \setminus (\mathcal{O}_{G'}(v) \cup \mathcal{T}_{G'}(v))| \leq 1$. If $\zeta \notin \mathcal{T}_{G'}(v) \cup \mathcal{U}_{G'}(v)$, then $\deg_G(v) \geq \deg_{G'}(v)$, $\mathcal{T}_{G'}(v) \subset \mathcal{T}_G(v)$ and

$$N_G(v) \setminus (\mathcal{O}_G(v) \cup \mathcal{T}_G(v)) \subset N_{G'}(v) \setminus (\mathcal{O}_{G'}(v) \cup \mathcal{T}_{G'}(v)).$$

This implies that v satisfies the condition (iii). Now suppose that $\zeta \in \mathcal{T}_{G'}(v) \cup \mathcal{U}_{G'}(v)$. If

- $\zeta \in \mathcal{T}_{G'}(v)$, then $N_{G'}(\zeta) \setminus \{v\} = \{u\}$ for some $u \in V(G')$ with $\deg_G(u) = 1$. Thus $\deg_G(\zeta) = 3$, $\mathcal{O}_G(\zeta) = \{\xi, u\}$ which means ζ satisfies the condition (i).
- $\zeta \in \mathcal{U}_{G'}(v)$, then there exists $w \in \mathcal{T}_{G'}(v)$ such that $N_{G'}(w) = \{\zeta, v\}$. Note that $\deg_G(w) = 2$ and $\mathcal{T}_G(w) = \{\zeta\}$ which means w satisfies the condition (ii).

This completes the proof of the lemma.

Let G be a tree on the vertex set $[n]$ with $n \geq 3$ and $I_2(G)$ be the 2-path ideal of G. For a monomial $M = x_{w_1}x_{w_2}x_{w_3} \in I_2(G)$ with $\deg_G(w_1) = 1$, we set $M^* = x_{w_1}x_{w_2}$ and $M_* = x_{w_2}x_{w_3}$. We use the notation $\Lambda[G]$ for the set

$$\{x_{w_1}x_{w_2}x_{w_3} : w_1, w_2, w_3 \text{ is a 2-path in } G, \deg_G(w_1) = 1\}$$

of monomial in $I_2(G)$. If $\chi^*, \chi_* \subset \Lambda[G]$ with $\chi^* \cap \chi_* = \emptyset$, then we set

$$I_G[\chi^*, \chi_*] = I_2(G) + \langle M^* : M \in \chi^* \rangle + \langle M_* : M \in \chi_* \rangle.$$

Note that $I_G[\emptyset, \emptyset] = I_2(G)$. Furthermore, if M is a square-free monomial in R, then we set $\sigma_M = \{w \in [n] : x_w | M\}$.

Remark 3.3 With the above notation, we consider the sets $\mathcal{N}_1 = \{\sigma_{M^*} : M \in \chi^*\}$, $\mathcal{N}_2 = \{\sigma_{M_*} : M \in \chi_*\}$ and

$$\mathcal{N}_3 = \{\{w_1, w_2, w_3\} : w_1, w_2, w_3 \text{ is a 2-path in } G, \sigma \not\subset \{w_1, w_2, w_3\} \text{ for all } \sigma \in \mathcal{N}_1 \cup \mathcal{N}_2\}.$$

(i) We have $\mathcal{N}(\Delta_{I_G[\chi^*, \chi_*]}) = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$.
(ii) If $\{w_1, w_2\} \in \mathcal{N}_1 \cup \mathcal{N}_2$, then $\{w_1, w_2\} \in E(G)$.
(iii) Let $w_1, w_2 \in [n]$ with $\deg_G(w_1) = 1$ and $w_2 \in N_G(w_1)$. Then $w_2 \in \sigma$ for all $\sigma \in \mathcal{N}(\Delta_{I_G[\chi^*, \chi_*]})$ with $w_1 \in \sigma$.

We need the following lemma for the proof of Theorem 3.5.

Lemma 3.4 *Let G be a tree on the vertex set $[n]$ with $n \geq 3$ and $\chi^*, \chi_* \subset \Lambda[G]$. Further, let v be a vertex of G.*

(i) *If $|\mathcal{O}_G(v)| \geq 2$, then v is a shedding vertex of $\Delta_{I_G[\chi^*,\chi_*]}$.*
(ii) *Let $w \in \mathcal{T}_G(v)$ and $N_G(w) = \{u, v\}$. If $\{u, w\} \in \mathcal{N}(\Delta_{I_G[\chi^*,\chi_*]})$, then w is a shedding vertex of $\Delta_{I_G[\chi^*,\chi_*]}$. Otherwise, v is a shedding vertex of $\Delta_{I_G[\chi^*,\chi_*]}$.*

Proof For convenience, we write $\mathcal{D} = \Delta_{I_G[\chi^*,\chi_*]}$.

