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The Relaxed Stochastic Maximum Principle in Singular Optimal Control of Jump Diffusions

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Abstract

This paper deals with optimal control of systems driven by stochastic differential equations (SDEs), with controlled jumps, where the control variable has two components, the first being absolutely continuous and the second singular. We study the corresponding relaxed-singular problem, in which the first part of the admissible control is a measure-valued process and the state variable is governed by a SDE driven by a relaxed Poisson random measure, whose compensator is a product measure. We establish a stochastic maximum principle for this type of relaxed control problems extending existing results. The proofs are based on Ekeland's variational principle and stability properties of the state process and adjoint variable with respect to the control variable.

Keywords Stochastic differential equation · Optimal control · Jump process · Relaxed control · Singular control · Stochastic maximum principle

Mathematics Subject Classification 49JXX · 49KXX · 60H10 · 93EXX

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1 Introduction

We consider mixed relaxed-singular stochastic control problems of systems governed by stochastic differential equations (SDEs) with controlled jumps (1.1), where the control variable has two components, the first being a measure-valued process and the second a bounded variation process, called the singular part. More precisely, the system evolves according to the SDE,

$$\begin{cases} dx_t^{\mu} = \int_{A_1} b(t, x_t^{\mu}, a)\mu_t(da)dt + \sigma(t, x_t^{\mu})dB_t + \int_{A_1}\int_{\Gamma} f(t, x_{t^-}^{\mu}, \theta, a)\widetilde{N}^{\mu}(dt, d\theta, da) + G_t d\zeta_t \\ x_0^{\mu} = 0, \end{cases} \tag{1.1}$$

on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, such that $\mathcal{F}_0$ contains the $P-$null sets. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is generated by a standard Brownian motion B and an independent Poisson random measure $\widetilde{N}^{\mu}$, with compensator $\mu_t \otimes \upsilon(da, d\theta)$, where μ is the relaxed control and υ is the compensator of Poisson measure N^{μ}. The control variable is (μ, ζ), where μ is a $\mathcal{P}(A_1)-$valued process, progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\zeta : [0; T] \times \Omega \longrightarrow A_2$ is an adapted process of bounded variation, nondecreasing, left-continuous with right limits such that $\zeta_0 = 0$.

The expected cost to be minimized over the class of admissible controls has the form

$$J(\mu, \zeta) = E\left[g(x_T^{\mu}) + \int_{A_1}\int_0^T h(t, x_t^{\mu}, a)\mu_t(da)dt + \int_0^T k_t d\zeta_t \right].$$

A control process that solves this problem is called optimal.

Singular stochastic control problems have been extensively studied in the literature. We refer to Botius [9], for a complete survey. This problem was first introduced by Bather and Chernoff [4] in 1967, by considering a simplified model for the control of a spaceship, known as the monotone follower for Brownian motion. The authors have noted that this model of singular control has a connection with some optimal stopping problem. It was proved, in particular, that the value function of the singular control problem is equal to the gradient of value function of the corresponding optimal stopping problem. After this seminal article, this connection has been deeply investigated in different contexts. Singular control problems find many applications in different areas of engineering such as mathematical finance, manufacturing systems and queuing theory [11, 27]. Two approaches were used to handle singular control problems. The first one is based on dynamic programming, which leads to a variational inequality. This approach has been studied by many authors including Beněs, Sheep, and Witsenhausen [6], Chow, Menaldi, and Robin [13], Karatzas and Shreve [22, 24], Davis and Norman [14], and Haussmann and Suo [18–20]. Probabilistic methods have been used to solve the dynamic programming equation, see, e.g., [16, 22] and the references therein. The second approach to solve singular control problems is the so-called stochastic maximum principle. This approach leads to necessary conditions for optimality satisfied by an optimal control and is based on some adjoint process and a variational inequality between hamiltonians. The first version of the stochastic maximum principle that covers singular control problems was obtained by Cadenillas and Haussman [10] for linear systems. General second-order necessary conditions for optimality for nonlinear SDEs with a controlled diffusion matrix were obtained by S. Bahlali and B. Mezerdi [3], extending Peng's second-order stochastic maximum principle to singular control problems. We refer also to [1] for systems with nonsmooth coefficients. The stochastic maximum principle for relaxed-singular control problem has been studied by S. Bahlali, B. Djehiche, and B. Mezerdi [2]. The relationship between the dynamic programming principle and the maximum principle has been investigated in [12].

Our main goal in this paper is to derive a stochastic maximum principle for mixed relaxed-singular control problems. This means that the first component of the control is a measure valued process and the second component is a process with increasing sample paths. By using the same weak convergence techniques as in [18, 20], it is not difficult to show the existence of an optimal control for the relaxed-singular control problem and that the value functions of the relaxed problem and the strict control problem are the same. The proof of our stochastic maximum principle is divided into three steps. We first establish necessary conditions for optimality satisfied by an optimal strict control. The second step is devoted to the necessary conditions for near optimality satisfied by a sequence of near optimal controls, by using Ekeland's variational principle. This auxiliary result is in itself one of the novelties of this paper. Indeed, in most practical situations, it is sufficient to characterize and compute nearly optimal controls. In the third step, we prove the relaxed stochastic maximum principle by passing to the limit in the adjoint processes and the variational inequalities. These properties are based on the stability of the state process and the adjoint processes with respect to the control variable. The novelty of our result is that our maximum principle is given for a relaxed optimal control, which exists and the dynamics involves controlled jumps. In particular, our main result extends the stochastic maximum principle proved in [2, 10] to systems of SDE with controlled jumps (1.1). On the other hand, it extends [5] to systems involving a singular component. The idea of the proof is to use spike variation of the absolutely continuous part of the control and a convex perturbation of the singular part. The principal result is given via an adjoint process of first order and two variational inequalities. The main result is obtained by using some stability properties of the state and adjoint processes with respect to the control variable.

The rest of the paper is organized as follows. In section 2, we formulate the control problem and introduce the assumptions of the model. Section 3 is devoted to the proof of the approximation result. In the last section, we state and prove a maximum principle for our relaxed control problem, which is the main result of this paper.

