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An Inverse Three Spectra Problem for Parameter-Dependent and Jumps Conformable Sturm–Liouville Operators

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Abstract

In this manuscript, we study the parameter-dependent conformable Sturm–Liouville problem (PDCSLP) in which its transmission conditions are arbitrary finite numbers at an interior point in $[0, \pi]$. Also, we prove the uniqueness theorems for inverse second order of conformable differential operators by applying three spectra with jumps and eigen-parameter-dependent boundary conditions. To this end, we investigate the PDCSLP in three intervals $[0, \pi]$, $[0, p]$, and $[p, \pi]$ where $p \in (0, \pi)$ is an interior point.

Keywords Conformable Sturm–Liouville problem · Internal discontinuities · Three spectra · Parameter-dependent boundary conditions

Mathematics Subject Classification 34A55 · 34B24 · 26A33 · 47A10

1 Introduction

Sturm–Liouville problem is one of the most classical and important problems in mathematics, physics and engineering. This problem arises in the modeling of many systems in vibration theory, quantum mechanics, hydrodynamics, etc. [15, 36].

In 2014, Khalil et al. [18] defined a well-behaved conformable derivative called conformable fractional derivative (CFD) that depends only on the basic limit definition of the derivative. Unlike other definitions of fractional derivative such as Riemann–

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Liouville and Caputo, this definition enables us to prove many properties similar to derivatives of integer order, for more information about the CFD, refer to [1, 4]. However, the CFD has its drawbacks. Its derivative has some disadvantages and some unusual properties, e.g., the zeroth derivative of a function does not return the function.

Fractional Sturm–Liouville problems (FSLPs) have attracted much attention as an important branch of fractional derivative research [19, 20, 28]. In our opinion, the most important useful property of the conformable derivative is the possibility of defining the inner product in the integral form. This capability makes the conformable Sturm–Liouville problem (CSLP) and PDCSLP well investigated in different situations. In [23], the authors investigated the existence of infinity of real eigenvalues of CSLP. So that the eigenvalues of CSLP are simple and the corresponding eigenfunctions are orthogonal.

The inverse three spectra problems to reconstruction of the potential function in the SLP were firstly discussed in [14, 26, 27]; it was shown that if these spectra are pairwise distinct, the potential function can be uniquely determined by applying the three spectra to the problems defined in the three intervals $[0, 1]$, $[0, d]$, and $[d, 1]$, $(d \in (0, 1))$. Also, in [14] the authors gave a violation example to demonstrate that the pairwise disjoint conditions are necessary. Recently, in [6, 7, 9–13, 31], the authors discussed the inverse three spectra problems in the several cases such as reconstruction of the potential function with different boundary and transmissions conditions, with one or some turning point, and some uniqueness results.

The main purpose of this manuscript is to study the PDCSLP with an arbitrary finite number of transmission conditions at an interior point in $[0, \pi]$, which is considered to formulated the inverse PDCSLP by using three spectra. One may consider the results of this paper as an extension of [12–14, 26, 27, 31] to the PDCSLP. For some related result in the inverse problems in SLP, FSLP, CSLP, PDSLP, we refer to [2, 3, 5, 8, 17, 24, 25, 29, 30, 32–34, 37]

2 Asymptotic Forms of PDCSLP

In this section, before introducing the asymptotic forms of PDCSLP, we give several important content of the CFD. In [18], Khalil and et al. defined the CFD as follows:

Definition 2.1 For the function $h : [0, \infty) \to \mathbb{R}$, the CF derivative of order $\alpha \in (0, 1]$ defined by:

$$T_\alpha h(x) = \lim_{\varepsilon \to 0} \frac{h(x + \varepsilon x^{1-\alpha}) - h(x)}{\varepsilon},$$

for all $x > 0$, and

$$T_\alpha h(0) = \lim_{x \to 0^+} T_\alpha h(x).$$

If h is a differentiable function, then

$$T_\alpha h(x) = x^{1-\alpha} h'(x).$$

If $T_\alpha h(x_0)$ exists and finite, then the function h is α-differentiable at x_0.

Definition 2.2 The CF integral of order $\alpha \in (0, 1]$ for a function $h : [0, \infty) \to \mathbb{R}$ defined by:

$$J^\alpha h(x) = \int_0^x h(t) \mathrm{d}_\alpha t = \int_0^x t^{\alpha-1} h(t) \mathrm{d}t, \quad x > 0.$$

However, the integral is the Riemann improper integral.

We use some important CFD for conformable Sturm–Liouville problems (CSLPs) in [1, 18, 34].

Let us consider the following three PDCSLPs

I:

$$\ell_0 y := -T_\alpha T_\alpha y + q y = \mu y \tag{2.1}$$

with parameter-dependent boundary conditions

$$\mathrm{B}_1(y) := \mu(T_\alpha y(0) + h_1 y(0)) - h_2 T_\alpha y(0) - h_3 y(0) = 0, \tag{2.2}$$
$$\mathrm{B}_2(y) := \mu(T_\alpha y(\pi) + H_1 y(\pi)) - H_2 T_\alpha y(\pi) - H_3 y(\pi) = 0, \tag{2.3}$$

and the following jump conditions

$$\begin{aligned} U_k(y) &:= y(p_k + 0) - b_k y(p_k - 0) = 0, \quad k = 1, 2, \ldots, m-1 \\ V_k(y) &:= T_\alpha y(p_k + 0) - c_k T_\alpha y(p_k - 0) - d_k y(p_k - 0) = 0, \end{aligned} \tag{2.4}$$

II:

$$\ell_1 y := -T_\alpha T_\alpha y + q_1 y = \mu y \tag{2.5}$$

with the following conditions

$$\mathrm{B}_1(y) = 0, \quad \mathrm{B}_3(y) := T_\alpha y(p) + \kappa_1 \, y(p) = 0, \tag{2.6}$$

by the jump conditions

$$U_k(y) = 0, \quad V_k(y) = 0, \ \text{for } k = 1, 2, \ldots, p-1, \tag{2.7}$$

and

III:

$$\ell_2 y := -T_\alpha T_\alpha y + q_2 y = \mu y \tag{2.8}$$

with Robin and parameter-dependent boundary conditions

$$\mathrm{B}_4(y) := T_\alpha y(p) + \kappa_2\, y(p) = 0,\ \mathrm{B}_2(y) = 0, \tag{2.9}$$

by the following jump conditions

$$U_k(y) = 0,\quad V_k(y) = 0,\ \text{for } k = p+1, 2, \ldots, m-1, \tag{2.10}$$

where T_α is the CFD of order $0 < \alpha \le 1$, $q(x) \in L^1_\alpha[0,\pi]$, $q_1 = q|_{[0,p)}$, and $q_2 = q|_{(p,\pi]}$ are real valued functions, and κ_1, κ_2, h_j, and H_j, $j = 1, 2, 3$ are real numbers, satisfying

$$r_1 := h_3 - h_1 h_2 > 0 \quad \text{and} \quad r_2 := H_1 H_2 - H_3 > 0.$$

Also, the numbers b_k, c_k, d_k, and p_k, with $k = 1, 2, \ldots, m-1$, $(m \ge 2)$ are real. The parameter μ is the spectral parameter. In this note, we suppose that $c_k b_k > 0$, $p_0 = 0 < p_1 < p_2 < \cdots < p_{m-1} < p_m = \pi$, and $p = p_s$ for $1 \le s \le m-1$. As well as, we use the notations $L_0 = L(q(x); h_j; H_j; p_k)$, $L_1 = L(q_1(x); h_j; \kappa_1; p_k)$, and $L_2 = L(q_2(x); \kappa_2; H_j; p_k)$ for the problems (2.1)–(2.10).
Using the jump conditions (2.4) in the transmission point $p = p_s$, $(1 \le s \le m-1)$, we must have $d_s = 0$ and

$$\kappa_2 = \frac{c_s}{b_s}\kappa_1,\ \text{for } \kappa_1 \in (0,\infty). \tag{2.11}$$