(i) Let $w, w' \in \mathcal{O}_G(v)$ with $w \neq w'$. Then $x_w x_v x_{w'} \in I_G[\chi^*, \chi_*]$, or equivalently $\{w, v, w'\}$ is a nonface of $\mathcal{D}$. It follows from Remark 3.3(i) and (ii) that $\{w, v\} \in \mathcal{N}(\mathcal{D})$, $\{w', v\} \in \mathcal{N}(\mathcal{D})$, or $\{u, v, w'\} \in \mathcal{N}(\mathcal{D})$. If $\{w, v\} \in \mathcal{N}(\mathcal{D})$, or $\{w', v\} \in \mathcal{N}(\mathcal{D})$, then it follows from Lemma 3.1 and Remark 3.3(iii) that v is a shedding vertex of $\mathcal{D}$. Now assume that $\{w, v, w'\} \in \mathcal{N}(\mathcal{D})$. Let $\sigma \in \text{link}_{\mathcal{D}}(v)$. Then $\{w, v, w'\} \not\subset \sigma \cup \{v\}$, i.e. $\{w, w'\} \not\subset \sigma$. Let $z \in \{w, w'\} \backslash \sigma$. We claim that $\sigma \cup \{z\} \in \mathcal{D}$. For a contradiction, suppose $\sigma \cup \{z\} \notin \mathcal{D}$. Then there exists $\sigma' \in \mathcal{N}(\mathcal{D})$ such that $\sigma' \subset \sigma \cup \{z\}$. Since $\sigma \in \mathcal{D}$, we obtain $z \in \sigma'$. Observe that $\deg_G(z) = 1$ and $v \in N_G(z)$. Thus, by Remark 3.3(iii), we obtain that $v \in \sigma'$. This implies that $v \in \sigma$, a contradiction. Thus we must have $\sigma \cup \{z\} \in \mathcal{D}$. Since $v \notin \sigma \cup \{z\}$, we obtain that $\sigma \cup \{z\} \in \text{del}_{\mathcal{D}}(v)$, and hence v is a shedding vertex of $\mathcal{D}$.

(ii) If $\{u, w\} \in \mathcal{N}(\mathcal{D})$, then it follows from Lemma 3.1 and Remark 3.3(iii) that w is a shedding vertex of $\mathcal{D}$. Now assume that $\{u, w\} \notin \mathcal{N}(\mathcal{D})$. Since $\{u, w, v\}$ is a nonface of $\mathcal{D}$, it follows from Remark 3.3(i) and (ii) that $\{w, v\} \in \mathcal{N}(\mathcal{D})$ or $\{u, w, v\} \in \mathcal{N}(\mathcal{D})$. Let $\sigma \in \text{link}_{\mathcal{D}}(v)$. Then $\{u, w, v\} \not\subset \sigma \cup \{v\}$. Let $z \in \{u, w\}$ be such that $z \notin \sigma$. We claim that $\sigma \cup \{z\} \in \mathcal{D}$. For a contradiction, suppose $\sigma \cup \{z\} \notin \mathcal{D}$. Then there exists $\sigma' \in \mathcal{N}(\mathcal{D})$ such that $\sigma' \subset \sigma \cup \{z\}$. Since $\sigma \in \mathcal{D}$, we obtain $z \in \sigma'$. Further, since $\sigma' \in \mathcal{N}(\mathcal{D})$, by Remark 3.3(i), we obtain that $\sigma' \in \mathcal{N}_1 \cup \mathcal{N}_2$, or $\sigma' \in \mathcal{N}_3$. In the former case, it follows from Remark 3.3(ii) that $\sigma' = \{z, z'\}$ for some $z' \in N_G(w) = \{u, v\}$. Since $\{u, w\} \notin \mathcal{N}(\mathcal{D})$, we get $z' = v$, i.e $v \in \sigma'$. In the later case, σ' contains precisely vertices of a 2-path in G. Since very 2-path in G passing through $z \in \{u, w\}$ contains v, we obtain that $v \in \sigma'$. Thus in both cases, we get $v \in \sigma'$ which implies that $v \in \sigma$, a contradiction. Thus $\sigma \cup \{z\} \in \mathcal{D}$. Now the fact $v \notin \sigma \cup \{z\}$ implies that $\sigma \cup \{z\} \in \text{del}_{\mathcal{D}}(v)$, and hence v is a shedding vertex of $\mathcal{D}$.