2 Formulation of the Problem

We consider optimal control problems of systems governed by stochastic differential equations, on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, such that $\mathcal{F}_0$ contains the $P-$null sets. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is a complete filtration generated by a standard Brownian motion B and an independent Poisson measure N. Assume that the compensator of N has the form $\upsilon(d\theta)dt$, where the jumps are confined to a compact set Γ and set

$$\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \upsilon(d\theta)dt.$$

Consider the following sets, A_1, is a nonempty compact subset of $\mathbb{R}^k$ and $A_2 = ([0; \infty))^m$, let U_1 the class of measurable, adapted processes $u : [0; T] \times \Omega \longrightarrow A_1$, and U_2 the class of measurable, adapted processes $\zeta : [0; T] \times \Omega \longrightarrow A_2$.

Definition 2.1 An admissible strict control is a term $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P, u_t, \zeta_t, W_t, X_t)$ such that

(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

(2) u_t is a A_1-valued process, progressively measurable with respect to $(\mathcal{F}_t)$.

(3) W_t is a $(\mathcal{F}_t, P)$- Brownian motion, $\widetilde{N}(dt, d\theta)$ is a compensated Poisson random measure independent from (W_t) and $(W_t, \widetilde{N}(dt, d\theta), X_t)$ satisfies (2.1).

4) ζ is of bounded variation, nondecreasing left-continuous with right limits and $\zeta_0 = 0$

5) (u, ζ) satisfies

$$E\left[\sup_{t\in[0;T]} |u_t|^2 + |\zeta_T|^2\right] < \infty.$$

We denote by $\mathcal{U}$ the space of strict controls. The controls as defined in the last definition are called weak controls, because of the possible change of the probability space, the Brownian motion and the Poisson random measure with u_t.

For any $(u, \zeta) \in \mathcal{U}$, we consider the following stochastic differential equation (SDE)

$$\begin{cases} dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \int_\Gamma f(t, x_{t^-}, \theta, u_t)\widetilde{N}(dt, d\theta) + G_t d\zeta_t \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} b &: [0; T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n \\ \sigma &: [0; T] \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R}) \\ f &: [0; T] \times \mathbb{R}^n \times \Gamma \times A \longrightarrow \mathbb{R}^n \\ G &: [0; T] \longrightarrow \mathcal{M}_{n\times m}(\mathbb{R}), \end{aligned}$$

are continuous functions.

The expected cost is given by:

$$J(u,\zeta)=E\left[g(x_T)+\int_0^T h(t,x_t,u_t)dt+\int_0^T k_t d\zeta_t,\right] \tag{2.2}$$

where

$$\begin{aligned} g &: \mathbb{R}^n \longrightarrow \mathbb{R}, \\ h &: [0;T]\times\mathbb{R}^n\times A \longrightarrow \mathbb{R}, \\ k &: [0;T] \longrightarrow ([0;\infty))^m , \end{aligned}$$

are continuous functions.

The strict optimal control problem is to minimize the functional $J(.,.)$ over $\mathcal{U}$. A control that solves this problem is called optimal.

The following assumptions will be in force throughout this paper

($\mathbf{H}_1$) The maps b, σ, f and h are continuously differentiable with respect to x. They and their derivatives b_x σ_x f_x and h_x are continuous in (x,u).

($\mathbf{H}_2$) The derivatives σ_x, f_x and g_x are bounded, and $b_x$and h_x are uniformly bounded in u.

($\mathbf{H}_3$) There exist $K>0$ such that, b, σ and f are bounded by $K(1+|x|+|u|)$, and g is bounded by $K(1+|x|)$.

($\mathbf{H}_4$) G and k are continuous and bounded.

Under the above hypothesis, (2.1) has a unique strong solution and the cost functional (2.2) is well defined from $\mathcal{U}$ into $\mathbb{R}$.

2.1 Examples

Singular control problems were first studied in connection with the so-called monotone follower problem and finite fuel problems. This class of problems is highly relevant in many branches of applied science, such as operation research, insurance problems, mathematical finance. In what follows, we give two examples of singular control problems.

2.1.1 The finite Fuel Problem

The finite fuel problem is a well-known control problem, in which the controller tracks a Brownian motion $x+B_t$ starting at x, by an adapted process $\varsigma(t)=\varsigma^+(t)-\varsigma^-(t)$ with bounded variation. $\varsigma^+(t)$ and $\varsigma^-(t)$ are increasing processes. We assume that the total variation of $\varsigma^-(t)$ is bounded, $|\varsigma|_t=\varsigma^+(t)+\varsigma^-(t)\le M$.

The objective is to minimize the discounted cost functional

$$J(\varsigma)=E\left(\int_0^\tau \exp(-\alpha t)\lambda X^2(t)dt+\int_{[0,\tau]}\exp(-\alpha t)d\,|\varsigma|_t+\exp(-\alpha\tau)\delta X^2(\tau)\right)$$

over such bounded variation processes $\varsigma(t)$ and stopping times τ.

This problem has been studied extensively by many authors under different assumptions; we can refer to [14] for more details.

2.1.2 The Portfolio Selection Under Transaction Costs

Assume that the investor has two instruments, a bank account $S_0(t)$ paying a fixed interest rate r and a risky asset (stock) $S_1(t)$, whose price evolves according to a geometric Brownian motion. The investor consumes at a rate $c(t)$ from the bank account, under the constraint that the total wealth should remain positive.

The dynamics for $S_0(t)$ and $S_1(t)$ are given by

$$\begin{cases} dS_0(t) = (rS_0(t) - c(t))\,dt - (1+\lambda)\,dL_t + (1-\mu)\,dU_t; \quad S_0(0) = x \\ dS_1(t) = \alpha S_1(t)S_1(t)dt + \sigma S_1(t)dB_t + dL_t - dU_t; \quad S_1(0) = y \end{cases}$$

A policy of investment and consumption is a triple (c, L, U), where L_t, U_t are right continuous nondecreasing processes, which represent the cumulative purchases and sales of stock, respectively.

The investor objective is to maximize the utility

$$J_{x,y}(c, L, U) = E_{x,y}\left(\int_0^{+\infty} e^{-\delta t}u(c(t))dt\right)$$

over admissible policies (c, L, U). For more details, see [23]

2.2 The Relaxed-Singular Control Problem

It is a well known that even in simple cases, there is no optimal control in the space of strict controls. The idea is to embed the space of strict controls into a wider space with good compactness properties. The idea of relaxed control is to replace the A_1-valued process (u_t) with a $\mathcal{P}(A_1)$-valued process (μ_t), where $\mathcal{P}(A_1)$ is the space of probability measures equipped with the topology of weak convergence.

