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On Certain Determinants and Related Legendre Symbols

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Abstract

Let p be an odd prime. For $b, c \in \mathbb{Z}$, we study the Legendre symbol $\left(\frac{D_p^*(b,c)}{p}\right)$, where $D_p^*(b, c)$ denotes the determinant of the matrix $[(i^2 + bij + cj^2)^{p-3}]_{1\le i,j\le p-1}$. For example, we prove that if $p \equiv 2 \pmod 3$ then

$$D_p^*(1, 1) \equiv \det\left[\frac{1}{(i^2 + ij + j^2)^2}\right]_{1\le i,j\le p-1} \equiv -x^2 \pmod p$$

for some integer $x \not\equiv 0 \pmod p$. We also show that

$$\left(\frac{D_p^*(2, 2)}{p}\right) = \left(\frac{p}{3}\right)(-1)^{(p+1)/8}$$

if $p \equiv 7 \pmod 8$.

Keywords Determinant · Legendre symbol · Generalized trinomial coefficient · Lucas sequence

Mathematics Subject Classification Primary 11C20, 15A15; Secondary 11A15, 11B39

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1 Introduction

For an $n \times n$ matrix $[a_{ij}]_{1\le i,j\le n}$ over a commutative ring with identity, we use $\det|a_{ij}|_{1\le i,j\le n}$ or $|a_{ij}|_{1\le i,j\le n}$ to denote its determinant.

Let p be an odd prime, and let $b, c \in \mathbb{Z}$. Sun [4] introduced

$$(b,c)_p = \det\left[\left(\frac{i^2+bij+cj^2}{p}\right)\right]_{1\le i,j\le p-1}$$

and

$$[b,c]_p = \det\left[\left(\frac{i^2+bij+cj^2}{p}\right)\right]_{0\le i,j\le p-1},$$

and proved the following results:

$$\left(\frac{c}{p}\right) = -1 \Longrightarrow (b,c)_p = 0, \tag{1.1}$$

and

$$\left(\frac{c}{p}\right) = 1 \Longrightarrow [b,c]_p = \begin{cases} \frac{p-1}{2}(b,c)_p & \text{if } p \nmid b^2-4c, \\ \frac{1-p}{p-2}(b,c)_p & \text{if } p \mid b^2-4c, \end{cases}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. Grinberg, Sun and Zhao [1, Theorem 1.3] determined $(\frac{S_c(b,p)}{p})$ in the case $p \nmid bc$, where

$$S_c(b,p) = \det\left[\left(\frac{i^2+bj^2+c}{p}\right)\right]_{1\le i,j\le (p-1)/2}.$$

For each prime $p \equiv 5 \pmod 6$, Sun [4] conjectured that

$$2\det\left[\frac{1}{i^2-ij+j^2}\right]_{1\le i,j\le p-1}$$

is a quadratic residue modulo p. This was confirmed by Wu et al. [8].

For any odd prime p and a p-adic integer $x \not\equiv 0 \pmod p$, clearly

$$\frac{1}{x} \equiv x^{p-2} \pmod p \quad \text{and} \quad \frac{1}{x^2} \equiv x^{p-3} \pmod p.$$

Let p be an odd prime, and let $b, c \in \mathbb{Z}$. Sun [5] showed that for any integer n with $(p-1)/2 < n < p-1$ we have

$$\det[(i^2+bij+cj^2)^n]_{0\le i,j\le p-1} \equiv 0 \pmod p.$$

Sun [5] also introduced

$$D_p(b,c)=\det[(i^2+bij+cj^2)^{p-2}]_{1\le i,j\le p-1},$$

and proved that for any prime $p>3$ with $p\equiv 3 \pmod 4$ we have

$$D_p(b,-1)\equiv D_p(2,2)\equiv 0 \pmod p.$$

By Wu et al. [8], we actually have $\left(\frac{D_p(1,1)}{p}\right)=\left(\frac{-2}{p}\right)$ if $p\equiv 2 \pmod 3$. Recently, Luo and Sun [3] have proved that

$$\left(\frac{D_p(1,1)}{p}\right)=\begin{cases}0 & \text{if } p\equiv 7 \pmod 9,\\ 1 & \text{if } p\equiv 1,4 \pmod 9,\end{cases} \tag{1.2}$$

and that

$$\left(\frac{D_p(2,2)}{p}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 8,\\ 0 & \text{if } p\equiv 5 \pmod 8.\end{cases} \tag{1.3}$$

Their tools include generalized trinomial coefficients and Lucas sequences. Similar to (1.1), Wu and She [7] extended a result of Sun [5] by proving that $D_p(b,c)\equiv 0 \pmod p$ if $(\frac{c}{p})=-1$.

We first present a basic result which is similar to (1.1) and Sun [5, Theorem 1.2].

Theorem 1.1 *Let p be an odd prime, and let $b,c\in\mathbb{Z}$ with $(\frac{c}{p})=-1$. For any integer n in the interval $[1,p-1]$, we have*

$$\det[(i^2+bij+cj^2)^n]_{1\le i,j\le p-1}\equiv 0 \pmod p. \tag{1.4}$$

Proof For $j=1,\ldots,p-1$, let $\pi_c(j)=\{cj\}_p$, the least nonnegative residue of cj modulo p. By Zolotarev's Lemma (cf. [9]), the sign of $\pi_c\in S_{p-1}$ is exactly the Legendre symbol $(\frac{c}{p})$. Observe that

$$\begin{aligned}
&c^{n(p-1)}\det[(i^2+bij+cj^2)^n]_{1\le i,j\le p-1}\\
&\quad=\det\left[(ci^2+bi(cj)+(cj)^2)^n\right]_{1\le i,j\le p-1}\\
&\quad=\det\left[(ci^2+bi\pi_c(j)+\pi_c(j)^2)^n\right]_{1\le i,j\le p-1}\\
&\quad=\sum_{\sigma\in S_{p-1}}\operatorname{sign}(\sigma)\prod_{i=1}^{p-1}(ci^2+bi\pi_c(\sigma(i))+\pi_c(\sigma(i))^2)^n\\
&\quad=\operatorname{sign}(\pi_c)\sum_{\tau\in S_{p-1}}\operatorname{sign}(\tau)\prod_{i=1}^{p-1}(ci^2+bi\tau(i)+\tau(i)^2)^n
\end{aligned}$$

$$= \left(\frac{c}{p}\right) \det[(i^2 + bij + cj^2)^n]_{1\le i,j\le p-1} = -\det[(i^2 + bij + cj^2)^n]_{1\le i,j\le p-1}.$$

