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Finite Fractal Dimensional Pullback Attractors for a Class of 2D Magneto-Viscoelastic Flows

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Abstract

In this article, the long-time behaviors of weak solutions for the 2D non-autonomous magneto-viscoelastic flows are considered. Unlike the results established by Liu and Liu (Politeh Univ Buchar Sci Bull Ser A Appl Math Phys 81(4):155–166, 2019), utilizing the method of ℓ-trajectories introduced by Málek and Pražák (J Differ Equ 181(2):243–279, 2002), we first justify the existence of finite-dimensional pullback attractors for the process $\{L(t, \tau)\}_{t\geq\tau}$ in the ℓ-trajectories space X_ℓ. Then we obtain the corresponding finite-dimensional pullback attractors for the process $\{U(t, \tau)\}_{t\geq\tau}$ in the original phase space $\mathbb{H}$.

Keywords Magneto-viscoelastic flows · Pullback attractors · Fractal dimension · The method of ℓ-trajectories

Mathematics Subject Classification 35Q35 · 35B41 · 37L30 · 76A10

1 Introduction

Magneto-viscoelastic flow is a class of important and complex non-Newtonian fluid, which has a broad application prospect in technological applications. The magneto-viscoelastic model has received extensive interest in the past years. A general magneto-

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viscoelastic model describing magnetoelastic materials was established by Forster in [11], which is based on an energetic variational approach (see, e.g., [14]). Since the magnetoelastic materials are extremely affected by the phenomenon of converting applied changes of the magnetic field and vice versa, they can be regarded as smart materials. For instance, the various magnetic materials can be found in sensors to measure the torque of a force, and can also be used in magnetic actuators and generators for ultrasonic sounds (see, e.g., [4, 5, 11, 13, 17, 33]).

Forster [11] established the following incompressible magneto-viscoelastic fluid model:

$$\begin{cases} v_t - \mu \Delta v + v \cdot \nabla v + \nabla p = \nabla \cdot (FF^T - \nabla^T M \nabla M) + \nabla^T H_{\text{ext}} M, \\ F_t + v \cdot \nabla F - \nabla v F = \kappa \Delta F, \\ M_t + v \cdot \nabla M = \Delta M - \frac{1}{\gamma^2}(|M|^2 - 1)M + H_{\text{ext}}, \\ \text{div } v = 0. \end{cases} \tag{1.1}$$

System (1.1) consists of the incompressible Navier–Stokes equations coupled with balance equations for the deformation gradient F and the magnetization M, where the magnetization M is a simplification of the Landau–Lifshitz–Gilbert equations with convection (see, e.g., [3, 11]). In technological applications, we sometimes need to investigate the perturbations of the external magnetic field to the magneto-viscoelastic fluid. That is, the magneto-viscoelastic fluid is exposed to an external effective magnetic field H_{ext}. Considering the coupling of magnetic and elastic effects, the study of magnetoelastic materials has attracted more attention from various technological applications and the view of mathematical modeling (see e.g., [2, 9, 17, 18, 34, 42]).

Note that if $M = 0$, system (1.1) is a model for incompressible viscoelastic flows (see e.g., [21, 24]). If $F = 0$, it reduces to the simplified Ericksen–Leslie system for incompressible liquid crystal flows (see e.g., [25, 26]). If $M = F = 0$, it translates into the standard Navier–Stokes equations (see e.g., [23, 38]).

In this paper, taking $H_{\text{ext}} = 0$, then we can obtain the following simplified 2D incompressible magneto-viscoelastic fluid model with non-autonomous external force term $g(x,t)$:

$$\begin{cases} v_t - \mu \Delta v + v \cdot \nabla v + \nabla p = \nabla \cdot (FF^T - \nabla^T M \nabla M) + g(x,t), \\ F_t + v \cdot \nabla F - \nabla v F = \kappa \Delta F, \\ M_t + v \cdot \nabla M = \Delta M - \frac{1}{\gamma^2}(|M|^2 - 1)M, \\ \text{div } v = 0, \end{cases} \tag{1.2}$$