Let $\mathcal{P}(A_1)$ be the space of probability measures on the control set A_1. Let $\mathbb{V}$ be the space of measurable transformations $\mu : [0, T] \longrightarrow \mathcal{P}(A_1)$, then μ can be identified as a nonnegative measure on the product $[0, T] \times A_1$, by putting for $C \in \mathcal{B}([0, T])$ and $D \in \mathcal{B}(A_1)$

$$\overline{\mu}(C \times D) = \int_C \mu_t(da)dt.$$

$\overline{\mu}$ may be extended uniquely to an element of $\mathbb{M}_+([0, T] \times A_1)$ the space of Radon measures on $[0, T] \times A_1$, equipped with the topology of stable convergence. This topology is the weakest topology such that the mapping

$$\overline{\mu} \longrightarrow \int_0^T \int_{A_1} \phi(t, a).\overline{\mu}(dt, da)$$

is continuous for all bounded measurable functions ϕ which are continuous in a.

Equipped with this topology, $\mathbb{M}_+([0, T] \times A_1)$ is a compact separable metrizable space. Therefore, $\mathbb{V}$ as a closed subspace of $\mathbb{M}_+([0, T] \times A_1)$ is also compact (see [15]) for more details.

Notice that $\mathbb{V}$ can be identified as the space of positive Radon measures on $[0, T] \times A_1$, whose projections on $[0, T]$ coincide with Lebesgue measure.

Let us define the Borel $\sigma-$field $\overline{\mathbb{V}}$ as the smallest $\sigma-$field such that the mappings

$$\int_0^T \int_{A_1} \phi(t, u).\mu_t(du)dt,$$

are measurable, where ϕ is a bounded measurable function which is continuous in a.

Let us also introduce the filtration $\left(\overline{\mathbb{V}}_t\right)$ on $\mathbb{V}$, where $\overline{\mathbb{V}}_t$ is generated by $\left\{1_{[0,t]}\mu, \quad \mu \in \mathbb{V}\right\}$.

Definition 2.2 A measure-valued control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable μ with values in $\mathbb{V}$ such that $\mu(\omega, t, da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t-$measurable.

The state variable is governed by a counting measure valued process, called the *relaxed Poisson measure*, as described in the following definition [5, 25, 26].

Definition 2.3 A relaxed Poisson measure N^μ is a counting measure valued process such that its compensator is the product measure $\mu \otimes \upsilon$ of the relaxed control μ with the compensator υ of N, such that for any Borel set $\Gamma_0 \subset \Gamma$ and $A_0 \subset A$, the processes

$$\widetilde{N}^\mu(t, A_0, \Gamma_0) = N^\mu(t, A_0, \Gamma_0) - \mu(t, A_0)\nu(\Gamma_0),$$

are $\mathcal{F}_t-$martingales and are orthogonal for disjoint $\Gamma_0 \times A$.

Now let us introduce the precise definitions of a relaxed control.

Definition 2.4 A relaxed control is a term $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P, \mu_t, W_t, \widetilde{N}^\mu, X_t)$ such that

(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.
(2) μ is a measure-valued control on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.
(3) W_t is a $(\mathcal{F}_t, P)$- Brownian motion, $\widetilde{N}^\mu(dt, d\theta)$ is a relaxed Poisson random measure, independent from (W_t) and $(W_t, \widetilde{N}^\mu(dt, d\theta), X_t)$ satisfies (2.3).
(4) ζ is of bounded variation, nondecreasing left-continuous with right limits and $\zeta_0 = 0$
(5) ζ satisfies $E\left[|\zeta_T|^2\right] < \infty$.

Accordingly, the relaxed cost functional will be given by

$$J(\mu, \zeta) = E\left[g(x_T^\mu) + \int_{A_1}\int_0^T h(t, x_t^\mu, a)\mu_t(da)dt + \int_0^T k_t d\zeta_t\right].$$

Let us denote by $\mathcal{R}$ the set of relaxed-singular controls.

For any $(\mu, \zeta) \in \mathcal{R}$, write the stochastic differential equation with controlled jumps in terms of relaxed Poisson measure as follows:

$$\begin{cases} dx_t^{\mu} = \int\limits_{A_1} b(t, x_t^{\mu}, a)\mu_t(da)dt + \sigma(t, x_t^{\mu})dB_t + \int\limits_{A_1}\int\limits_{\Gamma} f(t, x_{t-}^{\mu}, \theta, a)\widetilde{N}^{\mu}(dt, d\theta, da) + G_t d\zeta_t \\ x_0^{\mu} = 0 \end{cases} \tag{2.3}$$

It is well known that an optimal control exists in the class of relaxed-singular controls and that the value functions of the strict and relaxed control problems are equal. This important result is based on the continuity of the state process and the corresponding cost functional with respect to the control variable [18, 20].

3 Approximation of the Relaxed State Process

In order for the relaxed-singular control problem to be truly an extension of the strict control problem, the infimum of the expected cost over relaxed-singular controls must be equal to the infimum over strict controls. This result is based on the approximation of a relaxed control by a sequence of strict controls and the convergence of corresponding state processes.

Let us recall the so-called chattering lemma [8, 15, 17], whose proof is given here for the sake of completeness.

Lemma 3.1 (Chattering lemma) *Let μ be a relaxed control. Then there exists a sequence of adapted processes (u^n) with values in A_1, such that the sequence of random measures $\left(\delta_{u_t^n}(da)\,dt\right)$ converges in $\mathbb{V}$ to $dt.\mu_t(da)$, $P-a.s.$, that is for any f continuous in $[0,T]\times A_1$, we have:*
$\lim\limits_{n\to+\infty} \int_0^T f(s, u_s^n)ds = \int_0^T \int_{A_1} f(s,a)\mu_t(da)$ *uniformly in $t \in [0,T]$, $P-a.s.$*

Proof Suppose that $\mu(t, da)$ has continuous sample paths. Let $n \geq 1$, let us divide the interval $[0, T]$ into subintervals (T_i) of the form $[t_i, s_i[$ of length not exceeding 2^{-n}. Cover A_1 by finitely many disjoint sets $\left(A_j\right)$ such that diameter$(A_j) \leq 2^{-n}$. Choose a point (t_i, a_{ij}) in $T_i \times A_j$ for each i, j, t_i being as before. Let $\lambda_{ij} = \mu(t_i, A_j)$, then $\sum_j \lambda_{ij} = 1$. Subdivide each T_i further into disjoint left-closed, right-open intervals T_{ij} of length $\lambda_{ij}\times$ the length of T_i. Let $\varepsilon > 0$. Since f is uniformly continuous, then for n large enough we have

$$\left|f(t,a) - f(t_i, a_{ij})\right| < \varepsilon \quad \text{for } (t,a) \in T_i \times A_j,$$
$$\sup_a |f(t,a) - f(t_i, a)| < \varepsilon \quad \text{for } t \in T_i.$$