Thus, with the aid of Fermat's little theorem, we obtain (1.4). □

Let p be an odd prime, and let $b, c \in \mathbb{Z}$. In contrast to the notation $D_p(b, c)$, we introduce

$$D_p^*(b, c) = \det[(i^2 + bij + cj^2)^{p-3}]_{1\le i,j\le p-1}. \tag{1.5}$$

If $(\frac{b^2-4c}{p}) = -1$, then $i^2 + bij + cj^2 \not\equiv 0 \pmod p$ for all $i, j = 1, \ldots, p-1$, and hence,

$$D_p^*(b, c) \equiv \det\left[\frac{1}{(i^2 + bij + cj^2)^2}\right]_{1\le i,j\le p-1} \pmod p.$$

The notations $D_p(b, c)$ and $D_p^*(b, c)$ are motivated by Wolstenholme's congruences (cf. [6])

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod p$$

provided $p > 3$.

Now we state our main results.

Theorem 1.2 *Let p be an odd prime. Then*

$$\left(\frac{D_p^*(1, 1)}{p}\right) = \begin{cases} (\frac{-1}{p}) & \text{if } p \equiv 2 \pmod 3, \\ (\frac{p}{5}) \text{ or } 0 & \text{if } p \equiv 1 \pmod 3. \end{cases} \tag{1.6}$$

Consequently, when $p \equiv 2 \pmod 3$ the p-adic integer

$$-\det\left[\frac{1}{(i^2 + ij + j^2)^2}\right]_{1\le i,j\le p-1}$$

is a quadratic residue modulo p.

Theorem 1.3 *Let p be an odd prime. Then*

$$\left(\frac{D_p^*(2, 2)}{p}\right) = \begin{cases} 0 \text{ or } 1 & \text{if } p \equiv 1 \pmod 8 \\ (\frac{p}{3})\,(-1)^{\frac{p+1}{8}} & \text{if } p \equiv 7 \pmod 8, \\ 0 & \text{if } p \equiv \pm 3 \pmod 8. \end{cases} \tag{1.7}$$

Remark 1.1 Note that for any prime $p \equiv 3 \pmod 4$ we have

$$D_p^*(2,2) \equiv \det\left[\frac{1}{((i+j)^2+j^2)^2}\right]_{1\le i,j\le p-1} \pmod p.$$

Let $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $b, c \in \mathbb{Z}$. The generalized trinomial coefficients

$$\binom{n}{k}_{b,c} \quad (k \in \mathbb{Z})$$

are given by

$$\left(x + b + \frac{c}{x}\right)^n = \sum_{k\in\mathbb{Z}} \binom{n}{k}_{b,c} x^k. \tag{1.8}$$

We will make use of generalized trinomial coefficients to prove Theorems 1.2 and 1.3 in Sects. 2 and 3, respectively. Note that Theorems 1.2 and 1.3 cannot be deduced from Luo and Sun's results (1.2) and (1.3), and their proofs are somewhat sophisticated.

2 Proof of Theorem 1.2

Lemma 2.1 ([2, Lemma 10]) *Let R be a commutative ring with identity, and let $P(x) = \sum_{i=0}^{n-1} a_i x^i \in R[x]$. Then*

$$\det[P(X_iY_j)]_{1\le i,j\le n} = a_0a_1\cdots a_{n-1} \prod_{1\le i,j\le n} (X_i - X_j)(Y_i - Y_j).$$

Lemma 2.2 (Luo and Sun [3, (3.2)]) *For any odd prime p, we have*

$$\prod_{1\le i,j\le p-1} (i-j)\left(\frac{1}{i} - \frac{1}{j}\right) = (-1)^{(p+1)/2} \prod_{j=1}^{p-2} (j!)^2. \tag{2.1}$$

Lemma 2.3 ([3, Lemma 2.1]) *Let p be an odd prime, and let $b, c \in \mathbb{Z}$. For $k \in \{-p+2, \ldots, p-2\}$, we have*

$$(4c^2 - b)\binom{p-2}{k}_{b,c} \equiv \begin{cases} \binom{p-1}{-1}_{b,c} + c\binom{p-1}{1}_{b,c} - b \pmod p & \text{if } k = 0, \\ (k+1)\binom{p-1}{k-1}_{b,c} - (k-1)c\binom{p-1}{k+1}_{b,c} \pmod p & \text{otherwise.} \end{cases} \tag{2.2}$$

Lemma 2.4 *Let p be an odd prime, and let $b, c \in \mathbb{Z}$. For $k \in \{-p+3, \ldots, p-3\}$, we have*

$$(k+3)\binom{p-2}{k-1}_{b,c} - (k-3)c\binom{p-2}{k+1}_{b,c} - 2\binom{p-1}{k}_{b,c}$$

$$\equiv 2(4c-b^2)\binom{p-3}{k}_{b,c} \pmod p. \tag{2.3}$$

Proof For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we simply write $\left[{n \atop k}\right]$ for $\binom{n}{k}_{b,c}$.

Taking derivatives of both sides of the following identity

$$\sum_{k=-p+1}^{p-1} \begin{bmatrix} p-1 \\ k \end{bmatrix} x^k = \left(x+b+\frac{c}{x}\right)^{p-1},$$

we get

$$\sum_{k=-p+1}^{p-1} k\begin{bmatrix} p-1 \\ k \end{bmatrix} x^{k-1} = (p-1)\left(x+b+\frac{c}{x}\right)^{p-2}\left(1-\frac{c}{x^2}\right). \tag{2.4}$$

Comparing the coefficients of x^{k-1} on both sides of (2.4), we obtain

$$k\begin{bmatrix} p-1 \\ k \end{bmatrix} = (p-1)\left(\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix}\right). \tag{2.5}$$

