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Traveling Waves for a Sign-Changing Nonlocal Evolution Equation with Delayed Nonlocal Response

Juan He[1] · **Guo-Bao Zhang**[1]

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Abstract
This paper deals with the existence of traveling wave solutions for a nonlocal evolution equation with delayed nonlocal response and sign-changing kernel. By constructing a new pair of upper-lower solutions and applying Schauder's fixed point theorem, we first prove that there exists a number $c^{\#} > 0$ such that when $c > c^{\#}$, the nonlocal evolution equation admits a semi-wave solution with wave speed c, which connects the trivial equilibrium 0 at negative infinity. Then, we analyze the asymptotic behavior of wave profile at positive infinity and obtain the existence of a traveling wave solution with speed c and connecting the trivial equilibrium 0 and the positive equilibrium 1, when the wave speed c is large.

Keywords Nonlocal evolution equations · Traveling wave solutions · Nonlocal response · Sign-changing kernel

Mathematics Subject Classification 35K57 · 35C07 · 92D25

1 Introduction

Due to its important application in many subjects such as population biology, epidemiology, phase transition, signal propagation in neural networks [1, 2, 10, 15, 20, 31, 32], over the past decades, there are many works devoted to the study of the following nonlocal dispersal equation

$$u_t(x,t) = \mathcal{D}_1 u(x,t) + f(u(x,t)), \tag{1.1}$$

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✉ Guo-Bao Zhang
zhanggb2011@nwnu.edu.cn

1 College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, Gansu, China

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, and

$$\mathcal{D}_1 u(x, t) := \int_{\mathbb{R}} J(x - y)u(y, t)\mathrm{d}y - u(x, t)$$

models the nonlocal dispersal. The kernel function J is nonnegative, symmetric and unit integral. In mathematical biology, nonlocal dispersal $\mathcal{D}_1 u(x, t)$ is usually used to model the so-called long-distance effect, for example, the rapid spread of infectious disease across countries by air-traffic and the spread of small living things such as seeds, microbes or algae by wind or sea currents [1, 16, 17, 20, 21].

Notice that the drift of some individuals depends on their present positions from all possible positions at previous time. The reaction term that involves a weighted spatial averaging over the whole of the infinite domain is more realistic. Thus, the nonlocal delayed response has been incorporated into nonlocal dispersal model (1.1), that is,

$$u_t(x, t) = \mathcal{D}_1 u(x, t) + f\left(u(x, t), \int_{-\infty}^{+\infty} h(x - y)u(y, t - \tau)\mathrm{d}y\right). \tag{1.2}$$

The nonlocal dispersal equations like (1.1) and (1.2) with symmetric and asymmetric kernel functions J and h have been extensively studied. We refer readers to [4, 6, 7, 12, 17, 22–24, 26, 27, 30, 32] for the study of traveling wave solutions and entire solutions. In particular, for the model (1.2), when J and h are both nonnegative symmetric kernel functions, Yu and Yuan [29] investigated existence, asymptotic and uniqueness of traveling wave solutions. Later on, Cheng and Yuan [5] studied the global stability of traveling wave solutions by using the squeezing technique based on the comparison principle as well as super- and subsolutions. When J and h are nonnegative asymmetric kernel functions, Zhang and Li [33] proved the existence of traveling wave solutions by using super- and subsolutions, monotone iteration method and a limiting argument. The asymptotic behavior of the traveling wave solution and its derivative at minus infinity were also obtained in [33]. We refer readers to [13, 14, 19, 25, 28] for the study of traveling waves to various nonlocal evolution equations.

It is pointed out in [3] that in general, J clearly can change sign. For example, J can have a "mexican-hat" shape, where the kernel J is with negative parts [9, 10]. A situation also arises in some biological systems, see [11]. Hence, it is meaningful to study the nonlocal evolution equations with sign-changing kernels. In a recent paper [8], Ei et al. studied the nonlocal scalar equation (1.1) with sign-changing kernel J, and proved the existence of traveling wave solutions. In the current paper, we study the following nonlocal evolution equation with delayed nonlocal response and sing-changing kernel

$$u_t(x, t) = \mathcal{D}_2 u(x, t) + f\left(u(x, t), \int_{-\infty}^{+\infty} h(x - y)u(y, t - \tau)\mathrm{d}y\right), \tag{1.3}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, $\tau \geq 0$, and

$$\mathcal{D}_2 u(x, t) := \int_{\mathbb{R}} J(x - y)u(y, t)\mathrm{d}y - \alpha u(x, t)$$

with $\alpha \geq 0$. The kernel functions $J(\cdot)$ and $h(\cdot)$ are continuous in $\mathbb{R}$ and satisfy

(H1) $J(-x) = J(x), \forall x \in \mathbb{R}, \int_{\mathbb{R}} J(x)\mathrm{d}x = \alpha$, J is sign-changing and has a compact support.

(H2) $h(-x) = h(x) \geq 0$, $\forall x \in \mathbb{R}$, $\int_{\mathbb{R}} h(x)\mathrm{d}x = 1$, and for any $\lambda > 0$, $\int_{\mathbb{R}} h(y)e^{-\lambda y}\mathrm{d}y < +\infty$.

The nonlinear function $f(u, v)$ is a locally Lipschitz continuous function for $(u, v) \in \mathbb{R}^2$ satisfying

(H3) $f(0, 0) = f(1, 1) = 0$, $f(u, u) > 0$ for $u \in (0, 1)$.

(H4) $f(u, v) \leq \partial_1 f(0, 0)u + \partial_2 f(0, 0)v$ for $(u, v) \in [0, u^+]^2$ with some constant u^+ satisfying $1 < u^+ < +\infty$, and $\partial_1 f(1, 1) + \partial_2 f(1, 1) < 0$, where ∂_1 and ∂_2 denote the partial derivatives with respect to u and v, respectively.

(H5) $\partial_2 f(u, v) \geq 0$ for $(u, v) \in [0, u^+]^2$.

Remark 1.1 **i)** (H1) shows that the kernel J has both positive and negative values. If J is nonnegative and $\alpha = 1$, then (1.3) reduces to (1.2).

ii) (H3) implies that the homogeneous system of (1.3) admits two equilibria 0 and 1.

iii) (H3) together with (H4) implies that $\partial_1 f(0, 0) + \partial_2 f(0, 0) \geq 2f(\frac{1}{2}, \frac{1}{2}) > 0$, and hence, 0 is unstable and 1 is stable.

