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Schwarz Type Lemmas for Generalized Harmonic Functions

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Abstract

Let $\alpha, \beta \in (-1, \infty)$ such that $\alpha + \beta > -1$. Given two continuous functions $g \in \mathcal{C}(\overline{\mathbb{D}})$ and $f \in \mathcal{C}(\mathbb{T})$, we provide various Schwarz type lemmas for mappings u satisfying the inhomogeneous (α, β)-harmonic equation $L_{\alpha,\beta}u = g$ in $\mathbb{D}$ and $u = f$ in $\mathbb{T}$, where $\mathbb{D}$ is the unit disc of the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. The obtained results provide a significant improvement over previous research on the subject.

Keywords Schwarz lemma · (α, β)-Harmonic mapping · Schwarz–Pick lemma · Weighted Green function · (α, β)-Harmonic equation

Mathematics Subject Classification Primary: 31A30; Secondary: 31A05 · 35J25

1 Introduction and Main Results

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle, and denote by

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

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For a differentiable function u on $\mathbb{D}$, we denote

$$\|Du(z)\| = |\partial_z u(z)| + |\partial_{\overline{z}} u(z)|.$$

1.1 (α, β)-Harmonic Functions

We consider $L_{\alpha,\beta}$, the family of differential operators on $\mathbb{D}$, defined by

$$L_{\alpha,\beta} := (1-|z|^2)^{-(\alpha+\beta+1)}\Big((1-|z|^2)\partial_z\partial_{\overline{z}} + \alpha z\partial_z + \beta\overline{z}\partial_{\overline{z}} - \alpha\beta\Big),$$

where α, β are real numbers. There operators are essentially introduced by Geller [9], see also [15]. In fact, Geller introduced the following family of operators on $\mathbb{D}$

$$\Delta_{\alpha,\beta} := (1-|z|^2)\Big((1-|z|^2)\partial_z\partial_{\overline{z}} + \alpha z\partial_z + \beta\overline{z}\partial_{\overline{z}} - \alpha\beta\Big) = (1-|z|^2)^{\alpha+\beta+2} L_{\alpha,\beta}.$$

We found that it is more convenient to work with the operators $L_{\alpha,\beta}$ than $\Delta_{\alpha,\beta}$ as for $\alpha = \beta = 0$, the operator $L_{0,0}$ is the classical Laplacian operator. In addition, it can be shown that

$$L_{\alpha,0} = \partial_{\overline{z}}\omega_\alpha(z)^{-1}\partial_z,$$

the operator for weighted harmonic functions with the standard weight $\omega_\alpha(z) = (1-|z|^2)^\alpha$ studied in [4, 8, 16, 22, 24]. Recall that a function u is α-harmonic if $L_{\alpha,0}u = 0$.

A function $u \in \mathcal{C}^2(\mathbb{D})$ is called (α, β)-harmonic if $L_{\alpha,\beta}(u) = 0$.

Let

$$T_\alpha := -\frac{\alpha^2}{4}\omega_{\alpha+1}^{-1} + \frac{1}{2}(L_{\alpha,0} + \overline{L_{\alpha,0}}),$$

that is,

$$T_\alpha = -\frac{\alpha^2}{4}(1-|z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1-|z|^2)^{-\alpha-1}\overline{z}\partial_{\overline{z}} + \frac{\alpha}{2}(1-|z|^2)^{-\alpha-1}z\partial_z$$
$$+(1-|z|^2)^{-\alpha}\partial_z\partial_{\overline{z}}.$$

Thus

$$T_\alpha = L_{\frac{\alpha}{2},\frac{\alpha}{2}}.$$

Recall that a function u is T_α-harmonic if $T_\alpha u = 0$. Remark that T_α-harmonic functions are exactly $(\frac{\alpha}{2}, \frac{\alpha}{2})$-harmonic functions, for more details, we refer the reader to [6, 13, 14, 23].

Let $g \in \mathcal{C}(\mathbb{D})$ and $u \in \mathcal{C}^2(\mathbb{D})$, we consider the associated Dirichlet boundary value problem

$$\begin{cases} L_{\alpha,\beta}\, u = g & \text{in} \quad \mathbb{D}, \\ u = f & \text{on} \quad \mathbb{T}. \end{cases} \tag{1.1}$$

If $\beta = 0$, Eq. (1.1) is the inhomogeneous α-harmonic equation studied in [4, 16, 17].

Theorem A [9] *The Dirichlet problem*

$$L_{\alpha,\beta}\, u = 0, \quad u = f \text{ on } \mathbb{T}, \quad f \in \mathcal{C}(\mathbb{T}),$$

has a solution for all $f \in \mathcal{C}(\mathbb{T})$ if and only if $\alpha + \beta > -1$ and $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Z}^-$. In this case the solution is unique and is given by

$$u(z) := \mathcal{P}_{\alpha,\beta}[f](z) = \frac{1}{2\pi} \int_0^{2\pi} P_{\alpha,\beta}(ze^{-i\gamma}) f(e^{i\gamma}) \mathrm{d}\gamma, \tag{1.2}$$

with

$$P_{\alpha,\beta}(z) = c_{\alpha,\beta} \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1-z)^{\alpha+1}(1-\overline{z})^{\beta+1}}, \quad c_{\alpha,\beta} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}. \tag{1.3}$$

Denote

$$\mathcal{P}_{\alpha}[f](z) := \mathcal{P}_{\frac{\alpha}{2},\frac{\alpha}{2}}[f](z) = \frac{1}{2\pi} \int_0^{2\pi} P_{\alpha}(ze^{-i\gamma}) f(e^{i\gamma}) \mathrm{d}\gamma, \tag{1.4}$$

with

$$P_{\alpha}(z) = c_{\alpha} \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}, \quad c_{\alpha} = \frac{\Gamma^2(\frac{\alpha}{2}+1)}{\Gamma(\alpha+1)}. \tag{1.5}$$

The Hardy theory of (α, β)-harmonic functions is studied in detail in [1]. The authors proved the following

Theorem B [1] *Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Z}^-$ such that $\alpha + \beta > -1$ and let u be an (α, β)-harmonic function. Then:*

(i) $u = \mathcal{P}_{\alpha,\beta}[f]$ *for some $f \in L^p(\mathbb{T})$, $1 < p < \infty$ if and only if*

$$\sup_{0<r<1} \int_0^{2\pi} |u(re^{i\theta})|^p \mathrm{d}\theta < +\infty.$$

(ii) $u = \mathcal{P}_{\alpha,\beta}[\mu]$ *for some measure μ if and only if*

$$\sup_{0<r<1} \int_0^{2\pi} |u(re^{i\theta})| \mathrm{d}\theta < +\infty.$$

1.2 Hypergeometric Functions and Some Inequalities

Let us recall the Gaussian hypergeometric function defined by

$$F(a,b;c;z)=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\qquad z\in\mathbb{D},$$

for $a,b,c\in\mathbb{C}$ such that $c\neq 0,-1,-2,\dots$ where

$$(a)_n=\begin{cases} a(a+1)\dots(a+n-1) & \text{for } \ n=1,2,3,\dots \\ 1 & \text{for } \ n=0, \end{cases}$$

which are called Pochhammer symbols.

