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Interior Pointwise Gradient Estimates for Quasilinear Elliptic Equations in Heisenberg Group

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Abstract

Let $n \in \mathbb{N}$ and $\mathbb{H}^n$ be the Heisenberg group of dimension $2n+1$. Let Ω be a bounded open subset of $\mathbb{H}^n$ and $p \in (1, Q)$, where Q is the homogeneous dimension of $\mathbb{H}^n$. Within an appropriate framework, we prove an interior pointwise gradient estimate for a weak solution to the problem

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}(\mathbf{A}(x, \mathfrak{X}u)) = g - \operatorname{div}_{\mathbb{H}} f & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathfrak{X}$ represents the horizontal gradient on the Heisenberg group.

Keywords Quasilinear elliptic equation · Mixed data · Pointwise gradient estimate · Regularity · Heisenberg group

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1 Introduction

This paper concerns a regularity estimate for an elliptic equation in the setting of Heisenberg group. While much is known for the regularity theory in the Euclidean spaces, its Heisenberg counterpart is less developed. Recently, there is a drastic movement toward the latter. Namely, we refer the readers to [5, 8–10, 14, 15, 17] and the references therein. Here, we wish to add a result in this direction to the existing literature. The result is in the spirit of the Euclidean pointwise estimates in [2–4, 11]. Also see [1, 6, 7, 12–14] and the references therein for more new trends in elliptic equations and systems.

Before delivering the main content, we quickly review the Heisenberg group. Let $n \in \mathbb{N}$ and $\mathbb{H}^n$ be the Heisenberg group of dimension $2n+1$. That is, $\mathbb{H}^n$ is a two-step nilpotent Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$. The group operation is given by

$$(z, t) \cdot (w, s) := \big(z + w, t + s + 2\ \mathrm{Im}\langle z, w\rangle\big)$$

for all $(z, t), (w, s) \in \mathbb{H}^n$, where $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$ and

$$\langle z, w\rangle := \sum_{j=1}^{n} z_j\, \overline{w_j}.$$

For each $u = (z, t) \in \mathbb{H}^n$, the inverse element is

$$u^{-1} = (-z, -t)$$

and the identity is $0 = (0, 0)$.

By writing

$$(z, t) = (x_1 + iy_1, \ldots, x_n + iy_n, t) \qquad \text{for each } (z, t) \in \mathbb{H}^n,$$

the corresponding Lie algebra of the left-invariant vector fields on $\mathbb{H}^n$ is spanned by

$$X_j := \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad X_{n+j} := \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \qquad \text{and} \qquad T := \frac{\partial}{\partial t},$$

where $j \in \{1, \ldots, n\}$.

All non-trivial commutation relations are given by

$$[X_j, Y_j] = -4T, \qquad j \in \{1, \ldots, n\}$$

with $[\cdot, \cdot]$ being the usual Lie bracket.

The horizontal gradient $\mathfrak{X}$ is defined by

$$\mathfrak{X} := (X_1, \ldots, X_{2n}).$$

Let $\Omega \subset \mathbb{H}^n$. We obtain the horizontal Sobolev spaces $HW^{1,p}(\Omega)$ and $HW_0^{1,p}(\Omega)$ by replacing the usual gradient ∇ with $\mathfrak{X}$ in the definitions of $W^{1,p}$ and $W_0^{1,p}$ spaces. The horizontal divergence operator is then given by

$$\mathrm{div}_{\mathbb{H}}\, f := \mathfrak{X} \cdot f \qquad \text{for each } f \in W^{1,p}(\Omega; \mathbb{R}^{2n}).$$

For each $a > 0$ and $(z, t) \in \mathbb{H}^n$, a dilation on $\mathbb{H}^n$ is defined by

$$\delta_a(z, t) := (az, a^2 t),$$

which is also an automorphism of $\mathbb{H}^n$.

The homogeneous norm of $u = (z, t) \in \mathbb{H}^n$ is given by

$$|u| = (|z|^4 + t^2)^{1/4}.$$

With this in mind,

$$|u^{-1}| = |u| \qquad \text{and} \qquad |\delta_a(u)| = a\,|u|,$$

where $a > 0$. The homogeneous norm enjoys the triangle inequality and hence gives rise to a left-invariant distance

$$d(u, v) = |u^{-1} \cdot v|$$

for each $u, v \in \mathbb{H}^n$. Then, we may define the open ball with center $u \in \mathbb{H}^n$ and radius $r \in (0, \infty)$ by

$$B_r(u) := \{v \in \mathbb{H}^n : d(u, v) < r\}.$$

Both left and right Haar measures on $\mathbb{H}^n$ coincide with the Lebesgue measure $dz\,dt$ on $\mathbb{C}^n \times \mathbb{R}$. We denote the Lebesgue measure of a measurable set $E \subset \mathbb{H}^n$ by $|E|$. Then,

$$|B_r(u)| = r^Q\,|B_1(0)| \tag{1}$$

for all $u \in \mathbb{H}^n$ and $r \in (0, \infty)$, where

$$Q := 2n + 2$$

is called the homogeneous dimension of $\mathbb{H}^n$. Moreover,

$$|B_1(0)| = \frac{2\,\pi^{n+\frac{1}{2}}\,\Gamma(\frac{n}{2})}{(n+1)\,\Gamma(n)\,\Gamma(\frac{n+1}{2})}$$

and $\Gamma(\cdot)$ is the usual Gamma function.

In the sequel, we write

$$\lambda B_r(u) := B_{\lambda r}(u).$$

Then, it follows from (1) that

$$|\lambda B_r(u)| = \lambda^Q |B_r(u)|.$$

Next we formulate our problem precisely. Let $n \in \mathbb{N}$ and Ω be a bounded open subset of $\mathbb{H}^n$. Let $p \in (1, Q)$. Consider

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}(\mathbf{A}(x, \mathfrak{X}u)) = g - \operatorname{div}_{\mathbb{H}} f & \text{in} \quad \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

$$g \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}) \quad \text{and} \quad f \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^{2n}). \tag{3}$$

We assume that

$$\mathbf{A} : \mathbb{H}^n \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \text{ is a Caratheodory function} \tag{4}$$

This means that $\mathbf{A}(x, z)$ is measurable in x for every z and is continuous in z for a.e. x. Moreover, $\mathbf{A}(x, z)$ is differentiable in $z \neq 0$ for a.e. x. The following structural conditions are also imposed on $\mathbf{A}$:

$$|\mathbf{A}(x, z)| \le \Lambda |z|^{p-1} \quad \text{and} \quad |D_2\mathbf{A}(x, z)| \le \Lambda |z|^{p-2}, \tag{5}$$

$$\langle D_2\mathbf{A}(x, z)\, \eta, \eta\rangle \ge \Lambda^{-1} |z|^{p-2} |\eta|^2, \tag{6}$$