A traveling wave solution (for short, traveling wave) of (1.3) is a pair (ψ, c), where ψ is a real-valued function on $\mathbb{R}$ and c is a constant, such that $u(x, t) := \psi(x + ct)$ is a solution of (1.3). ψ and c are called the wave profile and the wave speed, respectively. Let $\xi := x + ct$. Then, the wave profile $\psi(\xi)$ satisfies the equation

$$c\psi'(\xi) = \mathcal{D}_2\psi(\xi) + f\left(\psi(\xi), \int_{-\infty}^{+\infty} h(y)\psi(\xi - y - c\tau)\mathrm{d}y\right), \tag{1.4}$$

where $\mathcal{D}_2\psi(\xi) = \int_{\mathbb{R}} J(y)\psi(\xi - y)\mathrm{d}y - \alpha\psi(\xi)$ and the prime denotes the derivative with respect to ξ. We want to find solutions of (1.4) with the following asymptotic boundary conditions

$$\lim_{\xi \to -\infty} \psi(\xi) = 0 \quad \text{and} \quad \lim_{\xi \to +\infty} \psi(\xi) = 1.$$

If the wave profile ψ only satisfies $\lim_{\xi \to -\infty} \psi(\xi) = 0$ and $\liminf_{\xi \to +\infty} \psi(\xi) > 0$, then we call $\psi(\cdot)$ a semi-wave solution.

Our aim of this paper is to establish the existence of monostable traveling wave solutions of (1.3). Since the kernel function J has both positive and negative values, the equation (1.3) does not possess the comparison principle. Hence, the monotone iteration method used in [33] and the theory of monotone semiflow [12] cannot be applied. In this paper, we shall apply the upper and lower solutions and Schauder's fixed point theorem to derive the existence of monostable traveling wave solutions $\psi(x + ct)$, see, e.g., [8, 18, 22]. For such solutions, a continuum of wave speeds is expected. We should point out that the wave profile ψ obtained may take both positive and negative values in general. From the biological point of view, we are more interested in the traveling wave solutions in a nonnegative situation. Thus, a more stringent condition (see (H6) in Sect. 2) is added to exclude this sign-changing nature of wave profile. Comparing with [29, 33], there are two notable differences. The first one is the definition and construction of upper and lower solutions. The second one is the establishment of asymptotic behavior of ψ at positive infinity.

The rest of the paper is organized as follows. In Sect. 2, we give some preliminaries and state the main result. In Sect. 3, we first establish a general result on the existence of traveling wave solutions, then construct a pair of upper and lower solutions to obtain the existence of traveling wave solutions of (1.3), and finally, investigate the behavior of the traveling wave solutions at positive infinity.

2 Preliminaries and Main Results

We set $J^+(x) := \max\{J(x), 0\}$ and $J^-(x) := \max\{-J(x), 0\}$. It is easy to see that $J(x) = J^+(x) - J^-(x)$. By (H1) and (H2), the functions

$$\mathcal{R}^\pm(\lambda) := \int_{\mathbb{R}} J^\pm(x)e^{-\lambda x}\mathrm{d}x \quad \text{and} \quad S(\lambda) := \int_{-\infty}^{+\infty} h(x)e^{-\lambda x}\mathrm{d}x$$

are well defined for all $\lambda \in [0, \infty)$. For $\lambda > 0$ and $c \in \mathbb{R}$, we define

$$\mathcal{K}(c, \lambda) := \tilde{\mathcal{K}}(\lambda) - c\lambda \quad \text{and} \quad \mathcal{N}(c, \lambda) := \tilde{\mathcal{N}}(\lambda) - c\lambda,$$

where

$$\tilde{\mathcal{K}}(\lambda) := \mathcal{R}^+(\lambda) - \mathcal{R}^-(\lambda) - \alpha + \partial_1 f(0, 0) + \partial_2 f(0, 0)e^{-c\tau\lambda}S(\lambda),$$
$$\tilde{\mathcal{N}}(\lambda) := \mathcal{R}^+(\lambda) + \mathcal{R}^-(\lambda) - \alpha + \partial_1 f(0, 0) + \partial_2 f(0, 0)e^{-c\tau\lambda}S(\lambda).$$

It is easily seen that $\mathcal{K}(c, \lambda) < \mathcal{N}(c, \lambda)$ for all $\lambda \in (0, +\infty)$ and $c \in \mathbb{R}$. We define

$$c^* := \inf_{\lambda\in(0,\hat{\lambda})} \frac{\tilde{\mathcal{K}}(\lambda)}{\lambda} \quad \text{and} \quad c^\sharp := \inf_{\lambda\in(0,+\infty)} \frac{\tilde{\mathcal{N}}(\lambda)}{\lambda},$$

where $\hat{\lambda}$ is defined to be the first positive zero of $\tilde{\mathcal{K}}(\lambda)$, if it exists; otherwise, we set $\hat{\lambda} := +\infty$. Note that $\tilde{\mathcal{K}}(0) = \partial_1 f(0, 0) + \partial_2 f(0, 0) > 0$. Then by the continuity of $\tilde{\mathcal{K}}(\lambda)$, we have $\tilde{\mathcal{K}}(\lambda) > 0$ for $\lambda > 0$ small. Thus, c^* is well defined and $c^* \geq 0$. By computation, we see that $\tilde{\mathcal{N}}(0) > \partial_1 f(0, 0) + \partial_2 f(0, 0) > 0$ and $\frac{\partial^2}{\partial\lambda^2}\tilde{\mathcal{N}}(\lambda) > 0$ for $\lambda \geq 0$. Hence, $c^\sharp$ is well defined and $c^\sharp > 0$. It is obvious that $c^* < c^\#$. Furthermore, we have the following lemma.

Lemma 2.1 *The following assertions hold.*

(i) *When $c > c^*$, the equation $\mathcal{K}(c, \lambda) = 0$ admits the smallest positive root $\lambda_1 := \lambda_1(c) \in (0, \hat{\lambda})$ such that*

$$\mathcal{K}(c, \lambda_1) = 0, \quad \mathcal{K}(c, \lambda) > 0, \;\; \forall\lambda \in [0, \lambda_1). \tag{2.1}$$

(ii) *When $c > c^{\#}$, the equation $\mathcal{N}(c, \lambda) = 0$ has two positive roots $\lambda_2 := \lambda_2(c)$ and $\lambda_3 := \lambda_3(c)$ with $\lambda_2 < \lambda_3$ such that*

$$\mathcal{N}(c, \lambda) < 0, \quad \forall \lambda \in (\lambda_2, \lambda_3); \quad \mathcal{N}(c, \lambda) > 0, \quad \forall \lambda \in [0, \lambda_2) \cup (\lambda_3, +\infty). \tag{2.2}$$

Remark 2.2 It can be seen that $\lambda_2 > \lambda_1$. In fact, by contradiction, we assume that $\lambda_2 \leq \lambda_1$. Then by (2.1) and (2.2), we have

$$\mathcal{N}(c, \lambda_2) = 0 \leq \mathcal{K}(c, \lambda_2),$$

which contradicts to $\mathcal{K}(c, \lambda) < \mathcal{N}(c, \lambda)$ for all $\lambda \in (0, +\infty)$.