Theorem 3.5 *Let G be a tree on the vertex set $[n]$ with $n \geq 3$. Then $\mathcal{D} = \Delta_{I_G[\chi^*,\chi_*]}$ is vertex decomposable for all $\chi^*, \chi_* \subset \Lambda[G]$. In particular, $\Delta_{I_2(G)}$ is vertex decomposable.*

Proof We prove the result by using induction on $n = |V(G)|$. For $n = 3$, the result follows from [2, Lemma 3.9]. Assume that $n > 3$ and that the result is true for all $n' < n$. Let v be a vertex of G established in Lemma 3.2. Now, we have the following two cases.

Case 1. When $|\mathcal{O}_G(v)| \geq 2$ and $|N_G(v) \backslash \mathcal{O}_G(v)| \leq 1$. In this case, it follows from Lemma 3.4 that v is a shedding vertex of $\mathcal{D}$.

Let $\Upsilon^* = \{M \in \chi^* : \sigma_M \subset V((G \backslash \{v\})^\circ)\}$ and $\Upsilon_* = \{M \in \chi_* : \sigma_M \subset V((G \backslash \{v\})^\circ)\}$. Then we have $\Upsilon^*, \Upsilon_* \subset \Lambda[(G \backslash \{v\})^\circ]$ with $\Upsilon^* \cap \Upsilon_* = \emptyset$. Using Remark 2.3, we obtain $\text{del}_{\mathcal{D}}(v) = \Delta_I$, where

$$I = \langle x_v \rangle + I_{(G \backslash \{v\})^\circ}[\Upsilon^*, \Upsilon_*].$$

Thus, by induction hypothesis, $\mathrm{del}_{\mathcal{D}}(v)$ is vertex decomposable.

Now assume that $N_G(v) = \{w_1, w_2, \ldots, w_r\}$ with $\deg_G(w_i) = 1$ for all $1 \leq i \leq s$, where $s \in \{r-1, r\}$. Let

$$\Theta^* = \{M \in \chi^* : \sigma_M \subset V((G \backslash N_G(v))^\circ)\}$$

and

$$\Theta_* = \{M \in \chi_* : \sigma_M \subset V((G \backslash N_G(v))^\circ)\}.$$

Then $\Theta^*, \Theta_* \subset \Lambda[(G \backslash N_G(v))^\circ]$ with $\Theta^* \cap \Theta_* = \emptyset$. Further, let $A = \{w \in N_G(v) : x_w x_v \in I_G[\chi^*, \chi_*]\}$ and K be the complete graph on the vertex set $N_G(v) \backslash A$. If either $s = r$, or $s = r - 1$ and $w_r \in A$, then by Remark 2.3, we get $\mathrm{link}_{\mathcal{D}}(v) = \Delta_J$, where

$$J = \langle x_w : w \in A \cup \{v\} \rangle + I(K) + I_{(G \backslash N_G(v))^\circ}[\Theta^*, \Theta_*].$$

Thus, by [12, Theorem 3.6] and induction hypothesis, $\mathrm{link}_{\mathcal{D}}(v)$ is vertex decomposable. On the other hand, suppose $s = r-1$ and $w_r \notin A$. If $N_G(v) \backslash A = \{w_{i_1}, \ldots, w_{i_t} = w_r\}$, then we set $\Omega_1 = \mathrm{link}_{\mathcal{D}}(v)$, $\Omega_m = \mathrm{del}_{\Omega_{m-1}}(w_{i_m})$ and $\Phi_m = \mathrm{link}_{\Omega_{m-1}}(w_{i_m})$ for all $2 \leq m \leq t$. Let $2 \leq m \leq t$ and K_{m-1} be the complete graph on the vertex set $\{w_{i_1}, w_{i_m}, \ldots, w_{i_t}\}$. Then by Remark 2.3, we get $\Omega_{m-1} = \Delta_{J_{m-1}}$, where

$$\begin{aligned} J_{m-1} = {} & \langle x_w : w \in A \cup \{v\} \rangle + \langle x_{w_{i_2}}, \ldots, x_{w_{i_{m-1}}} \rangle + I(K_{m-1}) \\ & + \langle x_{w_r} x_u : u \in N_G(w_r) \backslash \{v\} \rangle + I_{(G \backslash N_G(v))^\circ}[\Theta^*, \Theta_*]. \end{aligned}$$