We list few properties, see for instance ([3], Chapter 2)

$$\lim_{x\to 1}F(a,b;c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\qquad \text{if}\quad c-a-b>0, \tag{1.6}$$

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x), \tag{1.7}$$

$$\frac{d}{dx}F(a,b;c;x)=\frac{ab}{c}F(1+a,1+b;1+c;x). \tag{1.8}$$

$$F(a,b;a+b;x)\sim-\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\log(1-x)\quad(x\to 1^-), \tag{1.9}$$

The asymptotic relation (1.9) is due to Gauss, and its refined form is due to Ramanujan [5, p. 71].

$$F(a,b;a+b;x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}[R-\log(1-x)+O((1-x)\log(1-x))]\text{ as }x\to 1^-,$$

where $R=-\psi(a)-\psi(b)-2\gamma$, $\psi(a)=\frac{\Gamma'(a)}{\Gamma(a)}$, and γ denotes the Euler-Mascheroni constant.

For $\lambda>0$, let

$$I_\lambda(z):=\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\gamma}{|1-ze^{-i\gamma}|^\lambda},\quad z\in\mathbb{D}.$$

Using hypergeometric functions, one can see that

$$I_\lambda(z)\approx\begin{cases} 1, & \lambda<1; \\ \log\frac{1}{1-|z|^2}, & \text{if }\ \lambda=1; \\ (1-|z|^2)^{\lambda-1}, & \text{if }\ \lambda>1. \end{cases}$$

The notation $a(z)\approx b(z)$ means that the ratio $a(z)/b(z)$ is bounded from above and below by two positive constants as $|z|\to 1^-$.

Sharp estimates of I_λ are given in the following lemma.

Lemma 1.1 [13] *Let $\lambda > 0$. Then for all $z \in \mathbb{D}$, we have*

(i) *If $\lambda > 1$, then*

$$I_\lambda(z) \leq \frac{\Gamma(\lambda - 1)}{\Gamma(\lambda/2)^2}(1 - |z|^2)^{1-\lambda}.$$

(ii) *If $0 < \lambda < 1$, then*

$$I_\lambda(z) \leq \frac{\Gamma(1 - \lambda)}{\Gamma(1 - \lambda/2)^2}.$$

(iii) *If $\lambda = 1$, then*

$$I_1(z) \leq 1 + \frac{1}{\pi} \log\left(\frac{1}{1 - |z|^2}\right).$$

For $t > -1, \ c \in \mathbb{R}$, let

$$J_{c,t}(z) := \int_{\mathbb{D}} \frac{(1 - |\omega|^2)^t}{|1 - z\overline{\omega}|^{2+t+c}} \, dA(\omega).$$

where $\mathrm{d}A(w)$ is the normalized measure of the unit disc, given by $\mathrm{d}A(w) = \dfrac{\mathrm{d}u\mathrm{d}v}{\pi}$, $w = u + iv$.
Recall that the well-known Forelli–Rudin [26] estimates states that

$$J_{c,t}(t) \approx \begin{cases} 1, & \text{if } \ c < 0; \\ \log \frac{1}{1-|z|^2}, & \text{if } \ c = 0; \\ (1 - |z|^2)^{-c}, & \text{if } \ c > 0. \end{cases}$$

In [18], the author provided an estimate of the previous integral over the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ for $n \geq 1$. By taking $n = 1$, we get the following sharp estimates

Lemma 1.2 [18]

(i) *If $c < 0$, then for all $z \in \mathbb{D}$,*

$$\frac{\Gamma(1 + t)}{\Gamma(2 + t)} \leq J_{c,t}(z) \leq \frac{\Gamma(1 + t)\Gamma(-c)}{\Gamma^2(\frac{2+t-c}{2})}. \tag{1.10}$$

(ii) *If $c > 0$, then for all $z \in \mathbb{D}$,*

$$\frac{\Gamma(1 + t)}{\Gamma(2 + t)} \leq (1 - |z|^2)^c J_{c,t}(z) \leq \frac{\Gamma(1 + t)\Gamma(c)}{\Gamma^2(\frac{2+t+c}{2})}.$$

(iii) *If* $c = 0$, *then for all* $z \in \mathbb{D}$,

$$\frac{\Gamma(1+t)}{\Gamma^2(1+\frac{t}{2})} \leq |z|^2 \left(\log \frac{1}{1-|z|^2} \right)^{-1} J_{0,t}(z) \leq \frac{1}{1+t}. \tag{1.11}$$

The monotonicity of the hypergeometric function $F(a, b; a+b; x)$, $a, b > 0$ is studied in [2, Theorem 1.3], extending the complete elliptic integrals of the first kind. The authors proved the following

Lemma 1.3 [2, Theorem 1.3] *For* $a, b \in (0, \infty)$, *the function*

$$x \mapsto \frac{1 - F(a, b; a+b, x)}{\log(1-x)}$$

is increasing from $(0, 1)$ *into* $(ab/(a+b), 1/\mathrm{B(a, b)})$, *where* $\mathrm{B(a, b)}$ *is the Euler beta function.*

1.3 Green's Formula for $L_{\alpha,\beta}$

Let

$$g_{\alpha,\beta}(x) := \mathrm{B}(\alpha+1, \beta+1)(1-x)^{1+\alpha+\beta} F(1+\alpha, 1+\beta; 2+\alpha+\beta; 1-x), \tag{1.12}$$

where B is Euler beta function. In [1], the authors showed that $g_{\alpha,\beta}(|z|^2)$ is a radial (α, β)-harmonic away from zero and playing the role of the Green's function in the classical potential theory, and the weighted Green function $G_{\alpha,\beta}$ of the differential operator $L_{\alpha,\beta}$ could be written as

$$G_{\alpha,\beta}(z, w) := (1 - \overline{z}w)^{\alpha}(1 - z\overline{w})^{\beta} g_{\alpha,\beta}(|\varphi_z(w)|^2), \tag{1.13}$$

where φ_z is the Möbius transformation of the unit disc given by

$$\varphi_z(w) = \frac{z-w}{1-\overline{z}w}.$$

Remark that

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}.$$

The weighted potential of a function g can be represented by

$$\mathcal{G}_{\alpha,\beta}[g](z) = \int_{\mathbb{D}} G_{\alpha,\beta}(z, w) g(w) \mathrm{d}A(w). \tag{1.14}$$

Following Riesz-type decomposition formula [1], we see that all solutions $u \in \mathcal{C}^2(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ of (1.1) such that

$$\int_{\mathbb{D}} |g(w)|(1-|w|^2)^{\alpha+\beta+1} \mathrm{d}A(w) < +\infty, \tag{1.15}$$

are given by

$$u(z) = \mathcal{P}_{\alpha,\beta}[f](z) - \mathcal{G}_{\alpha,\beta}[g](z), \tag{1.16}$$

where $\mathcal{P}_{\alpha,\beta}[f]$ and $\mathcal{G}[g]$ are given, respectively, by (1.2) and (1.14). Clearly the condition (1.15) is satisfied if $u \in \mathcal{C}^2(\overline{\mathbb{D}})$.