In order to obtain the existence of traveling wave solutions of (1.3), we need the following additional technical assumption.

(H6) f satisfies $\partial_1 f(0, 0) > \alpha$, and there is a small constant $\sigma \in (0, 1)$ such that

$$f(u, v) = \partial_1 f(0, 0)u + \partial_2 f(0, 0)v \quad \text{for} \quad (u, v) \in [0, \sigma]^2. \tag{2.3}$$

Moreover, there exists a constant $\delta \in (0, \infty)$ such that

$$f(1 + \delta, 1 + \delta) < 0, \quad 1 + \delta \leq u^+, \tag{2.4}$$

and J^- satisfies

$$\int_{\mathbb{R}} J^-(x)\mathrm{d}x \leq \min\left\{\frac{-f(1 + \delta, 1 + \delta)}{1 + \delta}, \frac{(\partial_1 f(0, 0) - \alpha)\sigma}{1 + \delta}\right\}. \tag{2.5}$$

Define

$$\chi_i := \int_{\mathbb{R}} |x^i J(x)|\mathrm{d}x, \quad i = 0, 1, 2.$$

Since J has a compact support, χ_i is well defined. Now, we are ready to give our main result.

Theorem 2.3 *Assume that (H1)-(H6) hold. Then, the following assertions hold.*

(i) *If $c > c^{\#}$, then (1.4) has a positive bounded solution ψ such that $\psi(-\infty) = 0$, and*

$$\lim_{\xi \to -\infty} \psi(\xi)e^{-\lambda_1 \xi} = 1 \quad \text{and} \quad \lim_{\xi \to -\infty} \psi'(\xi)e^{-\lambda_1 \xi} = \lambda_1, \tag{2.6}$$

where λ_1 is defined by (2.1).

(ii) *Assume further that $f \in C^1(\mathbb{R} \times \mathbb{R})$. If $c > \max\{c^{\#}, (\chi_0 \chi_2)^{\frac{1}{2}}\}$, then (1.4) admits a positive bounded solution ψ such that $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. Moreover, (2.6) also holds.*

Remark 2.4 i) When the kernel J is nonnegative, whether it is symmetric or asymmetric, the traveling wave solutions of (1.3) are monotone, see [29, 33]. Unfortunately, when the kernel J is sign-changing, the monotonicity of traveling wave solution ψ cannot be obtained. The traveling wave solutions of (1.3) may be monotone or non-monotone. We conjecture that the monotone traveling wave solutions exist under some additional assumptions on the nonlinearity.

ii) By Theorem 2.3, we just know the existence of traveling wave solutions with speed $c > c^{\#}$. Whether the traveling wave solutions exist for some speed c between c^* and $c^{\#}$ is unknown. In other words, we are unable to determine the minimal wave speed, if it exists. We leave this problem for further study.

3 Existence of Traveling Wave Solutions

In this section, we are devoted to proving the existence of traveling wave solutions to (1.3), i.e., solutions of (1.4).

3.1 A General Result

In this subsection, we present a general existence result for solutions of (1.4). Define the integral operator $\mathcal{G} : C(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ by

$$\mathcal{G}[\psi](\xi) := \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi - y)} \mathcal{H}(\psi)(y)\mathrm{d}y,$$

where $\beta > \alpha$ is some large constant, and

$$\mathcal{H}(\psi)(\xi) = (J * \psi)(\xi) + (\beta - \alpha)\psi(\xi) + f\left(\psi(\xi), \int_{-\infty}^{+\infty} h(y)\psi(\xi - y - c\tau)\mathrm{d}y\right),$$

where $(J * \psi)(\xi) = \int_{\mathbb{R}} J(y)\psi(\xi - y)\mathrm{d}y$. It is easy to see that $\mathcal{G}$ is well defined, and a fixed point ψ of $\mathcal{G}$ is a solution of (1.4).

Let $\kappa := \max_{(u,v)\in[0,u^+]^2} |\partial_1 f(u, v)|$. Since $\partial_2 f(u, v) \geq 0$ for $(u, v) \in [0, u^+]^2$, the function f satisfies the following quasimonotone condition.

Lemma 3.1 *Assume that (H5) holds. Then, there is a positive constant $\beta > \kappa + \alpha$ such that*

$$\begin{aligned} & f\left(\psi_1(\xi), \int_{-\infty}^{+\infty} h(y)\psi_1(\xi - y - c\tau)\mathrm{d}y\right) \\ & - f\left(\psi_2(\xi), \int_{-\infty}^{+\infty} h(y)\psi_2(\xi - y - c\tau)\mathrm{d}y\right) \\ & + (\beta - \alpha)(\psi_1(\xi) - \psi_2(\xi)) \geq 0, \end{aligned}$$

where $\psi_1, \psi_2 \in C(\mathbb{R}, \mathbb{R})$ with $0 \leq \psi_2(\xi) < \psi_1(\xi) \leq u^+$ for all $\xi \in \mathbb{R}$.

The inequality in the above lemma can be easily proved by the mean value theorem. Thus, we omit the details here, see also [29, Lemma 2.1].

The existence of the fixed point will be proved by Schauder's fixed point theorem together with upper and lower solutions. We first introduce the concept of upper and lower solutions of (1.4).

Definition 3.2 A pair of continuous functions $\{\bar{\psi}, \underline{\psi}\}$ are called upper and lower solutions of (1.4) if

$$
\begin{aligned}
c\bar{\psi}'(\xi) \geq{}& (J^+ * \bar{\psi})(\xi) - (J^- * \underline{\psi})(\xi) - \alpha\bar{\psi}(\xi) \\
&+ f\left(\bar{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y\right), \qquad (3.1)\\
c\underline{\psi}'(\xi) \leq{}& (J^+ * \underline{\psi})(\xi) - (J^- * \bar{\psi})(\xi) - \alpha\underline{\psi}(\xi) \\
&+ f\left(\underline{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y\right), \qquad (3.2)
\end{aligned}
$$

for $\xi \in \mathbb{R} \setminus \mathbb{T}$ with some finite set $\mathbb{T} \subset \mathbb{R}$.

Now, we are in position to state the general existence result of solutions of (1.4) based on a pair of upper and lower solutions.