Taking derivatives of both sides of (2.4), we get

$$\begin{aligned}\sum_{k=-p+1}^{p-1} k(k-1)\begin{bmatrix} p-1 \\ k \end{bmatrix} x^{k-2} &= (p-1)(p-2)\left(x+b+\frac{c}{x}\right)^{p-3}\left(1-\frac{c}{x^2}\right)^2 \\ &\quad + \frac{2c}{x^3}(p-1)\left(x+b+\frac{c}{x}\right)^{p-2}. \end{aligned} \tag{2.6}$$

Comparing the coefficients of x^{k-2} on both sides of (2.6), we deduce that

$$\begin{aligned} k(k-1)\begin{bmatrix} p-1 \\ k \end{bmatrix} &= (p-1)(p-2)\left(\begin{bmatrix} p-3 \\ k-2 \end{bmatrix} - 2c\begin{bmatrix} p-3 \\ k \end{bmatrix} + c^2\begin{bmatrix} p-3 \\ k+2 \end{bmatrix}\right) \\ &\quad + 2c(p-1)\begin{bmatrix} p-2 \\ k+1 \end{bmatrix}. \end{aligned} \tag{2.7}$$

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $k \in \mathbb{Z}$, we have the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + b\begin{bmatrix} n-1 \\ k \end{bmatrix} + c\begin{bmatrix} n-1 \\ k+1 \end{bmatrix}$$

by Luo and Sun [3, (2.3)]. With the aid of this, we have

$$\begin{bmatrix} p-3 \\ k-2 \end{bmatrix} - 2c\begin{bmatrix} p-3 \\ k \end{bmatrix} + c^2\begin{bmatrix} p-3 \\ k+2 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - b\begin{bmatrix} p-3 \\ k-1 \end{bmatrix} - c\begin{bmatrix} p-3 \\ k \end{bmatrix} - 2c\begin{bmatrix} p-3 \\ k \end{bmatrix} \\
&\quad + c\left(\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} - b\begin{bmatrix} p-3 \\ k+1 \end{bmatrix} - \begin{bmatrix} p-3 \\ k \end{bmatrix}\right) \\
&= \begin{bmatrix} p-2 \\ k-1 \end{bmatrix} + c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} - 4c\begin{bmatrix} p-3 \\ k \end{bmatrix} - b\left(\begin{bmatrix} p-3 \\ k-1 \end{bmatrix} + c\begin{bmatrix} p-3 \\ k+1 \end{bmatrix}\right) \\
&= \begin{bmatrix} p-2 \\ k-1 \end{bmatrix} + c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} - 4c\begin{bmatrix} p-3 \\ k \end{bmatrix} - b\left(\begin{bmatrix} p-2 \\ k \end{bmatrix} - b\begin{bmatrix} p-3 \\ k \end{bmatrix}\right) \\
&= \begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - b\begin{bmatrix} p-2 \\ k \end{bmatrix} + c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} + (b^2-4c)\begin{bmatrix} p-3 \\ k \end{bmatrix} \\
&= \begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - \left(\begin{bmatrix} p-1 \\ k \end{bmatrix} - \begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix}\right) \\
&\quad + c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} + (b^2-4c)\begin{bmatrix} p-3 \\ k \end{bmatrix},
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&\begin{bmatrix} p-3 \\ k-2 \end{bmatrix} - 2c\begin{bmatrix} p-3 \\ k \end{bmatrix} + c^2\begin{bmatrix} p-3 \\ k+2 \end{bmatrix} \\
&\quad = -\begin{bmatrix} p-1 \\ k \end{bmatrix} + 2\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} + 2c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} + (b^2-4c)\begin{bmatrix} p-3 \\ k \end{bmatrix}.
\end{aligned} \tag{2.8}
$$

Combining (2.5), (2.7) and (2.8), we get

$$
\begin{aligned}
&(k-1)\left(\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix}\right) - 2c(p-1)\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} \\
&\quad = \frac{k(k-1)}{p-1}\begin{bmatrix} p-1 \\ k \end{bmatrix} - 2c(p-1)\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} \\
&\quad = (p-2)\left(\begin{bmatrix} p-3 \\ k-2 \end{bmatrix} - 2c\begin{bmatrix} p-3 \\ k \end{bmatrix} + c^2\begin{bmatrix} p-3 \\ k+2 \end{bmatrix}\right) - 2c(p-2)\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} \\
&\quad = (p-2)\left(-\begin{bmatrix} p-1 \\ k \end{bmatrix} + 2\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} + (b^2-4c)\begin{bmatrix} p-3 \\ k \end{bmatrix}\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&(k-1)\left(\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix}\right) + 2c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} \\
&\quad \equiv 2\left(\begin{bmatrix} p-1 \\ k \end{bmatrix} - 2\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - (b^2-4c)\begin{bmatrix} p-3 \\ k \end{bmatrix}\right) \pmod{p},
\end{aligned}
$$

that is,

$$(k+3)\begin{bmatrix} p-2 \\ k-1 \end{bmatrix} - (k-3)c\begin{bmatrix} p-2 \\ k+1 \end{bmatrix} - 2\begin{bmatrix} p-1 \\ k \end{bmatrix} \equiv 2(4c-b^2)\begin{bmatrix} p-3 \\ k \end{bmatrix} \pmod{p}. \tag{2.9}$$

This concludes the proof. □

Proof of Theorem 1.2 Let $b, c \in \mathbb{Z}$. By Luo and Sun [3, (2.2)], we have

$$\binom{n}{-k}_{b,c} = c^k \binom{n}{k}_{b,c} \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad k \in \mathbb{Z}. \tag{2.10}$$

Thus,

$$\begin{aligned}
&(x^2+bx+c)^{p-3} - \binom{p-3}{0}_{b,c} x^{p-3} \\
&= \sum_{\substack{k=-p-3 \\ k\neq 0}}^{p-3} \binom{p-3}{k}_{b,c} x^{p-3+k} \\
&= \sum_{k=1}^{p-3} \left(\binom{p-3}{k}_{b,c} x^{p-3+k} + \binom{p-3}{k}_{b,c} c^k x^{p-3-k} \right) \\
&= \sum_{k=2}^{p-3} \left(\binom{p-3}{k}_{b,c} x^{p-1} + \binom{p-3}{p-1-k}_{b,c} c^{p-1-k} \right) x^{k-2} \\
&\quad + \binom{p-3}{1}_{b,c} x^{p-2} + \binom{p-3}{1}_{b,c} c x^{p-4}.
\end{aligned} \tag{2.11}$$