In the case of $(0, \alpha)$-harmonic functions, Behm [4] showed that the weighted Green function G_α of $\Delta_{0,\alpha}$ could be written as

$$G_\alpha(z, w) := (1-\bar{z}w)^\alpha h\left(1 - |\varphi_z(w)|^2\right),$$

where

$$h(x) = \int_0^x \frac{t^\alpha}{1-t} \mathrm{d}t. \quad 0 \le x < 1.$$

Using the zero-balanced Gauss's hypergeometric function. One can see that

$$\begin{aligned} G_\alpha(z, w) = & \frac{1}{\alpha+1} \frac{(1-|z|^2)^{\alpha+1}(1-|w|^2)^{\alpha+1}}{(1-\bar{z}w)(1-z\overline{w})^{\alpha+1}} \\ & \times F\left(1, \alpha+1; \alpha+2; \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}\right). \end{aligned}$$

Hence $G_\alpha = G_{0,\alpha}$.

1.4 Schwarz and Schwarz–Pick Lemma

The Schwarz lemma for analytic functions plays a vital role in complex analysis, and it has been generalized in various settings; see [10, 13, 14, 16, 19–21] and the references therein.

Heinz [10] generalized it to the class of complex-valued harmonic functions. That is, if u is a complex-valued harmonic function from $\mathbb{D}$ into itself with $u(0) = 0$, then for $z \in \mathbb{D}$,

$$|u(z)| \le \frac{4}{\pi} \arctan |z|.$$

Moreover, this inequality is sharp for each point $z \in \mathbb{D}$.

Hethcote [11] and Pavlović [25] improved the above result of Heinz by removing the assumption $u(0) = 0$ and showed that for harmonic function u from $\mathbb{D}$ to $\mathbb{D}$, then

$$\left|u(z) - \frac{1-|z|^2}{1+|z|^2}u(0)\right| \le \frac{4}{\pi}\arctan|z|, \tag{1.17}$$

holds for all $z \in \mathbb{D}$.
Recently, Chen and Kalaj [7] established a Heinz-Hethcote type theorem for the solutions of the Dirichlet boundary value problem of the laplacian operator. In [13], we established a Heinz-Hethcote theorem for T_α-harmonic functions.
Let $\alpha > -1$, define

$$U_\alpha(z) := \mathcal{P}_{\frac{\alpha}{2},\frac{\alpha}{2}}[\chi_{\mathbb{T}^r} - \chi_{\mathbb{T}^l}](z), \tag{1.18}$$

where

$$\mathbb{T}^r = \{z \in \mathbb{T} : \operatorname{Re} z > 0\}, \text{ and } \mathbb{T}^l = \{z \in \mathbb{T} : \operatorname{Re} z < 0\}.$$

Notice that U_α is a T_α-harmonic function on $\mathbb{D}$ with values in $(-1, 1)$ such that $U_\alpha(0) = 0$.

Theorem C *Let $\alpha > -1$ and $u : \mathbb{D} \longrightarrow \mathbb{D}$ be a T_α-harmonic function, then*

$$\left|u(z) - \frac{(1-|z|^2)^{\alpha+1}}{(1+|z|^2)^{\frac{\alpha}{2}+1}}u(0)\right| \le U_\alpha(|z|),$$

for all $z \in \mathbb{D}$, where U_α is the function defined in (1.18).

This theorem extends the estimate (1.17), indeed, for $\alpha = 0$, we have $U_0(|z|) = \frac{4}{\pi}\arctan|z|$. Recently, a Heinz-Hethcote type theorem is proved for α-harmonic functions, see [12].

Theorem D [12] *Let $\alpha > -1$ and $u : \mathbb{D} \to \mathbb{D}$ be an α-harmonic function. Then*

(1) *If $\alpha \ge 0$, then*

$$\left|u(z) - \frac{(1-|z|^2)^{\alpha+1}}{1+|z|^2}u(0)\right| \le \frac{2^{\alpha+2}}{\pi}\arctan|z| + 2^{\alpha+1}(1-|z|)\left(1-(1-|z|)^\alpha\right).$$

(2) *If $\alpha < 0$, then*

$$\left|u(z) - \frac{(1-|z|^2)^{\alpha+1}}{1+|z|^2}u(0)\right| \le \frac{4}{\pi}(1-|z|)^\alpha\arctan|z| + \left((1-|z|)^\alpha - 1\right).$$

Other variants of Schwarz lemma for α-harmonic functions are considered in Li and Chen [16] for mappings u in $\mathbb{D}$ satisfying the α-harmonic equation $L_{\alpha,0}u = g$, extending previous results of Li et al. [17].

Here, we should point out that the inequalities obtained in the case $\alpha < 0$ are not convenient due to the factor $(1-|z|)^\alpha$ which goes to infinity as $|z| \to 1$.

In this paper, we extend and improve the above estimates and obtain a Schwarz type lemma for solutions to the (α, β)-harmonic equation (1.1). Our first main result is the following theorem.

Theorem 1.1 *Suppose that $g \in \mathcal{C}(\overline{\mathbb{D}})$ and $f \in \mathcal{C}(\mathbb{T})$. If $u \in \mathcal{C}^2(\mathbb{D})$ satisfies the (α, β)-harmonic equation (1.1) for $\alpha, \beta \in (-1, \infty)$ such that $\alpha + \beta > -1$, for $z \in \mathbb{D}$,*

(i) *If $\alpha + \beta > 0$, then*

$$\left| u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{1+|z|^2} \mathcal{P}_{\alpha,\beta}[f](0) \right| \\ \leq 2^{\alpha+\beta+1} |c_{\alpha,\beta}| \left[\frac{2}{\pi} \arctan |z| + \frac{|\alpha| + |\beta|}{\alpha+\beta} (1-|z|) \left(1 - (1-|z|)^{\alpha+\beta}\right) \right] \|f\|_\infty \\ + d_{\alpha,\beta} (1-|z|^2)^{\alpha+\beta+1} \|g\|_\infty.$$

(ii) *If $\alpha + \beta = 0$, then*

$$\left| u(z) - \frac{1-|z|^2}{1+|z|^2} \mathcal{P}_{\alpha,\beta}[f](0) \right| \leq |c_{\alpha,-\alpha}| \left[\frac{4}{\pi} \arctan |z| + \frac{|\alpha|\pi}{4} (1-|z|) \right] \|f\|_\infty \\ + d_{\alpha,-\alpha} (1-|z|^2) \|g\|_\infty.$$

(iii) *If $\alpha + \beta < 0$, then*

$$\left| u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1+|z|^2)^{\frac{\alpha+\beta}{2}+1}} \mathcal{P}_{\alpha,\beta}[f](0) \right| \\ \leq |c_{\alpha,\beta}| \left[\frac{U_{\alpha+\beta}(|z|)}{c_{\alpha+\beta}} + \frac{|\alpha-\beta|\pi}{4} (1-|z|^2)^{\alpha+\beta+1} \right] \|f\|_\infty \\ + d_{\alpha,\beta} (1-|z|^2)^{\alpha+\beta+1} \|g\|_\infty,$$

where $\|f\|_\infty = \sup_{\xi \in \mathbb{T}} |f(\xi)|$, $\|g\|_\infty = \sup_{z \in \mathbb{D}} |g(z)|$, $d_{\alpha,\beta} := 2^{|\alpha+\beta|-2} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma^2(\frac{\alpha+\beta+4}{2})}$, *$U_{\alpha+\beta}$ is defined by (1.18) and $c_{\alpha,\beta}$, $c_{\alpha+\beta}$ are defined in (1.3) and (1.5).*