Proposition 3.3 *Assume that* (1.4) *has a pair of upper and lower solutions* $\{\bar{\psi}, \underline{\psi}\}$ *with range in* $[0, u^+]$ *such that* $\underline{\psi} \leq \bar{\psi}$ *in* $\mathbb{R}$. *Then for each* $c > 0$, (1.4) *admits a solution* ψ *satisfying* $\underline{\psi}(\xi) \leq \psi(\xi) \leq \bar{\psi}(\xi)$ *for any* $\xi \in \mathbb{R}$.

Proof For any $\mu \in (0, \frac{\beta}{c})$, define

$$
\mathcal{B}_\mu(\mathbb{R}) = \{\psi \in C(\mathbb{R}, \mathbb{R}) \mid \|\psi\|_\mu < \infty\}, \quad \|\psi\|_\mu := \sup_{\xi\in\mathbb{R}} |\psi(\xi)|e^{-\mu|\xi|}.
$$

Then, $(\mathcal{B}_\mu(\mathbb{R}), \|\cdot\|_\mu)$ is a Banach space. Let $\Gamma := \{\psi \in C(\mathbb{R}, \mathbb{R}) \mid \underline{\psi}(\xi) \leq \psi(\xi) \leq \bar{\psi}(\xi), \forall \xi \in \mathbb{R}\}$. It is easy to see that Γ is a nonempty convex bounded closed set with respect to the weighted norm $\|\cdot\|_\mu$. We claim that (i) $\mathcal{G}(\Gamma) \subset \Gamma$; (ii) $\mathcal{G} : \Gamma \to \Gamma$ is completely continuous with respect to the weighted norm $\|\psi\|_\mu$. Then, the proposition can be proved by Schauder's fixed point theorem.

We only prove the claim (i). The claim (ii) can be proved by a similar argument as that in [29, Lemma 2.4], we omit the details here. For any given $\psi \in \Gamma$, by the fact that $J(x) = J^+(x) - J^-(x)$ and Lemma 3.1, we obtain

$$
\begin{aligned}
\mathcal{G}[\psi](\xi) ={}& \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-y)}\left[(J^+ * \psi)(y) - (J^- * \psi)(y) + (\beta - \alpha)\psi(y)\right]\mathrm{d}y \\
&+ \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-y)} f\left(\psi(y), \int_{-\infty}^{+\infty} h(s)\psi(y - s - c\tau)\mathrm{d}s\right)\mathrm{d}y \\
\leq{}& \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-y)}\left[(J^+ * \bar{\psi})(y) - (J^- * \underline{\psi})(y) + (\beta - \alpha)\bar{\psi}(y)\right]\mathrm{d}y
\end{aligned}
$$

$$+\frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-y)} f\left(\bar{\psi}(y), \int_{-\infty}^{+\infty} h(s)\bar{\psi}(y-s-c\tau)\mathrm{d}s\right)\mathrm{d}y.$$

Furthermore, by (3.2), we have

$$\mathcal{G}[\psi](\xi) \leq \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-y)}[c\bar{\psi}'(y)+\beta\bar{\psi}(y)]\mathrm{d}y = \bar{\psi}(\xi)$$

for all $\xi \in \mathbb{R}$. Similarly, we can get that $\mathcal{G}[\psi](\xi) \geq \underline{\psi}(\xi)$ for all $\xi \in \mathbb{R}$. Hence, the claim (i) holds. The proof is complete. □

By Proposition 3.3, we see that in order to prove the existence of solutions of (1.4), it suffices to construct a pair of suitable upper and lower solutions.

3.2 Proof of Theorem 2.3

It follows from Remark 2.2 that $\lambda_1 < \lambda_2$. We choose $\eta > 1$ such that $\eta\lambda_1 \in (\lambda_2, \lambda_3)$. For a given constant $q > 1$, we define a continuous function

$$\varphi(\xi) := e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}.$$

It is easy to verify that $\varphi(\xi)$ has a unique zero point $\xi_0 := \frac{-\ln q}{(\eta-1)\lambda_1}$, i.e., $\varphi(\xi_0) = 0$, such that $\varphi(\xi) > 0$ for $\xi < \xi_0$ and $\varphi(\xi) < 0$ for $\xi \in (\xi_0, \infty)$. Moreover, $\varphi(\xi)$ has a maximum point $\xi_M := \frac{-\ln \eta q}{(\eta-1)\lambda_1}$, i.e., $\varphi(\xi_M) = \max_{\xi\in\mathbb{R}}\varphi(\xi)$. In what follows, we choose q large enough such that $\varphi(\xi_M) = \sigma$, where σ is defined in (2.3).

Based on the above numbers $\lambda_1, \eta, q, \delta$ and σ, we define two continuous functions

$$\bar{\psi}(\xi) = \begin{cases} e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}, & \xi \leq \xi_1, \\ 1+\delta, & \xi \geq \xi_1, \end{cases} \quad \underline{\psi}(\xi) = \begin{cases} e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}, & \xi \leq \xi_M, \\ \sigma, & \xi \geq \xi_M, \end{cases} \tag{3.3}$$

where the constant ξ_1 is chosen so that $e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi} = 1+\delta$.

We are going to prove that the functions $\bar{\psi}$ and $\underline{\psi}$ are upper and lower solutions of (1.4), respectively. In order to simplify notations, we define

$$\begin{aligned} \mathcal{L}_1(\xi) =& -c\bar{\psi}'(\xi) + (J^+ * \bar{\psi})(\xi) - (J^- * \underline{\psi})(\xi) - \alpha\bar{\psi}(\xi) \\ &+ f\left(\bar{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi-y-c\tau)\mathrm{d}y\right), \\ \mathcal{L}_2(\xi) =& -c\underline{\psi}'(\xi) + (J^+ * \underline{\psi})(\xi) - (J^- * \bar{\psi})(\xi) - \alpha\underline{\psi}(\xi) \\ &+ f\left(\underline{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi-y-c\tau)\mathrm{d}y\right). \end{aligned}$$

Lemma 3.4 *Assume that (H1)-(H6) hold. If $c > c^{\#}$, then the functions $\bar{\psi}$ and $\underline{\psi}$ defined by (3.3) are upper and lower solutions of (1.4), respectively.*

Proof We first show that $\mathcal{L}_1(\xi) \le 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_1\}$. When $\xi < \xi_1$, we have $\bar{\psi}(\xi) = e^{\lambda_1 \xi} + qe^{\eta\lambda_1 \xi}$. Since $0 \le \bar{\psi}(\xi) \le 1 + \delta \le u^+$ for all $\xi \in \mathbb{R}$, by (H4), we have

$$\begin{aligned} &f\left(\bar{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y\right) \\ &\quad \le \partial_1 f(0,0)\bar{\psi}(\xi) + \partial_2 f(0,0) \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y. \end{aligned}$$