Let $k \in \{-p+3, \ldots, p-3\}$. Taking $b = c = 1$ in Lemma 2.4, we get

$$6\binom{p-3}{k}_{1,1} \equiv (k+3)\binom{p-2}{k-1}_{1,1} - (k-3)\binom{p-2}{k+1}_{1,1} - 2\binom{p-1}{k}_{1,1} \pmod{p}. \tag{2.12}$$

Putting $b = c = 1$ in (2.2) and noting (2.10), we obtain

$$3\binom{p-2}{k}_{1,1} \equiv \begin{cases} 2\binom{p-1}{1}_{1,1} - 1 \pmod{p} & \text{if } k = 0, \\ (k+1)\binom{p-1}{k-1}_{1,1} - (k-1)\binom{p-1}{k+1}_{1,1} \pmod{p} & \text{if } k \neq 0. \end{cases} \tag{2.13}$$

Combining (2.12) with (2.13), we see that

$$\begin{aligned}
&18\binom{p-3}{k}_{1,1} \\
&\equiv 3(k+3)\binom{p-2}{k-1}_{1,1} - 3(k-3)\binom{p-2}{k+1}_{1,1} - 6\binom{p-1}{k}_{1,1}
\end{aligned}$$

$$\equiv \begin{cases} -2\binom{p-1}{-3}_{1,1}+8\binom{p-1}{1}_{1,1}-4 \pmod p & \text{if } k=-1,\\ -2\binom{p-1}{3}_{1,1}+8\binom{p-1}{1}_{1,1}-4 \pmod p & \text{if } k=1,\\ k(k+3)\binom{p-1}{k-2}_{1,1}-2(k^2-3)\binom{p-1}{k}_{1,1}+k(k-3)\binom{p-1}{k+2}_{1,1} \pmod p & \text{if } k\neq \pm 1.\end{cases}$$

For each $k \in \{0,\ldots,p-3\}$, we have

$$\binom{p-1}{p-k}_{1,1} \equiv \left(\frac{k}{3}\right) \pmod p$$

by Luo and Sun [3, (2.14)], and hence,

$$\begin{aligned} 18\binom{p-3}{k}_{1,1} &\equiv 3(k+3)\binom{p-2}{k-1}_{1,1}-3(k-3)\binom{p-2}{k+1}_{1,1}-6\binom{p-1}{k}_{1,1}\\ &\equiv \begin{cases} -2\left(\frac{p}{3}\right)+8\left(\frac{p-1}{3}\right)-4 \pmod p & \text{if } k=1,\\ k(k+3)\left(\frac{p-k+2}{3}\right)-2(k^2-3)\left(\frac{p-k}{3}\right)+k(k-3)\left(\frac{p-k-2}{3}\right) \pmod p & \text{if } 2\le k\le p-3.\end{cases} \end{aligned} \tag{2.14}$$

When $2 \le k \le p-3$, we have

$$\begin{aligned} &18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\\ &\equiv k(k+3)\left(\frac{p-k+2}{3}\right)+(p-1-k)(p+2-k)\left(\frac{k+3}{3}\right)\\ &\quad-2(k^2-3)\left(\frac{p-k}{3}\right)-2((k+1)^2-3)\left(\frac{k+1}{3}\right)\\ &\quad+k(k-3)\left(\frac{p-k-2}{3}\right)+(p-1-k)(p-4-k)\left(\frac{k-1}{3}\right) \pmod p, \end{aligned}$$

and hence,

$$\begin{aligned} &18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\\ &\equiv \begin{cases} (-3k^2+3k+6)\left(\frac{k+1}{3}\right)+(3k^2+9k)\left(\frac{k+2}{3}\right) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ -6k\left(\frac{k+1}{3}\right)+6\left(\frac{k+2}{3}\right) \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases} \end{aligned} \tag{2.15}$$

In view of (2.11) and (2.14), we obtain

$$\begin{aligned} &18(x^2+x+1)^{p-3}\\ &\equiv 18\binom{p-3}{0}_{1,1}x^{p-3}+\sum_{k=2}^{p-3}\left(18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\right)x^{k-2}\\ &\quad+18\binom{p-3}{1}_{1,1}(x^{p-2}+x^{p-4})\\ &\equiv 6\left(\frac{p}{3}\right)x^{p-3}+\sum_{k=2}^{p-3}\left(18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\right)x^{k-2} \end{aligned}$$

$$+\left(-2\left(\frac{p}{3}\right)+8\left(\frac{p+2}{3}\right)-4\right)(x^{p-2}+x^{p-4}) \pmod p.$$

Thus, with the aid of (2.15), we have

$$\begin{aligned}
&18(x^2+x+1)^{p-3}\\
&\equiv 6\left(\frac{p}{3}\right)(x^{p-3}-x^{p-2}-x^{p-4})\\
&\quad+\begin{cases}18\sum_{k=2}^{p-3}\left(-\frac{(k+1)(k-2)}{6}\left(\frac{k+1}{3}\right)+\frac{k(k+3)}{6}\left(\frac{k+2}{3}\right)\right)x^{k-2} \pmod p & \text{if } p\equiv 1 \pmod 3,\\ 18\sum_{k=2}^{p-3}\left(-\frac{k}{3}\left(\frac{k+1}{3}\right)+\frac{1}{3}\left(\frac{k+2}{3}\right)\right)x^{k-2} \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}
\end{aligned} \tag{2.16}$$

Let $F(x)=(x^2+x+1)^{p-3}$. For $1\le i,j\le p-1$, we have

$$\frac{(i^2+ij+j^2)^{p-3}}{j^{2(p-3)}}=\left(\frac{i^2}{j^2}+\frac{i}{j}+1\right)^{p-3}=F\left(\frac{i}{j}\right),$$

and hence,

$$\left(\frac{D_p^*(1,1)}{p}\right)=\left(\frac{|F(i/j)|_{1\le i,j\le p-1}}{p}\right) \tag{2.17}$$

by Fermat's little theorem.

Case 1. $p\equiv 1 \pmod 3$.