Next, we give the Schwarz–Pick inequality for the solutions of the (α, β)-harmonic equation extending [16, Theorem 1.2]

Theorem 1.2 *Suppose that $g \in \mathcal{C}(\mathbb{D})$ and $f \in \mathcal{C}(\mathbb{T})$. If $u \in \mathcal{C}^2(\mathbb{D})$ satisfies the (α, β)-harmonic equation (1.1) for $\alpha, \beta \in (-1, \infty)$ such that $\alpha + \beta > -1$, then for $z \in \mathbb{D}$,*

$$\|Du(z)\| \leq \tau_{\alpha,\beta} \frac{\|f\|_\infty}{1-|z|^2} + \sigma_{\alpha,\beta} (1-|z|^2)^{\alpha+\beta} \|g\|_\infty,$$

where

$$\tau_{\alpha,\beta} := (|\alpha| + |\alpha+1| + |\beta| + |\beta+1|) \frac{|c_{\alpha,\beta}|}{c_{\alpha+\beta}},$$

and

$$\sigma_{\alpha,\beta} := 5\,(|\alpha| + |\beta|)\,2^{|\alpha+\beta|-1} + 2^{\alpha+\beta+2}(\alpha+\beta+1))\mathrm{B}(\alpha+1, \beta+1)$$
$$+2^{\alpha+\beta+2}(\alpha+\beta+2).$$

2 Schwarz Lemma for (α, β)-Harmonic Functions

To prove Schwarz lemma, we will distinguish two cases:

2.1 Case $\alpha + \beta \geq 0$

In this case, we write the generalized Poisson kernel $P_{\alpha,\beta}$ in the following form:

$$P_{\alpha,\beta}(z) = h_{\alpha,\beta}(z)P(z),$$

where

$$h_{\alpha,\beta}(z) := c_{\alpha,\beta}\frac{(1-|z|^2)^{\alpha+\beta}}{(1-z)^{\alpha}(1-\overline{z})^{\beta}},$$

and P is the Poisson kernel. As $\alpha + \beta \geq 0$, we have

$$\|h_{\alpha,\beta}(r)\|_{\infty} := \sup_{0\leq\theta\leq 2\pi} |h_{\alpha,\beta}(re^{i\theta})| \leq |c_{\alpha,\beta}|2^{\alpha+\beta}. \tag{2.1}$$

Let u be an (α, β)-harmonic mapping from the unit disc to itself. Then, we can write

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{\alpha,\beta}(ze^{-i\theta})f(e^{i\theta})\mathrm{d}\theta,$$

where f is the boundary function of u.
Let

$$H_{\alpha,\beta}(z) = h_{\alpha,\beta}[f](z) = \frac{1}{2\pi}\int_0^{2\pi} h_{\alpha,\beta}(ze^{-i\theta})f(e^{i\theta})\mathrm{d}\theta. \tag{2.2}$$

As in [12], we prove the following lemma.

Lemma 2.1 *Let $\alpha, \beta \in \mathbb{R}\setminus\mathbb{Z}^-$ such that $\alpha+\beta \geq 0$ and u be an (α, β)-harmonic function from the unit disc to itself. Then*

$$\left|u(z) - \frac{1-|z|^2}{1+|z|^2}H_{\alpha,\beta}(z)\right| \leq \frac{|c_{\alpha,\beta}|2^{\alpha+\beta+2}}{\pi}\arctan|z|.$$

Proof We have

$$\begin{aligned}\left|u(z)-\frac{1-|z|^2}{1+|z|^2}H_{\alpha,\beta}(z)\right| &\le \frac{1}{2\pi}\int_0^{2\pi}\left|P(ze^{-i\theta})-\frac{1-|z|^2}{1+|z|^2}\right||h_{\alpha,\beta}(ze^{-i\theta})|f(e^{i\theta})|\mathrm{d}\theta.\\ &\le \frac{4}{\pi}\|h_{\alpha,\beta}(|z|)\|_\infty \arctan|z|,\end{aligned}$$

and the conclusion follows from (2.1) and (1.17). □

Next, we prove

Lemma 2.2 *Let* $\alpha,\beta\in\mathbb{R}$ *and* $r\in[0,1)$. *Then*

(1) *If* $\alpha+\beta\neq 0$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi}|(1-re^{it})^{-\alpha}(1-re^{-it})^{-\beta}-1|\,\mathrm{d}t\le\frac{|\alpha|+|\beta|}{|\alpha+\beta|}\left|(1-r)^{-\alpha-\beta}-1\right|.$$

(2) *If* $\alpha+\beta=0$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi}|(1-re^{it})^{-\alpha}(1-re^{-it})^{-\beta}-1|\,\mathrm{d}t\le\frac{|\alpha|\pi}{4}.$$

Proof Let

$$g(r,t)=(1-re^{it})^{-\alpha}(1-re^{-it})^{-\beta}.$$

Differentiating g with respect r, we get

$$\partial_r g(r,t)=(1-re^{it})^{-\alpha-1}(1-re^{-it})^{-\beta-1}(\alpha e^{it}+\beta e^{-it}-r(\alpha+\beta)).$$

(1) For $\alpha+\beta\neq 0$, we have

$$|\partial_r g(r,t)|\le(|\alpha|+|\beta|)|1-re^{it}|^{-(\alpha+\beta)-1}\le(|\alpha|+|\beta|)(1-r)^{-(\alpha+\beta)-1}.$$

Therefore, we have

$$g(r,t)-g(0,t)=\int_0^r\partial_x g(x,t)\mathrm{d}x.$$

Then

$$\begin{aligned}|g(r,t)-g(0,t)|&\le\int_0^r|\partial_x g(x,t)|\mathrm{d}x\\ &\le\int_0^r(|\alpha|+|\beta|)(1-x)^{-(\alpha+\beta)-1}\mathrm{d}x\\ &=\frac{|\alpha|+|\beta|}{\alpha+\beta}((1-r)^{-(\alpha+\beta)}-1).\end{aligned}$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}|(1-re^{it})^{-\alpha}(1-re^{-it})^{-\beta}-1|\mathrm{d}t \leq \frac{|\alpha|+|\beta|}{|\alpha+\beta|}|(1-r)^{-\alpha-\beta}-1|.$$

(2) For $\alpha+\beta=0$, we have

$$|\partial_r g(r,t)| \leq \frac{2|\alpha||\sin t|}{|1-re^{it}|^2} = \frac{2|\alpha||\sin t|}{1-2r\cos t+r^2}.$$

Thus

$$|g(r,t)-g(0,t)| \leq 2|\alpha||\sin t|\int_0^1 \frac{\mathrm{d}x}{1-2x\cos t+x^2}.$$

One can check the following two integrals

$$\int_0^{2\pi}\frac{|\sin t|}{1-2x\cos t+x^2}\mathrm{d}t = \frac{1}{x}\ln\left(\frac{1+x}{1-x}\right),$$
$$\int_0^1 \frac{1}{x}\ln\left(\frac{1+x}{1-x}\right)\mathrm{d}x = \mathrm{Li}_2(1)-\mathrm{Li}_2(-1) = \frac{\pi^2}{4},$$

where Li_2 is the polylogarithm function. By Fubini theorem, it yields

$$\frac{1}{2\pi}\int_0^{2\pi}|(1-re^{it})^{-\alpha}(1-re^{-it})^{\alpha}-1|\mathrm{d}t \leq \frac{|\alpha|\pi}{4}.$$

□

We establish the following Schwarz lemma for (α,β)-harmonic functions in the case $\alpha+\beta\geq 0$.