Since $x_{w_{i_1}} x_{w_{i_m}} \in J_{m-1}$ and $(x_{w_{i_m}} M)/x_{w_{i_1}} \in J_{m-1}$ for all monomials M in J_{m-1} with $x_{w_{i_1}} | M$, one can easily prove that w_{i_m} is a shedding vertex of Ω_{m-1}. Again, it follows from Remark 2.3 that $\Phi_m = \Delta_{J'_m}$ and $\Omega_t = \Delta_{J_t}$, where

$$J'_m = \langle x_w : w \in A \cup \{v\} \rangle + \langle x_{w_{i_1}}, \ldots, x_{w_{i_t}} \rangle + I_{(G \backslash N_G(v))^\circ}[\Theta^*, \Theta_*]$$

and

$$J_t = \langle x_w : w \in A \cup \{v\} \rangle + \langle x_{w_{i_2}}, \ldots, x_{w_{i_t}} \rangle + I_{(G \backslash N_G(v))^\circ}[\Theta^*, \Theta_*].$$

By induction hypothesis, both Ω_t and Φ_m are vertex decomposable for all $2 \leq m \leq t$. Using Lemma 2.2, we obtain that $\mathrm{link}_{\mathcal{D}}(v)$ is vertex decomposable.

Case 2. When $\deg_G(v) \geq 2$, $\mathcal{T}_G(v) \neq \emptyset$ and $|N_G(v) \backslash (\mathcal{O}_G(v) \cup \mathcal{T}_G(v))| \leq 1$. If $|\mathcal{O}_G(v)| \geq 2$, then the result holds in view of Case 1. Therefore, we assume that $|\mathcal{O}_G(v)| \leq 1$. Consider the sets of vertices $A = \{w \in N_G(v) : x_w x_v \in I_G[\chi^*, \chi_*]\}$ and

$$U = \{w \in \mathcal{T}_G(v) : x_u x_w \in I_G[\chi^*, \chi_*] \text{ for some } u \in N_G(w) \backslash \{v\}\}.$$

We split the proof into the following two subcases.

Subcase 2(a). When $U = \emptyset$. In this case, it follows from Lemma 3.4 that v is a shedding of $\mathcal{D}$. Let $\Upsilon^* = \{M \in \chi^* : \sigma_M \subset V((G\backslash\{v\})^\circ)\}$ and $\Upsilon_* = \{M \in \chi_* : \sigma_M \subset V((G\backslash\{v\})^\circ)\}$. Then $\Upsilon^*, \Upsilon_* \subset \Lambda[(G\backslash\{v\})^\circ]$ and $\Upsilon^* \cap \Upsilon_* = \emptyset$. By Remark 2.3, we get $\mathrm{del}_{\mathcal{D}}(v) = \Delta_I$, where

$$I = \langle x_v \rangle + I_{(G\backslash\{v\})^\circ}[\Upsilon^*, \Upsilon_*].$$

Thus, by induction hypothesis, $\mathrm{del}_{\mathcal{D}}(v)$ is vertex decomposable.

Let $\Theta^* = \{M \in \chi^* : \sigma_M \subset V((G\backslash N_G(v))^\circ)\}$ and $\Theta_* = \{M \in \chi_* : \sigma_M \subset V((G\backslash N_G(v))^\circ)\}$. Then $\Theta^*, \Theta_* \subset \Lambda[(G\backslash N_G(v))^\circ]$ and $\Theta^* \cap \Theta_* = \emptyset$. We set $B = \mathcal{T}_G(v) \cup \mathcal{O}_G(v) \cup A$. If K is the complete graph on the vertex set $N_G(v)\backslash A$, then by Remark 2.3, $\mathrm{link}_{\mathcal{D}}(v) = \Delta_J$, where

$$\begin{aligned} J = \ & \langle x_w : w \in A \cup \{v\}\rangle + \langle x_u x_z : z \in \mathcal{T}_G(v)\backslash A, u \in N_G(z)\backslash\{v\}\rangle \\ & + \langle x_\xi x_\phi : \xi \in N_G(v)\backslash B, \phi \in N_G(\xi)\backslash\{v\}\rangle + I(K) + I_{(G\backslash N_G(v))^\circ}[\Theta^*, \Theta_*]. \end{aligned}$$