Applying Lemma 2.1 with $P(x)=F(x)$, $X_i=i$ and $Y_j=1/j$, and noting the identity (2.1), we get

$$\begin{aligned}
&\left|F\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1}\\
&\equiv \frac{1}{27}\prod_{k=2}^{p-3}\left(-\frac{(k+1)(k-2)}{6}\left(\frac{k+1}{3}\right)+\frac{k(k+3)}{6}\left(\frac{k+2}{3}\right)\right)\\
&\quad\times\prod_{1\le i<j\le p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right)\\
&\equiv \frac{1}{27}(-1)^{\frac{p+1}{2}}(-1)^{\frac{p-4}{3}}\prod_{j=1}^{p-2}(j!)^2\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-7}{3}\}}\frac{k(k+3)}{6}\\
&\quad\times\prod_{k\in\{4+3a:\ 0\le a\le\frac{p-7}{3}\}}\frac{(k+1)(k-2)}{6}\times\prod_{k\in\{3+3a:\ 0\le a\le\frac{p-7}{3}\}}\frac{k^2+k-1}{3}\\
&\equiv \frac{1}{27}(-1)^{\frac{p+1}{2}+\frac{p-4}{3}}\prod_{j=1}^{p-2}(j!)^2\times\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-7}{3}\}}\left(\frac{k(k+3)}{3}\right)^2
\end{aligned}$$

$$\times\ 3^{-\frac{p-4}{3}} \prod_{\substack{3\le k\le p-4\\ 3|k}} (k^2+k-1)$$

$$\equiv 3^{-\frac{p+5}{3}}(-1)^{\frac{p+1}{2}+\frac{p-4}{3}} \prod_{j=1}^{p-2}(j!)^2 \times \prod_{k\in\{2+3a:\ 0\le a\le \frac{p-7}{3}\}} \left(\frac{k(k+3)}{3}\right)^2$$

$$\times \prod_{\substack{3\le k<\frac{p-1}{2}\\ 3|k}} (k^2+k-1)((p-1-k)^2+(p-1-k)-1)$$

$$\times \left(\left(\frac{p-1}{2}\right)^2+\frac{p-1}{2}-1\right),$$

and hence,

$$\left|F\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} \equiv 3^{-\frac{p+5}{3}}(-1)^{\frac{p+1}{2}+\frac{p-4}{3}}\frac{p^2-5}{4}\prod_{j=1}^{p-2}(j!)^2 \times \prod_{k\in\{2+3a:\ 0\le a\le \frac{p-7}{3}\}} \left(\frac{k(k+3)}{3}\right)^2 \times \prod_{\substack{3\le k<\frac{p-1}{2}\\ 3|k}} (k^2+k-1)^2 \pmod p. \tag{2.18}$$

Observe that

$$\left(\frac{3}{p}\right)^{-\frac{p+5}{3}}\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-4}{3}}\left(\frac{-5}{p}\right) = \left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-4}{3}+1}\left(\frac{5}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{p+1}{2}}\left(\frac{5}{p}\right) = \left(\frac{5}{p}\right). \tag{2.19}$$

Combining (2.17)–(2.19), we obtain

$$\left(\frac{D_p^*(1,1)}{p}\right) = \left(\frac{|F(i/j)|_{1\le i,j\le p-1}}{p}\right) = \left(\frac{5}{p}\right) \text{ or } 0.$$

Case 2. $p\equiv 2 \pmod 3$.
By Lemma 2.1 and the identity (2.1), we have

$$\left|F\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} \equiv -\frac{1}{27}\prod_{k=2}^{p-3}\left(-\frac{k}{3}\left(\frac{k+1}{3}\right)+\frac{1}{3}\left(\frac{k+2}{3}\right)\right)\prod_{1\le i<j\le p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right)$$

$$\equiv -\frac{1}{27}(-1)^{\frac{p+1}{2}}(-1)^{\frac{p-5}{3}}\prod_{j=1}^{p-2}(j!)^2 \times \prod_{k\in\{2+3a:\ 0\le a\le\frac{p-5}{3}\}}\frac{1}{3}$$

$$\times \prod_{k\in\{4+3a:\ 0\le a\le \frac{p-8}{3}\}} \frac{k}{3} \times \prod_{k\in\{3+3a:\ 0\le a\le \frac{p-8}{3}\}} \frac{k+1}{3}$$
$$\equiv 3^{-(p-4)-3}(-1)^{\frac{p+1}{2}+\frac{p-5}{3}+1} \prod_{j=1}^{p-2}(j!)^2 \times \prod_{k\in\{4+3a:\ 0\le a\le \frac{p-8}{3}\}} k^2.$$

Combining this with (2.17), we finally obtain

$$\left(\frac{D_p^*(1,1)}{p}\right) = \left(\frac{|F(i/j)|_{1\le i,j\le p-1}}{p}\right) = \left(\frac{3}{p}\right)^{-p+1}\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-5}{3}+1}$$
$$=\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-5}{3}+1} = \left(\frac{-1}{p}\right).$$

In view of the above, we have completed our proof of Theorem 1.2. □

3 Proof of Theorem 1.3

Lemma 3.1 *Let $p > 5$ be a prime, and let $b, c \in \mathbb{Z}$. Then*

$$\left(\frac{D_p^*(b,c)}{p}\right) = \left(\frac{c}{p}\right)^{\frac{(p-1)(p-3)}{8}} \left(\frac{\binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2}{p}\right)\left(\frac{W(\frac{p-1}{2})\prod_{k=2}^{(p-3)/2} W(k)^2}{p}\right), \tag{3.1}$$

where

$$W(k) = \binom{p-3}{k}_{b,c} + \binom{p-3}{p-1-k}_{b,c} c^{p-1-k}.$$

Proof Let $G(x) = (x^2 + bx + c)^{p-3}$. For $1 \le i, j \le p-1$, we have

$$\frac{(i^2 + bij + cj^2)^{p-3}}{j^{2(p-3)}} = \left(\frac{i^2}{j^2} + b\frac{i}{j} + c\right)^{p-3} = G\left(\frac{i}{j}\right),$$

and hence,

$$\left(\frac{D_p^*(b,c)}{p}\right) = \left(\frac{|G(i/j)|_{1\le i,j\le p-1}}{p}\right)$$

by Fermat's little theorem. In view of (2.11), and Lemmas 2.1 and 2.2, we see that

$$\begin{aligned}\left|G\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} &= \binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2 \\ &\quad\times c\prod_{k=2}^{p-3}\left(\binom{p-3}{k}_{b,c}+\binom{p-3}{p-1-k}_{b,c}c^{p-1-k}\right)\\ &\quad\times\prod_{1\le i<j\le p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right)\\ &=\binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2 c\prod_{k=2}^{p-3}W(k)\times(-1)^{\frac{p+1}{2}}\prod_{j=1}^{p-2}(j!)^2\\ &=\binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2 cW\left(\frac{p-1}{2}\right)\prod_{k=2}^{\frac{p-3}{2}}(W(k)W(p-1-k))\\ &\quad\times(-1)^{\frac{p+1}{2}}\prod_{j=1}^{p-2}(j!)^2.\end{aligned}$$