Define the weighted inner products as follows:

$$\begin{aligned} \langle F, G\rangle_{\mathcal{H}_0} &:= \int_0^\pi f\overline{g} w_0 \mathrm{d}_\alpha x + \frac{w_0(0)}{r_1} f_1\overline{g_1} + \frac{w_0(\pi)}{r_2} f_2\overline{g_2}, \\ F &= \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix},\ G = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix}, \end{aligned} \tag{2.12}$$

$$\langle F_1, G_1\rangle_{\mathcal{H}_1} := \int_0^d f\overline{g} w_1 \mathrm{d}_\alpha x + \frac{w_1(0)}{r_1} f_1\overline{g_1}, \quad F_1 = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix},\ G_1 = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix},$$

and

$$\langle F_2, G_2\rangle_{\mathcal{H}_2} := \int_d^\pi f\overline{g} w_2 \mathrm{d}_\alpha x + \frac{w_2(\pi)}{r_2} f_2\overline{g_2}, \quad F_2 = \begin{pmatrix} f(x) \\ f_2 \end{pmatrix},\ G_2 = \begin{pmatrix} g(x) \\ g_2 \end{pmatrix}.$$

where

$$w_0(x) = \begin{cases} 1, & 0 \le x < p_1, \\ \frac{1}{b_1 c_1}, & p_1 < x < p_2, \\ \vdots & \\ \frac{1}{b_1 c_1 \cdots b_{m-1} c_{m-1}}, & p_{m-1} < x \le \pi. \end{cases}$$

Also, $w_1 = w_0|_{[0,p)}$, and $w_2 = w_0|_{(p,\pi]}$. We note that $\mathcal{H}_0 := L^2_\alpha((0,\pi); w_0) \oplus \mathbb{C}^2$ and $\mathcal{H}_i := L^2_\alpha((0,\pi); w_i) \oplus \mathbb{C}$ $(i = 1, 2)$, are Hilbert spaces with norms $\|F\|_{\mathcal{H}_i} = \langle F, F\rangle^{1/2}_{\mathcal{H}_i}$, (i=0,1,2).

Next we introduce

$$\begin{aligned} R_1(f) &:= T_\alpha f(0) + h_1 f(0), \quad R_1'(f) := h_2 T_\alpha f(0) + h_3 f(0), \\ R_2(f) &:= T_\alpha f(\pi) + H_1 f(\pi), \;\; R_2'(f) := H_2 T_\alpha f(\pi) + H_3 f(\pi). \end{aligned} \tag{2.13}$$

In these spaces, we have the following operators

$$A_i : \mathcal{H}_i \to \mathcal{H}_i \quad i = 0, 1, 2$$

with domains

$$\operatorname{dom}(A_0) = \left\{ F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \middle| \begin{array}{c} f, T_\alpha f \in AC\left(\cup_0^{m-1} (p_k, p_{k+1})\right), \ \ell_0 f \in L^2_\alpha(0,\pi) \\ U_k(f) = V_k(f) = 0, \ f_1 = R_1(f), \ f_2 = R_2(f) \end{array} \right\},$$

$$\operatorname{dom}(A_1) = \left\{ F_1 = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \middle| \begin{array}{c} f, T_\alpha f \in AC\left(\cup_0^{s-1} (p_k, p_{k+1})\right), \ \ell_1 f \in L^2_\alpha(0,p) \\ U_i(f) = V_i(f) = 0, \ f_1 = R_1(f) \end{array} \right\},$$

and

$$\operatorname{dom}(A_2) = \left\{ F_2 = \begin{pmatrix} f(x) \\ f_2 \end{pmatrix} \middle| \begin{array}{c} f, T_\alpha f \in AC\left(\cup_s^{m-1} (p_k, p_{k+1})\right), \ \ell_2 f \in L^2_\alpha(p,\pi) \\ U_k(f) = V_k(f) = 0, \ f_2 = R_2(f) \end{array} \right\}$$

by

$$A_0 F = \begin{pmatrix} \ell f \\ R_1'(f) \\ R_2'(f) \end{pmatrix} \quad \text{with } F = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} \in \operatorname{dom}(A_0). \tag{2.14}$$

and

$$A_i F = \begin{pmatrix} \ell f \\ R_i'(f) \end{pmatrix} \quad \text{with } F_i = \begin{pmatrix} f(x) \\ R_i(f) \end{pmatrix} \in \operatorname{dom}(A_i) \ \ (i = 1, 2).$$

By construction, the eigenvalue problems A_0 and A_i,

$$A_0 Y = \mu Y, \qquad Y := \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in \operatorname{dom}(A_0),$$
$$A_i Y_i = \mu Y_i, \qquad Y_i := \begin{pmatrix} y(x) \\ R_i(y) \end{pmatrix} \in \operatorname{dom}(A_i),$$

are equivalent to the problems (2.1)–(2.4) for the operator L, and (2.5)–(2.7) or (2.8)–(2.10) for L_i, $(i = 1, 2)$, respectively.

Considering the linear differential equations, we obtain the modified fractional Wronskian as follows

$$W_\alpha(\theta, \vartheta) = w_0(x)\big(\theta(x) T_\alpha \vartheta(x) - T_\alpha \theta(x) \vartheta(x)\big)$$

We get that this function is constant on $x \in [0, p_1) \cup \big(\cup_1^{m-2} (p_k, p_{k+1})\big) \cup (p_{m-1}, \pi]$ for two solutions $\ell_0 \theta = \mu \theta$ and $\ell_0 \vartheta = \mu \vartheta$ satisfying the discontinuous conditions (2.4).

Lemma 2.3 *For $0 < \alpha \leq 1$, the operators A_i, $i = 0, 1, 2$, are symmetric.*

Proof We prove this lemma for $i = 0$. After using α-integration by parts twice, it follows immediately and by direct calculation by (2.12)–(2.14):

$$\langle A_0 F, G \rangle = W_\alpha\big(f, \bar{g}\big)\big|_{x=\pi} - W_\alpha\big(f, \bar{g}\big)\big|_{x=0} + \langle F, A_0 G \rangle.$$

So, from Eqs. (2.2)–(2.4) we have:

$$W_\alpha\big(f, \bar{g}\big)\big|_{x=\pi} - W_\alpha\big(f, \bar{g}\big)\big|_{x=0} = 0.$$

Then A_0 is symmetric operator on $L^2_\alpha((0, \pi); w_0) \oplus \mathbb{C}^2$. Similarly, the operators A_1 and A_2 are also symmetric. □

By applying Lemma 2.3, the eigenvalues of the problems A_i and hence of L_i are simple and real. Since problem (2.1) with initial conditions $g(\nu \pm 0) = g_0$ and $g'(\nu \pm 0) = g_1$ (with $\nu \in (0, \pi)$)) is a Cauchy problem, then it has a unique solution.