Theorem 2.1 *Let $\alpha,\beta\in\mathbb{R}\setminus\mathbb{Z}^-$ such that $\alpha+\beta\geq 0$ and u be an (α,β)-harmonic function from the unit disc $\mathbb{D}$ into itself, then*

(1) *If $\alpha+\beta>0$, then*

$$\left|u(z)-\frac{(1-|z|^2)^{\alpha+\beta+1}}{1+|z|^2}u(0)\right| \leq \frac{|c_{\alpha,\beta}|2^{\alpha+\beta+2}}{\pi}\arctan|z| + \frac{2^{\alpha+\beta+1}|c_{\alpha,\beta}|(|\alpha|+|\beta|)}{\alpha+\beta}(1-|z|)\left(1-(1-|z|)^{\alpha+\beta}\right). \tag{2.3}$$

(2) *If $\alpha+\beta=0$, then*

$$\left|u(z)-\frac{1-|z|^2}{1+|z|^2}u(0)\right| \leq \frac{4}{\pi}|c_{\alpha,-\alpha}|\arctan|z| + \frac{|\alpha|\pi}{4}(1-|z|^2).$$

Proof Let

$$\Psi_{\alpha,\beta}(z) := \frac{H_{\alpha,\beta}(z)}{(1-|z|^2)^{\alpha+\beta}} = \frac{c_{\alpha,\beta}}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{(1-ze^{-i\theta})^{\alpha}(1-\overline{z}e^{i\theta})^{\beta}}\mathrm{d}\theta,$$

where $H_{\alpha,\beta}$ is defined by (2.2) and

$$\Phi_{\alpha,\beta}(z) := \Psi_{\alpha,\beta}(z) - u(0) = \frac{c_{\alpha,\beta}}{2\pi}\int_0^{2\pi} \left((1-ze^{-i\theta})^{-\alpha}(1-\overline{z}e^{i\theta})^{-\beta} - 1\right) f(e^{i\theta})\mathrm{d}\theta. \tag{2.4}$$

It yields that

$$u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{1+|z|^2}u(0) = \left(u(z) - \frac{1-|z|^2}{1+|z|^2}H_{\alpha,\beta}(z)\right) + \frac{(1-|z|^2)^{\alpha+\beta+1}}{1+|z|^2}\Phi_{\alpha,\beta}(z).$$

Since $\alpha+\beta \geq 0$, by Lemma 2.1, we get

$$\left|u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{1+|z|^2}u(0)\right| \leq \frac{|c_{\alpha,\beta}|2^{\alpha+\beta+2}}{\pi}\arctan|z| + (1-|z|^2)^{\alpha+\beta+1}|\Phi_{\alpha,\beta}(z)|,$$

and the conclusion follows from Lemma 2.2 to estimate $\Phi_{\alpha,\beta}$. □

2.2 Case $\alpha+\beta \in (-1,0)$

In the case $-1 < \alpha+\beta < 0$, we write the kernel $P_{\alpha,\beta}$ in the following form

$$P_{\alpha,\beta}(z) = \frac{k_{\alpha,\beta}(z)}{c_{\alpha+\beta}}P_{\alpha+\beta}(z), \tag{2.5}$$

where

$$k_{\alpha,\beta}(z) := c_{\alpha,\beta}(1-z)^{\frac{\beta-\alpha}{2}}(1-\overline{z})^{\frac{\alpha-\beta}{2}} \tag{2.6}$$

and $P_{\alpha+\beta}$ is the Poisson kernel for $T_{\alpha+\beta}$-harmonic functions defined by equations (1.4) and (1.5). Remark that $|k_{\alpha,\beta}(z)| = |c_{\alpha,\beta}|$.
Let

$$K_{\alpha,\beta}(z) := k_{\alpha,\beta}[f](z) = \frac{1}{2\pi}\int_0^{2\pi} k_{\alpha,\beta}(ze^{-i\theta})f(e^{i\theta})\mathrm{d}\theta.$$

First, we prove the following lemma.

Lemma 2.3 *Let $\alpha, \beta \in \mathbb{R}\backslash\mathbb{Z}^-$ such that $\alpha+\beta \in (-1,0)$ and u be an (α,β)-harmonic function from the unit disc to itself. Then*

$$\left|u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1+|z|^2)^{\frac{\alpha+\beta}{2}+1}}K_{\alpha,\beta}(z)\right| \leq \frac{|c_{\alpha,\beta}|}{c_{\alpha+\beta}}U_{\alpha+\beta}(|z|).$$

We omit the proof as it is similar to Lemma 2.1 and uses Theorem C, a Heinz-Hethcote theorem of T_α-harmonic functions.
Next, we show

Theorem 2.2 *Let* $\alpha, \beta \in \mathbb{R}\backslash\mathbb{Z}^-$ *such that* $\alpha+\beta \in (-1, 0)$ *and* u *be an* (α, β)*-harmonic function from the unit disc to itself. Then*

$$\begin{aligned}&\left|u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1+|z|^2)^{\frac{\alpha+\beta}{2}+1}}u(0)\right| \\ &\quad\le |c_{\alpha,\beta}|\left[\frac{1}{c_{\alpha+\beta}}U_{\alpha+\beta}(|z|) + \frac{|\alpha-\beta|\pi}{4}(1-|z|^2)^{\alpha+\beta+1}\right].\end{aligned}$$

Proof Using the triangle inequality, we have

$$\begin{aligned}\left|u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1+|z|^2)^{\frac{\alpha+\beta}{2}+1}}u(0)\right| &\le \left|u(z) - \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1+|z|^2)^{\frac{\alpha+\beta}{2}+1}}K_{\alpha,\beta}(z)\right| \\ &\quad + (1-|z|^2)^{\alpha+\beta+1}\left|K_{\alpha,\beta}(z) - u(0)\right|. \end{aligned} \tag{2.7}$$

We observe that

$$|\Phi_{\frac{\beta-\alpha}{2},\frac{\alpha-\beta}{2}}(z)| = \left|K_{\alpha,\beta}(z) - u(0)\right|,$$

where $\Phi_{\alpha,\beta}$ is given by (2.4). Thus we can use the second case in Lemma 2.2 to obtain

$$\left|K_{\alpha,\beta}(z) - u(0)\right| \le |c_{\alpha,\beta}|\frac{|\alpha-\beta|\pi}{4}. \tag{2.8}$$

Then, with an immediate consequence from Lemma 2.3 and the inequality (2.8) we obtain the desired result. □

2.3 Estimates of $\mathcal{G}_{\alpha,\beta}[g]$ and Its Derivatives

Lemma 2.4 *Let* $\gamma \in \mathbb{R}$ *and* z, w *in* $\mathbb{D}$*. Then*

$$\frac{(1-|z|^2)(1-|w|^2)^{\gamma+1}}{|1-\overline{z}w|^{\gamma+2}} \le 2^{|\gamma|}.$$

Proof Let us denote by $F_\gamma(z, w) := \dfrac{(1-|z|^2)(1-|w|^2)^{\gamma+1}}{|1-\overline{z}w|^{\gamma+2}}$.