in $\Omega \times [\tau, T]$, where Ω is a bounded regular domain with smooth boundary, $\tau \in \mathbb{R}, \tau \leq T$, $v(x,t) : \Omega \times [\tau, T] \to \mathbb{R}^2$ is the velocity of the fluid, $p = p(x,t)$ is the scalar pressure, and $F : \Omega \times [\tau, T] \to \mathbb{R}^{2\times 2}$ is the deformation gradient, $M : \Omega \times [\tau, T] \to \mathbb{R}^3$ is the magnetization vector. $\mu, \kappa > 0$ are viscosity coefficients, $\gamma > 0$ stands for the parameter that controls the strength of penalization on the deviation of $|M|$ from 1. $g = g(x,t)$ is time dependent external force term.

System (1.2) is given the Dirichlet boundary conditions for v, F and the Neumann boundary condition for M

$$v(x,t)|_{\partial\Omega} = 0, \; F(x,t)|_{\partial\Omega} = 0, \; \frac{\partial M}{\partial \mathbf{n}}|_{\partial\Omega} = 0, \quad t \geq \tau, \tag{1.3}$$

and the initial conditions

$$v(x,\tau) = v_\tau(x), \text{with div } v_\tau = 0, \qquad x \in \Omega, \tag{1.4}$$

$$F(x,\tau) = F_\tau(x) = I, \; M(x,\tau) = M_\tau(x), \qquad x \in \Omega, \tag{1.5}$$

where I is the 2×2 identity matrix and $\mathbf{n}$ is the unit outward normal vector.

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a function and $G(r) = \int_0^r h(s)\mathrm{d}s$ be a potential function as follows

$$h(M) = (|M|^2 - 1)M, \quad G(M) = \frac{1}{4}(|M|^2 - 1)^2,$$

and define the basic energy

$$\mathcal{E}(t) = \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|F\|_{L^2}^2 + \frac{1}{2}\|\nabla M\|_{L^2}^2 + \int_\Omega G(M)\mathrm{d}x.$$

It is well known that the long-time asymptotic behavior of dynamical systems is one of the most important problems for nonlinear dissipative evolution systems. Until now, one effective way to deal with this problem for a given evolution system is to study the existence and structure of its attractor. In particular, the non-autonomous evolution systems, which can well describe the intrinsic properties of many natural phenomena, are rather more complicated than autonomous ones. In the past decades, many scholars have focused on the research of more general non-autonomous differential equations. The pullback attractors can well describe the pullback asymptotic behavior of non-autonomous dynamical systems (see [6, 10, 20, 32, 37] and references therein), which is a minimal family of compact invariant sets under the process and pullback attracts any bounded subset of the phase space.

In this paper, we aim to establish the existence of pullback attractors with finite fractal dimension in the original phase space $\mathbb{H}$ for the magneto-viscoelastic system (1.2)–(1.5). For the autonomous case of system (1.2)–(1.5), i.e., when the non-autonomous external force term $g = 0$, the solution operator defines a semigroup. In [28], the authors proved the existence of global attractors for the autonomous case of system (1.2)–(1.5). However, to the best of our knowledge, there are no results about the existence of the pullback attractors with finite fractal dimension for the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) in the phase space. As Liu and Liu in [28] pointed out, since the strong coupling nonlinear terms and the Neumann boundary conditions for problem (1.2)–(1.5), it is difficult to justify the smooth property of the difference of two solutions and the differentiability of the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) on the pullback attractors. Unlike the results obtained in [28], to overcome these difficulties, we utilize the novel idea of the method of

ℓ-trajectories in [1, 27, 30]. This novel method is based on an observation that the limit behavior of solutions for a given dynamical system in an original phase space can be equivalently captured by the limit behavior of ℓ-trajectories space (see [30] for more details). By virtue of this method, many scholars have studied a large class of nonlinear dissipation problems, especially for the problems of lack of good regularity properties or uniqueness of solutions (see [7, 12, 19, 22, 29, 35, 41, 43] and references therein). Furthermore, our method can also be used to establish the existence of global attractor with finite fractal dimension for the autonomous case of system (1.2)–(1.5).

In the sequel, we make the following assumption.