Since

$$W(p-1-k)=\binom{p-3}{p-1-k}_{b,c}+c^k\binom{p-3}{k}_{b,c}\equiv c^kW(k)\ (\mathrm{mod}\ p)$$

for all $k=2,\dots,(p-3)/2$, we have

$$\begin{aligned}\prod_{k=2}^{\frac{p-3}{2}}W(k)W(p-1-k)&=\prod_{k=2}^{\frac{p-3}{2}}\left(\frac{c^kW(k)^2}{p}\right)=\left(\frac{c}{p}\right)^{\sum_{k=2}^{\frac{p-3}{2}}k}\left(\frac{\prod_{k=2}^{(p-3)/2}W(k)^2}{p}\right)\\ &=\left(\frac{c}{p}\right)^{\frac{p^2-4p-5}{8}}\left(\frac{\prod_{k=2}^{(p-3)/2}W(k)^2}{p}\right).\end{aligned}$$

Combining the above, we immediately obtain the desired identity (3.1). □

Proof of Theorem 1.3 Applying Theorem 1.1 with $c=2$, we see that $(\frac{D_p^*(2,2)}{p})=0$ if $p\equiv\pm3\ (\mathrm{mod}\ 8)$. Below we assume that $p\equiv\pm1\ (\mathrm{mod}\ 8)$.

Let $k\in\{0,\dots,p-3\}$. Taking $b=c=2$ in Lemma 2.4 and (2.2), we get

$$8\binom{p-3}{k}_{2,2}\equiv(k+3)\binom{p-2}{k-1}_{2,2}-2(k-3)\binom{p-2}{k+1}_{2,2}-2\binom{p-1}{k}_{2,2}\ (\mathrm{mod}\ p) \tag{3.2}$$

and

$$4\binom{p-2}{k}_{2,2}\equiv\begin{cases}\binom{p-1}{-1}_{2,2}+2\binom{p-1}{1}_{2,2}-2\ (\mathrm{mod}\ p) & \text{if } k=0,\\ (k+1)\binom{p-1}{k-1}_{2,2}-2(k-1)\binom{p-1}{k+1}_{2,2}\ (\mathrm{mod}\ p) & \text{otherwise.}\end{cases} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$32\binom{p-3}{k}_{2,2} \equiv 4(k+3)\binom{p-2}{k-1}_{2,2} - 8(k-3)\binom{p-2}{k+1}_{2,2} - 8\binom{p-1}{k}_{2,2}$$
$$\equiv \begin{cases} 20\binom{p-1}{1}_{2,2} - 8\binom{p-1}{3} - 8 \pmod{p} & \text{if } k=1,\\ k(k+3)\binom{p-1}{k-2}_{2,2} - 4(k+2)(k-2)\binom{p-1}{k}_{2,2} + 4k(k-3)\binom{p-1}{k+2}_{2,2} \pmod{p} & \text{otherwise,} \end{cases} \tag{3.4}$$

and also

$$32\binom{p-3}{p-1-k}_{2,2} \equiv (k^2-k-2)\binom{p-1}{p-3-k}_{2,2} - 4(k^2+2k-3)\binom{p-1}{p-1-k}_{2,2} + 4(k^2+5k+4)\binom{p-1}{p+1-k}_{2,2} \pmod{p} \tag{3.5}$$

if $2 \le k \le p-3$.

For any $2 \le k \le p-3$, define

$$W(k) = \binom{p-3}{k}_{2,2} + \binom{p-3}{p-1-k}_{2,2} 2^{p-1-k}.$$

Then

$$\begin{aligned} 32W(k) &= 32\binom{p-3}{k}_{2,2} + 2^{p-1-k} \times 32\binom{p-3}{p-1-k}_{2,2} \\ &= k(k+3)\binom{p-1}{k-2}_{2,2} - 4(k+2)(k-2)\binom{p-1}{k}_{2,2} + 4k(k-3)\binom{p-1}{k+2}_{2,2} \\ &\quad + 2^{p-1-k}\bigg((k^2-k-2)\binom{p-1}{p-3-k}_{2,2} - 4(k^2+2k-3)\binom{p-1}{p-1-k}_{2,2} \\ &\quad + 4(k^2+5k+4)\binom{p-1}{p+1-k}_{2,2}\bigg). \end{aligned}$$

Define the sequence $(u_n)_{n\ge 0}$ by

$$u_0 = 0,\ u_1 = 1, \quad \text{and} \quad u_{n+1} = -2u_n - 2u_{n-1} \quad \text{for } n = 1, 2, 3, \ldots.$$

By Luo and Sun [3, (4.3)],

$$\binom{p-1}{p-k}_{2,2} \equiv u_k \pmod{p} \tag{3.6}$$

for all $k = 0, 1, \ldots, p-1$. Thus,

$$\begin{aligned} 32W(k) &= k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2} \\ &+ 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right) \end{aligned}$$

(3.7)

for all $k = 2, \dots, p-3$.