Remark 2.4 We will denote the restriction of any function g with $g \in \operatorname{dom}(A_i)$, by g_k, $1 \leq k \leq m$, to the subinterval (p_{k-1}, p_k). Also, we will set $g_k(p_{k-1}) = g(p_{k-1} + 0)$ and $g_k(p_k) = g(p_k - 0)$.

Remark 2.5 Without losing of generality of the problem (2.1)–(2.10), by [33, Lemma 2.3], we can take $b_k c_k = 1$, for $k = 1, 2, \ldots, m$.

Suppose $u(x, \mu)$ and $v(x, \mu)$ are solutions of (2.1) with the jump conditions (2.4) and the following conditions, respectively

$$u(0, \mu) = \mu - h_2, \;\; T_\alpha u(0, \mu) = h_3 - \mu h_1,$$

and

$$v(\pi,\mu)=H_2-\mu,\ T_\alpha v(\pi,\mu)=\mu H_1-H_3.$$

The functions $u(x,\mu)$, $T_\alpha u(x,\mu)$, $v(x,\mu)$, and $T_\alpha v(x,\mu)$ for any fixed $x\in[0,\pi]$ are entire functions with respect to μ of order $\frac{1}{2}$ [35]. The asymptotic form of solutions and characteristic function $\Delta(\mu)$ are discussed as follows:

Theorem 2.6 *Let $\mu=\varrho^2$ and $\varrho:=\sigma+i\tau$. The solutions $u(x,\mu)$ and $T_\alpha u(x,\mu)$ for PDCSLP (2.1)–(2.4) as $|\mu|\to\infty$, have the following asymptotic forms:*

$$u(x,\mu)=\begin{cases}\varrho^2\left[\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)\right]+O\left(\varrho\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & 0\le x<p_1,\\ \varrho^2\left[a_1\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+a_1'\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)\right]+O\left(\varrho\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_1<x<p_2,\\ \varrho^2\left[a_1a_2\cos\varrho\left(\frac{\varrho}{\alpha}x^\alpha\right)+a_1'a_2\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+a_1a_2'\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_2^\alpha)\right)\right.\\ \quad\left.+a_1'a_2'\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)\right]+O\left(\varrho\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_2<x<p_3,\\ \quad\vdots\\ \varrho^2\left[a_1a_2\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+\right.\\ \quad+a_1'a_2\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+\cdots\\ \quad+a_1a_2\ldots a_{m-1}'\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_{m-1}^\alpha)\right)+\\ \quad+a_1'a_2'a_3\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)+\cdots\\ \quad+a_1\ldots a_i'\ldots a_j'\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2p_i^\alpha-2p_j^\alpha)\right)\\ \quad+a_1\ldots a_i'\ldots a_j'\ldots a_k'\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_i^\alpha+2p_j^\alpha-2p_k^\alpha)\right)+\cdots\\ \quad\left.+a_1'a_2'\ldots a_{m-1}'\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2(-1)^{m-1}p_1^\alpha+2(-1)^{m-2}p_2^\alpha-2p_m^\alpha)\right)\right]\\ \quad+O\left(\varrho\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_{m-1}<x\le\pi,\end{cases}\tag{2.15}$$

$$T_\alpha u(x,\mu)=\begin{cases}\varrho^3\left[-\sin\left(\frac{\varrho}{\alpha}x^\alpha\right)\right]+O\left(\varrho^2\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & 0\le x<p_1,\\ \varrho^3\left[-a_1\sin\left(\frac{\varrho}{\alpha}x^\alpha\right)-a_1'\sin\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)\right]+O\left(\varrho^2\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_1<x<p_2,\\ \varrho^3\left[-a_1a_2\sin\varrho\left(\frac{\varrho}{\alpha}x^\alpha\right)-a_1'a_2\sin\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)-\alpha_1\alpha_2'\sin\left(\frac{\varrho}{\alpha}(x^\alpha-2p_2^\alpha)\right)\right.\\ \quad\left.-a_1'a_2'\sin\left(\frac{\varrho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)\right]+O\left(\varrho^2\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_2<x<p_3,\\ \quad\vdots\\ \varrho^3\left[-a_1a_2\ldots\alpha_{m-1}\sin\left(\frac{\varrho}{\alpha}x^\alpha\right)\right.\\ \quad-a_1'a_2\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+\cdots\\ \quad-a_1a_2\ldots a_{m-1}'\sin\left(\frac{\varrho}{\alpha}(x^\alpha-2p_{m-1}^\alpha)\right)+\\ \quad-a_1'a_2'a_3\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)+\cdots\\ \quad-a_1\ldots a_i'\ldots a_j'\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(x^\alpha+2p_i^\alpha-2p_j^\alpha)\right)\\ \quad-a_1\ldots a_i'\ldots a_j'\ldots a_k'\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(x^\alpha-2p_i^\alpha+2p_j^\alpha-2p_k^\alpha)\right)+\cdots\\ \quad\left.-a_1'a_2'\ldots a_{m-1}'\sin\left(\frac{\varrho}{\alpha}(x^\alpha+2(-1)^{m-1}p_1^\alpha+2(-1)^{m-2}p_2^\alpha-2p_m^\alpha)\right)\right]\\ \quad+O\left(\varrho^2\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_{m-1}<x\le\pi,\end{cases}\tag{2.16}$$

where

$$a_k=\frac{1}{2}(b_k+c_k),\qquad a_k'=\frac{1}{2}(b_k-c_k),\tag{2.17}$$

for $k=1,2,\ldots,m-1$.

Proof Let $\mathcal{S}(x,\mu)$ and $\mathcal{C}(x,\mu)$ be the solutions of (2.1) and (2.4) with the following conditions

$$\mathcal{S}(0,\mu)=0,\ T_\alpha\mathcal{S}(0,\mu)=1,\ \mathcal{C}(0,\mu)=1,\ \text{and } T_\alpha\mathcal{C}(0,\mu)=0.$$

Using (2.4) for $\mathcal{C}(x,\mu)$, we obtain

$$\mathcal{C}(x,\mu)=\begin{cases}\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+O\left(\frac{1}{\varrho}\exp\frac{|\tau|}{\alpha}x^\alpha\right), & 0\le x<p_1,\\ b_1\mathcal{C}_1(p_1,\mu)\cos\left(\frac{\varrho}{\alpha}(x^\alpha-p_1^\alpha)\right)+\frac{c_1}{\varrho}\mathcal{C}_1'(p_1,\mu)\sin\left(\frac{\varrho}{\alpha}(x^\alpha-p_1^\alpha)\right) & \\ \qquad +O\left(\frac{1}{\varrho}\exp\frac{\tau}{\alpha}\left(x^\alpha-p_1^\alpha\right)\right), & p_1<x<p_2,\\ b_2\mathcal{C}_2(p_2,\mu)\cos\left(\frac{\varrho}{\alpha}(x^\alpha-p_2^\alpha)\right)+\frac{c_2}{\varrho}\mathcal{C}_2'(p_2,\mu)\sin\left(\frac{\varrho}{\alpha}(x^\alpha-p_2^\alpha)\right) & \\ \qquad +O\left(\frac{1}{\varrho}\exp\frac{\tau}{\alpha}\left(x^\alpha-p_2^\alpha\right)\right), & p_2<x<p_3,\\ \vdots & \\ b_{m-1}\mathcal{C}_{m-1}(p_{m-1},\mu)\cos\left(\frac{\varrho}{\alpha}(x^\alpha-p_{m-1}^\alpha)\right)+ & \\ \quad +\frac{c_{m-1}}{\varrho}\mathcal{C}_{m-1}'(p_{m-1},\mu)\sin\left(\frac{\varrho}{\alpha}(x^\alpha-p_{m-1}^\alpha)\right)+ & \\ \quad +O\left(\frac{1}{\varrho}\exp\frac{\tau}{\alpha}\left(x^\alpha-p_{m-1}^\alpha\right)\right), & p_{m-1}<x\le\pi.\end{cases}$$