$D_2\mathbf{A}(x, z)$ is symmetric and

$$|D_2\mathbf{A}(x, z) - D_2\mathbf{A}(x, \eta)| \le \begin{cases} \Lambda |z|^{p-2} |\eta|^{p-2} (|z|^2 + |\eta|^2)^{(2-p-\alpha)/2} |z - \eta|^\alpha & \text{if } p < 2, \\ \Lambda (|z|^2 + |\eta|^2)^{(p-2-\alpha)/2} |z - \eta|^\alpha & \text{if } p \ge 2 \end{cases}$$

for some $\alpha \in (0, |2 - p|)$ if $p \neq 2$ and $\alpha \in (0, 1]$ if $p = 2$, (7)

$$|\mathbf{A}(x, z) - \mathbf{A}(x_0, z)| \le \Lambda\, \omega(|x - x_0|)\, |z|^{p-1} \tag{8}$$

for some constant $\Lambda \ge 1$ as well as for all $x, x_0 \in \mathbb{H}^n$ and $(z, \eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \backslash \{(0, 0)\}$, where $D_2\mathbf{A}(x, z)$ denotes the Jacobian matrix of $\mathbf{A}$ with respect to the second variable $z \in \mathbb{R}^{2n} \backslash \{0\}$. In (8), the function $\omega : [0, \infty) \longrightarrow [0, 1]$ is required to be non-decreasing and satisfies

$$\lim_{r \downarrow 0} \omega(r) = \omega(0) = 0$$

together with Dini's condition

$$W_0 := \int_0^1 \omega(r)^{\tau_0} \frac{dr}{r} < \infty \tag{9}$$

with $\tau_0 := \frac{2}{p} \wedge 1$.

Note that (2) encapsulates well the p-Laplace equation with mixed data

$$-\Delta_p u := -\mathrm{div}_{\mathbb{H}}(|\mathfrak{X}u|^{p-2}\mathfrak{X}u) = g - \mathrm{div}_{\mathbb{H}} f \quad \text{in } \Omega.$$

Our aim here is to derive an interior pointwise estimate for the gradient of a weak solution to (2). To state our main result, we first need some definitions.

Definition 1.1 A function $u \in HW_0^{1,p}(\Omega)$ is a weak solution to (2) if

$$\int_\Omega \mathbf{A}(x, \mathfrak{X}u) \cdot \mathfrak{X}\varphi \, dx = \int_\Omega f \cdot \mathfrak{X}\varphi \, dx + \int_\Omega g\, \varphi \, dx$$

for all $\varphi \in HW_0^{1,p}(\Omega)$.

In what follows, for each measurable function $h : \Omega \longrightarrow \mathbb{R}$ denote

$$(h)_B = \fint_B h(x)\, dx = \frac{1}{|B|} \int_B h(x)\, dx,$$

where $B \subset \Omega$ is a ball. The oscillation of h on a set $A \subset \Omega$ is defined by

$$\operatorname*{osc}_A h := \sup_{x,y \in A} \big(h(x) - h(y)\big) = \sup_{x \in A} h(x) - \inf_{x \in A} h(x).$$

Also set

$$\mathbf{F}_q^R(f, g)(x) := \int_0^R \left[\left(\fint_{B_\rho(x_0)} |f - (f)_{B_\rho(x_0)}|^{q'} \, dx \right)^{\frac{1}{q}} + \rho^{\frac{1}{q-1}} \left(\fint_{B_\rho(x_0)} |g|^{\frac{Qq}{Qq-Q+q}} \, dx \right)^{\frac{Qq-Q+q}{(Qq-Q)q}} \right] \frac{d\rho}{\rho}$$

for each $q \in (1, n)$ and $R > 0$, where q' is the conjugate index of q.

Our main result is as follows.

Theorem 1.2 *Let $n \in \mathbb{N}$ and Ω be a bounded open subset of $\mathbb{H}^n$. Let $p \in (1, Q)$. Assume (3), (4), (5), (6), (7) and (8). Suppose that $u \in C^1(\Omega)$ is a weak solution to (2). Then, there exists a constant $C = C(n, p, \Lambda, W_0) > 0$ such that*

$$|\mathfrak{X}u(x)| \le C \left[\mathbf{F}_p^R(f, g)(x) + \left(\fint_{B_R(x)} |\mathfrak{X}u(y)|^p \, dy \right)^{\frac{1}{p}} \right]$$

for all ball $B_R(x) \subset \Omega$.

A technical remark is worth mentioning.

Remark 1.3 Working with Heisenberg group has its own intrinsic difficulties. See Lemma 3.1 below. Therein the appearance of the term M_r is new, which does not happen in the Euclidean setting. This results in weaker estimates compared to the corresponding Euclidean versions. The phenomenon was also observed in [10, Remark 3.1].

Despite all these, we still manage to achieve the pointwise estimate by carefully handling the new term M_r.

The procedure for proving Theorem 1.2 involves the construction of certain comparison estimates in Sect. 2, an iteration argument in Sect. 3. With these in place, we prove Theorem 1.2 in Sect. 4.

Throughout assumptions. In the whole paper, let $n \in \mathbb{N}$ and Ω be a bounded open subset of $\mathbb{H}^n$. Let $p \in (1, Q)$. We always assume the set of conditions (3), (4), (5), (6), (7) and (8). If further assumptions are required, they will be explicitly stated in the corresponding statements.

2 Comparison Estimates

In this section, we construct two comparison estimates that serve to prove Theorem 1.2. The second one in Lemma 2.4 is new as it adapts our general setting proposed in this paper.

For later use, we draw some nice consequences from the aforementioned structural conditions. The first inequality in (5) and the Caratheodory property together yield

$$\mathbf{A}(x, 0) = 0 \quad \text{for a.e. } x \in \mathbb{H}^n.$$

Meanwhile (6) implies the strict monotonicity condition

$$\langle \mathbf{A}(x, z) - \mathbf{A}(x, \eta), z - \eta \rangle \geq C(n, p, \Lambda)\, \Phi(z, \eta) \tag{10}$$

for all $(z, \eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus \{(0, 0)\}$ and for a.e. $x \in \mathbb{H}^n$, where $\Phi : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is defined by

$$\Phi(z, \eta) := \begin{cases} \left(|z|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |z - \eta|^2 & \text{if } p \leq 2, \\ |z - \eta|^p & \text{if } p > 2 \end{cases}$$

due to [16, Lemma 1].