Note that $|N_G(v)\backslash B| \le 1$. First suppose that $\mathcal{O}_G(v) \not\subset A$ and $\mathcal{O}_G(v)\backslash A = \{\zeta\}$. If $N_G(v)\backslash A = \{\zeta, w_1, \ldots, w_r\}$ (with assumption $w_r = \xi$ when $N_G(v)\backslash B = \{\xi\}$), then we set $\Omega_0 = \mathrm{link}_{\mathcal{D}}(v)$, $\Omega_k = \mathrm{del}_{\Omega_{k-1}}(w_k)$ and $\Phi_k = \mathrm{link}_{\Omega_{k-1}}(w_k)$ for all $1 \le k \le r$. By proceeding as in Case 1, we can show that Ω_r and Φ_k are vertex decomposable for all $1 \le k \le r$, and w_k is a shedding vertex of Ω_{k-1} for all $1 \le k \le r$. Using Lemma 2.2, we obtain that $\mathrm{link}_{\mathcal{D}}(v)$ is vertex decomposable.

Now suppose that $\mathcal{O}_G(v) \subset A$. Let $\mathcal{T}_G(v)\backslash A = \{w_1, \ldots, w_r\}$. Then we set $\Omega_0 = \mathrm{link}_{\mathcal{D}}(v)$, $\Omega_m = \mathrm{del}_{\Omega_{m-1}}(w_m)$ and $\Phi_m = \mathrm{link}_{\Omega_{m-1}}(w_m)$ for all $1 \le m \le r$. Further, let $1 \le m \le r$ and K_{m-1} be the complete graph on the vertex set $N_G(v)\backslash(A \cup \{w_1, \ldots, w_{m-1}\})$, where $\{w_1, \ldots, w_{m-1}\} = \emptyset$ for $m = 1$. Then, by Remark 2.3, we get $\Omega_{m-1} = \Delta_{J_{m-1}}$, where

$$\begin{aligned} J_{m-1} = \ & \langle x_w : w \in A \cup \{v, w_1, \ldots, w_{m-1}\}\rangle \\ & + \langle x_u x_z : z \in \mathcal{T}_G(v)\backslash(A \cup \{v, w_1, \ldots, w_{m-1}\}), u \in N_G(z)\backslash\{v\}\rangle \\ & + \langle x_\xi x_\phi : \xi \in N_G(v)\backslash B, \phi \in N_G(\xi)\backslash\{v\}\rangle + I(K_{m-1}) + I_{(G\backslash N_G(v))^\circ}[\Theta^*, \Theta_*]. \end{aligned}$$

Suppose $u \in N_G(w_m)\backslash\{v\}$. Then $x_u x_{w_m} \in J_{m-1}$ and $x_{w_m}|M$ for all monomials M in J_{m-1} with $x_u|M$. Thus, it follows from Lemma 3.1 that w_m is a shedding vertex of Ω_{m-1}. In view of Remark 2.3, we also conclude that $\Phi_m = \Delta_{J'_m}$, where

$$J'_m = \langle x_w : w \in N_G(v) \cup \{u\}\rangle + I_{(G\backslash N_G(v))^\circ}[\Theta^*, \Theta_*].$$

By induction hypothesis, Φ_m is vertex decomposable. Finally, using Remark 2.3, we see that $\Omega_r = \Delta_{J_r}$, where

$$J_r = \langle x_w : w \in B \cup \{v\}\rangle + \langle x_\xi x_\phi : \xi \in N_G(v)\backslash B, \phi \in N_G(\xi)\backslash\{v\}\rangle + I_{(G\backslash B)^\circ}[\Theta^*, \Theta_*].$$

To verify that Ω_r is vertex decomposable, let $v' \notin V(G)$ be a vertex and G' be the tree on the vertex set $V(G') = V((G\backslash B)^\circ) \cup \{v'\}$ and the edge set $E(G') =$

$E((G\backslash B)^{\circ}) \cup \{\{v', \xi\}\}$, where $\xi \in N_G(v)\backslash B$. Then we can write

$$J_r = \langle x_w : w \in B \cup \{v\}\rangle + I_{G'}[\Theta^*, \Theta_* \cup \{x_{v'}x_{\xi}x_{\phi} : \phi \in N_G(\xi)\backslash\{v\}\}].$$

Thus, by induction hypothesis, Ω_r is vertex decomposable. By Lemma 2.2, we deduce that $\text{link}_{\mathcal{D}}(v)$ is vertex decomposable.