If $\gamma \ge 0$, then

$$F_\gamma(z, w) = (1-|\varphi_w(z)|^2)\frac{(1-|w|^2)^\gamma}{|1-\overline{z}w|^\gamma} \le (1+|w|)^\gamma \le 2^\gamma.$$

If $\gamma < 0$, then

$$F_\gamma(z,w) = (1-|\varphi_w(z)|^2)^{\gamma+1}\frac{|1-\bar{z}w|^\gamma}{(1-|z|^2)^\gamma} \le (1+|z|)^{-\gamma} \le 2^{-\gamma}.$$

□

First, we estimate the Green functions $g_{\alpha,\beta}$ and $G_{\alpha,\beta}$.

Lemma 2.5 *Let $\alpha, \beta \in (-1,\infty)$ such that $\alpha+\beta > -1$. Then, the functions $g_{\alpha,\beta}$ and $G_{\alpha,\beta}$ satisfy the estimates*

$$0 \le g_{\alpha,\beta}(x) \le (1-x)^{\alpha+\beta+1}\left(\mathrm{B}(\alpha+1,\beta+1) + \log\frac{1}{x}\right), \quad x \in (0,1]. \tag{2.9}$$

and

$$|G_{\alpha,\beta}(z,w)| \le \frac{(1-|z|^2)^{\alpha+\beta+1}(1-|w|^2)^{\alpha+\beta+1}}{|1-\bar{z}w|^{\alpha+\beta+2}}\left(\log\left|\frac{1-\bar{z}w}{z-w}\right|^2 + \mathrm{B}(\alpha+1,\beta+1)\right). \tag{2.10}$$

The estimates (2.9) and (2.10) extend [16, Lemma B] and [16, Lemma 4.2], respectively.

Proof From Lemma 1.3, we observe that the function

$$x \mapsto \frac{F(\alpha+1,\beta+1;\alpha+\beta+2,x)-1}{\log\frac{1}{1-x}}$$

is increasing from $(0,1)$ into $\left(\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+2}, \frac{1}{\mathrm{B}(\alpha+1,\beta+1)}\right)$. Then

$$F(\alpha+1,\beta+1;\alpha+\beta+2,x) \le \frac{1}{\mathrm{B}(\alpha+1,\beta+1)}\log\frac{1}{1-x} + 1. \tag{2.11}$$

Hence

$$g_{\alpha,\beta}(x) \le (1-x)^{\alpha+\beta+1}\left(\log\frac{1}{x} + \mathrm{B}(\alpha+1,\beta+1)\right).$$

The estimate of $G_{\alpha,\beta}$ follows immediately from (2.9)

Remark The inequalities (2.9) and (2.10) hold for $\alpha, \beta \in \mathbb{R}\backslash\mathbb{Z}^-$ such that $\alpha+\beta > -1$ where the constant $\frac{1}{\mathrm{B}(\alpha+1,\beta+1)}$ in (2.11) should be replaced by

$$M_{\alpha,\beta} := \sup_{x\in(0,1)} \frac{F(\alpha+1,\beta+1;\alpha+\beta+2,x)-1}{\log\frac{1}{1-x}}.$$

$M_{\alpha,\beta}$ is finite as the function $\dfrac{F(\alpha+1,\beta+1;\alpha+\beta+2,x)-1}{\log\frac{1}{1-x}}$ is continuous on $(0,1)$ having finite limits at 0 and 1, see (1.9). Lemma 1.3 says simply that if $\alpha,\beta>-1$ and $\alpha+\beta>-1$, then $M_{\alpha,\beta}=\frac{1}{\mathrm{B}(\alpha+1,\beta+1)}$.

Proposition 2.1 *Let $\alpha,\beta\in(-1,\infty)$ such that $\alpha+\beta>-1$ and $g\in\mathcal{C}(\overline{\mathbb{D}})$. Then*

$$\left|\mathcal{G}_{\alpha,\beta}[g](z)\right|\leq d_{\alpha,\beta}(1-|z|^2)^{\alpha+\beta+1}\|g\|_\infty,$$

where

$$d_{\alpha,\beta}:=2^{|\alpha+\beta|-2}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma^2(\frac{\alpha+\beta+4}{2})}.$$

Proof Combining (2.10) and Lemma 2.4, we get

$$\begin{aligned}|G_{\alpha,\beta}(z,w)|\leq{}&2^{|\alpha+\beta|}(1-|z|^2)^{\alpha+\beta}\left(\log\left|\frac{1-\overline{z}w}{z-w}\right|^2\right)\\&+\mathrm{B}(\alpha+1,\beta+1)\frac{(1-|z|^2)^{\alpha+\beta+1}(1-|w|^2)^{\alpha+\beta+1}}{|1-\overline{z}w|^{\alpha+\beta+2}}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\mathcal{G}_{\alpha,\beta}[g](z)\right|&\leq\int_{\mathbb{D}}|G_{\alpha,\beta}(z,w)||g(w)|\mathrm{d}A(w)\\&\leq\|g\|_\infty2^{|\alpha+\beta|}(1-|z|^2)^{\alpha+\beta}\int_{\mathbb{D}}\log\left|\frac{1-\overline{z}w}{z-w}\right|^2\mathrm{d}A(w)\\&\quad+\|g\|_\infty\mathrm{B}(\alpha+1,\beta+1)(1-|z|^2)^{\alpha+\beta+1}\int_{\mathbb{D}}\frac{(1-|w|^2)^{\alpha+\beta+1}}{|1-\overline{z}w|^{\alpha+\beta+2}}\mathrm{d}A(w).\end{aligned}$$

Let denote by

$$\mathcal{I}:=\int_{\mathbb{D}}\log\left|\frac{1-z\overline{w}}{z-w}\right|^2\mathrm{d}A(w)$$

and

$$\mathcal{J}:=\int_{\mathbb{D}}\frac{(1-|w|^2)^{\alpha+\beta+1}}{|1-\overline{z}w|^{\alpha+\beta+2}}\mathrm{d}A(w).$$

As $\mathcal{I}$ is the Green function of the Laplacian operator, we deduce that

$$\mathcal{I}=\frac{(1-|z|^2)}{4}. \tag{2.12}$$

Now we estimate $\mathcal{J}$. By using the estimate (1.10) in Theorem 1.2 for $t = \alpha + \beta + 1$ and $c = -1$, we have

$$\mathcal{J} = J_{-1,\alpha+\beta+1} \leq \frac{\Gamma(\alpha+\beta+2)}{\Gamma^2(\frac{\alpha+\beta+4}{2})},$$

and, we reach our conclusion. □

Next, we estimate $|\partial_z G_{\alpha,\beta}(z,w)|$.