Let $u \in HW_0^{1,p}(\Omega)$ be a weak solution to (2). Suppose $B_{2r}(x_0) \Subset \Omega$. Hereafter we write

$$B_\rho := B_\rho(x_0)$$

for each $\rho > 0$. Let $w \in u + HW_0^{1,p}(B_{2r})$ be the unique weak solution to the problem

$$\begin{cases} -\operatorname{div}_{\mathbb{H}} (\mathbf{A}(x, \mathfrak{X} w)) = 0 & \text{in } B_{2r}, \\ w = u & \text{on } \partial B_{2r} \end{cases} \tag{11}$$

and let $v \in w + HW_0^{1,p}(B_r)$ be the unique solution of

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}(\mathbf{A}(x_0, \mathfrak{X}v)) = 0 & \text{in} \quad B_r, \\ \qquad\qquad v = w & \text{on} \quad \partial B_r. \end{cases} \tag{12}$$

Recall the following uniform estimate from [10, (2.12)].

Lemma 2.1 *There exists a constant $C = C(n, p, \Lambda) > 0$ such that*

$$\|\mathfrak{X}v\|_{L^\infty(B_{\delta r})} \leq C\,(1-\delta)^{-\frac{Q}{t}} \left(⨍_{B_r} |\mathfrak{X}v|^t dx \right)^{1/t}$$

for all $1 < t < \infty$, $r > 0$ and $0 < \delta < 1$.

As a direct consequence of Lemma 2.1, we obtain the following useful estimate.

Corollary 2.2 *There exists a constant $C = C(n, p, \Lambda) > 0$ such that*

$$⨍_{B_\rho} |\mathfrak{X}u|^t\, dx \leq C ⨍_{B_r} |\mathfrak{X}u|^t\, dx$$

for all $1 < t < \infty$ and $0 < \rho \leq \frac{r}{2}$.

The first comparison estimate relates v to w.

Lemma 2.3 *There exists a constant $C = C(n, p, \Lambda) > 0$ such that*

$$⨍_{B_r} |\mathfrak{X}v - \mathfrak{X}w|^p\, dx \leq C\omega(r)^{\tau_0 p} ⨍_{B_r} |\mathfrak{X}w|^p\, dx.$$

Proof Using $v - w$ as a test function in (12), we obtain

$$\int_{B_r} \mathbf{A}(x_0, \mathfrak{X}v) \cdot \mathfrak{X}(v - w)\, dx = 0 \tag{13}$$

or equivalently

$$\int_{B_r} \mathbf{A}(x_0, \mathfrak{X}v) \cdot \mathfrak{X}v\, dx = \int_{B_r} \mathbf{A}(x_0, \mathfrak{X}v) \cdot \mathfrak{X}w\, dx.$$

Then, (10), (5) and Holder's inequality together imply

$$\int_{B_r} |\mathfrak{X}v|^p\, dx \leq C(n, p, \Lambda) \int_{B_r} |\mathfrak{X}w|^p\, dx. \tag{14}$$

Next we use $v - w$ as a test function in (11) and then combine with (13) to see that

$$\int_{B_r} \mathbf{A}(x, \mathfrak{X}w) \cdot \mathfrak{X}(v - w)\, dx = \int_{B_r} \mathbf{A}(x_0, \mathfrak{X}v) \cdot \mathfrak{X}(v - w)\, dx$$

or equivalently

$$\int_{B_r} \big(\mathbf{A}(x, \mathfrak{X}w) - \mathbf{A}(x_0, \mathfrak{X}w)\big) \cdot \mathfrak{X}(v-w)\, dx$$
$$= \int_{B_r} \big(\mathbf{A}(x_0, \mathfrak{X}v) - \mathbf{A}(x_0, \mathfrak{X}w)\big) \cdot \mathfrak{X}(v-w)\, dx.$$

Now we apply (10) and (8) to derive

$$\begin{aligned}
\Phi(\mathfrak{X}v, \mathfrak{X}w) &\le C(n,p,\Lambda) \int_{B_r} \big(\mathbf{A}(x_0, \mathfrak{X}v) - \mathbf{A}(x_0, \mathfrak{X}w)\big) \cdot \mathfrak{X}(v-w)\, dx \\
&= C(n,p,\Lambda) \int_{B_r} \big(\mathbf{A}(x, \mathfrak{X}w) - \mathbf{A}(x_0, \mathfrak{X}w)\big) \cdot \mathfrak{X}(v-w)\, dx \\
&\le C(n,p,\Lambda)\, \omega(r) \int_{B_r} |\mathfrak{X}w|^{p-1}\, |\mathfrak{X}v - \mathfrak{X}w|\, dx \\
&\le C(n,p,\Lambda)\, \omega(r) \int_{B_r} (|\mathfrak{X}v|^2 + |\mathfrak{X}w|^2)^{\frac{p-2}{2}}\, |\mathfrak{X}w|\, |\mathfrak{X}v - \mathfrak{X}w|\, dx \\
&\le C(n,p,\Lambda)\, \omega(r)^2 \int_{B_r} (|\mathfrak{X}v|^2 + |\mathfrak{X}w|^2)^{\frac{p-2}{2}}\, |\mathfrak{X}w|^2\, dx \\
&\quad + \frac{1}{2} \int_{B_r} (|\mathfrak{X}v|^2 + |\mathfrak{X}w|^2)^{\frac{p-2}{2}}\, |\mathfrak{X}v - \mathfrak{X}w|^2\, dx.
\end{aligned}$$

Consequently,

$$\fint_{B_r} |\mathfrak{X}v - \mathfrak{X}w|^p\, dx \le \Phi(\mathfrak{X}v, \mathfrak{X}w) \le C(n,p,\Lambda)\, \omega(r)^{\tau_0 p} \int_{B_r} |\mathfrak{X}w|^p\, dx$$

due (14) and the fact that $0 \le \omega \le 1$. □

The second comparison estimate is between w and u.

Lemma 2.4 *There exists a constant $C = C(n, p, \Lambda) > 0$ such that*

$$\begin{aligned}
&\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \\
&\le C \left[\inf_{m\in\mathbb{R}^n} \fint_{B_{2r}} |f-m|^{p'} dx + r^{p'} \left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp-Q}} \right] \\
&\quad + C \left[\inf_{m\in\mathbb{R}^n} \left(\fint_{B_{2r}} |f-m|^{p'} dx \right)^{p-1} \right. \\
&\quad \left. + r^p \left(\fint_{B_{2r}} |g|^{\frac{np}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Q}} \right] \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{2-p}
\end{aligned}$$

for $p \in (1, 2)$ *and*

$$\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \leq C \left[\inf_{m \in \mathbb{R}^n} \fint_{B_{2r}} |f - m|^{p'} dx + r^{p'} \left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp-Q}} \right]$$

for $p \in [2, Q)$.