Subcase 2(b). When $U \neq \emptyset$. Let $w' \in U$ and $u' \in N_G(w')\backslash\{v\}$. Then $x_{u'}x_{w'} \in I_G[\chi^*, \chi_*]$. It follows from Lemma 3.4 that w' is a shedding vertex of $\mathcal{D}$. By using similar arguments as in Subcase 2(a), we can prove that $\text{del}_{\mathcal{D}}(w')$ is vertex decomposable.

Let $\Theta^* = \{M \in \chi^* : \sigma_M \subset V((G\backslash W)^{\circ})\}$ and $\Theta_* = \{M \in \chi_* : \sigma_M \subset V((G\backslash W)^{\circ})\}$, where $W = \mathcal{O}_G(v) \cup \mathcal{T}_G(v)$. Then $\Theta^*, \Theta_* \subset \Lambda[(G\backslash W)^{\circ}]$ and $\Theta^* \cap \Theta_* = \emptyset$. Using Remark 2.3, we get $\text{link}_{\mathcal{D}}(w') = \Delta_J$, where

$$\begin{aligned} J = & \langle x_{u'}, x_{w'}\rangle + \langle x_u x_w : w \in U\backslash\{w'\}, u \in N_G(w)\backslash\{v\}\rangle \\ & + \langle x_v x_{\phi} : \phi \in N_G(v)\backslash\{w'\}\rangle + I_{(G\backslash W)^{\circ}}[\Theta^*, \Theta_*]. \end{aligned}$$

First suppose that $U = \{w'\}$. Then, we have

$$J = \langle x_{u'}, x_{w'}\rangle + \langle x_v x_{\phi} : \phi \in N_G(v)\backslash\{w'\}\rangle + I_{(G\backslash W)^{\circ}}[\Theta^*, \Theta_*].$$

We further distinguish between two possibilities:

- If $N_G(v)\backslash\{w'\} = \{\xi\}$ for some $\xi \in [n]$, then $J = \langle x_{u'}, x_{w'}\rangle + \langle x_v x_{\xi}\rangle + I_{(G\backslash W)^{\circ}}[\Theta^*, \Theta_*]$. Choose $\zeta \in N_G(\xi)$ such that $x_v x_{\xi} x_{\zeta} \notin \chi^*$. Then we can rewrite

$$J = \langle x_u, x_w\rangle + I_{(G\backslash\{w\})^{\circ}}[\Theta^* \cup \{x_v x_{\xi} x_{\zeta}\}, \Theta_*].$$

  Thus, by induction hypothesis, $\text{link}_{\mathcal{D}}(w')$ is vertex decomposable.
- If $\deg_G(v) > 2$, then it follows from Lemma 3.1 that v is a shedding vertex of $\text{link}_{\mathcal{D}}(w')$. Write $\Omega = \text{link}_{\mathcal{D}}(w')$. Using Remark 2.3, we obtain $\text{del}_{\Omega}(v) = \Delta_{L_1}$ and $\text{link}_{\Omega}(v) = \Delta_{L_2}$, where

$$L_1 = \langle x_{u'}, x_{w'}, x_v\rangle + I_{(G\backslash W)^{\circ}}[\Theta^*, \Theta_*]$$

  and

$$L_2 = \langle x_{u'}, x_{w'}, x_v\rangle + \langle x_{\phi} : \phi \in N_G(v)\backslash\{w'\}\rangle + I_{(G\backslash N_G(v))^{\circ}}[\Theta^*, \Theta_*].$$

  By induction hypothesis, both $\text{del}_{\Omega}(v)$ and $\text{link}_{\Omega}(v)$ are vertex decomposable, and hence Ω is vertex decomposable.