So, we insert the k'th statement into the $(k+1)$'th statement to get

$$\mathcal{C}(x,\mu)=\begin{cases}\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+O\left(\frac{1}{\varrho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0\le x<p_1,\\ a_1\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+a_1'\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+O\left(\frac{1}{\varrho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & p_1<x<p_2,\\ a_1a_2\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+a_1'a_2\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+a_1a_2'\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_2^\alpha)\right) & \\ \quad +a_1'a_2'\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)+O\left(\frac{1}{\varrho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & p_2<x<p_3,\\ \quad\vdots & \\ a_1a_2\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}x^\alpha\right)+ & \\ \quad +a_1'a_2\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+\cdots & \\ \quad +a_1a_2\ldots a_{m-1}'\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_{m-1}^\alpha)\right)+ & \\ \quad +a_1'a_2'a_3\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)+\cdots & \\ \quad +a_1\ldots a_j'\ldots a_k'\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2p_j^\alpha-2p_k^\alpha)\right) & \\ \quad +a_1\ldots a_j'\ldots a_k'\ldots a_s'\ldots a_{m-1}\cos\left(\frac{\varrho}{\alpha}(x^\alpha-2p_j^\alpha+2p_k^\alpha-2p_s^\alpha)\right)+\cdots & \\ \quad +a_1'a_2'\ldots a_{m-1}'\cos\left(\frac{\varrho}{\alpha}(x^\alpha+2(-1)^{m-1}p_1^\alpha+2(-1)^{m-2}p_2^\alpha-2p_m^\alpha)\right) & \\ \quad +O\left(\frac{1}{\varrho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & p_{m-1}<x\le\pi,\end{cases}$$

where a_i and a_i' are defined in (2.17) and $j<k<s$, $j,k,s=1,2,\ldots,m-1$. Similarly, we can obtain the asymptotic formula for $\mathcal{S}(x,\mu)$. Applying Definition 2.1,

we calculate the asymptotic form of $T_\alpha \mathcal{S}(x,\mu)$ and $T_\alpha \mathcal{C}(x,\mu)$. This completes the proof by using $u(x,\mu) = (\mu - h_2)\mathcal{C}(x,\mu) + (h_3 - \mu h_1)\,\mathcal{S}(x,\mu)$. □

From Theorem 2.6 and Definition 2.1, we get that

$$|u(x,\mu)| = O\left(|\varrho|^2 \exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right),$$
$$|T_\alpha u(x,\mu)| = |x^{1-\alpha}u'(x,\mu)| = O\left(|\varrho|^3 \exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right),\ 0 \le x \le \pi.$$

By changing x to $\pi - x$ in (2.1) and using the jump condition (2.4), we get a new problem. Applying the definition of 2.1, we obtain the asymptotic form of $w(x,\mu)v(x,\tau)$ and $T_\alpha v(x,\mu)$. Specially,

$$|v(x,\mu)| = O\left(|\varrho|^2 \exp\left(\frac{|\tau|}{\alpha}(\pi - x)^\alpha\right)\right),$$
$$|T_\alpha v(x,\mu)| = |x^{1-\alpha}v'(x,\mu)| = O\left(|\varrho|^3 \exp\left(\frac{|\tau|}{\alpha}(\pi - x)^\alpha\right)\right),\quad 0 \le x \le \pi.$$

In addition, using (2.2) and Remark 2.4, we obtain

$$\begin{aligned}\Delta(\mu) :&= W_\alpha(u(\mu), v(\mu))\\ &= B_1(v(\mu))\\ &= -w_0(\pi)B_2(u(\mu))\\ &= w_0(p)\left(c_s u(p,\mu)T_\alpha v(p,\mu) - b_s T_\alpha u(p,\mu)v(p,\mu)\right).\end{aligned} \tag{2.18}$$

It follows from equation (2.18) that the characteristic function $\Delta(\mu)$ is the combination of the solutions and from [16] it is clear that every solution is an entire function of order $\frac{1}{2}$. As a result, $\Delta(\mu)$ is an entire function of order $\frac{1}{2}$, so that its roots are, μ_n, the eigenvalues of L. The asymptotic form of the characteristic function will be:

$$\begin{aligned}\Delta(\mu) = \varrho^5 w_0(\pi)\Big[&a_1a_2\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}\pi^\alpha\right) + a_1'a_2\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(\pi^\alpha - 2p_1^\alpha)\right) + \cdots\\ &+ a_1a_2\ldots a_{m-1}'\sin\left(\frac{\varrho}{\alpha}(\pi^\alpha - 2p_{m-1}^\alpha)\right) + a_1'a_2'a_3\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(\pi^\alpha + 2p_1^\alpha - 2p_2^\alpha)\right)\\ &+ \cdots + a_1\ldots a_i'\ldots a_j'\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(\pi^\alpha + 2p_i^\alpha - 2p_j^\alpha)\right)\\ &+ a_1\ldots a_i'\ldots a_j'\ldots a_k'\ldots a_{m-1}\sin\left(\frac{\varrho}{\alpha}(\pi^\alpha - 2p_i^\alpha + 2p_j^\alpha - 2p_k^\alpha)\right) + \cdots\\ &+ a_1'a_2'\ldots a_{m-1}'\sin\left(\frac{\varrho}{\alpha}(\pi^\alpha + 2(-1)^{m-1}d_1^\alpha + 2(-1)^{m-2}p_2^\alpha - 2p_m^\alpha)\right)\Big]\\ &+ O\left(\varrho^4\exp\left(\frac{|\tau|}{\alpha}\pi^\alpha\right)\right).\end{aligned} \tag{2.19}$$

As a result of Valiron's theorem ([22, Thm. 13.4]) and (2.19), we obtain the following asymptotic form.

Table 1 Eigenvalues and asymptotic results for Example 2.9

n	$\varrho_{n,\alpha}$				$\xi_{n,\alpha}$			
	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=0.99$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=0.99$
1	0.6902	0.6801	0.6679	0.6558	0.699	0.676	0.662	0.655
2	1.3864	1.4247	1.4202	1.3939	0.702	0.708	0.704	0.696
3	1.8690	1.9207	2.0068	2.0897	0.631	0.637	0.663	0.696
4	3.1125	3.0263	2.9211	2.8390	0.788	0.752	0.724	0.709
10	8.9998	9.0220	8.9560	9.2168	0.892	0.897	0.887	0.920
20	18.9288	19.3153	19.2437	18.8692	0.939	0.960	0.953	0.942
30	28.6067	29.0718	29.4662	29.2323	0.971	0.972	0.973	0.973

Theorem 2.7 *Let $\mu_n = \varrho_n^2$ be the eigenvalues of the problem L_α, then we have the following asymptotic formula*

$$\varrho_n = \alpha\pi^{1-\alpha}n + O(1) \tag{2.20}$$

as $n \to \infty$.