Clearly, (3.5) with $k = (p-1)/2$ yields that

$$32W\left(\frac{p-1}{2}\right) = \frac{p-1}{2}\cdot\frac{p+5}{2}u_{\frac{p+5}{2}} - 4\frac{p+3}{2}\cdot\frac{p-5}{2}u_{\frac{p+1}{2}} + 4\frac{p-1}{2}\cdot\frac{p-7}{2}u_{\frac{p-3}{2}} + \left(\frac{2}{p}\right)\left(\frac{p+1}{2}\cdot\frac{p-5}{2}u_{\frac{p+5}{2}} - 4\frac{p-3}{2}\cdot\frac{p+5}{2}u_{\frac{p+1}{2}} + 4\frac{p+1}{2}\cdot\frac{p+7}{2}u_{\frac{p-3}{2}}\right). \tag{3.8}$$

By Luo and Sun [3, (4.9)], for any $k \in \mathbb{N}$ we have

$$u_k = (-4)^{\lfloor\frac{k}{4}\rfloor} \times \begin{cases} 0 & \text{if } k \equiv 0 \pmod 4, \\ 1 & \text{if } k \equiv 1 \pmod 4, \\ -2 & \text{if } k \equiv 2 \pmod 4, \\ 2 & \text{if } k \equiv 3 \pmod 4. \end{cases} \tag{3.9}$$

Case 1. $p \equiv 1 \pmod 8$.

By (3.4) and (3.6), we have

$$\binom{p-3}{0}_{2,2} = \frac{1}{2}\binom{p-1}{0}_{2,2} = \frac{u_p}{2} = \frac{1}{2}(-4)^{\lfloor\frac{p}{4}\rfloor} \equiv \frac{1}{2}\cdot 2^{\frac{p-1}{2}} = \frac{1}{2}\left(\frac{2}{p}\right) = \frac{1}{2} \pmod p. \tag{3.10}$$

Write $p = 8q+1$ with $q \in \mathbb{N}$. In view of (3.8) and (3.9), we have

$$\begin{aligned} 32W\left(\frac{p-1}{2}\right) &= 2\left(4q(4q+3)u_{4q+3} - 4(4q+2)(4q-2)u_{4q+1} + 4(4q)(4q-3)u_{4q-1}\right) \\ &= 8q(4q+3)\times 2(-4)^q - 8(4q+2)(4q-2)(-4)^q \\ &\quad + 32q(4q-3)\times 2(-4)^{q-1} \end{aligned}$$

and hence,

$$\begin{aligned} W\left(\frac{p-1}{2}\right) &= (-1)^q\left(2q(4q+3)2^{2q-2} - (4q+2)(4q-2)2^{2q-2} - 2q(4q-3)2^{2q-2}\right) \\ &= (-1)^q 2^{2q-2}\left(8q^2+6q-16q^2+4-8q^2+6q\right) \\ &= (-1)^q 2^{2q}(-q+1)(4q+1) = (-1)^{\frac{p+7}{8}} 2^{2q}\cdot\frac{p-9}{8}\cdot\frac{p+1}{2}. \end{aligned}$$

Therefore,

$$\left(\frac{W(\frac{p-1}{2})}{p}\right) = \left(\frac{-1}{p}\right)^{\frac{p+7}{8}}\left(\frac{-1}{p}\right) = 1. \tag{3.11}$$

Note that

$$\left(\frac{\binom{p-3}{0}_{2,2}}{p}\right) = \left(\frac{2}{p}\right) = 1$$

by (3.10). Combining this with Lemma 3.1, (3.12) and (3.11), we obtain

$$\left(\frac{D_p^*(2,2)}{p}\right) \neq -1.$$

Case 2. $p \equiv 7 \pmod 8$.

In this case, we write $p = 8q+7$ with $q \in \mathbb{N}$. For $2 \le k \le p-3$, write $k = 4s+r$ with $s \in \mathbb{N}$ and $r \in \{0, 1, 2, 3\}$. We will first show that $W(k) \not\equiv 0 \pmod p$ for any $k \in \{2, 3, \dots, p-3\}$.

Subcase 2.1. $r = 0$.

In this subcase, by (3.7), (3.9) and Fermat's little theorem, we have

$$\begin{aligned}
&32W(k)\\
&\quad\equiv k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&\qquad + 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
&\quad\equiv k(k+3)(-4)^{\lfloor\frac{p-k+2}{4}\rfloor} - 8(k+2)(k-2)(-4)^{\lfloor\frac{p-k}{4}\rfloor} + 4k(k-3)(-4)^{\lfloor\frac{p-k-2}{4}\rfloor}\\
&\qquad + 2^{-k}\Big(2(k+1)(k-2)(-4)^{\lfloor\frac{k+3}{4}\rfloor} - 4(k-1)(k+3)(-4)^{\lfloor\frac{k+1}{4}\rfloor}\\
&\quad + 8(k+1)(k+4)(-4)^{\lfloor\frac{k-1}{4}\rfloor}\Big)\\
&\quad\equiv 4s(4s+3)(-4)^{2q-s+2} - 8(4s+2)(4s-2)(-4)^{2q-s+1} + 16s(4s-3)(-4)^{2q-s+1}\\
&\qquad + 2^{-k}\left(2(4s+1)(4s-2)(-4)^{s} - 4(4s-1)(4s+3)(-4)^{s} + 8(k+1)(k+4)(-4)^{s-1}\right)\\
&\quad\equiv 2k(k+3)(-4)^{-s} \times \left(\frac{2}{p}\right) + 4(k+2)(k-2)(-4)^{-s}\left(\frac{2}{p}\right) - 2k(k-3)\left(\frac{2}{p}\right)(-4)^{-s}\\
&\qquad + 2^{-4s+1}(k+1)(k-2)(-4)^{s}\\
&\qquad - 2^{-4s+2}(k-1)(k+3)(-4)^{s} - 2^{-4s+1}(k+1)(k+4)(-4)^{s}\\
&\quad= 2(-4)^{-s}(-4k-8) \not\equiv 0 \pmod p.
\end{aligned}$$

Subcase 2.2. $r = 1$.