Note that $\bar{\psi}(\xi) \le e^{\lambda_1 \xi} + qe^{\eta\lambda_1 \xi}$ and $\underline{\psi}(\xi) \ge e^{\lambda_1 \xi} - qe^{\eta\lambda_1 \xi}$ for all $\xi \in \mathbb{R}$. Thus, for $\xi < \xi_1$, we obtain

$$\begin{aligned} \mathcal{L}_1(\xi) \le& -c[\lambda_1 e^{\lambda_1 \xi} + \eta\lambda_1 q e^{\eta\lambda_1 \xi}] + \int_{\mathbb{R}} J^+(y)[e^{\lambda_1(\xi - y)} + qe^{\eta\lambda_1(\xi - y)}]\mathrm{d}y \\ &- \int_{\mathbb{R}} J^-(y)[e^{\lambda_1(\xi - y)} - qe^{\eta\lambda_1(\xi - y)}]\mathrm{d}y - \alpha[e^{\lambda_1 \xi} + qe^{\eta\lambda_1 \xi}] \\ &+ f\left(\bar{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y\right) \\ \le& e^{\lambda_1 \xi}[-c\lambda_1 + \mathcal{R}^+(\lambda_1) - \mathcal{R}^-(\lambda_1) - \alpha] \\ &+ qe^{\eta\lambda_1 \xi}[-c\eta\lambda_1 + \mathcal{R}^+(\eta\lambda_1) + \mathcal{R}^-(\eta\lambda_1) - \alpha] \\ &+ \partial_1 f(0,0)\bar{\psi}(\xi) + \partial_2 f(0,0) \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y \\ =& e^{\lambda_1 \xi}[\mathcal{K}(c, \lambda_1) - \partial_1 f(0,0) - \partial_2 f(0,0)e^{-c\tau\lambda_1} S(\lambda_1)] \\ &+ qe^{\eta\lambda_1 \xi}[\mathcal{N}(c, \eta\lambda_1) - \partial_1 f(0,0) - \partial_2 f(0,0)e^{-c\tau\eta\lambda_1} S(\eta\lambda_1)] \\ &+ \partial_1 f(0,0)\bar{\psi}(\xi) + \partial_2 f(0,0) \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y. \end{aligned} \tag{3.4}$$

It can be seen from (2.1) and (2.2) that $\mathcal{K}(c, \lambda_1) = 0$ and $\mathcal{N}(c, \eta\lambda_1) < 0$. Hence, (3.4) becomes

$$\begin{aligned} \mathcal{L}_1(\xi) <& - \partial_1 f(0,0)(e^{\lambda_1 \xi} + qe^{\eta\lambda_1 \xi}) - \partial_2 f(0,0) \\ &\int_{-\infty}^{+\infty} h(y)(e^{\lambda_1(\xi - y - c\tau)} + qe^{\eta\lambda_1(\xi - y - c\tau)})\mathrm{d}y \\ &+ \partial_1 f(0,0)\bar{\psi}(\xi) + \partial_2 f(0,0) \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y \le 0. \end{aligned}$$

When $\xi > \xi_1$, we have $\bar{\psi}(\xi) = 1 + \delta$. Since $\bar{\psi}(\xi) \le 1 + \delta$ for all $\xi \in \mathbb{R}$, by (H5), we have

$$f\left(\bar{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\bar{\psi}(\xi - y - c\tau)\mathrm{d}y\right) \le f(1 + \delta, 1 + \delta), \quad \forall \xi > \xi_1.$$

Furthermore, by considering that fact that $J(x) = J^+(x) - J^-(x)$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} J(x)\mathrm{d}x = \alpha$, and $\underline{\psi}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, one has, for $\xi > \xi_1$,

$$\begin{aligned}\mathcal{L}_1(\xi) &\leq (1+\delta)\int_{\mathbb{R}} J^+(y)\mathrm{d}y - \alpha(1+\delta) + f(1+\delta, 1+\delta)\\ &= (1+\delta)\int_{\mathbb{R}} J^-(y)\mathrm{d}y + f(1+\delta, 1+\delta) \leq 0.\end{aligned}$$

The last inequality holds due to (2.5). Therefore, we obtain $\mathcal{L}_1(\xi) \leq 0$ for all $\xi \in \mathbb{R}\backslash\{\xi_1\}$.

Next, we prove that $\mathcal{L}_2(\xi) \geq 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_M\}$. When $\xi < \xi_M$, we have $\underline{\psi}(\xi) = e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}$. Note that $\underline{\psi}(\xi) \in (0, \sigma]$ for all $\xi \in \mathbb{R}$. Then by (2.3), we have

$$\begin{aligned}&f\left(\underline{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y\right)\\ &\quad= \partial_1 f(0,0)\underline{\psi}(\xi) + \partial_2 f(0,0)\int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y.\end{aligned}$$

In view of $\bar{\psi}(\xi) \leq e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}$ for all $\xi \in \mathbb{R}$, and by (2.1) and (2.2), we derive, for $\xi < \xi_M$,

$$\begin{aligned}\mathcal{L}_2(\xi) \geq &- c[\lambda_1 e^{\lambda_1\xi} - \eta\lambda_1 q e^{\eta\lambda_1\xi}] + \int_{\mathbb{R}} J^+(y)[e^{\lambda_1(\xi-y)} - qe^{\eta\lambda_1(\xi-y)}]\mathrm{d}y\\ &- \int_{\mathbb{R}} J^-(y)[e^{\lambda_1(\xi-y)} + qe^{\eta\lambda_1(\xi-y)}]\mathrm{d}y - \alpha[e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}]\\ &+ f\left(\underline{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y\right)\\ = &\, e^{\lambda_1\xi}[-c\lambda_1 + \mathcal{R}^+(\lambda_1) - \mathcal{R}^-(\lambda_1) - \alpha]\\ &- qe^{\eta\lambda_1\xi}[-c\eta\lambda_1 + \mathcal{R}^+(\eta\lambda_1) + \mathcal{R}^-(\eta\lambda_1) - \alpha]\\ &+ f\left(\underline{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y\right)\\ = &\, e^{\lambda_1\xi}[\mathcal{K}(c,\lambda_1) - \partial_1 f(0,0) - \partial_2 f(0,0)e^{-c\tau\lambda_1}S(\lambda_1)]\\ &- qe^{\eta\lambda_1\xi}[\mathcal{N}(c,\eta\lambda_1) - \partial_1 f(0,0) - \partial_2 f(0,0)e^{-\eta c\tau\lambda_1}S(\eta\lambda_1)]\\ &+ f\left(\underline{\psi}(\xi), \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y\right)\\ \geq &-\partial_1 f(0,0)(e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi})\\ &- \partial_2 f(0,0)\int_{-\infty}^{+\infty} h(y)(e^{\lambda_1(\xi-y-c\tau)} - qe^{\eta\lambda_1(\xi-y-c\tau)})\mathrm{d}y\\ &+ \partial_1 f(0,0)\underline{\psi}(\xi) + \partial_2 f(0,0)\int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y \geq 0.\end{aligned}$$