Hypothesis. Assume that the external force $g \in L^2_{loc}(\mathbb{R}; \mathbf{H})$ satisfies

$$(A_1): \quad R_g := \sup_{r\in\mathbb{R}} \left(\int_{r-1}^{r} \|g(s)\|^2_{L^2} \mathrm{d}s \right) < +\infty,$$

where $\mathbf{H}$ is given later.

In this paper, we obtain the main results as follows.

Theorem 1.1 *Assuming that (A_1) holds, then the following assertions are true:*

(i) There exists a pullback attractor $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\} = \{e_1(\mathcal{A}_\ell(t-\ell)) : t \in \mathbb{R}\}$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) in $\mathbb{H}$, where $\mathcal{A}_\ell(t-\ell)$ is the section of pullback attractor $\hat{\mathcal{A}}_\ell = \{\mathcal{A}_\ell(t) : t \in \mathbb{R}\}$ established in Theorem 3.3 for the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) in X_ℓ;

(ii) The pullback attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) in $\mathbb{H}$ has finite fractal dimension.

The rest of this article is arranged as follows. In the second part, we mainly introduce some basic notations, some useful results and global well-posedness for problem (1.2)–(1.5). In the third part, using the method of ℓ-trajectories in [27, 30, 41], we first define a process $\{L(t,\tau)\}_{t\geq\tau}$ on the ℓ-trajectories space X_ℓ with the topology of $L^2(\tau, \tau+\ell; \mathbb{H})$ induced by the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by system (1.2)–(1.5), then we prove the existence of pullback attractor $\hat{\mathcal{A}}_\ell$ for the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) in X_ℓ. In addition, analyzing the smoothing property of the process $\{L(t,\tau)\}_{t\geq\tau}$, we justify that the pullback attractor $\hat{\mathcal{A}}_\ell$ for the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) has a finite fractal dimension. Finally, using a Lipschitz continuous operator on the pullback attractor $\hat{\mathcal{A}}_\ell$, we obtain that the corresponding finite-dimensional pullback attractor $\hat{\mathcal{A}}$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) in $\mathbb{H}$.

2 Preliminaries

In this section, we will present some notations and useful results that are used all through paper. Let C be a positive constant, which may vary in different situations. Specifically, we also use C_0, C_1, $C(\cdot)$ to emphasize certain dependence. As usual, for any $1 \leq p < \infty$ and $k \in \mathbb{N}$, we denote the Lebesgue space and Sobolev space by

$L^p(\Omega)$ and $W^{k,p}(\Omega)$ endowed with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$, respectively, where

$$\|u\|_{L^p} := \left(\int_\Omega |u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad \|u\|_{W^{k,p}} := \left(\sum_{|\alpha|\le k}\int_\Omega |\partial^\alpha u|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

and we also denote the space $W_0^{k,p}(\Omega)$ as completions of $C_0^\infty(\Omega)$ in norms of $W^{k,p}(\Omega)$.

In particular, $H^k(\Omega) = W^{k,2}(\Omega)$, $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ with $k \in \mathbb{N}$ and $p = 2$, $H^{-k}(\Omega)$ is the dual space of $H_0^k(\Omega)$. The Lebesgue space $L^p(0,t;X)$ consists of all those functions u that take values in X for almost every $s \in (0,t)$, which satisfy $\left(\int_0^t \|u(s)\|_X^p \mathrm{d}s\right)^{\frac{1}{p}} < +\infty$, for any $1 \le p \le \infty$. For the sake of conciseness, we do not distinguish functional space when scalar-valued or vector-valued functions are involved. To deal with problem (1.2)–(1.5) in a proper setting, we also introduce some function spaces. Let

$$\mathcal{V} = \left\{v : \ v \in C_0^\infty(\Omega), \ \operatorname{div} v = 0\right\},$$

where $C_0^\infty(\Omega)$ is the space of any smooth functions v which are zero outside of some compact support depending on v. Denote the closure of $\mathcal{V}$ by $\mathbf{H}$ and $\mathbf{V}$ with respect to the $L^2(\Omega)$-norm and $H^1(\Omega)$-norm, respectively. Let