Lemma 2.6 *Let $\alpha, \beta \in (-1, \infty)$ such that $\alpha + \beta > -1$. Then*

$$|\partial_z G_{\alpha,\beta}(z,w)| \leq |\beta| 2^{|\alpha+\beta|+1}(1-|z|^2)^{\alpha+\beta-1} \log\left|\frac{1-\overline{z}w}{z-w}\right|^2 + \gamma_{\alpha,\beta}\frac{(1-|z|^2)^{\alpha+\beta}}{|z-w|}.$$

where

$$\gamma_{\alpha,\beta} := (|\beta| 2^{|\alpha+\beta|} + 2^{\alpha+\beta+1}(\alpha+\beta+1))\mathrm{B}(\alpha+1,\beta+1) + 2^{\alpha+\beta+1}(\alpha+\beta+2). \quad (2.13)$$

Proof Using the chain rule and (1.8), we get

$$\partial_z G_{\alpha,\beta}(z,w) = -\beta\overline{w}\frac{G_{\alpha,\beta}(z,w)}{1-z\overline{w}} + l(z,w)H\left(1-|\varphi_w(z)|^2\right),$$

where

$$l(z,w) := \frac{(1-|z|^2)^{\alpha+\beta}(1-|w|^2)^{\alpha+\beta+1}(\overline{w}-\overline{z})}{(1-\overline{z}w)^{\beta+1}(1-z\overline{w})^{\alpha+2}},$$

and

$$H(x) = \mathrm{B}(\alpha+1,\beta+1)\Big((\alpha+\beta+1)F(\alpha+1,\beta+1;\alpha+\beta+2;x)\Big) + \mathrm{B}(\alpha+1,\beta+1)\Big(\frac{x(\alpha+1)(\beta+1)}{\alpha+\beta+2}F(\alpha+2,\beta+2;\alpha+\beta+3;x)\Big).$$

Claim:

$$H(x) \leq (\alpha+\beta+2)\frac{x}{1-x} + (\alpha+\beta+1)\mathrm{B}(\alpha+1,\beta+1). \quad (2.14)$$

Indeed, using (1.7), we have

$$F(\alpha+2,\beta+2;\alpha+\beta+3;x) = \frac{1}{1-x}F(\alpha+1,\beta+1;\alpha+\beta+3;x).$$

As the function $F(\alpha+1,\beta+1;\alpha+\beta+3;.)$ is increasing on $(0,1)$, then by (1.6), we have

$$\begin{aligned}\frac{x(\alpha+1)(\beta+1)}{\alpha+\beta+2}&F(\alpha+2,\beta+2;\alpha+\beta+3;x)\\&\le\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+2}\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)}\frac{x}{1-x}\\&=\frac{1}{\mathrm{B}(\alpha+1,\beta+1)}\frac{x}{1-x}.\end{aligned}\tag{2.15}$$

On the other hand, by Lemma 1.3, we have

$$F(\alpha+1,\beta+1;\alpha+\beta+2;x)\le\left(\frac{1}{\mathrm{B}(\alpha+1,\beta+1)}\log\frac{1}{1-x}+1\right).\tag{2.16}$$

Hence, by combining (2.15) and (2.16), we obtain

$$H(x)\le(\alpha+\beta+1)\Big(\log\frac{1}{1-x}+\mathrm{B}(\alpha+1,\beta+1)\Big)+\frac{x}{1-x}.$$

Using $\log(t)\le t-1$ for all $t\ge 1$, one can see that $\log\dfrac{1}{1-x}\le\dfrac{x}{1-x}$ for all $x\in[0,1)$. Thus the proof of the claim is complete.
It follows from the inequality (2.10) and Lemma 2.4 that

$$\begin{aligned}\left|-\beta\overline{w}\frac{G_{\alpha,\beta}(z,w)}{1-z\overline{w}}\right|&\le|\beta|\frac{(1-|z|^2)^{\alpha+\beta+1}(1-|w|^2)^{\alpha+\beta+1}}{|1-z\overline{w}|^{\alpha+\beta+3}}\\&\qquad\left(\log\left|\frac{1-\overline{z}w}{z-w}\right|^2+\mathrm{B}(\alpha+1,\beta+1)\right)\\&\le|\beta|2^{|\alpha+\beta|+1}(1-|z|^2)^{\alpha+\beta-1}\log\left|\frac{1-\overline{z}w}{z-w}\right|^2\\&\qquad+|\beta|2^{|\alpha+\beta|}\mathrm{B}(\alpha+1,\beta+1)\frac{(1-|z|^2)^{\alpha+\beta}}{|1-z\overline{w}|}.\end{aligned}\tag{2.17}$$

Also, we have

$$\begin{aligned}|l(z,w)|&=\frac{(1-|z|^2)^{\alpha+\beta}(1-|w|^2)^{\alpha+\beta+1}|z-w|}{|1-z\overline{w}|^{\alpha+\beta+3}}\\&\le\frac{(1-|z|^2)^{\alpha+\beta}}{|1-z\overline{w}|}\left(\frac{1-|w|^2}{|1-z\overline{w}|}\right)^{\alpha+\beta+1}\left|\frac{z-w}{1-z\overline{w}}\right|\\&\le 2^{\alpha+\beta+1}\frac{(1-|z|^2)^{\alpha+\beta}}{|1-z\overline{w}|}.\end{aligned}\tag{2.18}$$

By the claim (2.14), we have

$$H\left(1-|\varphi(z,w)|^2\right) \le (\alpha+\beta+2)\frac{(1-|z|^2)(1-|w|^2)}{|z-w|^2} + (\alpha+\beta+1)\mathrm{B}(\alpha+1,\beta+1).$$

Therefore,

$$|l(z,w)|H(1-|\varphi(z,w)|^2) \le 2^{\alpha+\beta+1}(\alpha+\beta+2)\frac{(1-|z|^2)^{\alpha+\beta}}{|z-w|} + 2^{\alpha+\beta+1}(\alpha+\beta+1)\mathrm{B}(\alpha+1,\beta+1)\frac{(1-|z|^2)^{\alpha+\beta}}{|1-z\overline{w}|}.$$

The proof of the lemma is complete. □

Theorem E [27] *Suppose that X is an open subset of $\mathbb{R}$, and Ω is a measure space. Suppose, further, that a function $F : X \times \Omega \to \mathbb{R}$ satisfies the following conditions:*

(1) *$F(x,w)$ is a measurable function of x and w jointly, and is integrable with respect to w for almost every $x \in X$.*
(2) *For almost every $w \in \Omega$, $F(x,w)$ is an absolutely continuous function with respect to x. [This guarantees that $\frac{\partial F(x,w)}{\partial x}$ exists almost everywhere.]*
(3) *$\frac{\partial F}{\partial x}$ is locally integrable, that is, for all compact intervals $[a,b]$ contained in X:*

$$\int_a^b \int_\Omega \left|\frac{\partial}{\partial x}F(x,w)\right| \mathrm{d}w\mathrm{d}x < \infty.$$

Then, $\int_\Omega F(x,w)\mathrm{d}w$ is an absolutely continuous function with respect to x, and for almost every $x \in X$, its derivative exists, which is given by

$$\frac{\partial}{\partial x}\int_\Omega F(x,w)\mathrm{d}w = \int_\Omega \frac{\partial}{\partial x}F(x,w)\mathrm{d}w.$$

Proposition 2.2 *Let $\alpha, \beta \in (-1,\infty)$ such that $\alpha+\beta > -1$ and $g \in \mathcal{C}(\overline{\mathbb{D}})$. Then*

$$\left|D\mathcal{G}[g](z)\right| \le \delta_{\alpha,\beta}(1-|z|^2)^{\alpha+\beta}\|g\|_\infty.$$

where $\delta_{\alpha,\beta} = 2^{|\alpha+\beta|-1}(|\alpha|+|\beta|) + 2(\gamma_{\alpha,\beta}+\gamma_{\beta,\alpha})$ and $\gamma_{\alpha,\beta}$ is defined in Eq. (2.13).