In this subcase, by (3.7) and (3.9) we have

$$\begin{aligned}
32W(k) &\equiv k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&\quad + 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
&\equiv 8(k+2)(k-2)(-4)^{\lfloor\frac{p-k}{4}\rfloor} + 2^{-k} \times 8(k-1)(k+3)(-4)^{\lfloor\frac{k+1}{4}\rfloor}\\
&\equiv 8(k+2)(k-2)(-4)^{2q-s+1} + 2^{-k} \times 8(k-1)(k+3)(-4)^{s}\\
&\equiv -4(k+2)(k-2)(-4)^{-s}\left(\frac{2}{p}\right) + 2^{-4s+2}(k-1)(k+3)(-4)^{s}\\
&= (-4)^{-s+1}(-1-2k) \not\equiv 0 \pmod p.
\end{aligned}$$

Subcase 2.3. $r = 2$.
In view of (3.7) and (3.9), we have

$$\begin{aligned}
&32W(k)\\
&\equiv k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&\quad + 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
&\equiv 2k(k+3)(-4)^{\lfloor\frac{p-k+2}{4}\rfloor} - 4(k+2)(k-2)(-4)^{\lfloor\frac{p-k}{4}\rfloor} + 8k(k-3)(-4)^{\lfloor\frac{p-k-2}{4}\rfloor}\\
&\quad + 2^{-k}\Big((k+1)(k-2)(-4)^{\lfloor\frac{k+3}{4}\rfloor} - 8(k-1)(k+3)(-4)^{\lfloor\frac{k+1}{4}\rfloor}\\
&\quad + 4(k+1)(k+4)(-4)^{\lfloor\frac{k-1}{4}\rfloor}\Big)\\
&\equiv 2k(k+3)(-4)^{2q-s+1} - 4(k+2)(k-2)(-4)^{2q-s+1} + 8k(k-3)(-4)^{2q-s}\\
&\quad + 2^{-k}\left((k+1)(k-2)(-4)^{s+1} - 8(k-1)(k+3)(-4)^{s} + 4(k+1)(k+4)(-4)^{s}\right)\\
&\equiv -(-4)^{-s}k(k+3) + 2(-4)^{-s}(k+2)(k-2) + (-4)^{-s}k(k-3)\\
&\quad + 2^{-4s-2}\left(-4(k+1)(k-2)(-4)^{s} - 8(k-1)(k+3)(-4)^{s} + 4(k+1)(k+4)(-4)^{s}\right)\\
&= (-4)^{-s+1}(k-1) \not\equiv 0 \pmod{p}.
\end{aligned}$$

Subcase 2.4. $r = 3$.
In this subcase, by (3.7) and (3.9) we have

$$\begin{aligned}
32W(k) &\equiv k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&\quad + 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
&\equiv -2k(k+3)(-4)^{\lfloor\frac{p-k+2}{4}\rfloor} - 8k(k-3)(-4)^{\lfloor\frac{p-k-2}{4}\rfloor}\\
&\quad + 2^{-k}(-2(k+1)(k-2)(-4)^{\lfloor\frac{k+3}{4}\rfloor} - 8(k+1)(k+4)(-4)^{\lfloor\frac{k-1}{4}\rfloor})\\
&\equiv -2k(k+3)(-4)^{2q-s+1} - 8k(k-3)(-4)^{2q-s}\\
&\quad + 2^{-4s}\left((k+1)(k-2)(-4)^{s} - (k+1)(k+4)(-4)^{s}\right)\\
&\equiv k(k+3)(-4)^{-s} - k(k-3)(-4)^{-s} + (k+1)(k-2)(-4)^{-s}\\
&\quad - (k+1)(k+4)(-4)^{-s}\\
&= -6(-4)^{-s} \not\equiv 0 \pmod{p}.
\end{aligned}$$

In view of the discussions for all the four subcases, we do have

$$W(k) \not\equiv 0 \pmod{p} \quad \text{for any } 2 \le k \le p-3. \tag{3.12}$$

By (3.4) and (3.6), we have

$$\begin{aligned}32\binom{p-3}{1}_{2,2} &= 20\binom{p-1}{1}_{2,2} - 8\binom{p-1}{3}_{2,2} - 8\\ &\equiv 20u_{p-1} - 8u_{p-3} - 8 = -40(-4)^{\lfloor \frac{p-1}{4} \rfloor} - 8 = 10(-4)^{2q+2} - 8\\ &= 10 \times 2^{4q+4} - 8 = 20 \times 2^{\frac{p-1}{2}} - 8 = 20\left(\frac{2}{p}\right) - 8 = 12 \pmod p\end{aligned}$$

and

$$\begin{aligned}\binom{p-3}{0}_{2,2} &= \frac{1}{2}\binom{p-1}{0}_{2,2}\\ &\equiv \frac{u_p}{2} = (-4)^{\lfloor \frac{p}{4} \rfloor} = -\frac{1}{2}2^{\frac{p-1}{2}} \equiv -\frac{1}{2}\left(\frac{2}{p}\right) = -\frac{1}{2} \pmod p.\end{aligned}$$

Therefore,

$$\left(\frac{\binom{p-3}{1}_{2,2}}{p}\right) \neq 0 \quad \text{and} \quad \left(\frac{\binom{p-3}{0}_{2,2}}{p}\right) = -1. \tag{3.13}$$

By (3.8) and (3.9), we have

$$\begin{aligned}32W\left(\frac{p-1}{2}\right) &\equiv 2(4q+3)(4q+6)u_{4q+6} - 8(4q+5)(4q+1)u_{4q+4} + 8(4q+3)4qu_{4q+2}\\ &\equiv 16(4q+3)(4q+6)(-4)^q - 16(4q+3)4q(-4)^q \pmod p,\end{aligned}$$

and hence,

$$\begin{aligned}W\left(\frac{p-1}{2}\right) &\equiv (4q+3)(2q+3)(-4)^q - (4q+3)2q(-4)^q\\ &= (-1)^q((4q+3)(2q+3)2^{2q} - (4q+3)2q2^{2q}) = 3(-1)^q 2^{2q}\frac{p-1}{2} \pmod p.\end{aligned}$$

Therefore,

$$\left(\frac{W(\frac{p-1}{2})}{p}\right) = \left(\frac{3}{p}\right)(-1)^{\frac{p-7}{8}}\left(\frac{p-1}{p}\right)\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right)(-1)^{\frac{p+1}{8}}. \tag{3.14}$$

Combining Lemma 3.1, (3.12)–(3.14), we finally obtain

$$\left(\frac{D_p^*(2,2)}{p}\right) = \left(\frac{\binom{p-3}{0}_{2,2}}{p}\right)\left(\frac{W(\frac{p-1}{2})}{p}\right) = -1 \times \left(\frac{3}{p}\right)(-1)^{\frac{p+1}{8}} = \left(\frac{p}{3}\right)(-1)^{\frac{p+1}{8}}.$$

In view of the above, we have finished our proof of Theorem 1.3. □

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Declarations

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