$$H_0^k(\Omega) = \{v \in H^k(\Omega) : v|_{\partial\Omega} = 0\},$$
$$H_{\mathbf{n}}^k(\Omega) = \{v \in H^k(\Omega) : \frac{\partial v}{\partial \mathbf{n}}|_{\partial\Omega} = 0\},$$

and the phase space

$$\mathbb{H} = \mathbf{H} \times \mathbf{V} \times H_{\mathbf{n}}^1(\Omega).$$

Let $\mathbf{H}'$ and $\mathbf{V}'$ be the dual spaces of $\mathbf{H}$ and $\mathbf{V}$, respectively, and the injections $\mathbf{V} \hookrightarrow \mathbf{H} \equiv \mathbf{H}' \hookrightarrow \mathbf{V}'$ are dense and continuous. $\|\cdot\|_{\mathbf{V}'}$ and $< \cdot, \cdot >$ stand for the norm in $\mathbf{V}'$ and the duality product between $\mathbf{V}$ and $\mathbf{V}'$(also $\mathbf{H}$ and its dual space), respectively.

For simplicity, Einstein summation convention is used in our paper. We denote $a \cdot b = \sum_{i=1}^n a_i b_i := a_i b_i$, $(a \otimes b)_{ij} = a_i b_j$ for the vectors a and b, $A : B = A_{ij} B_{ij}$, $\nabla A \vdots \nabla B = \partial_k A_{ij} \partial_k B_{ij}$ for the matrices $A = (A_{ij})$, $B = (B_{ij})$.

Next, we also introduce the following some useful operators (see e.g., [1]):
The bilinear form $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is defined as

$$a(u, v) := \sum_{i=1,j=1}^{2} \int_\Omega \partial_{x_j} u_i \cdot \partial_{x_j} v_i \mathrm{d}x, \quad \text{forall } u, v \in \mathbf{V}.$$

Let $\mathbb{P}$ be the Helmholtz–Leray orthogonal projection operator from $L^2(\Omega)$ onto $\mathbf{H}$. Then we define the operator $A : \mathbf{V} \to \mathbf{V}'$ by $Au = -\mathbb{P}\Delta u$, which is the Stokes operator with the domain $D(A) = H^2(\Omega) \cap \mathbf{V}$, endowed with inner product and norm

$$\begin{aligned} < Au, v > &= \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i dx, \quad \|u\|_V^2 := \|\nabla u\|_{L^2}^2 \\ &= \sum_{i=1}^{2} \|\nabla u_i\|_{L^2}^2, \quad \text{for all } u, v \in \mathbf{V}. \end{aligned}$$

The mapping $B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}'$ is defined by

$$B(u, v) := \mathbb{P}((u \cdot \nabla)v), \quad \text{for all } u, v \in \mathbf{V},$$

then

$$b(u, v, w) = \langle B(u, v), w \rangle = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x,$$

with

$$b(u, v, v) = 0, \quad b(u, v, w) = -b(u, w, v).$$

Moreover, we shall use the following some identities (see e.g., [28])

$$\begin{aligned} &\mathrm{div}(\nabla^T M \nabla M) = \frac{\nabla|\nabla M|^2}{2} + \nabla^T M \Delta M, \\ &\int_\Omega (\nabla^T M \Delta M) \cdot v \mathrm{d}x = \int_\Omega (v \cdot \nabla M) \cdot \Delta M \mathrm{d}x, \\ &\int_\Omega \mathrm{div}(F F^T) \cdot v \mathrm{d}x = -\int_\Omega (\nabla v F) : F \mathrm{d}x. \end{aligned} \tag{2.1}$$

Now, applying the projection operator $\mathbb{P}$ to problem (1.2)–(1.5), we can get the following equivalent functional differential equation

$$\begin{cases} v_t + \mu A v + B(v, v) = \mathbb{P}(\nabla \cdot (F F^T - \nabla^T M \nabla M)) + \mathbb{P} g(x, t), \\ F_t + v \cdot \nabla F - \nabla v F = \kappa \Delta F, \\ M_t + v \cdot \nabla M = W, \\ W = \Delta M - \frac{1}{\gamma^2}(|M|^2 - 1)M. \end{cases} \tag{2.2}$$