Proof Let $m \in \mathbb{R}^n$. Then,

$$\int_{B_{2r}} \left[A(x, \mathfrak{X}u) - A(x, \mathfrak{X}w) \right] \cdot \mathfrak{X}\varphi \, dx = \int_{B_{2r}} (f - m) \cdot \mathfrak{X}\varphi \, dx + \int_{\mathbb{R}^n} g \, \varphi \, dx \tag{15}$$

for all $\varphi \in HW_0^{1,p}(B_{2r})$. Choosing $\varphi = u - w$ as a test function in (15) gives

$$\int_{B_{2r}} \left[A(x, \mathfrak{X}u) - A(x, \mathfrak{X}w) \right] \cdot \mathfrak{X}(u - w) dx = \int_{B_{2r}} (f - m) \cdot \mathfrak{X}(u - w) dx + \int_{\mathbb{R}^n} g \, (u - w) \, dx.$$

Using (10) and Holder's inequality, this leads to

$$\fint_{B_{2r}} \Phi(\mathfrak{X}w, \mathfrak{X}u) \, dx \leq C(n, p, \Lambda) \left(\fint_{B_{2r}} |f - m|^{p'} dx \right)^{1/p'} \left(\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \right)^{\frac{1}{p}} + C(n, p, \Lambda) \left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \left(\fint_{B_{2r}} |u - w|^{\frac{Qp}{Q-p}} dx \right)^{\frac{Q-p}{Qp}}.$$

Next we apply Sobolev's inequality (cf. [17, Theorem 2.1]) to obtain

$$\left(\fint_{B_{2r}} |u - w|^{\frac{Qp}{Q-p}} dx \right)^{\frac{Q-p}{Qp}} \leq C(n, p) \, r \left(\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \right)^{1/p}.$$

Consequently,

$$\fint_{B_{2r}} \Phi(\mathfrak{X}w, \mathfrak{X}u)\,dx \le C(n,p,\Lambda)\left[\left(\fint_{B_{2r}} |f-m|^{p'}dx\right)^{1/p'} + r\left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp}}\right] \times \left(\fint_{B_{2r}} |\mathfrak{X}u-\mathfrak{X}w|^p dx\right)^{\frac{1}{p}}. \tag{16}$$

Now we consider two cases.
Case 1: Suppose $p \in (1,2)$. Then rewrite

$$\begin{aligned} |\mathfrak{X}u-\mathfrak{X}w|^p &= (|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(p-2)}{2}} |\mathfrak{X}u-\mathfrak{X}w|^p (|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(2-p)}{2}} \\ &\le (|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(p-2)}{2}} |\mathfrak{X}u-\mathfrak{X}w|^p (|\mathfrak{X}u|+|\mathfrak{X}u-\mathfrak{X}w|)^{\frac{p(2-p)}{2}} \\ &\le (|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(p-2)}{2}} |\mathfrak{X}u-\mathfrak{X}w|^p |\mathfrak{X}u|^{\frac{p(2-p)}{2}} \\ &\quad + (|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(p-2)}{2}} |\mathfrak{X}u-\mathfrak{X}w|^p (|\mathfrak{X}u-\mathfrak{X}w|)^{\frac{p(2-p)}{2}}. \end{aligned} \tag{17}$$

Using Young's inequality in the form

$$ab^{\frac{2-p}{2}} \le \frac{p\epsilon^{\frac{p-2}{p}} a^{\frac{2}{p}}}{2} + \frac{(2-p)\epsilon b}{2}$$

with an appropriate $\epsilon > 0$, we arrive at

$$\begin{aligned} |\mathfrak{X}u-\mathfrak{X}w|^p &\le (|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(p-2)}{2}} |\mathfrak{X}u-\mathfrak{X}w|^p |\mathfrak{X}u|^{\frac{p(2-p)}{2}} \\ &\quad + C(p)\,(|\mathfrak{X}w|+|\mathfrak{X}u|)^{p-2}|\mathfrak{X}(u-w)|^2 + \frac{1}{2}|\mathfrak{X}u-\mathfrak{X}w|^p, \end{aligned}$$

whence it follows from (17) that

$$\begin{aligned} |\mathfrak{X}u-\mathfrak{X}w|^p &\le C(p)\,(|\mathfrak{X}w|+|\mathfrak{X}u|)^{p-2}|\mathfrak{X}(u-w)|^2 \\ &\quad + C(p)\,(|\mathfrak{X}u|+|\mathfrak{X}w|)^{\frac{p(p-2)}{2}} |\mathfrak{X}u-\mathfrak{X}w|^p |\mathfrak{X}u|^{\frac{p(2-p)}{2}}. \end{aligned}$$

By integrating both sides of the above estimate on B_{2r} and the apply Holder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p} > 1$, we obtain

$$
\begin{aligned}
\fint_{B_{2r}} |\mathfrak{X}(u-w)|^p dx \le C(p) \fint_{B_{2r}} (|\mathfrak{X}w| + |\mathfrak{X}u|)^{p-2} |\mathfrak{X}(u-w)|^2 dx \\
+ C(p) \left(\fint_{B_{2r}} (|\mathfrak{X}w| + |\mathfrak{X}u|)^{p-2} |\mathfrak{X}(u-w)|^2 dx \right)^{\frac{p}{2}} \\
\times \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{\frac{2-p}{2}} .
\end{aligned} \tag{18}
$$

Combining (16) with (18) yields

$$
\begin{aligned}
\fint_{B_{2r}} |\mathfrak{X}(u-w)|^p dx \le C(n,p,\Lambda) \Bigg[\left(\fint_{B_{2r}} |f-m|^{p'} dx \right)^{1/p'} \\
+ r \left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \Bigg] \\
\times \left(\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \right)^{\frac{1}{p}} \\
+ C(n,p,\Lambda) \Bigg[\left(\fint_{B_{2r}} |f-m|^{p'} dx \right)^{1/p'} \\
+ r \left(\fint_{B_{2r}} |g|^{\frac{np}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \Bigg]^{p/2} \\
\times \left(\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \right)^{\frac{1}{2}} \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{\frac{2-p}{2}} .
\end{aligned}
$$

Another application of Young's inequality yields

$$
\begin{aligned}
& \fint_{B_{2r}} |\mathfrak{X}(u-w)|^p dx \\
& \le C(n,p,\Lambda) \left[\fint_{B_{2r}} |f-m|^{p'} dx + r^{p'} \left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp-Q}} \right] \\
& \quad + C(n,p,\Lambda) \Bigg[\left(\fint_{B_{2r}} |f-m|^{p'} dx \right)^{p-1} \\
& \quad + r^p \left(\fint_{B_{2r}} |g|^{\frac{np}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Q}} \Bigg] \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{2-p}
\end{aligned}
$$

as required.