Remark 2.8 The asymptotic forms of the solutions and characteristic function for the operators L_1 and L_2 are similar to the operator L_0.

Example 2.9 Consider the following PDCSLP with $h_1 = 0,\ h_2 = 0,\ H_1 = 0,\ H_2 = 0$ and $h_3 = H_3 = 1$ with one jump point in $p = \frac{\pi}{4}$,

$$\begin{aligned}
&-T_\alpha T_\alpha y = \mu y\\
&\mu T_\alpha y(0) - y(0) = 0, \quad \mu T_\alpha y(\pi) + y(\pi) = 0,\\
&y(\frac{\pi}{4}+) - 2y(\frac{\pi}{4}-) = 0, \quad T_\alpha y(\frac{\pi}{4}+) - \frac{1}{2}T_\alpha y(\frac{\pi}{4}-) = 0.
\end{aligned} \tag{2.21}$$

The characteristic function and the eigenfunctions are

$$\begin{aligned}
\Delta(\mu) =& -\varrho^5\left[\frac{5}{4}\sin(\frac{\varrho}{\alpha}\pi^\alpha) + \frac{3}{4}\sin(\frac{\varrho}{\alpha}(\pi^\alpha - 2(\frac{\pi}{4})^\alpha))\right] + \frac{5}{2}\varrho^2\cos(\frac{\varrho}{\alpha}\pi^\alpha)\\
&+ \frac{1}{\varrho}\left(\frac{5}{4}\sin(\frac{\varrho}{\alpha}\pi^\alpha) - \frac{3}{4}\sin(\frac{\varrho}{\alpha}(\pi^\alpha - 2(\frac{\pi}{4})^\alpha))\right)
\end{aligned}$$

$$u_{n,\alpha}(x) = \begin{cases} \varrho_n^2\cos(\frac{\varrho_n}{\alpha}x^\alpha) + \frac{1}{\varrho_n}\sin(\frac{\varrho_n}{\alpha}x^\alpha), & 0 \le x < \frac{\pi}{4},\\ \varrho_n^2(\frac{5}{4}\cos(\frac{\varrho_n}{\alpha}x^\alpha) + \frac{3}{4}\cos(\frac{\varrho_n}{\alpha}(x^\alpha - 2(\frac{\pi}{4})^\alpha))) + \frac{5}{4\varrho_n}\sin(\frac{\varrho_n}{\alpha}x^\alpha), & \\ \quad -\frac{3}{4\varrho_n}\sin(\frac{\varrho_n}{\alpha}(x^\alpha - 2(\frac{\pi}{4})^\alpha)), & \frac{\pi}{4} \le x \le \pi. \end{cases}$$

Example 2.10 Consider the following PDCSLP with $h_1 = 0,\ h_2 = 0,\ H_1 = 0,\ H_2 = 0$ and $h_3 = H_3 = 1$ with one jump point in $p = \frac{\pi}{2}$

$$-T_\alpha T_\alpha y = \mu y$$

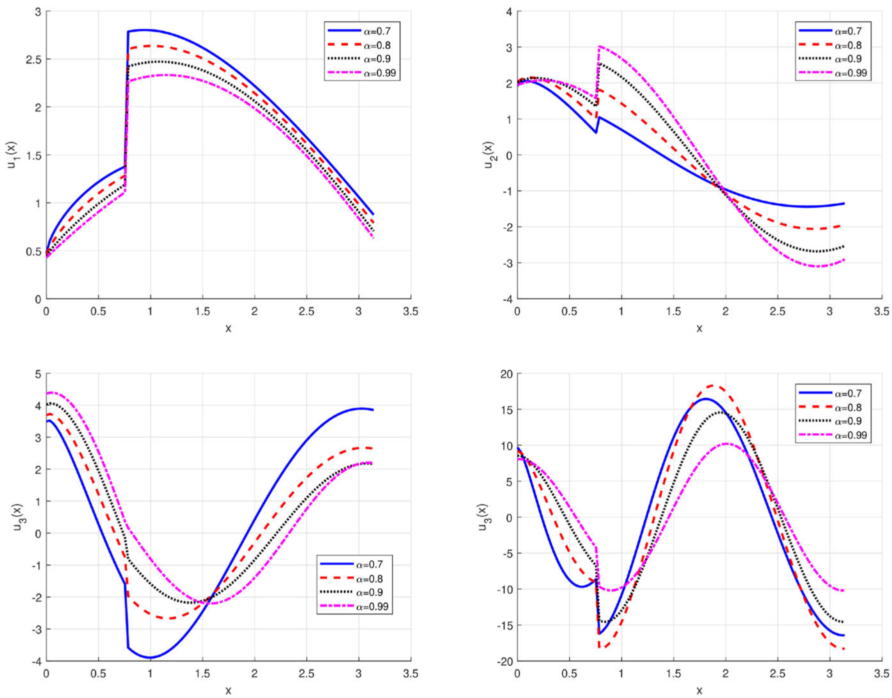

Fig. 1 Eigenfunctions of Example 2.9 for different values of n and α

$$\mu T_\alpha y(0) - y(0) = 0, \quad \mu T_\alpha y(\pi) + y(\pi) = 0,$$
$$y(\frac{\pi}{2}+) - 3y(\frac{\pi}{2}-) = 0, \quad T_\alpha y(\frac{\pi}{2}+) - \frac{1}{3} T_\alpha y(\frac{\pi}{2}-) = 0. \tag{2.22}$$

The characteristic function and the eigenfunctions are

$$\Delta(\mu) = -\varrho^5 \left[\frac{5}{3} \sin(\frac{\varrho}{\alpha}\pi^\alpha) + \frac{4}{3} \sin(\frac{\varrho}{\alpha}(\pi^\alpha - 2(\frac{\pi}{2})^\alpha)) \right] + \frac{10}{3} \varrho^2 \cos(\frac{\varrho}{\alpha}\pi^\alpha)$$
$$+ \frac{1}{\varrho} \left(\frac{5}{3} \sin(\frac{\varrho}{\alpha}\pi^\alpha) - \frac{4}{3} \sin(\frac{\varrho}{\alpha}(\pi^\alpha - 2(\frac{\pi}{2})^\alpha)) \right)$$
$$u_{n,\alpha}(x) = \begin{cases} \varrho_n^2 \cos(\frac{\varrho_n}{\alpha} x^\alpha) + \frac{1}{\varrho_n} \sin(\frac{\varrho_n}{\alpha} x^\alpha), & 0 \le x < \frac{\pi}{2}, \\ \varrho_n^2 (\frac{5}{3} \cos(\frac{\varrho_n}{\alpha} x^\alpha) + \frac{4}{3} \cos(\frac{\varrho_n}{\alpha}(x^\alpha - 2(\frac{\pi}{2})^\alpha))) + \frac{5}{3\varrho_n} \sin(\frac{\varrho_n}{\alpha} x^\alpha), & \\ \quad - \frac{4}{3\varrho_n} \sin(\frac{\varrho_n}{\alpha}(x^\alpha - 2(\frac{\pi}{2})^\alpha)), & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