Now, suppose that $\{w'\} \subsetneq U$ and $(U \setminus \{w'\}) \cup \{v\} = \{w_1, \ldots, w_r\}$ with $w_r = v$. For each $1 \leq k \leq r$, let $N_G(w_k) = \{u_k, v\}$. We set $\Omega_0 = \text{link}_{\mathcal{D}}(w')$, $\Omega_k = \text{del}_{\Omega_{k-1}}(w_k)$ and $\Phi_k = \text{link}_{\Omega_{k-1}}(w_k)$ for all $1 \leq k \leq r$. Let $1 \leq k \leq r$. Then, using Remark 2.3, we obtain $\Phi_k = \Delta_{J_k}$, where

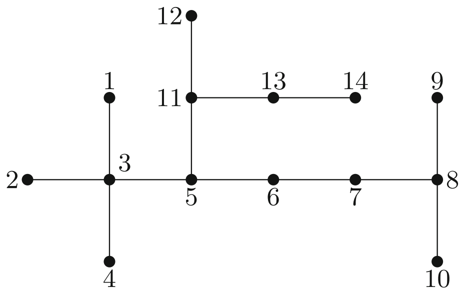

Fig. 1 $\Delta_{I_2(G)}$ is vertex decomposable

$$J_k = \langle x_{u'}, x_{w'}, x_{w_1}, \ldots, x_{w_k}, x_{u_k}, x_v\rangle + \langle x_{u_t}x_{w_t} : k < t \leq r\rangle + I_{(G\backslash W)^\circ}[\Theta^*, \Theta_*].$$

Note that $U \cap V((G\backslash W)^\circ) = \emptyset$. Thus, by induction hypothesis and [2, Lemma 3.9], Φ_k is vertex decomposable. Further, in view of Remark 2.3, we observe that $\Omega_r = \Delta_L$, where

$$L = \langle x_{u'}, x_{w'}, x_{w_1}, \ldots, x_{w_r}\rangle + \langle x_v x_\phi : \phi \in N_G(v)\backslash\{w', w_1, \ldots, w_r\}\rangle + I_{(G\backslash W)^\circ}[\Theta^*, \Theta_*].$$

By following the same method as above, we can prove that Ω_r is vertex decomposable. Also, it follows from Lemma 3.1 that w_k is a shedding vertex of Ω_{k-1} for all $1 \leq k \leq r$. Thus, it follows from Lemma 2.2 that $\mathrm{link}_{\mathcal{D}}(w')$ is vertex decomposable.

The following corollary is a direct consequence of Lemma 2.7 and Theorem 3.5.

Corollary 3.6 *Let G be a tree on the vertex set $[n]$ with $n \geq 3$. Then the Alexander dual $I_2(G)^\vee$ of $I_2(G)$ has linear quotients, and hence it is componentwise linear.*

We illustrate Theorem 3.5 with the help of following example.

Example 3.7 Let G be the tree as shown in Fig. 1. Then

$$I_2(G) = \left\langle \begin{array}{l} x_1x_2x_3, x_1x_3x_4, x_1x_3x_5, x_2x_3x_4, x_2x_3x_5, \\ x_3x_4x_5, x_3x_5x_6, x_3x_5x_{11}, x_5x_6x_{11}, x_5x_6x_7, \\ x_6x_7x_8, x_7x_8x_9, x_7x_8x_{10}, x_8x_9x_{10}, \\ x_5x_{11}x_{12}, x_5x_{11}x_{13}, x_{11}x_{12}x_{13}, x_{11}x_{13}x_{14} \end{array} \right\rangle.$$

Let $\mathcal{D} = \Delta_{I_G[\chi^*, \chi_*]}$, where $\chi^* = \{x_1x_3x_5\}$ and $\chi_* = \{x_{11}x_{13}x_{14}\}$. In view of Lemma 3.4, 3 is a shedding vertex of $\mathcal{D}$. In order to show that $\mathcal{D}$ is vertex decomposable, we have to show that both $\mathrm{del}_{\mathcal{D}}(3)$ and $\mathrm{link}_{\mathcal{D}}(3)$ are vertex decomposable.

By Remark 2.3, we obtain $\mathrm{del}_{\mathcal{D}}(3) = \Delta_I$ and $\mathrm{link}_{\mathcal{D}}(3) = \Delta_J$, where

$$I = \langle x_3, x_{11}x_{13}, x_5x_6x_{11}, x_5x_6x_7, x_6x_7x_8, x_7x_8x_9, x_7x_8x_{10}, x_8x_9x_{10}, x_5x_{11}x_{12}\rangle$$

and

$$J = \langle x_1, x_3, x_{11}x_{13}, x_2x_4, x_2x_5, x_4x_5, x_5x_6, x_5x_{11}, x_6x_7x_8, x_7x_8x_9, x_7x_8x_{10}, x_8x_9x_{10}\rangle .$$