Defined the sequence of predictable process $\mu_n(.)$ by $\mu_n(t, da) = \delta_{a_{ij}}(da)$ for $t \in T_{ij}$.Then

$$\left|\int_0^T \int_{A_1} f(t,a)\mu_n(t,da)dt - \int_0^T \int_{A_1} f(t,a)\mu(t,da)dt\right|$$

$$= \left|\sum_{i,j}\left(\int_{T_{ij}} f(t,a_{ij})dt - \int_{T_{ij}}\int_{A_1} f(t,a)\mu(t,da)dt\right)\right|$$
$$\leq 2\varepsilon T + \left|\sum_{i,j}\left(\int_{T_{ij}} f(t,a_{ij})dt - \int_{T_{ij}}\int_{A_1} f(t_i,a)\mu(t,da)dt\right)\right|.$$

By path-continuity of $u(.)$, we may increase n further if necessary to ensure that the above is bounded by

$$3\varepsilon T + \left|\sum_{i,j}\left(\int_{T_{ij}} f(t,s_{ij})dt - \int_{T_{ij}}\int_{A_1} f(t_i,s)\mu(t_i,da)dt\right)\right|$$
$$\leq 4\varepsilon T + \left|\sum_{i,j}\left(\int_{T_{ij}} f(t,s_{ij})dt - \int_{T_{ij}}\int_{A_1} f(t_i,s_{ij})\mu(t_i,da)dt\right)\right|$$
$$\leq 4\varepsilon T,$$

which achieves the proof. Now if $\mu(t,da)$ does not have continuous sample paths, approximate it by controls which do, e.g., by $\mu^n(.)$ defined for a continuous f by

$$\int_{\mathbb{A}} f d\mu^n(t) = k^{-1}\int_{(t-1/n)\vee 0}^{t}\int_{A_1} f d\mu(a)da,$$

where $k = [t - (t - 1/n) \vee 0]$. □

The next theorem gives the stability of the stochastic differential equations, with respect to the control variable, and that the two problems have the same infimum of the expected costs.

Theorem 3.2 *Let (μ, ζ) be a relaxed-singular control, and let x^μ be the corresponding trajectory. Then there exists a sequence (u^n, ζ) of strict controls, such that*

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T}\left|x_t^n - x_t^\mu\right|^2\right] = 0,$$

and

$$\lim_{n\to\infty} J(u^n,\zeta) = J(\mu,\zeta), \tag{3.1}$$

where x^n denote the trajectory associated with the strict control (u^n, ζ).

Proof See Theorem 1 in [5] □

4 The Maximum Principle for Relaxed Control Problems

Our main goal in this section is to establish optimality necessary conditions for relaxed-singular control problems, where the system is described by a SDE driven by a relaxed Poisson measure which is a martingale measure, of the form (2.3) and the admissible controls are measure-valued processes which are called relaxed controls. The proof is based on the chattering lemma 3.1, and Ekeland's variational principle (4.6). We first derive necessary conditions of near optimality satisfied by a sequence of strict controls. Then by using stability properties of the state equations and adjoint processes, we are able to obtain the maximum principle for our relaxed control problem.

4.1 The Maximum Principle for Strict Controls

The purpose of this subsection is to derive optimality necessary conditions, satisfied by an optimal strict control. The proof is based on strong perturbation for the absolutely continuous part, and the convex perturbation for the singular components of the optimal control (u^*, ζ^*), which is defined by:

$$(u^h, \zeta^*) = \begin{cases} (\nu, \zeta^*) & \text{if } t \in [t_0; t_0 + h] \\ (u^*, \zeta^*) & \text{otherwise,} \end{cases}, \tag{4.1}$$

$$(u^*, \zeta^h) = (u^*, \zeta^* + h(\xi - \zeta^*), \tag{4.2}$$

for some $(\nu, \xi) \in \mathcal{U}$.

4.1.1 The First Variational Inequality

To obtain the first variational inequality in the stochastic maximum principle, we use the strong perturbations (4.1). The first variational inequality is derived from the fact that

$$\left. \frac{dJ(u^h, \zeta^*)}{dh} \right|_{h=0} \geq 0.$$

Indeed since (u^*, ζ^*) is optimal, then $J(u^h, \zeta^*) \geq J(u^*, \zeta^*)$ and therefore if the derivative exists we get $\left. \frac{dJ(u^h,\zeta^*)}{dh} \right|_{h=0} \geq 0$.

Note that the singular part is not affected by the perturbation (4.1). So, it is easy to check by standard arguments that

$$\lim_{h \to 0} E\left[\sup_{t \in [0;T]} \left| x_t^{(u^h,\zeta^*)} - x_t^* \right|^2 \right] = 0, \tag{4.3}$$

where

$$\begin{cases} x_t^{(u^h,\zeta^*)} = x_t^*; \quad t \leq t_0 \\ dx_t^{(u^h,\zeta^*)} = b(t, x_t^{(u^h,\zeta^*)}, \nu)dt + \sigma(t, x_t^{(u^h,\zeta^*)})dB_t + \displaystyle\int_\Gamma f(t, x_{t-}^{(u^h,\zeta^*)}, \theta, \nu)\widetilde{N}(dt, d\theta) \\ +G_t d\zeta_t^*; \quad t_0 < t < t_0 + h \\ dx_t^{(u^h,\zeta^*)} = b(t, x_t^{(u^h,\zeta^*)}, u^*)dt + \sigma(t, x_t^{(u^h,\zeta^*)})dB_t + \displaystyle\int_\Gamma f(t, x_{t-}^{(u^h,\zeta^*)}, \theta, u^*)\widetilde{N}(dt, d\theta) \\ +G_t d\zeta_t^*; \quad t_0 + h < t < T. \end{cases}$$

Under our assumptions, one has

$$\left.\frac{dJ(u^h,\zeta^*)}{dh}\right|_{h=0} = E\left[g_x(x_T^*)z_T + \varsigma_T\right], \tag{4.4}$$

where

$$\begin{cases} d\varsigma_t = h_x(t,x_t^*,u_t^*)z_t dt \qquad t_0 \le t \le T \\ \varsigma_{t_0} = h(t_0,x_{t_0}^*,\nu) - h(t_0,x_{t_0}^*,u_{t_0}^*), \end{cases}$$

and the process z is the solution of the linear SDE

$$dz_t = \begin{cases} b_x(t,x_t^*,u_t^*)z_t dt + \sigma_x(t,x_t^*)z_t dB_t + \int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)z_{t^-}\widetilde{N}(dt,d\theta); \quad t_0 \le t \le T \\ z_{t_0} = \left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right]. \end{cases} \tag{4.5}$$

From ($\mathbf{H}_2$) the variational Eq. (4.5) has a unique solution. To prove Prop (4.1.1) we need the following estimates.