When $\xi > \xi_M$, we have $\underline{\psi}(\xi) = \sigma$. Note that $0 < \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y \le \sigma$, $\underline{\psi}(\xi) > 0$ and $\bar{\psi}(\xi) \le 1 + \delta$ for all $\xi \in \mathbb{R}$. Then for $\xi > \xi_M$, we have

$$\begin{aligned}
\mathcal{L}_2(\xi) &\ge -(1+\delta)\int_{\mathbb{R}} J^-(y)\mathrm{d}y - \alpha\sigma + f\left(\sigma, \int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y\right) \\
&= -(1+\delta)\int_{\mathbb{R}} J^-(y)\mathrm{d}y - \alpha\sigma + \partial_1 f(0,0)\sigma \\
&\quad + \partial_2 f(0,0)\int_{-\infty}^{+\infty} h(y)\underline{\psi}(\xi - y - c\tau)\mathrm{d}y \\
&\ge -(1+\delta)\int_{\mathbb{R}} J^-(y)\mathrm{d}y - \alpha\sigma + \partial_1 f(0,0)\sigma \ge 0.
\end{aligned}$$

The last inequality holds due to (2.5). Hence, we obtain that $\mathcal{L}_2(\xi) \ge 0$ for all $\xi \in \mathbb{R}\setminus\{\xi_M\}$. Therefore, $\bar{\psi}$ and $\underline{\psi}$ are upper and lower solutions to (1.4), respectively. The proof is complete. □

Proof of Theorem 2.3 (i) By Proposition 3.3 and Lemma 3.4, we obtain that there exists a solution ψ of (1.4). By the definition of $\bar{\psi}$ and $\underline{\psi}$ in (3.3), we see that $\bar{\psi}(-\infty) = \underline{\psi}(-\infty) = 0$. In view of $\underline{\psi}(\xi) \le \psi(\xi) \le \bar{\psi}(\xi)$ for $\xi \in \mathbb{R}$, we obtain that $\psi(-\infty) = 0$, and $\liminf_{\xi\to+\infty}\psi(\xi) > 0$. In addition, notice that

$$e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi} \le \psi(\xi) \le e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}, \quad \forall \xi \in \mathbb{R},$$

where $\lambda_1 > 0$ and $\eta > 1$. Then, we derive

$$\lim_{\xi\to-\infty}\psi(\xi)e^{-\lambda_1\xi} = 1. \tag{3.5}$$

In view of (H6) and (3.5), and by Lebesgue's dominated convergence theorem, we see

$$\begin{aligned}
&\lim_{\xi\to-\infty} f\left(\psi(\xi), \int_{-\infty}^{+\infty} h(y)\psi(\xi - y - c\tau)\mathrm{d}y\right)e^{-\lambda_1\xi} \\
&= \lim_{\xi\to-\infty}\left(\partial_1 f(0,0)\psi(\xi) + \partial_2 f(0,0)\int_{-\infty}^{+\infty} h(y)\psi(\xi - y - c\tau)\mathrm{d}y\right)e^{-\lambda_1\xi} \\
&= \partial_1 f(0,0) + \partial_2 f(0,0)e^{-\lambda_1 c\tau}\int_{-\infty}^{+\infty} h(y)e^{-\lambda_1 y}\mathrm{d}y.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\xi\to-\infty}\psi'(\xi)e^{-\lambda_1\xi} \\
&= \frac{1}{c}\lim_{\xi\to-\infty}\left\{\int_{\mathbb{R}} J(y)\psi(\xi - y)\mathrm{d}y - \alpha\psi(\xi)\right. \\
&\quad \left. + f\left(\psi(\xi), \int_{-\infty}^{+\infty} h(y)\psi(\xi - y - c\tau)\mathrm{d}y\right)\right\}e^{-\lambda_1\xi}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{c}\left\{\int_{-\infty}^{+\infty} J(y)e^{-\lambda_1 y}\mathrm{d}y - \alpha + \partial_1 f(0,0) + \partial_2 f(0,0)e^{-\lambda_1 c\tau}\int_{-\infty}^{+\infty} h(y)e^{-\lambda_1 y}\mathrm{d}y\right\}\\
&= \frac{1}{c}(\mathcal{K}(c,\lambda_1) + c\lambda_1)\\
&= \lambda_1,
\end{aligned}
$$

since $\mathcal{K}(c,\lambda_1) = 0$. Thus, the assertion of Theorem 2.3 (i) is true. The proof is complete. □

Next, we shall study the behavior of the solution to (1.4) at $\xi = +\infty$.

Proposition 3.5 *Let ψ be a solution to (1.4) obtained in Theorem 2.3 (i). If the limit $\rho := \lim_{\xi\to+\infty}\psi(\xi)$ exists, then $f(\rho,\rho) = 0$.*

Proof Since $\rho := \lim_{\xi\to+\infty}\psi(\xi)$ exists, there exists a sequence $\{\xi_n\}$ with $\{\xi_n\} \to +\infty$ such that $\psi(\xi_n) \to \rho$ and $\psi'(\xi_n) \to 0$ as $n \to \infty$. Taking ξ_n into (1.4) yields

$$
c\psi'(\xi_n) = (J*\psi)(\xi_n) - \alpha\psi(\xi_n) + f\left(\psi(\xi_n), \int_{-\infty}^{+\infty} h(y)\psi(\xi_n - y - c\tau)\mathrm{d}y\right).
$$

In order to prove that $f(\rho,\rho) = 0$, we just need to show that

$$
\lim_{n\to\infty}(J*\psi)(\xi_n) = \alpha\rho \quad \text{and} \quad \lim_{n\to\infty}\int_{-\infty}^{+\infty} h(y)\psi(\xi_n - y - c\tau)\mathrm{d}y = \rho. \tag{3.6}
$$