The eigenvalues and eigenfunctions are presented in Table 1 and Fig. 1. We use the **Roots** function in **Maple 2021**, to compute the zeros $\varrho_{n,\alpha}$ of the function $\Delta(\mu)$. We compared the eigenvalues with first term of asymptotic form (2.20) as $\xi_{n,\alpha} = \frac{\varrho_{n,\alpha}}{n\alpha\pi^{1-\alpha}}$. The eigenvalues and ratios $\xi_{n,\alpha}$ are presented in Tables 1 and 2. According to asymptotic form (2.20), the values of $\xi_{n,\alpha}$ must tend to one, that hold for results of $\xi_{n,\alpha}$ in Tables 1 and 2. The first four eigenfunctions for different values of α are plotted in

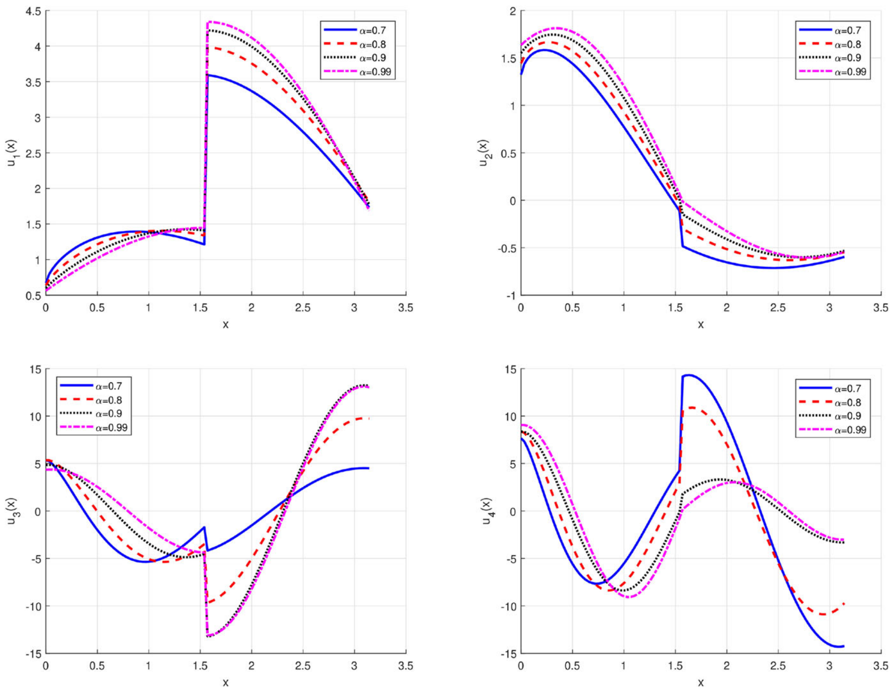

Fig. 2 Eigenfunctions of Example 2.10 for different values of n and α

Table 2 Eigenvalues and asymptotic results for Example 2.10

n	$\varrho_{n,\alpha}$				$\xi_{n,\alpha}$			
	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.99$
1	0.8190	0.7985	0.7743	0.7506	0.830	0.794	0.767	0.750
2	1.1503	1.2010	1.2451	1.2787	0.583	0.597	0.617	0.638
3	2.3121	2.3131	2.2028	2.0745	0.781	0.767	0.728	0.694
4	2.7374	2.7675	2.8925	3.0108	0.693	0.688	0.717	0.752
10	8.8289	9.3208	8.8171	8.9637	0.895	927	0.874	0.895
20	18.5084	18.9557	19.4448	18.9225	0.938	0.942	0.963	0.945
30	28.3659	28.9557	29.2035	28.8841	0.958	0.960	0.965	0.962

Figs. 1 and 2. It is well known that the nth eigenfunction of classical Sturm–Liouville problem defined on $[0, \pi]$, has $(n-1)$ zero in interval $(0, \pi)$. The graphs in Figs. 1 and 2 indicate that this result holds also for PDCSLP with jump conditions.

3 Uniqueness Result

In this section, we formulated the inverse PDCSLPs. To this end, first we necessity the following lemma about poles, residues, and asymptotic formulas to determine a meromorphic Herglotz–Nevanlinna function, see [14, Thm 2.3].

Lemma 3.1 *Suppose that the functions $h_1(z)$ and $h_2(z)$ are two meromorphic Herglotz–Nevanlinna function with the similar sets of residues and poles. If*

$$h_1(it) - h_2(it) \to 0, \quad as\ t \to \infty,$$

then $h_1 = h_2$.

Define the Weyl–Titchmarsh $\mathfrak{m}$-functions

$$\mathfrak{m}_-(\mu) = -\frac{T_\alpha u(p,\mu)}{u(p,\mu)}, \qquad \mathfrak{m}_+(\mu) = \frac{T_\alpha v(p,\mu)}{v(p,\mu)}. \tag{3.1}$$

As a consequence of theorem [14, Thms. 2.1 and 2.2] we obtain:

Lemma 3.2 *The functions $\mathfrak{m}_-(\mu)$ and $\mathfrak{m}_+(\mu)$ satisfy in the conditions of Herglotz–Nevanlinna's functions.*

Proof Suppose that the functions u and $\bar{u}$ are solutions of $\ell_1 u = \mu u$ and $\overline{\ell_1 u} = \ell_1 \bar{u} = \bar{\mu}\bar{u}$. It is easy to see that

$$(\mu - \bar{\mu}) \int_0^x u(t)\bar{u}(t)w_1(t)\mathrm{d}_\alpha t = W_\alpha(u,\bar{u})(x) - W_\alpha(u,\bar{u})(0).$$

From definition of $\mathfrak{m}_-(\mu)$ in the point $x = p$ and the condition (2.6), we get

$$\mathrm{Im}(\mu)\|u\|^2_{T_1} = \mathrm{Im}(\mathfrak{m}_-(\mu))|u(p)|^2.$$

Then, the function $\mathfrak{m}_-(\mu)$ is Herglotz–Nevanlinna function. Similarly the function $\mathfrak{m}_+(\mu)$ is also Herglotz–Nevanlinna function. □

Lemma 3.3 *For every arbitrary $\upsilon > 0$ with $\upsilon < \arg\mu < 2\pi - \upsilon$, the following asymptotic formula for $\mathfrak{m}_-(\mu)$ and $\mathfrak{m}_+(\mu)$ hold:*

$$\mathfrak{m}_+(\mu) = i\sqrt{\mu} + o(\sqrt{\mu}), \quad \mathfrak{m}_-(\mu) = i\sqrt{\mu} + o(\sqrt{\mu}), \quad as\ \mu \to \infty. \tag{3.2}$$

Specially, when $\mu \to -\infty$, we have

$$\mathfrak{m}_+(\mu) = -\sqrt{|\mu|} + o(\sqrt{|\mu|}), \quad \mathfrak{m}_-(\mu) = -\sqrt{|\mu|} + o(\sqrt{|\mu|}) \quad as\ \mu \to -\infty. \tag{3.3}$$