Proposition 4.1 *Under assumptions ($\boldsymbol{H}_1$) – ($\boldsymbol{H}_3$), it holds that*

$$1)\quad \lim_{h\to 0} E\left[\left|\frac{x_t^{(u^h,\zeta^*)} - x_t^*}{h} - z_t\right|^2\right] = 0.$$

$$2)\quad \lim_{h\to 0} E\left[\left|\frac{1}{h}\int_{t_0}^T \left[(h(t,x_t^*,u_t^h) - (h(t,x_t^*,u_t^*)\right] - \varsigma_T\right|^2\right] = 0.$$

Proof Since $x_t^{(u^h,\zeta^*)} - x_t^*$ does not depend on the singular part, the proof follows that of Lemma 6 in [5]. □

Let us introduce the adjoint process and the first variational inequality from (4.4). We proceed as in [7].

Let $\varphi(t,\tau)$ be the solution of the linear equation

$$\begin{cases} d\varphi(t,\tau) = \left[b_x(t,x_t^*,u_t^*)\varphi(t,\tau) + \sigma_x(t,x_t^*)\varphi(t,\tau)dB_t \right. \\ \qquad\qquad + \int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)\varphi(t^-,\tau)\widetilde{N}(dt,d\theta) \qquad 0 \le \tau \le t \le T \\ \varphi(\tau,\tau) = I_d \end{cases} \tag{4.6}$$

This equation is linear with bounded coefficients. Hence it admits a unique strong solution. Moreover, the process φ is invertible, with an inverse ψ satisfying suitable integrability conditions.

From Ito's formula, we can easily check that $d(\varphi(t,\tau)\psi(t,\tau)) = 0$, and $\varphi(\tau,\tau)\psi(\tau,\tau) = I_d$, where ψ is the solution of the following equation

$$\begin{cases} d\psi(t,\tau) = \Big[\sigma_x(t,x_t^*)\psi(t,\tau)\sigma_x(t,x_t^*) - b_x(t,x_t^*,u_t^*)\psi(t,\tau) \\ -\int\limits_\Gamma f_x(t,x_{t-}^*,\theta,u_t^*)\psi(t^-,\tau)\upsilon(d\theta)\Big] dt \\ -\sigma_x(t,x_t^*)\psi(t,\tau)dB_t \\ -\psi(t^-,\tau)\int\limits_\Gamma \left(f_x(t,x_{t-}^*,\theta,u_t^*)+I_d\right)^{-1} f_x(t,x_{t-}^*,\theta,u_t^*)N(dt,d\theta) \\ \psi(\tau,\tau) = I_d. \end{cases} \quad 0 \le \tau \le t \le T$$

If $\tau = 0$ we simply write $\varphi(t,0) = \varphi_t$ and $\psi(t,0) = \psi_t$.
By the uniqueness property, it is easy to check that

$$z_t = \varphi(t,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right];$$

then (4.4) will become

$$\begin{aligned} \left.\frac{\mathrm{d}J(u^h)}{\mathrm{d}h}\right|_{h=0} &= E\Bigg[\int\limits_{t_0}^T h_x(t,x_t^*,u_t^*)\varphi(t,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right] dt \\ &+g_x(x_T^*)\varphi(T,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right] \\ &+\left[h(t_0,x_{t_0}^*,\nu) - h(t_0,x_{t_0}^*,u_{t_0}^*)\right]\Bigg]. \end{aligned} \tag{4.7}$$

Now, if we define the adjoint process by

$$p_t = y_t\psi_t^*, \tag{4.8}$$

where

$$\begin{aligned} y_t &= E\left[g_x(x_T^*)\varphi_T^* + \int\limits_t^T h_x(s,x_s^*,u_s^*)\varphi_s^* dt \big/ \mathcal{F}_t\right] \\ &= E\left[X/\mathcal{F}_t\right] - \int\limits_0^t h_x(s,x_s^*,u_s^*)\varphi_s^* dt, \end{aligned}$$

with

$$X = g_x(x_T^*)\varphi_T^* + \int\limits_0^T h_x(s,x_s^*,u_s^*)\varphi_s^* dt.$$

It follows that

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[p_t\left[b(t_0, x^*_{t_0}, v) - b(t_0, x^*_{t_0}, u^*_{t_0})\right] + \left[h(t_0, x^*_{t_0}, v) - h(t_0, x^*_{t_0}, u^*_{t_0})\right]\right].$$

Defining the Hamiltonian H from $[0; T] \times \mathbb{R}^n \times A \times \mathbb{R}^n$ into $\mathbb{R}$ by

$$H(t, x, u, p) = h(t, x_t, u_t) + pb(t, x_t, u_t), \tag{4.9}$$

we get from the optimality of u^*

$$E\left[H(t_0, x_{t_0}, v, p_{t_0}) - H(t_0, x_{t_0}, u^*_{t_0}, p_{t_0})\right] \geq 0.$$

By the Ito representation theorem [21], there exist two processes $Q \in \mathcal{M}^2$ and $R \in \mathcal{L}^2$ satisfying

$$E\left[X \diagup \mathcal{F}_t\right] = E\left[X\right] + \int_0^t Q_s dB_s + \int_0^t \int_\Gamma R_s(\theta)\widetilde{N}(ds, d\theta);$$

hence,

$$y_t = E\left[X\right] - \int_0^t h_x(s, x^*_s, u^*_s)\varphi_s ds + \int_0^t Q_s dB_s + \int_0^t \int_\Gamma R_s(\theta)\widetilde{N}(ds, d\theta).$$

Let

$$\begin{aligned} q_t &= Q_t\psi_t - p_t\sigma_x(t, x^*_t) \\ r_t(\theta) &= R_t(\theta)\psi_t\left(f_x(t, x^*_{t-}, \theta, u^*_t) + I_d\right)^{-1} \\ &\quad + p_t\left[\left(f_x(t, x^*_{t-}, \theta, u^*_t) + I_d\right) - I_d\right]. \end{aligned}$$

The above discussion will allow us to introduce the next inequality which is the first variational inequality.

$$E\left[H(t, x^*_t, v, p_t) - H(t, x^*_t, u^*_t, p_t)\right] \geq 0.dt - a.e. \tag{4.10}$$

where the Hamiltonian H is defined by (4.9).