Case 2: Suppose $p \in [2, Q)$. Starting from (16), we have

$$\begin{aligned}\fint_{B_{2r}} |\mathfrak{X}(u-w)|^p dx \leq C(n,p,\Lambda) & \Bigg[\left(\fint_{B_{2r}} |f-m|^{p'} dx \right)^{1/p'} \\ & + r \left(\fint_{B_{2r}} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \Bigg] \\ & \times \left(\fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \right)^{\frac{1}{p}}. \\ & \leq \frac{1}{2} \fint_{B_{2r}} |\mathfrak{X}u - \mathfrak{X}w|^p dx \\ & + C(n,p,\Lambda) \Bigg[\fint_{B_{2r}} |f-m|^{p'} dx \\ & + r^{p'} \left(\fint_{B_{2r}} |g|^{\frac{np}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp-Q}} \Bigg]\end{aligned}$$

as required, where we used Young's inequality in the second step.

This completes the proof. □

3 Iteration Argument

Besides the comparison estimates, the proof of Theorem 1.2 requires an iteration argument given by Proposition 3.2 below. Proposition 3.2 in turn requires an auxiliary result stated in Lemma 3.1.

In what follows, let $u \in HW_0^{1,p}(\Omega)$ be a weak solution of (2). Also w and v are given by (11) and (12), respectively, with $B_{2r}(x_0) \Subset \Omega$.

Lemma 3.1 *There exist constants $C \geq 1$, $\kappa \in (0,1)$ and $\sigma_0 \in (0, 1/2]$, all of which depend on n, p and Λ only, such that*

$$\begin{aligned}& \left(\fint_{B_\rho(x_0)} |\mathfrak{X}v - (\mathfrak{X}v)_{B_\rho(x_0)}|^q \, dx \right)^{1/q} \\ & \leq C \left(\frac{\rho}{\sigma_0 r} \right)^{\kappa} \left[\fint_{B_{\sigma_0 r}} |\mathfrak{X}v - (\mathfrak{X}v)_{B_{\sigma_0 r}(x_0)}| \, dx + M_r \, \sigma_0^{\kappa} \right]\end{aligned}$$

for all $q \in [1, \infty)$ and $0 < \rho < \sigma_0 r$, where

$$M_r = \max_{1 \leq i \leq 2n} \sup_{B_r(x_0)} |X_i v|. \tag{19}$$

Proof Let $q \in [1, \infty)$. In view of [10, (3.5) and (3.6)], there exist constants $C \geq 1$, $\kappa \in (0, 1)$ and $\sigma_0 \in (0, 1/2]$, all of which depend on n, p and Λ only, such that

$$\operatorname*{osc}_{B_\rho(x_0)} \mathfrak{X} v \leq C \left(\frac{\rho}{\sigma_0 r}\right)^\kappa \left[\fint_{B_{\sigma_0 r}} |\mathfrak{X} v - (\mathfrak{X} v)_{B_{\sigma_0 r}(x_0)}|\, dx + M_r\, \sigma_0^\kappa \right] \tag{20}$$

for all $0 < \rho < \sigma_0 r$. Also observe that

$$\left(\fint_{B_\rho(x_0)} |\mathfrak{X} v - (\mathfrak{X} v)_{B_\rho(x_0)}|^q\, dx \right)^{1/q} \leq \operatorname*{osc}_{B_\rho(x_0)} \mathfrak{X} v$$

for all $\rho > 0$.

The claim is now justified by combining the above estimates together. ◻

Next define

$$\mathbf{I}(\rho) = \mathbf{I}(x_0, \rho) := \left(\fint_{B_\rho} |\mathfrak{X} u - (\mathfrak{X} u)_{B_\rho}|^p dx \right)^{1/p}$$

for a ball $B_\rho = B_\rho(x_0) \subset \Omega$.

Proposition 3.2 *There exist constants* $\kappa = \kappa(n, p, \Lambda) \in (0, 1)$ *and* $\sigma_0 = \sigma_0(n, p, \Lambda) \in (0, 1/2]$ *such that*

$$\begin{aligned}
&\mathbf{I}(\delta r) \\
&\leq C_0 \left(\frac{\delta}{\sigma_0}\right)^\kappa \mathbf{I}(\sigma_0 r) + C_0\, \delta^\kappa\, M_r \\
&\quad + C_0\, \delta^{-Q/p} \left[\inf_{m \in \mathbb{R}^n} \left(\fint_{B_r} |f - m|^{p'} dx \right)^{1/p} \right. \\
&\quad \left. + r^{1/(p-1)} \left(\fint_{B_r} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{(Qp-Q)p}} \right] \\
&\quad + C'\, \delta^{-Q/p} \left[\inf_{m \in \mathbb{R}^n} \left(\fint_{B_r} |f - m|^{p'} dx \right)^{1/p'} + r \left(\fint_{B_r} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \right] \\
&\quad \left(\fint_{B_r} |\mathfrak{X} u|^p dx \right)^{(2-p)/p} \\
&\quad + C_0\, \delta^{-Q/p}\, \omega(r)^{\tau_0} \left(\fint_{B_r} |\mathfrak{X} u|^p dx \right)^{1/p}
\end{aligned}$$

for all $\delta \in (0, \sigma_0]$ *and* $B_r(x_0) \Subset \Omega$, *where* $C_0 = C_0(n, p, \Lambda)$, $C' = C'(n, p, \Lambda)$ *and* M_r *is given by* (19).

Remark 3.3 We may choose $C' = 0$ in Proposition 3.2 if $p \in [2, Q)$.

Proof By virtue of Lemma 3.1, there exist constants $C \geq 1$, $\kappa \in (0, 1)$ and $\sigma_0 \in (0, 1/2]$, all of which depend on n, p and Λ only, such that

$$\begin{aligned}
&\fint_{B_{\delta r}} |\mathfrak{X}u - (\mathfrak{X}u)_{B_{\delta r}}|^p dx \\
&\leq C \fint_{B_{\delta r}} |\mathfrak{X}v - (\mathfrak{X}v)_{B_{\delta r}}|^p dx + C \fint_{B_{\delta r}} |\mathfrak{X}u - \mathfrak{X}v|^p dx \\
&\leq C \left(\frac{\delta}{\sigma_0}\right)^{p\kappa} \left[\fint_{B_{\sigma_0 r}} |\mathfrak{X}v - (\mathfrak{X}v)_{B_{\sigma_0 r}}|^p dx + M_r^p \sigma_0^{p\kappa}\right] + C\delta^{-Q} \fint_{B_r} |\mathfrak{X}u - \mathfrak{X}v|^p dx \\
&\leq C \left(\frac{\delta}{\sigma_0}\right)^{p\kappa} \left[\fint_{B_{\sigma_0 r}} |\mathfrak{X}u - (\mathfrak{X}u)_{B_{\sigma_0 r}}|^p dx + M_r^p \sigma_0^{p\kappa}\right] + C\delta^{-Q} \fint_{B_r} |\mathfrak{X}u - \mathfrak{X}v|^p dx
\end{aligned}$$

for all $\delta \in (0, \sigma_0]$.