One can use inductive argument to check that $\mathrm{del}_{\mathcal{D}}(3)$ is vertex decomposable. We can rewrite $I = \langle x_3\rangle + I_{(G\setminus\{3\})^\circ}[\emptyset, \chi_*]$ and

$$J = \langle x_1, x_3\rangle + I(K) + \langle x_5x_6, x_5x_{11}\rangle + I_{(G\setminus N_G(3))^\circ}[\emptyset, \chi_*],$$

where K is the complete graph on vertex set $\{2, 4, 5\}$. Now, set $\Omega_0 = \mathrm{link}_{\mathcal{D}}(3)$, $\Omega_k = \mathrm{del}_{\Omega_{k-1}}(w_k)$ and $\Phi_k = \mathrm{link}_{\Omega_{k-1}}(w_k)$, for $k = 1, 2$, where $w_1 = 4$ and $w_2 = 5$. One can check that w_k is a shedding vertex of Ω_{k-1}. Now, by Remark 2.3, we obtain that $\Phi_k = \Delta_{J_k}$, where

$$J_1 = \langle x_1, x_2, x_3, x_4, x_5\rangle + I_{(G\setminus N_G(3))^\circ}[\emptyset, \chi_*]$$

and

$$J_2 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_{11}\rangle + I_{(G\setminus N_G(3))^\circ}[\emptyset, \chi_*].$$

Also, we have $\Omega_2 = \Delta_L$, where $L = \langle x_1, x_3, x_4, x_5\rangle + I_{(G\setminus N_G(3))^\circ}[\emptyset, \chi_*]$. Again, by using inductive argument, we see that Ω_2, Φ_1 and Φ_2 are vertex decomposable. Using Lemma 2.2, we obtain that $\mathrm{link}_{\mathcal{D}}(3)$ is vertex decomposable.

The following example shows that $\Delta_{I_t(G)}$ does not need to be vertex decomposable for $t \geq 3$.

Example 3.8 Let $t \geq 3$. We write G_t for the tree on the vertex set $[t + 3]$ with edge set

$$E(G) = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \dots, \{t, t + 1\}, \{t + 1, t + 2\}, \{t + 1, t + 3\}\}.$$

We show that $\Delta_{I_t(G_t)}$ is not vertex decomposable. We use induction on t. One can check that $\Delta_{I_3(G_3)}$ is not vertex decomposable. Let $t > 3$ and $\mathcal{D} = \Delta_{I_t(G_t)}$. First we show that $w \in [t+3]$ is a shedding vertex of $\mathcal{D}$ if and only if $w \notin \{1, 2, t+2, t+3\}$. Suppose that $w \notin \{1, 2, t + 2, t + 3\}$. Let $\sigma \in \mathrm{link}_{\mathcal{D}}(w)$. Since $x_1x_3x_4 \cdots x_{t+1}x_{t+2} \in I_t(G_t)$, there exists $z \in \{1, 3, 4, \dots, t + 1, t + 2\}$ such that $z \notin \sigma$. The fact $x_w|M$ for all monomials $M \in I_t(G)$ implies that $\sigma \cup \{z\} \in \mathcal{D}$. Thus, w is a shedding vertex of $\mathcal{D}$. Conversely, suppose that $w \in \{1, 2, t + 2, t + 3\}$. By symmetry, we may assume that $w = 1$. Note that $\{2, \dots, t + 1\} \in \mathrm{link}_{\mathcal{D}}(w)$ and $\{2, \dots, t + 1\}$ is a facet in $\mathrm{del}_{\mathcal{D}}(w)$. This implies that w is not a shedding vertex of $\mathcal{D}$.

Now, let $w \in [t + 3]\setminus\{1, 2, t + 2, t + 3\}$. Then, by Remark 2.3, we obtain that $\mathrm{link}_{\mathcal{D}}(w) \simeq \Delta_J$, where $J = I_{t-1}(G_{t-1}) + \langle x_{t+3}\rangle$ (By considering $I_{t-1}(G_{t-1})$ as an ideal in $\mathbb{K}[x_1, \dots, x_{t+3}]$). By induction hypothesis, $\mathrm{link}_{\mathcal{D}}(w)$ is not vertex decomposable, and hence $\mathcal{D}$ is not vertex decomposable.

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Declarations

Conflict of interest The author declares that there is no conflict of interest.

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