4.1.2 The Second Variational Inequality

To obtain the second variational inequality of the stochastic maximum principle, we use the second perturbation (4.2) of the optimal control. Since (u^*, ζ^*) is optimal control, then we have

$$J(u^*,\zeta^h) - J(u^*,\zeta^*) \geq 0. \tag{4.11}$$

From this inequality, we will be able to derive the second variational inequality.

Lemma 4.2 *Let $x_t^{(u^*,\zeta^h)}$ be the trajectory associated with (u^*,ζ^h), and x_t^* the trajectory associated with (u^*,ζ^*), then the following estimate holds:*

$$\lim_{h\to 0} E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)} - x_t^*\right|^2\right] = 0. \tag{4.12}$$

Proof From the boundedness and continuity of b_x, σ_x, and f_x and by using the Burkholder–Davis–Gundy inequality for the martingale part, we get

$$E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)} - x_t^*\right|^2\right] \leq C_1\int_0^t E\left[\sup_{s\in[0;T]}\left|x_s^{(u^*,\zeta^h)} - x_s^*\right|^2\right]ds + C_2h^2d\left|\xi-\zeta^*\right|^2$$
$$+ C_3\int_0^t E\left[\sup_{s\in[0;T]}\int_\Gamma\left(\sup_{\theta\in\Gamma}\left|x_s^{(u^*,\zeta^h)} - x_s^*\right|^2\right)\upsilon(d\theta)ds\right],$$

which implies that

$$E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)} - x_t^*\right|^2\right] \leq C_1\int_0^t E\left[\sup_{s\in[0;T]}\left|x_s^{(u^*,\zeta^h)} - x_s^*\right|^2\right]ds + C_2h^2d\left|\xi-\zeta^*\right|^2$$
$$+ C_3\upsilon(\Gamma)\int_0^t E\left[\sup_{s\in[0;T]}\left(\sup_{\theta\in\Gamma}\left|x_s^{(u^*,\zeta^h)} - x_s^*\right|^2\right)ds\right].$$

Therefore,

$$E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)} - x_t^*\right|^2\right] \leq (C_1 + C_3\upsilon(\Gamma))\int_0^t E\left[\sup_{s\in[0;T]}\left|x_s^{(u^*,\zeta^h)} - x_s^*\right|^2\right]ds$$
$$+C_2h^2d\left|\xi-\zeta^*\right|^2.$$

Since $\upsilon(\Gamma) < \infty$, by the Gronwall inequality, the result follows immediately by letting h go to zero. □

Lemma 4.3 *Under assumptions $(\boldsymbol{H}_1) - (\boldsymbol{H}_4)$, it holds that*

$$\lim_{h\to 0} E\left[\left|\frac{x_t^{(u^*,\zeta^h)} - x_t^*}{h} - z_t\right|^2\right] = 0, \tag{4.13}$$

where z_t is the solution of the following equation:

$$z_t = \int_0^t b_x(s, x_s^*, u_s^*) z_s ds + \int_0^t \sigma_x(s, x_s^*) z_s dB_s + \int_0^t \int_\Gamma f_x(s, x_{s-}^*, \theta, u_s^*) z_{s-} \widetilde{N}(ds, d\theta)$$
$$+ \int_0^t G_s \, d(\xi - \zeta^*)_s.$$

Proof Let

$$y_t^h = \frac{x_t^{(u^*, \zeta^h)} - x_t^*}{h} - z_t,$$

then,

$$dy_t^h = \frac{1}{h}\left[b(t, x_t^{(u^*, \zeta^h)}, u_t^*) - b(t, x_t^*, u_t^*)\right] dt + \frac{1}{h}\left[\sigma(t, x_t^{(u^*, \zeta^h)}) - \sigma(t, x_t^*)\right] dB_t$$
$$+ \frac{1}{h}\int_\Gamma \left[f(t, x_{t-}^{(u^*, \zeta^h)}, \theta, u_{t-}^*,) - f(t, x_{t-}^*, \theta, u_{t-}^*)\right] \widetilde{N}(dt, d\theta)$$
$$- b_x(t, x_t^*, u_t^*) z_t dt - \sigma_x(t, x_t^*) z_t dB_t$$
$$- \int_\Gamma f_x(t, x_{t-}^*, \theta, u_{t-}^*) z_{t-} \widetilde{N}(dt, d\theta).$$

Hence

$$y_t^h = \int_0^t \int_0^1 b_x(s, x_t^* + \lambda(x_t^{(u^*, \zeta^h)} - x_t^*), u_s^*) y_s^h d\lambda ds$$
$$+ \int_0^t \int_0^1 \sigma_x(s, x_t^* + \lambda(x_t^{(u^*, \zeta^h)} - x_t^*)) y_s^h d\lambda dB_s$$
$$+ \int_0^t \int_0^1 \int_\Gamma f_x(s, x_{s-}^* + \lambda(x_{s-}^{(u^*, \zeta^h)} - x_{s-}^*), \theta, u_{s-}^*) y_{s-}^h d\lambda \widetilde{N}(ds, d\theta) + \rho_t^h,$$

where

$$\rho_t^h = \int_0^t \int_0^1 b_x(s, x_t^* + \lambda(x_t^{(u^*, \zeta^h)} - x_t^*), u_s^*) z_s d\lambda ds$$

$$+\int_0^t\int_0^1 \sigma_x(s,x_t^*+\lambda(x_t^{(u^*,\zeta^h)}-x_t^*))z_s d\lambda dB_s$$
$$+\int_0^t\int_0^1\int_\Gamma f_x(s,x_{s-}^*+\lambda(x_{s-}^{(u^*,\zeta^h)}-x_{s-}^*),\theta,u_{s-}^*)z_{s-}d\lambda\widetilde{N}(ds,d\theta)$$
$$-\int_0^t\int_\Gamma f_x(s,x_{s-}^*,\theta,u_{s-}^*)z_{s-}\widetilde{N}(ds,d\theta)$$
$$-\int_0^t b_x(s,x_s^*,u_s^*)z_s ds-\int_0^t \sigma_x(s,x_s^*)z_s dB_s.$$