Next we aim to bound the second term on the right-hand side of the above estimate. Observe that

$$\begin{aligned}
\fint_{B_r} |\mathfrak{X}u - \mathfrak{X}v|^p dx &\leq C \fint_{B_r} |\mathfrak{X}u - \mathfrak{X}w|^p \, dx + C \fint_{B_r} |\mathfrak{X}w - \mathfrak{X}v|^p \, dx \\
&\leq C \fint_{B_r} |\mathfrak{X}u - \mathfrak{X}w|^p \, dx + C\omega(r)^{\tau_0 p} \fint_{B_r} |\mathfrak{X}w|^p \, dx \\
&\leq C \fint_{B_r} |\mathfrak{X}u - \mathfrak{X}w|^p \, dx + C\omega(r)^{\tau_0 p} \fint_{B_r} |\mathfrak{X}u|^p \, dx,
\end{aligned}$$

where we used Lemma 2.3 in the second step and the fact that $0 \leq \omega \leq 1$ in the third step. Here $C = C(n, p, \Lambda)$.

It remains to bound the term

$$\fint_{B_r} |\mathfrak{X}u - \mathfrak{X}w|^p \, dx,$$

which can be done using Lemma 2.4.

This completes our proof. □

4 Interior Pointwise Gradient Estimates

Now we have enough preparation to prove Theorem 1.2.

Proof of Theorem 1.2 Let $B_R(x_0) \subset \Omega$. For short, we will write $B_\rho = B_\rho(x_0)$ for each $\rho > 0$ in the sequel. Set $\epsilon \in (0, \sigma_0/2) \subset (0, 1/4)$ be sufficiently small so that $C_0\epsilon^\kappa \leq \frac{1}{4}$, where C_0, κ and σ_0 are the constants given by Proposition 3.2.

Let w and v be given by (11) and (12), respectively, with $r = \frac{R}{4}$. Recall from Lemma 2.1 that

$$\|\mathfrak{X}v\|_{L^\infty(B_{r/2})} \leq C(n, p, \Lambda) \left(\fint_{B_r} |\mathfrak{X}v|^p dx \right)^{1/p}.$$

Also recall from (14) that

$$\int_{B_r} |\mathfrak{X}v|^p \, dx \leq C(n, p, \Lambda) \int_{B_r} |\mathfrak{X}w|^p \, dx.$$

Likewise,

$$\int_{B_{2r}} |\mathfrak{X}w|^p \, dx \leq C(n, p, \Lambda) \int_{B_{2r}} |\mathfrak{X}u|^p \, dx.$$

These three estimates together yield that there exists a constant $C_1 = C_1(n, p, \Lambda) > 0$ satisfying

$$\|\mathfrak{X}v\|_{L^\infty(B_{r/2})} \leq C_1 \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{1/p}. \tag{21}$$

For all $j \in \mathbb{N}$ set

$$r_j = \epsilon^j r, \quad B_j = B_{2r_j}(x_0), \quad \mathbf{I}_j = \mathbf{I}(r_j) \quad \text{and} \quad T_j := \left(\fint_{B_j} |\mathfrak{X}u|^p dx \right)^{1/p}.$$

It suffices to show that

$$\begin{aligned} &|\mathfrak{X}u(x_0)| \\ &\leq C \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{1/p} \\ &\quad + C \int_0^{2r_1} \left[\fint_{B_\rho(x_0)} |f - \beta|^{p'} dx + \rho^{p'} \left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp-Q}} \right] \frac{d\rho}{\rho}, \end{aligned} \tag{22}$$

where $C = C(n, p, \Lambda, W_0)$.

Let $\beta \in \mathbb{R}^n$. Then,

$$\begin{aligned} \mathbf{I}_{j+1} &\leq \frac{1}{4}\mathbf{I}_j + \frac{1}{4}\epsilon^{\kappa j} M_{r/2} \\ &\quad + C_2 \left[\left(\fint_{B_j} |f - \beta|^{p'} dx \right)^{1/p} + r_j^{1/(p-1)} \left(\fint_{B_j} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{(Qp-Q)p}} \right] \end{aligned}$$

$$
\begin{aligned}
&+ C_3 \left[\left(\fint_{B_j} |f-\beta|^{p'} dx \right)^{1/p'} + r_j \left(\fint_{B_j} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \right] T_j^{2-p} \\
&+ C_2\, \omega(r_j)^{\tau_0}\, T_j \\
&\le \frac{1}{4}\mathbf{I}_j + \frac{C_1}{4}\, \epsilon^{\kappa j} \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{1/p} \\
&+ C_2 \left[\left(\fint_{B_j} |f-\beta|^{p'} dx \right)^{1/p} + r_j^{1/(p-1)} \left(\fint_{B_j} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{(Qp-Q)p}} \right] \\
&+ C_3 \left[\left(\fint_{B_j} |f-\beta|^{p'} dx \right)^{1/p'} + r_j \left(\fint_{B_j} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \right] T_j^{2-p} \\
&+ C_2\, \omega(r_j)^{\tau_0}\, T_j,
\end{aligned}
$$

where we applied Proposition 3.2 with $C_j = C_j(n, p, \Lambda, \epsilon)$ for $j \in \{2, 3\}$ in the first step and (21) in the second step. Moreover, we may choose $C_3 = 0$ if $p \in [2, Q)$.

Let $j_0, m \in \mathbb{N}$ be such that $j_0 \ge 2$ and $m \ge j_0 + 1$, whose appropriate values will be chosen later. Summing the above estimate up over $j \in \{j_0, j_0+1, \dots, m-1\}$, we obtain

$$
\begin{aligned}
\sum_{j=j_0}^{m} \mathbf{I}_j \le \frac{4}{3}\mathbf{I}_{j_0} &+ \frac{C_1}{3} \left(\sum_{j=j_0}^{m-1} \epsilon^{\kappa j} \right) \left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx \right)^{1/p} \\
&+ \frac{4}{3} C_2 \sum_{j=j_0}^{m-1} \left[\left(\fint_{B_j} |f-\beta|^{p'} dx \right)^{1/p} \right. \\
&\left. + r_j^{1/(p-1)} \left(\fint_{B_j} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{(Qp-Q)p}} \right] \\
&+ \frac{4}{3} C_3 \sum_{j=j_0}^{m-1} \left[\left(\fint_{B_j} |f-\beta|^{p'} dx \right)^{1/p'} \right. \\
&\left. + r_j \left(\fint_{B_j} |g|^{\frac{Qp}{Qp-Q+p}} dx \right)^{\frac{Qp-Q+p}{Qp}} \right] T_j^{2-p} \\
&+ \frac{4}{3} C_2 \sum_{j=j_0}^{m-1} \omega(r_j)^{\tau_0}\, T_j. \qquad (23)
\end{aligned}
$$