For any $\varepsilon > 0$ sufficiently small, since $\lim_{\xi\to+\infty}\psi(\xi) = \rho$, $\int_{\mathbb{R}} J(x)\mathrm{d}x = \alpha$ and $\int_{\mathbb{R}} h(x)\mathrm{d}x = 1$, there exists a constant $M \gg 1$ such that

$$
|\psi(\xi) - \rho| < \min\left\{\frac{\varepsilon}{4(\mathcal{R}^+(0) + \mathcal{R}^-(0))}, \frac{\varepsilon}{4}\right\} \text{ for } \xi \geq M, \tag{3.7}
$$

$$
\int_{|y|\geq M} J^{\pm}(y)\mathrm{d}y < \frac{\varepsilon}{4\|\psi\|_\infty}, \tag{3.8}
$$

$$
\left|\int_{|y|\geq M} J(y)\mathrm{d}y\right| < \frac{\varepsilon}{4(|\rho|+1)} \quad \text{and} \quad \left|\int_{|y|\geq M} h(y)\mathrm{d}y\right| < \min\left\{\frac{\varepsilon}{2\|\psi\|_\infty}, \frac{\varepsilon}{4(|\rho|+1)}\right\}, \tag{3.9}
$$

where $\|\cdot\|_\infty$ mean the supremum norm. We choose $N \gg 1$ such that $\xi_n - c\tau \geq 2M$ for all $n > N$. Then, by (3.7), for any $n > N$ and $y \in [-M, M]$, one has

$$
|\psi(\xi_n - y - sc\tau) - \rho| < \min\left\{\frac{\varepsilon}{4(\mathcal{R}^+(0) + \mathcal{R}^-(0))}, \frac{\varepsilon}{4}\right\}, \tag{3.10}
$$

where $s \in \{0, 1\}$.

It is easy to see that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} J(y)\psi(\xi_n - y)\mathrm{d}y - \alpha\rho\right| \\
&\quad \le \left|\int_{|y|\ge M} J(y)\psi(\xi_n - y)\mathrm{d}y\right| + \left|\int_{-M}^{M} J(y)[\psi(\xi_n - y) - \rho]\mathrm{d}y\right| \\
&\quad + \left|\int_{-M}^{M} J(y)\mathrm{d}y - \alpha\right| \cdot |\rho|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} h(y)\psi(\xi_n - y - c\tau)\mathrm{d}y - \rho\right| \\
&\quad \le \left|\int_{|y|\ge M} h(y)\psi(\xi_n - y - c\tau)\mathrm{d}y\right| + \left|\int_{-M}^{M} h(y)[\psi(\xi_n - y - c\tau) - \rho]\mathrm{d}y\right| \\
&\quad + \left|\int_{-M}^{M} h(y)\mathrm{d}y - 1\right| \cdot |\rho|.
\end{aligned}
$$

For any $n \ge N$, by (3.8), (3.9) and (3.10), one has

$$
\begin{aligned}
&\left|\int_{|y|\ge M} J(y)\psi(\xi_n - y)\mathrm{d}y\right| \le \left(\int_{|y|\ge M} J^{+}(y)\mathrm{d}y + \int_{|y|\ge M} J^{-}(y)\mathrm{d}y\right) \|\psi\|_\infty < \frac{\varepsilon}{2}, \\
&\left|\int_{-M}^{M} J(y)[\psi(\xi_n - y) - \rho]\mathrm{d}y\right| \le \int_{-M}^{M} |J(y)|\mathrm{d}y \cdot \frac{\varepsilon}{4[\mathcal{R}^{+}(0) + \mathcal{R}^{-}(0)]} \le \frac{\varepsilon}{4}, \\
&\left|\int_{|y|\ge M} h(y)\psi(\xi_n - y - c\tau)\mathrm{d}y\right| \le \left|\int_{|y|\ge M} h(y)\mathrm{d}y\right| \|\psi\|_\infty < \frac{\varepsilon}{2},
\end{aligned}
$$

and

$$
\left|\int_{-M}^{M} h(y)[\psi(\xi_n - y - c\tau) - \rho]\mathrm{d}y\right| \le \left|\int_{-M}^{M} h(y)\mathrm{d}y\right| \cdot \frac{\varepsilon}{4} \le \frac{\varepsilon}{4}.
$$

Since $\int_{\mathbb{R}} J(x)\mathrm{d}x = \alpha$ and $\int_{\mathbb{R}} h(x)\mathrm{d}x = 1$, by (3.9), we have

$$
\left|\int_{-M}^{M} J(y)\mathrm{d}y - \alpha\right| \cdot |\rho| = \left|\int_{|y|\ge M} J(y)\mathrm{d}y\right| \cdot |\rho| < \frac{\varepsilon}{4}
$$

and

$$
\left|\int_{-M}^{M} h(y)\mathrm{d}y - 1\right| \cdot |\rho| = \left|\int_{|y|\ge M} h(y)\mathrm{d}y\right| \cdot |\rho| < \frac{\varepsilon}{4}.
$$

Hence, (3.6) holds. The proof is complete. □

Define the set

$$C_b^k(\mathbb{R}) := \{\varphi \in C^k(\mathbb{R}) \mid \|\varphi^{(j)}\|_\infty < \infty,\ j = 0, 1, \ldots, k\}.$$

Lemma 3.6 *Let* $\psi \in C_b^2(\mathbb{R})$ *be a solution to* (1.4). *If* $c > (\chi_0\chi_2)^{\frac{1}{2}}$, *then* $\psi' \in L^2(\mathbb{R})$ *and* $\lim_{\xi\to+\infty} \psi'(\xi) = 0$.

Proof Since $\psi \in C_b^2(\mathbb{R})$, we denote $C_0 := \|\psi\|_\infty$ and $C_1 = \|\psi'\|_\infty$. Moreover, we define

$$\tilde{f}(u) := \int_0^u f(s, g(s))\mathrm{d}s \quad \text{and} \quad C_2 := \max_{u\in[-C_0,C_0]} |\tilde{f}(u)|,$$

where $g(s)$ is continuous function in $\mathbb{R}$. It is easy to see that $\tilde{f}'(u) = f(u, g(u))$ and $[\tilde{f}(u(\xi))]' = f(u(\xi), g(u(\xi)))u'(\xi)$. Let $A, B > 0$. Multiplying (1.4) by ψ' and then integrating from $-A$ to B, and applying the Cauchy–Schwarz inequality, we derive

$$\begin{aligned}
&c\int_{-A}^{B}(\psi')^2(\xi)\mathrm{d}\xi \\
&\quad= \int_{-A}^{B} \psi'(\xi)\,[(J*\psi)(\xi) - \alpha\psi(\xi) \\
&\qquad+ f\left(\psi(\xi), \int_{-\infty}^{+\infty} h(y)\psi(\xi - y - c\tau)\mathrm{d}y\right)\Bigg]\mathrm{d}\xi \\
&\quad= \int_{-A}^{B} \psi'(\xi)[(J*\psi)(\xi) - \alpha\psi(\xi)]\mathrm{d}\xi + \int_{-A}^{B}[\tilde{f}(\psi(\xi))]'\mathrm{d}\xi \\
&\quad\le \left(\int_{-A}^{B}(\psi')^2(\xi)\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{-A}^{B}[(J*\psi)(\xi) - \alpha\psi(\xi)]^2\mathrm{d}\xi\right)^{\frac{1}{2}} + 2C_2,
\end{aligned} \tag{3.11}$$

where we used $g(\psi(\xi)) = \int_{\mathbb{R}} h(y)\psi(\xi - y - c\tau)\mathrm{d}y$. By a similar argument as that in [8, Lemma 4.2], we obtain

$$\int_{-A}^{B}[(J*\psi)(\xi) - \alpha\psi(\xi)]^2\mathrm{d}\xi \le \chi_2\left\{\chi_0\int_{-A}^{B}(\psi')^2(\xi)\mathrm{d}\xi + C_1^2\chi_1\right\}. \tag{3.12}$$