Therefore,

$$E\left|y_t^h\right|^2\le KE\int_0^t\left|\int_0^1 b_x(s,x_t^*+\lambda(x_t^{(u^*,\zeta^h)}-x_t^*),u_s^*)y_s^h d\lambda\right|^2 ds$$
$$+KE\int_0^t\left|\int_0^1 \sigma_x(s,x_t^*+\lambda(x_t^{(u^*,\zeta^h)}-x_t^*))y_s^h d\lambda\right|^2 ds$$
$$+KE\int_0^t\int_\Gamma\left|\int_0^1 f_x(s,x_s^*+\lambda(x_s^{(u^*,\zeta^h)}-x_s^*),\theta,u_s^*)y_s^h d\lambda\right|^2 \upsilon(d\theta)ds+KE\left|\rho_t^h\right|^2.$$

Since b_x, σ_x, and f_x are bounded, then

$$E\left|y_t^h\right|^2\le CE\int_0^t\left|y_s^h\right|^2 ds+KE\left|\rho_t^h\right|^2.$$

We conclude by using the boundedness and continuity of b_x, σ_x, and f_x, the dominated convergence theorem and $\lim_{h\to 0} E\left|\rho_t^h\right|^2=0$. Hence, by Gronwall's lemma, we get $\lim_{h\to 0} E\left|y_t^h\right|^2=0$. □

Lemma 4.4 *The following inequality holds:*

$$E\left[g_x(x_T^*)z_T+\int_0^T h_x(t,x_t^*,u_t^*)z_t dt+\int_0^T k_t d(\xi-\zeta^*)_t\right]\ge 0.$$

Proof From (4.11), we have

$$0 \leq \frac{1}{h}\left[E\left[g(x_T^{(u^*,\zeta^h)}) - g(x_T^*)\right]\right.$$
$$\left. +E\int_0^T \left[h(t, x_t^{(u^*,\zeta^h)}, u_t^*) - h(t, x_t^*, u_t^*)\right] dt\right] + E\int_0^T k_t d(\xi - \zeta^*)_t$$
$$= E\int_0^1 \left[\left(\frac{x_T^{(u^*,\zeta^h)} - x_T^*}{h}\right) g_x\left[x_T^* + \lambda(x_T^{(u^*,\zeta^h)} - x_T^*)\right] d\lambda\right]$$
$$+ E\int_0^T\int_0^1 \left[\left(\frac{x_t^{(u^*,\zeta^h)} - x_t^*}{h}\right) h_x\left[x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*), u_t^*\right]\right] d\lambda dt + E\int_0^T k_t d(\xi - \zeta^*)_t.$$

From the continuity and boundedness of g_x and h_x, by letting h go to zero, we can deduce the result from (4.12) and (4.13).

Now, we are able to derive the second variational inequality from (4.11). If $\varphi(t, s)$ denotes the solution of (4.6), it is easy to check that z_t is given by

$$z_t = \int_0^T \varphi(t, s) G_t d(\xi - \zeta^*)_t.$$

Replacing z_t by its value, we obtain the second variational inequality

$$E\left[\int_0^T (k_t + G_t^* p_t) d(\xi - \zeta^*)_t\right] \geq 0, \tag{4.14}$$

where p is the adjoint process defined by (4.8). □

Combining the first and second variational inequalities (4.10) and (4.14), we are able to state the maximum principle for strict controls.

Theorem 4.5 *(*the maximum principle for strict control problem*) Let* (u^*, ζ^*) *be the optimal strict control minimizing the cost functional* J *over* $\mathcal{U}$ *and denote by* x^* *the corresponding optimal trajectory, then the following variational inequalities hold:*

1) $E\left[H(t, x_t^*, v, p_t) - H(t, x_t^*, u_t^*, p_t)\right] \geq 0.dt - a.e,$

2) $\int_0^T \{k_t + G_t p_t\} d(\zeta - \zeta^*)_t \geq 0.$

where the Hamiltonian H is defined by (4.9).

4.2 The Maximum Principle for Near Optimal Controls

In this section, we establish necessary conditions of near optimality satisfied by a sequence of nearly optimal strict controls; this result is based on Ekeland's variational principle, which is given by the next lemma.

Lemma 4.6 *Let (E, d) be a complete metric space and $f : E \to \overline{\mathbb{R}}$ be lower semi-continuous and bounded from below. Given $\varepsilon > 0$, suppose $u^{\varepsilon} \in E$ satisfies $f(u^{\varepsilon}) \leq \inf(f) + \varepsilon$. Then for any $\lambda > 0$, there exists $\nu \in E$ such that*

- $f(\nu) \leq f(u^{\varepsilon})$
- $d(u^{\varepsilon}, \nu) \leq \lambda$
- $f(\nu) \leq f(\omega) + \frac{\varepsilon}{\lambda} d(\omega, \nu)$ *for all* $\omega \neq \nu$.

To apply Ekeland's variational principle, we have to endow the set $\mathcal{U}$ of strict controls with an appropriate metric. For any (u, ζ) and $(\nu, \xi) \in \mathcal{U}$, we set

$$d_1(u, \nu) = P \otimes dt \{(\omega, t) \in \Omega \times [0; T], \quad u(\omega, t) \neq \nu(\omega, t)\}$$

$$d_2(\zeta, \xi) = E \left(\sup_{t \in [0;T]} |\zeta_t - \xi_t|^2 \right)^{\frac{1}{2}}$$

$$d[(u, \zeta), (\nu, \xi)] = d_1(u, \nu) + d_2(\zeta, \xi)$$

where $P \otimes dt$ is the product measure of P with the Lebesgue measure dt.

Remark 4.7 According to [28, 29], $(\mathcal{U}, d)$ is a complete metric space and the cost functional J is continuous from $\mathcal{U}$ into $\mathbb{R}$.

Let $(\mu^*, \zeta^*) \in \mathcal{R}$ be an optimal relaxed-singular control and denote by x^* the trajectory of the system controlled by (μ^*, ζ^*). From Lemma (3.1), there exists a sequence (u^n) of strict controls such that

$$\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t^*(da)dt \quad P - a.s$$

and

$$\lim_{n \to \infty} E\left[\left| x_t^n - x_t^{\mu^*} \right|^2 \right] = 0$$

where (x^n) is the solution of (2.3) corresponding to μ^n.