Proof Using Lemma (2.6), we have

$$\int_{\mathbb{D}} |\partial_z G_{\alpha,\beta}(z,w)|\mathrm{d}A(w) \le |\beta|2^{|\alpha+\beta|+1}(1-|z|^2)^{\alpha+\beta-1}\int_{\mathbb{D}} \log\left|\frac{1-\overline{z}w}{z-w}\right|^2 \mathrm{d}A(w) + \gamma_{\alpha,\beta}(1-|z|^2)^{\alpha+\beta}\int_{\mathbb{D}} \frac{1}{|z-w|}\mathrm{d}A(w).$$

Using [28, proof of theorem 1.1], we have

$$\int_{\mathbb{D}} \frac{1}{|z-w|} \mathrm{d}A(w) \leq 2,$$

and by (2.12), it yields

$$\int_{\mathbb{D}} \log \left| \frac{1-z\overline{w}}{z-w} \right|^2 \mathrm{d}A(w) = \frac{1-|z|^2}{4}.$$

Thus $\partial_z G_{\alpha,\beta}(z,w)$ is integrable on $\mathbb{D} \times \mathbb{D}$ and by Theorem E, we have

$$\partial_z \mathcal{G}[g](z) = \int_{\mathbb{D}} \partial_z G_{\alpha,\beta}(z,w) g(w) \mathrm{d}A(w).$$

We conclude that

$$\begin{aligned} \left|\partial_z \mathcal{G}[g](z)\right| &\leq \|g\|_\infty \int_{\mathbb{D}} \left|\partial_z G_{\alpha,\beta}(z,w)\right| \mathrm{d}A(w) \\ &\leq (|\beta| 2^{|\alpha+\beta|-1} + 2\gamma_{\alpha,\beta})(1-|z|^2)^{\alpha+\beta} \|g\|_\infty. \end{aligned}$$

Similarly we obtain

$$\left|\partial_{\overline{z}} \mathcal{G}[g](z)\right| \leq \left(|\alpha| 2^{|\alpha+\beta|-1} + 2\gamma_{\beta,\alpha}\right)(1-|z|^2)^{\alpha+\beta} \|g\|_\infty.$$

Thus, the proof is complete. □

3 Proofs of Main Results

Proof of Theorem 1.1
The proof of Theorem 1.1 follows immediately from Theorems 2.1, 2.2 and Proposition 2.1.

Proof of Theorem 1.2
Differentiating $P_{\alpha,\beta}$ with respect to z and $\overline{z}$, we get

$$\partial_z P_{\alpha,\beta}(z) = \left(-(\alpha+\beta+1) \frac{\overline{z}}{1-|z|^2} + (\alpha+1) \frac{1}{1-z} \right) P_{\alpha,\beta}, \tag{3.1}$$

and

$$\partial_{\overline{z}} P_{\alpha,\beta}(z) = \left(-(\alpha+\beta+1) \frac{z}{1-|z|^2} + (\beta+1) \frac{1}{1-\overline{z}} \right) P_{\alpha,\beta}. \tag{3.2}$$

Therefore

$$\partial_z \mathcal{P}_{\alpha,\beta}[f](z) = \frac{1}{2\pi} \int_0^{2\pi} \partial_z P_{\alpha,\beta}(ze^{-i\theta}) e^{-i\theta} f(e^{i\theta}) \, \mathrm{d}\theta,$$

and

$$\partial_{\overline{z}}\mathcal{P}_{\alpha,\beta}[f]([z) = \frac{1}{2\pi}\int_0^{2\pi} \partial_{\overline{z}}P_{\alpha,\beta}(ze^{-i\theta})e^{i\theta}f(e^{i\theta})\mathrm{d}\theta.$$

Hence, by using (3.1) and (3.2), we obtain

$$\begin{aligned}
|\partial_z\mathcal{P}_{\alpha,\beta}[f](z)| &\le \frac{1}{2\pi}\int_0^{2\pi}\left|\partial_z P_{\alpha,\beta}(ze^{-i\theta})e^{-i\theta}f(e^{i\theta})\right|\mathrm{d}\theta \\
&\le \frac{\|f\|_\infty}{2\pi}\int_0^{2\pi}\left|\frac{(\alpha+1)(1-\overline{z}e^{i\theta})-\beta\overline{z}e^{i\theta}(1-ze^{-i\theta})}{(1-ze^{-i\theta})(1-|z|^2)}\right||P_{\alpha,\beta}(ze^{-i\theta})|\mathrm{d}\theta \\
&\le \frac{\|f\|_\infty}{2\pi}\frac{1}{1-|z|^2}\int_0^{2\pi}(|\alpha+1|+|\beta|)|P_{\alpha,\beta}(ze^{-i\theta})|\mathrm{d}\theta \\
&\le \frac{\|f\|_\infty\big((|\alpha+1|+|\beta|\big)}{1-|z|^2}\frac{|c_{\alpha,\beta}|}{2\pi}\int_0^{2\pi}\frac{(1-|z|^2)^{\alpha+\beta+1}}{|1-ze^{-i\theta}|^{\alpha+\beta+2}}\mathrm{d}\theta. \qquad (3.3)
\end{aligned}$$

By using the first inequality in Lemma 1.1, we obtain

$$\|\partial_z\mathcal{P}_{\alpha,\beta}[f](z)\| \le (|\alpha+1|+|\beta|)\frac{|c_{\alpha,\beta}|}{c_{\alpha+\beta}}\frac{\|f\|_\infty}{1-|z|^2}. \qquad (3.4)$$

Similarly,

$$\begin{aligned}
\|\partial_{\overline{z}}\mathcal{P}_{\alpha,\beta}[f](z)\| &\le (|\beta+1|+|\alpha|)\frac{|c_{\alpha,\beta}|}{c_{\alpha+\beta}}\frac{\|f\|_\infty}{1-|z|^2}. \\
\|Du(z)\| &\le \|D\mathcal{P}_{\alpha,\beta}[f](z)\| + \|D\mathcal{G}[g](z)\|.
\end{aligned} \qquad (3.5)$$

Combining Proposition 2.2 and (3.4) and (3.5), we get our conclusion and the proof of Theorem 1.2 is complete.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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