Proof Using the asymptotic forms of $u(x,\mu)$ and $T_\alpha u(x,\mu)$ in (2.15) and (2.16) and similar asymptotic forms for $v(x,\mu)$ and $T_\alpha v(x,\mu)$, it can be checked by direct calculations that the asymptotic forms of $\mathfrak{m}_-(\mu)$ and $\mathfrak{m}_+(\mu)$ are satisfying in (3.2)–(3.3). □

Suppose that the eigenvalues of the PDCSLPs (2.5)–(2.7) and PDCSLPs (2.6)–(2.10) are denoted by $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$, respectively. In this part, we express the fundamental theorem of uniqueness result for the problems (2.1)–(2.10). For the uniqueness theorem we need using the similar operators $\tilde{L}_i$, with operators L_i but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$, $\tilde{H}_1$, $\tilde{b}_k$, $\tilde{c}_k$, $\tilde{d}_k$, $\tilde{p}_k$. Given a function

$$f(\mu) := \begin{cases} -\frac{\Delta(\mu)}{w_0(p)u(p,\mu)v(p,\mu)}, & H_1 = \infty, \\ -\frac{\Delta(\mu)}{w_0(p)[T_\alpha u(p,\mu)+H_1 u(p,\mu)][T_\alpha v(p,\mu)+H_2 v(p,\mu)]}, & H_1 \neq \infty. \end{cases} \tag{3.4}$$

It is easy to check that $f(\mu)$ is a meromorphic function and the set of poles of $f(\mu)$ is all values of $\{\mu_n\}_{n=1}^{\infty} \cup \{\nu_n\}_{n=1}^{\infty}$. Using Eq. (2.18) and $H_2 = \frac{c_s}{b_s} H_1$, we have

$$\begin{aligned} f(\mu) &= \begin{cases} -c_s \frac{T_\alpha v(p,\mu)}{v(p,\mu)} + b_s \frac{T_\alpha u(p,\mu)}{u(p,\mu)}, & H_1 = \infty, \\ -c_s \frac{u(p,\mu)}{T_\alpha u(p,\mu)+H_1\, u(p,\mu)} + b_s \frac{v(p,\mu)}{T_\alpha v(p,\mu)+H_2\, v(p,\mu)}, & H_1 \neq \infty, \end{cases} \\ &:= M_+(\mu) + M_-(\mu), \end{aligned}$$

where from (3.1)

$$M_+(\mu) = \begin{cases} -c_s \mathfrak{m}_+(\mu), & H_2 = \infty, \\ \frac{b_s}{H_2+\mathfrak{m}_+(\mu)}, & H_2 \in \mathbb{R}, \end{cases} \quad M_-(\mu) = \begin{cases} -b_s \mathfrak{m}_-(\mu), & H_1 = \infty, \\ \frac{c_s}{\mathfrak{m}_-(\mu)-H_1}, & H_1 \in \mathbb{R}. \end{cases} \tag{3.5}$$

Lemma 3.4 *Fixed $H_1 \in \mathbb{R} \cup \{\infty\}$. For every arbitrary $\upsilon > 0$ with $\upsilon < \arg \mu < 2\pi - \upsilon$, the following asymptotic formula for $M_-(\mu)$ and $M_+(\mu)$ hold:*

$$M_-(\mu) = \begin{cases} i\, b_s \sqrt{\mu} + o(\sqrt{\mu}), & H_1 = \infty, \\ \frac{i\, c_s}{\sqrt{\mu}} + o\left(\frac{1}{\sqrt{\mu}}\right), & H_1 \in \mathbb{R}, \end{cases}$$

and

$$M_+(\mu) = \begin{cases} i\, c_s \sqrt{\mu} + o(\sqrt{\mu}), & H_2 = \infty, \\ \frac{i\, b_s}{\sqrt{\mu}} + o\left(\frac{1}{\sqrt{\mu}}\right), & H_2 \in \mathbb{R}. \end{cases}$$

Proof The proof is similar to Lemma 3.3. □

Theorem 3.5 *If $\mu_n = \tilde{\mu}_n$, $\omega_n = \tilde{\omega}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, and $w_0(x) = \tilde{w}_0(x)$, $h_j = \tilde{h}_j$, and $H_j = \tilde{H}_j$, $(j = 1, 2, 3)$ and if $\{\omega_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

Proof From Lemma 3.2, $\mathfrak{m}_-(\mu)$ and $\mathfrak{m}_+(\mu)$ are Herglotz–Nevanlinna functions. Therefore, one can easily check that the function $M_+(\mu)$ and $M_-(\mu)$ are Herglotz–Nevanlinna functions. The functions $\tilde{\mathfrak{m}}_-(\mu)$, $\tilde{M}_-(\mu)$, $\tilde{\mathfrak{m}}_+(\mu)$, $\tilde{M}_+(\mu)$, and $\tilde{f}(\mu)$

defined by a similar way by replacing L to $\tilde{L}$. We define

$$\mathcal{G}(\mu) := \frac{f(\mu)}{\tilde{f}(\mu)}.$$

Since the functions $f(\mu)$ and $\tilde{f}(\mu)$ have the same poles and zeros, $\mathcal{G}(\mu)$ is an entire function. Applying Lemmas 3.3 and 3.4, we have

$$\mathcal{G}(\mu) = \frac{f(\mu)}{\tilde{f}(\mu)} = 1 + o(1)$$

for any $\upsilon > 0$ in the area of $\upsilon \leq \arg \mu \leq 2\pi - \upsilon$. By applying Liouville's theorem, we get

$$\mathcal{G}(\mu) = 1$$

then

$$f(\mu) = \tilde{f}(\mu).$$

From (3.4) and (3.5), the poles of $M_-(\mu)$ and $M_+(\mu)$ are exactly the same $\{\omega_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$, respectively. Then, we get

$$\operatorname*{Res}_{\mu=\omega_n} M_-(\mu) = \operatorname*{Res}_{\mu=\omega_n} f(\mu) \text{ and } \operatorname*{Res}_{\mu=\nu_n} M_+(\mu) = \operatorname*{Res}_{\mu=\nu_n} f(\mu), \quad \text{for } n = 1, 2, 3, \ldots.$$

which means that

$$\operatorname*{Res}_{\mu=\omega_n} M_-(\mu) = \operatorname*{Res}_{\mu=\omega_n} \tilde{M}_-(\mu) \text{ and } \operatorname*{Res}_{\mu=\nu_n} M_+(\mu) = \operatorname*{Res}_{\mu=\nu_n} \tilde{M}_+(\mu), \quad \text{for } n = 1, 2, 3, \ldots.$$

Applying the Borg's theorem [21] for the M-Weyl–Titchmarsh functions $M_+(\mu)$ and $M_-(\mu)$, we get

$$L = \tilde{L}.$$

□

Assuming $b_s = c_s = 1$ in Eq. (2.11) we have $\kappa_1 = \kappa_2$. From this assumption, the main result (Theorem 3.5) can be extended to the case $p \in (p_s - 1, p_{s+1})$.

Corollary 3.6 *Let $\mu_n = \tilde{\mu}_n$, $\omega_n = \tilde{\omega}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, and $w_0(x) = \tilde{w}_0(x)$, $h_j = \tilde{h}_j$, $H_j = \tilde{H}_j$, $(j = 0, 1, 2)$, $b_s = c_s = 1$, and if $\{\omega_n\}_{n=0}^{+\infty}$ and $\{\nu_n\}_{n=0}^{+\infty}$ are separate in pairs, then $L_0 = \tilde{L}_0$.*

Let $b_k = c_k = 1$, $d_k = 0$ for $k = 1, 2, \ldots, m - 1$ in Eqs. (2.4), then our PDCSLP changes to the continuous case equation.