Taking (3.12) into (3.11) yields

$$c\int_{-A}^{B}(\psi')^2(\xi)\mathrm{d}\xi \le (\chi_2)^{\frac{1}{2}}\left\{\chi_0\left(\int_{-A}^{B}(\psi')^2(\xi)\mathrm{d}\xi\right)^2 + C_1^2\chi_1\int_{-A}^{B}(\psi')^2(\xi)\mathrm{d}\xi\right\}^{\frac{1}{2}} + 2C_2,$$

which implies that if $c > (\chi_0\chi_2)^{\frac{1}{2}}$, then $\left\{\int_{-A}^{B}(\psi')^2 | A > 0, B > 0\right\}$ is uniformly bounded, and hence, we obtain that $\psi' \in L^2(\mathbb{R})$. Since $\psi \in C_b^2(\mathbb{R})$, it is clear that

ψ' is uniformly continuous on $\mathbb{R}$. Thus, we have $\lim_{\xi\to+\infty}\psi'(\xi)=0$. The proof is complete. □

Lemma 3.7 *Let $\psi \in C_b^2(\mathbb{R})$ be a solution to (1.4), and assume further that $f \in C^1(\mathbb{R}\times\mathbb{R})$. If $c > (\chi_0\chi_2)^{\frac{1}{2}}$ and*

$$\{u \in [-\|\psi\|_\infty, \|\psi\|_\infty] | f(u,u)=0\} = \{0,1\},$$

then $\psi(+\infty)$ exists and belongs to $\{0,1\}$.

Proof Let $\mathcal{S}$ be the set of accumulation points of ψ at $+\infty$. Note that $\psi \in C_b^2(\mathbb{R})$. Thus, $\mathcal{S}$ is not empty. Let $\rho \in \mathcal{S}$. Then, there exists a sequence $\xi_n \to +\infty$ such that $\psi(\xi_n) \to \rho$ as $n \to +\infty$. Let $\varphi_n(\xi) := \psi(\xi+\xi_n)$. Then by (1.4), $\varphi_n(\xi)$ solves

$$c\varphi_n'(\xi) = (J*\varphi_n)(\xi) - \alpha\varphi_n(\xi) + f\left(\varphi_n(\xi), \int_{-\infty}^{+\infty} h(y)\varphi_n(\xi-y-c\tau)\mathrm{d}y\right), \quad \forall \xi \in \mathbb{R}.$$

For all $L>0$ and all $1<p<\infty$, the sequence $\{\varphi_n\}$ is bounded in $W^{2,p}([-L,L])$. Then by the Sobolev embedding theorem, there exists a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ such that $\{\varphi_{n_k}\} \to \varphi$ as $k\to\infty$ strongly in $C_{loc}^1(\mathbb{R})$ and weakly in $W_{loc}^{1,p}(\mathbb{R})$. Thus, φ satisfies

$$c\varphi'(\xi) = (J*\varphi)(\xi) - \alpha\varphi(\xi) + f\left(\varphi(\xi), \int_{-\infty}^{+\infty} h(y)\varphi(\xi-y-c\tau)\mathrm{d}y\right), \quad \forall \xi \in \mathbb{R}.$$

In addition, by Lemma 3.6, we obtain

$$\varphi'(\xi) = \lim_{k\to\infty} \psi'(\xi+\xi_{n_k}) = 0, \quad \forall \xi \in \mathbb{R},$$

which implies that $\varphi(\xi)$ is a constant function of $\xi \in \mathbb{R}$. Hence, we derive

$$\varphi(\xi) \in \{u \in [-\|\psi\|_\infty, \|\psi\|_\infty] | f(u,u)=0\}, \ \forall \xi \in \mathbb{R}.$$

In particular,

$$\rho = \lim_{k\to\infty} \psi(\xi_{n_k}) = \varphi(0) \in \{u \in [-\|\psi\|_\infty, \|\psi\|_\infty] | f(u,u)=0\}.$$

By the assumption of the lemma, we obtain that $\rho \in \{0,1\}$. Since ψ is a continuous function, $\mathcal{S}$ is connected. Therefore, $\psi(+\infty)$ exists and belongs to $\{0,1\}$. The proof is complete. □

Proof of Theorem 2.3 (ii) In order to apply Lemma 3.7, we need to show that the traveling wave solution ψ obtained in Theorem 2.3 (i) satisfies $\psi \in C_b^2(\mathbb{R})$. Since $\underline{\psi}(\xi) \le \psi(\xi) \le \bar{\psi}(\xi)$ for all $\xi \in \mathbb{R}$, and ψ satisfies (1.4), we obtain that ψ is continuous and bounded. It then follows from (1.4) and (H4) that ψ' is also continuous and bounded for any fixed c. In view of $f \in C^1(\mathbb{R} \times \mathbb{R})$, we can see that the right side of (1.4) is differentiable. Then, we have

$$c\psi''(\xi) = \mathcal{D}_2\psi'(\xi) + \partial_1 f \cdot \psi'(\xi) + \partial_2 f \cdot \int_{-\infty}^{+\infty} h(y)\psi'(\xi - y - c\tau)\mathrm{d}y,$$

which implies that ψ'' is continuous and bounded in $\mathbb{R}$. Therefore, $\psi \in C_b^2(\mathbb{R})$. Note from (3.3) that when $\xi \geq \xi_M$, $\psi(\xi) \geq \sigma > 0$. It then follows from Lemma 3.7 that when $c > \max\{c^\#, (\chi_0\chi_2)^{\frac{1}{2}}\}$, $\psi(+\infty) = 1$. Thus, the assertion (ii) of Theorem 2.3 is true. The proof is complete. □

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Declarations

Conflict of interest This work does not have any conflicts of interest.

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