Observe that

$$\sum_{j=j_0}^{m} \mathbf{I}_j \geq \epsilon^Q \sum_{j=j_0}^{m} |(\mathfrak{X}u)_{B_{j+1}} - (\mathfrak{X}u)_{B_j}| \geq \epsilon^Q |(\mathfrak{X}u)_{B_{m+1}} - (\mathfrak{X}u)_{B_{j_0}}|.$$

Therefore, (23) implies

$$\begin{aligned}
&|(\mathfrak{X}u)_{B_{m+1}}| \\
&\leq \frac{4}{3}\epsilon^{-Q}\mathbf{I}_{j_0} + \epsilon^{-Q}\frac{C_1}{3}\left(\sum_{j=j_0}^{m-1}\epsilon^{\kappa j}\right)\left(\fint_{B_{2r}}|\mathfrak{X}u|^p dx\right)^{1/p} + |(\mathfrak{X}u)_{B_{j_0}}| \\
&\quad + \frac{4}{3}C_2\epsilon^{-Q}\sum_{j=j_0}^{m-1}\left[\left(\fint_{B_j}|f-\beta|^{p'}dx\right)^{1/p}\right. \\
&\quad \left. + r_j^{1/(p-1)}\left(\fint_{B_j}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{(Qp-Q)p}}\right] \\
&\quad + \frac{4}{3}C_3\epsilon^{-Q}\sum_{j=j_0}^{m-1}\left[\left(\fint_{B_j}|f-\beta|^{p'}dx\right)^{1/p'}\right. \\
&\quad \left. + r_j\left(\fint_{B_j}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp}}\right]T_j^{2-p} \\
&\quad + \frac{4}{3}C_2\epsilon^{-Q}\sum_{j=j_0}^{m-1}\omega(r_j)^{\tau_0}T_j.
\end{aligned} \tag{24}$$

Estimating between an integral and its partial sum reveals that

$$\begin{aligned}
&\sum_{j=j_0}^{m-1}\left[\left(\fint_{B_j}|f-\beta|^{p'}dx\right)^{1/p} + r_j^{1/(p-1)}\left(\fint_{B_j}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{(Qp-Q)p}}\right] \\
&\quad \leq \epsilon^{-1}\int_0^{2R_1}\left[\left(\fint_{B_\rho(x_0)}|f-\beta|^{p'}dx\right)^{1/p}\right. \\
&\quad \left. + \rho^{1/(p-1)}\left(\fint_{B_\rho(x_0)}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{(Qp-Q)p}}\right]\frac{d\rho}{\rho},
\end{aligned} \tag{25}$$

and

$$\sum_{j=j_0}^{m-1}\left[\left(\fint_{B_j}|f-\beta|^{p'}dx\right)^{1/p'}+r_j\left(\fint_{B_j}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp}}\right]$$
$$\le \epsilon^{-1}\int_0^{2R_1}\left[\left(\fint_{B_\rho(x_0)}|f-\beta|^{p'}dx\right)^{1/p'}+\rho\left(\fint_{B_\rho(x_0)}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp}}\right]\frac{d\rho}{\rho}. \tag{26}$$

In what follows, choose a $j_0 = j_0(\epsilon, C_2, W_0, \Omega)$ such that

$$\frac{8}{3}C_2\,\epsilon^{-2Q}\sum_{j=j_0}^{\infty}\omega(r_j)^{\tau_0} < \frac{1}{10}, \tag{27}$$

where C_2 is given in (23) and (24). Note that this choice is possible due to (9).

Now we consider three cases.

Case 1: Suppose $|\mathfrak{X}u(x_0)| \le T_{j_0}$. Then, (22) trivially follows.

Case 2: Suppose there exists a $j_1 \in \mathbb{N}$ such that $j_1 \ge j_0$ and

$$T_j \le |\mathfrak{X}u(x_0)| \quad \text{and} \quad |\mathfrak{X}u(x_0)| < T_{j_1+1} \tag{28}$$

for all $j \in \{j_0, j_0+1, \dots, j_1\}$.

Then,

$$|\mathfrak{X}u(x_0)| < \left(\fint_{B_{j_1+1}}|\mathfrak{X}u|^p dx\right)^{1/p} \le \mathbf{I}_{j_1+1} + |(\mathfrak{X}u)_{B_{j_1+1}}|$$
$$\le \epsilon^{-Q}\left(\mathbf{I}_{j_1} + |(\mathfrak{X}u)_{B_{j_1+1}}|\right).$$

Now applying (23) and (24) with $m = j_1$, we derive

$$|\mathfrak{X}u(x_0)|$$
$$\le \frac{8}{3}\epsilon^{-2Q}\,\mathbf{I}_{j_0} + \epsilon^{-2Q}\frac{2C_1}{3}\left(\sum_{j=j_0}^{j_1-1}\epsilon^{\kappa j}\right)\left(\fint_{B_{2r}}|\mathfrak{X}u|^p dx\right)^{1/p} + \epsilon^{-Q}|(\mathfrak{X}u)_{B_{j_0}}|$$
$$+\frac{8}{3}C_2\,\epsilon^{-2Q}\sum_{j=j_0}^{j_1-1}\left[\left(\fint_{B_j}|f-\beta|^{p'}dx\right)^{1/p}\right.$$
$$\left.+r_j^{1/(p-1)}\left(\fint_{B_j}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{(Qp-Q)p}}\right]$$
$$+\frac{8}{3}C_3\,\epsilon^{-2Q}\sum_{j=j_0}^{j_1-1}\left[\left(\fint_{B_j}|f-\beta|^{p'}dx\right)^{1/p'}\right.$$