Corollary 3.7 *If $\mu_n = \tilde{\mu}_n$, $\omega_n = \tilde{\omega}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, $h_j = \tilde{h}_j$, $H_j = \tilde{H}_j$, $(j = 0, 1, 2)$, and $b_k = c_k = 1$, $d_k = 0$ for $k = 1, 2, \ldots, m-1$, if $\{\omega_n\}_{n=0}^{+\infty}$ and $\{\nu_n\}_{n=0}^{+\infty}$ are separate in pairs, then $L_0 = \tilde{L}_0$.*

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Declarations

Conflict of interest The author declare no conflicts of interest in this research paper.

References

1. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
2. Adalar, I.: On Mochizuki–Trooshin theorem for Sturm–Liouville operators. Cumhuriyet Sci. J. **40**(1), 108–116 (2019)
3. Adalar, I., Ozkan, A.S.: Inverse problems for a conformable fractional Sturm–Liouville operator. J. Inverse Ill-Posed Probl. **28**(6), 775–782 (2020)
4. Atangana, A., Baleanu, D., Alsaedi, A.: New properties of conformable derivative. Open Math. **13**(1), 889–898 (2015)
5. Binding, P.A., Browne, P.J., Watson, B.A.: Sturm–Liouville problems with boundary conditions rationally dependent on the eigenparameter, II. J. Comput. Appl. Math. **148**(1), 147–168 (2002)
6. Boyko, O., Martinyuk, O., Pivovarchik, V.: Higher order Nevanlinna functions and the inverse three spectra problem. Opuscula Math. **36**(3), 301 (2016)
7. Boyko, O., Pivovarchik, V., Yang, C.F.: On solvability of three spectra problem. Math. Nachr. **289**(14–15), 1727–1738 (2016)
8. Çakmak, Y., Keskin, B.: Uniqueness theorems for Sturm–Liouville operator with parameter dependent boundary conditions and finite number of transmission conditions. Cumhuriyet Sci. J. **38**(3), 535–543 (2017)
9. Drignei, M.C.: Inverse Sturm-Liouville Problems Using Multiple Spectra. Iowa State University, Ames (2008)
10. Drignei, M.C.: Uniqueness of solutions to inverse Sturm–Liouville problems with L2 (0, a) potential using three spectra. Adv. Appl. Math. **42**(4), 471–482 (2009)
11. Drignei, M.C.: Constructibility of an solution to an inverse Sturm–Liouville problem using three Dirichlet spectra. Inverse Prob. **26**(2), 025003 (2009)
12. Fu, S., Xu, Z., Wei, G.: Inverse indefinite Sturm–Liouville problems with three spectra. J. Math. Anal. Appl. **381**(2), 506–512 (2011)
13. Fu, S., Xu, Z., Wei, G.: The interlacing of spectra between continuous and discontinuous Sturm–Liouville problems and its application to inverse problems. Taiwan. J. Math. **16**(2), 651–663 (2012)
14. Gesztesy, F., Simon, B.: On the determination of a potential from three spectra. Differ. Oper. Spect. Theory **189**, 85–92 (1999)
15. Gladwell, G.M.: Inverse Problems in Vibration. Kluwer academic publishers, New York (2004)
16. Halvorsen, S.G.: A function-theoretic property of solutions of the equation $x'' + (w - q)x = 0$. Q. J. Math. **38**(1), 73–76 (1987)
17. Keskin, B.: Inverse problems for one dimensional conformable fractional Dirac type integro differential system. Inverse Prob. **36**(6), 065001 (2020)
18. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
19. Khosravian-Arab, H., Dehghan, M., Eslahchi, M.R.: Fractional Sturm-Liouville boundary value problems in unbounded domains: theory and applications. J. Comput. Phys. **299**, 526–560 (2015)
20. Klimek, M., Agrawal, O.P.: Fractional Sturm–Liouville problem. Comput. Math. Appl. **66**(5), 795–812 (2013)
21. Levitan, B.M.: Inverse Sturm–Liouville Problems. VNU Science Press, De Gruyter (1987)

22. Levin, B.Y.: Lectures on Entire Functions, vol. 150. American Mathematical Society, Providence (1996)
23. Mortazaasl, H., Jodayree Akbarfam, A.: Trace formula and inverse nodal problem for a conformable fractional Sturm–Liouville problem. Inverse Probl. Sci. Eng. **28**(4), 524–555 (2020)
24. Mehrabov, V.A.: Spectral properties of a fourth-order differential operator with eigenvalue parameter-dependent boundary conditions. Bull. Malays. Math. Sci. Soc. **45**, 741–766 (2022)
25. Ozkan, A.S., Keskin, B.: Inverse nodal problems for Sturm-Liouville equation with eigenparameter-dependent boundary and jump conditions. Inverse Probl. Sci. Eng. **23**(8), 1306–1312 (2015)
26. Pivovarchik, V.N.: An inverse Sturm–Liouville problem by three spectra. Integr. Eqn. Oper. Theory **34**, 234–243 (1999)
27. Pivovarchik, V.: A special case of the Sturm–Liouville inverse problem by three spectra: uniqueness results. Proc. R. Soc. Edinb. Sect. A Math. **136**(1), 181–187 (2006)
28. Rivero, M., Trujillo, J., Velasco, M.: A fractional approach to the Sturm–Liouville problem. Open Phys. **11**(10), 1246–1254 (2013)
29. Shahriari, M.: Inverse Sturm–Liouville problem with eigenparameter dependent boundary and transmission conditions. Azerb. J. Math. **4**(2), 16–30 (2014)
30. Shahriari, M.: Inverse Sturm-Liouville problems with transmission and spectral parameter boundary conditions. Comput. Methods Differ. Equ. **2**(3), 123–139 (2014)
31. Shahriari, M.: Inverse Sturm–Liouville problems using three spectra with finite number of transmissions and parameter dependent conditions. Bull. Iran. Math. Soc. **43**(5), 1341–1355 (2017)
32. Shahriari, M., Akbari, R.: Inverse Conformable Sturm-Liouville Problems with a Transmission and Eigen-Parameter Dependent Boundary Conditions. Sahand Commun. Math. Anal. **20**(4), 87–104 (2023)
33. Shahriari, M., Akbarfam, A.J., Teschl, G.: Uniqueness for inverse Sturm–Liouville problems with a finite number of transmission conditions. J. Math. Anal. Appl. **395**(1), 19–29 (2012)
34. Shahriari, M., Mirzaei, H.: Inverse Sturm–Liouville problem with conformable derivative and transmission conditions. Hacettepe J. Math. Stat. **52**(3), 753–767 (2023)
35. Titchmarsh, E.C., Weiss, G.: Eigenfunction expansions associated with second-order differential equations, part 1. Phys. Today **15**(8), 52–52 (1962)
36. Teschl, G.: Mathematical Methods in Quantum Mechanics, vol. 157. American Mathematical Society, Providence (2014)
37. Yan-Hsiou, C.: The dual eigenvalue problems of the conformable fractional Sturm–Liouville problems. Bound. Value Probl. **2021**(1), 1–10 (2021)

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