According to the optimality of μ^* and (4.6), there exists a sequence (ε_n) of positive numbers with $\lim_{n \to \infty} \varepsilon_n = 0$ such that

$$J(u^n, \zeta^*) = J(\mu^n, \zeta^*) \leq J(\mu^*, \zeta^*) + \varepsilon_n = \inf_{u \in U} J(u, \zeta) + \varepsilon_n.$$

A suitable version of Lemma (4.6) implies that, given any $\varepsilon_n > 0$, there exists $(u^n, \zeta^*) \in \mathcal{U}$ such that

$$J(u^n, \zeta^*) \leq J(\nu, \xi) + \varepsilon_n d\left[(u^n, \zeta^*), (\nu, \xi)\right], \forall (\nu, \xi) \in \mathcal{U}. \tag{4.15}$$

Let us define the perturbations

$$(u^{n,h},\zeta^*)=\begin{cases}(\nu,\zeta^*) & \text{if } t\in[t_0;t_0+h]\\ (u^n,\zeta^*) & \text{otherwise,}\end{cases}$$
$$(u^n,\zeta^h)=\big(u^n,\zeta^*+h(\xi-\zeta^*)\big)$$

From (4.15) we have

$$0\le J(u^{n,h},\zeta^*)-J(u^n,\zeta^*)+\varepsilon_n d\left[(u^{n,h},\zeta^*),(u^n,\zeta^*)\right]$$
$$0\le J(u^n,\zeta^h)-J(u^n,\zeta^*)+\varepsilon_n d\left[(u^n,\zeta^h),(u^n,\zeta^*)\right]$$

Using the definition of the distance d, it holds that

$$0\le J(u^{n,h},\zeta^*)-J(u^n,\zeta^*)+\varepsilon_n d_1(u^{n,h},u^n),$$
$$0\le J(u^n,\zeta^h)-J(u^n,\zeta^*)+\varepsilon_n d_2(\zeta^h,\zeta^*).$$

Finally, using the definition of d_1 and d_2, it holds that

$$0\le J(u^{n,h},\zeta^*)-J(u^n,\zeta^*)+\varepsilon_n C_1 h,$$
$$0\le J(u^n,\zeta^h)-J(u^n,\zeta^*)+\varepsilon_n C_2 h. \tag{4.16}$$

where C_i is a positive constant.
Now, we can introduce the next theorem, which is the main result of this subsection.

Theorem 4.8 *For each $\varepsilon_n>0$, there exists a nearly optimal strict control $(u^n,\zeta)\in\mathcal{U}$ such that there exists a unique triple of square integrable adapted processes (p^n,q^n,r^n) solution of the backward SDE*

$$\begin{cases} dp_t^n=-\Big[h_x(t,x_t^n,u_t^n)+p_t^n b_x(t,x_t^n,u_t^n)+q_t^n\sigma_x(t,x_t^n)\\ \qquad +\displaystyle\int_\Gamma r_t^n(\theta)f(t,x_{t-}^n,\theta,u_t^n)\upsilon(d\theta)\Big]dt\\ \qquad +q_t^n dB_t+\displaystyle\int_\Gamma r_t^n(\theta)\widetilde{N}(dt,d\theta)\\ p_T^n=g_x(x_T^n),\end{cases} \tag{4.17}$$

such that for all $(u,\zeta)\in\mathcal{U}$

$$E\int_0^T\Big[\big[H(t,x_t^n,\nu,p_t^n)-H(t,x_t^n,u_t^n,p_t^n)\big]+C_1\varepsilon_n\Big]dt\ge 0,$$

$$E\int_0^T \left[(k_t + G_t p_t^n) d(\zeta_t - \zeta_t^*) + C_2 \varepsilon_n\right] \geq 0. \tag{4.18}$$

where C_i is a positive constant.

Proof From inequality (4.16), we use the same method as in the previous subsection, to obtain the desired result (4.18). □

Remark 4.9 The preceding theorem is interesting by itself, in the sense that in many practical situations in engineering characterizing and computing nearly optimals is sufficient.

4.3 The Relaxed Stochastic Maximum Principle

Now, we can introduce the next theorem, which is the main result of this paper.

Theorem 4.10 (The relaxed stochastic maximum principle) *Let (μ^*, ζ^*) be an optimal relaxed-singular control minimizing the functional $J\ (.,.)$ over $\mathcal{R}$, and let x_t^*be the corresponding optimal trajectory. Then there exists a unique triple of square integrable and adapted processes (p^*, q^*, r^*), solution of the backward SDE*

$$\begin{cases} dp_t^* = -\left[\int_{A_1} h_x(t, x_t^*, a)\mu_t^*(da) + \int_{A_1} p_t^* b_x(t, x_t^*, a)\mu_t^*(da) \right. \\ \left. +q_t^*\sigma_x(t, x_t^{\mu^*}) + \int_{A_1}\int_{\Gamma} r_t^*(\theta) f(t, x_{t-}^{\mu^*}, \theta, a)\mu_t^* \otimes \upsilon(da, d\theta)\right] dt \\ \qquad +q_t^* dB_t + \int_{\Gamma} r_t^*(\theta)\widetilde{N}^*(dt, d\theta, da) \\ p_T^* = g_x(x_T^*), \end{cases} \tag{4.19}$$

such that for all $(u, \zeta) \in \mathcal{U}$:

i) $E\int_0^T \left[H(t, x_t^*, u_t, p_t^*, q^*, r_t^*(.)) - \int_{A_1} H(t, x_t^*, a, p_t^*, q^*, r_t^*(.))\mu_t^*(da)\right] dt \quad \geq 0,$

ii) $E\left[\int_0^T (k_t + G_t p_t^*) d(\zeta_t - \zeta_t^*)\right] \geq 0.$

For the proof of the above theorem, we need the following stability lemma.

Lemma 4.11 *Let* (p^n, q^n, r^n) *and* (p^*, q^*, r^*), *be the unique solutions of (4.17) and (4.19), respectively. Then we have*

$$\lim_{n\to\infty}\left[E\left|p^n - p^*\right|^2 + E\int_t^T \left|q^n - q^*\right|^2 ds + E\int_t^T\int_\Gamma \left|r^n - r^*\right|^2 \upsilon(d\theta)ds\right] = 0.$$

Proof Since the singular term does not affect the adjoint processes, the proof is the same as the proof of Lemma 8 in [5]. □

Proof of Theorem 4.9. The result follows immediately by letting n go to infinity in inequalities of 4.18 and using Lemma (4.11). □

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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