$$
\begin{aligned}
&+r_j\left(\fint_{B_j}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp}}\Bigg]\,T_j^{2-p}\\
&+\frac{8}{3}\,C_2\,\epsilon^{-2Q}\sum_{j=j_0}^{j_1-1}\omega(r_j)^{\tau_0}\,T_j\\
\leq\ &\frac{8}{3}\,\epsilon^{-2Q}\,\mathbf{I}_{j_0}+\epsilon^{-2Q}\,\frac{2C_1}{3}\left(\sum_{j=j_0}^{j_1-1}\epsilon^{\kappa j}\right)\left(\fint_{B_{2r}}|\mathfrak{X}u|^p dx\right)^{1/p}+\epsilon^{-2Q}\,|\,(\mathfrak{X}u)_{B_{j_0}}\,|\\
&+\frac{8}{3}\,C_2\,\epsilon^{-2Q-1}\int_0^{2R_1}\Bigg[\left(\fint_{B_\rho(x_0)}|f-\beta|^{p'}dx\right)^{1/p}\\
&+\rho^{1/(p-1)}\left(\fint_{B_\rho(x_0)}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{(Qp-Q)p}}\Bigg]\,\frac{d\rho}{\rho}\\
&+\frac{8}{3}\,C_3\,\epsilon^{-2Q-1}\,|\mathfrak{X}u(x_0)|^{2-p}\int_0^{2R_1}\Bigg[\left(\fint_{B_\rho(x_0)}|f-\beta|^{p'}dx\right)^{1/p'}\\
&+\rho\left(\fint_{B_\rho(x_0)}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp}}\Bigg]\,\frac{d\rho}{\rho}\\
&+\frac{|\mathfrak{X}u(x_0)|}{10},
\end{aligned}
$$

where we used the fact that $0<\epsilon<1$ in the first step as well as (25), (26), (27) and (28) to estimate the last three terms in the second step.

It remains to estimate the fifth term on the right-hand side of the above inequality. Since $C_3=0$ when $p\in[2,n)$, we need only focus on $p\in(1,2)$. In this case, it follows from Young's inequality that

$$
\begin{aligned}
|\mathfrak{X}u(x_0)|\leq\ &\frac{8}{3}\,\epsilon^{-2Q}\,\mathbf{I}_{j_0}+\epsilon^{-2Q}\,\frac{2C_1}{3}\\
&\left(\sum_{j=j_0}^{j_1-1}\epsilon^{\kappa j}\right)\left(\fint_{B_{2r}}|\mathfrak{X}u|^p dx\right)^{1/p}+\epsilon^{-2Q}\,|\,(\mathfrak{X}u)_{B_{j_0}}\,|+\frac{1}{5}|\mathfrak{X}u(x_0)|\\
&+C(n,p,\Lambda,\epsilon)\int_0^{2R_1}\Bigg[\left(\fint_{B_\rho(x_0)}|f-\beta|^{p'}dx\right)^{1/p}\\
&+\rho^{1/(p-1)}\left(\fint_{B_\rho(x_0)}|g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{(Qp-Q)p}}\Bigg]\,\frac{d\rho}{\rho}.
\end{aligned}
$$

Either way we always have

$$
\begin{aligned}
|\mathfrak{X}u(x_0)| \le 40\epsilon^{-2Q}\, T_{j_0} + \epsilon^{-2Q}\,\frac{10C_1}{3}\left(\sum_{j=j_0}^{j_1-1}\epsilon^{\kappa j}\right)\left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx\right)^{1/p} \\
+ C(n,p,\Lambda,\epsilon)\int_0^{2R_1}\left[\fint_{B_\rho(x_0)} |f-\beta|^{p'} dx\right. \\
\left. +\rho^{p'}\left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}} dx\right)^{\frac{Qp-Q+p}{Qp-Q}}\right]\frac{d\rho}{\rho} \\
\le C(n,p,\Lambda,\epsilon)\left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx\right)^{1/p} \\
+ C(n,p,\Lambda,\epsilon)\int_0^{2R_1}\left[\fint_{B_\rho(x_0)} |f-\beta|^{p'} dx\right. \\
\left. +\rho^{p'}\left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}} dx\right)^{\frac{Qp-Q+p}{Qp-Q}}\right]\frac{d\rho}{\rho},
\end{aligned}
$$

where we used Corollary 2.2 in the second step. This is (22) as desired.

Case 3: Suppose $T_j \le |\mathfrak{X}u(x_0)|$ for all $j \in \{2,3,4,\ldots\}$. Then, we deduce from (24), (25) and (26) that

$$
\begin{aligned}
&|\,(\mathfrak{X}u)_{B_{k+1}}\,| \\
&\le \frac{4}{3}\,\epsilon^{-Q}\,\mathbf{I}_{j_0} + \epsilon^{-Q}\,\frac{C_1}{3}\left(\sum_{j=j_0}^{k-1}\epsilon^{\kappa j}\right)\left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx\right)^{1/p} + |\,(\mathfrak{X}u)_{B_{j_0}}\,| \\
&\quad + \frac{4}{3}\,C_1\,\epsilon^{-Q}\int_0^{2R_1}\left[\fint_{B_\rho(x_0)} |f-\beta|^{p'} dx\right. \\
&\quad \left. +\rho^{p'}\left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}} dx\right)^{\frac{Qp-Q+p}{Qp-Q}}\right]\frac{d\rho}{\rho} \\
&\quad + \frac{4}{3}\,C_2\,\epsilon^{-Q}\,|\mathfrak{X}u(x_0)|^{2-p}\int_0^{2R_1}\left[\left(\fint_{B_\rho(x_0)} |f-\beta|^{p'} dx\right)^{p-1}\right. \\
&\quad \left. +\rho^{p}\left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}} dx\right)^{\frac{Qp-Q+p}{Q}}\right]\frac{d\rho}{\rho} \\
&\quad + \frac{|\mathfrak{X}u(x_0)|}{10}
\end{aligned}
$$

for all $k \in \{j_0, j_0+1, j_0+2, \ldots\}$.

Simplifying the above estimate further and then letting $k \longrightarrow \infty$, we arrive at

$$
\begin{aligned}
&|\mathfrak{X}u(x_0)| \\
&\quad \le C(n,p,\Lambda,\epsilon)\left(\fint_{B_{2r}} |\mathfrak{X}u|^p dx\right)^{1/p} \\
&\quad + \frac{4}{3}C_2\,\epsilon^{-Q}\int_0^{2R_1}\left[\fint_{B_\rho(x_0)} |f-\beta|^{p'}dx\right. \\
&\quad \left. +\rho^{p'}\left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Qp-Q}}\right]\frac{d\rho}{\rho} \\
&\quad + \frac{4}{3}C_3\,\epsilon^{-Q}\,|\mathfrak{X}u(x_0)|^{2-p}\int_0^{2R_1}\left[\left(\fint_{B_\rho(x_0)} |f-\beta|^{p'}dx\right)^{p-1}\right. \\
&\quad \left. +\rho^{p}\left(\fint_{B_\rho(x_0)} |g|^{\frac{Qp}{Qp-Q+p}}dx\right)^{\frac{Qp-Q+p}{Q}}\right]\frac{d\rho}{\rho}.
\end{aligned}
$$

Now (22) follows after an application of Young's inequality as we did in the last part of Case 2.

Thus, the proof is complete. □

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