Definition 2.1 Let $(v_\tau, F_\tau, M_\tau) \in \mathbb{H}$ and $g \in L^2_{loc}(\mathbb{R}; \mathbf{H})$. The triple (v, F, M) is called a weak solution of problem (1.2)–(1.5), for any $T \geq \tau$, $\tau \in \mathbb{R}$, provided that

$$\begin{aligned}&v \in L^\infty(\tau, T; \mathbf{H}) \cap L^2(\tau, T; \mathbf{V}),\\&F \in L^\infty(\tau, T; L^2(\Omega; \mathbb{R}^{2\times 2})) \cap L^2(\tau, T; H^1(\Omega; \mathbb{R}^{2\times 2})),\\&M \in L^\infty(\tau, T; H^1(\Omega; \mathbb{R}^3)) \cap L^2(\tau, T; H^2(\Omega; \mathbb{R}^3))\end{aligned}$$

with $v_t \in L^2(\tau, T; \mathbf{V}')$, $F_t \in L^2(\tau, T; H^{-1}(\Omega))$, $M_t \in L^2(\tau, T; L^2(\Omega))$ such that $v(x, \tau) = v_\tau$, $F(x, \tau) = F_\tau$, $M(x, \tau) = M_\tau$, and if for test functions $\varphi \in W^{1,\infty}(\tau, T; \mathbb{R})$ with $\varphi(T) = 0$, $\psi \in \mathbf{V}$, $\omega \in H^1_0(\Omega; \mathbb{R}^{2\times 2})$, $\phi \in H^1(\Omega; \mathbb{R}^3)$, satisfy

$$\begin{aligned}&\int_\tau^T \int_\Omega -v \cdot (\varphi_t \psi) + (v \cdot \nabla) v \cdot (\varphi\psi) + (FF^T - \nabla^T M \nabla M) : (\varphi \nabla \psi) \mathrm{d}x\mathrm{d}t\\&= \int_\Omega v_\tau (\varphi(\tau)\psi) \mathrm{d}x - \mu \int_\tau^T \int_\Omega \nabla v : (\varphi \nabla \psi) \mathrm{d}x\mathrm{d}t,\\&\int_\tau^T \int_\Omega -F : (\varphi_t \omega) + (v \cdot \nabla F) : (\varphi\omega) - (\nabla v F) : (\varphi\omega) \mathrm{d}x\mathrm{d}t\\&= \int_\Omega F_\tau(\varphi(\tau)\omega) \mathrm{d}x - \kappa \int_\tau^T \int_\Omega \nabla F \vdots (\varphi \nabla \omega) \mathrm{d}x\mathrm{d}t,\\&\int_\tau^T \int_\Omega -M \cdot (\varphi_t \phi) + (v \cdot \nabla) M \cdot (\varphi\phi) \mathrm{d}x\mathrm{d}t - \int_\Omega v_\tau(\varphi(\tau)\psi)\mathrm{d}x\\&= \int_\tau^T \int_\Omega -\nabla M : (\varphi \nabla \phi) - \frac{1}{\gamma^2}(|M|^2 - 1) M \cdot (\varphi\phi) \mathrm{d}x\mathrm{d}t.\end{aligned}$$

For the given problem (1.2)–(1.5), the global well-posedness of weak solutions for the two dimensional magneto-viscoelastic flows on a bounded smooth domain was obtained in [11, 34]. For convenience, we also briefly present the following results.

Theorem 2.1 *[11, 34] Assume that $g \in L^2_{loc}(\mathbb{R}; \mathbf{H})$, then for any $(v_\tau, F_\tau, M_\tau) \in \mathbb{H}$, there exists a unique weak solution $(v(t), F(t), M(t))$ to problem (1.2)–(1.5) satisfying the conditions of Definition 2.1 such that $(v(x,\tau), F(x,\tau), M(x,\tau)) = (v_\tau, F_\tau, M_\tau)$, which depends continuously on the initial data (v_τ, F_τ, M_τ) with respect to the norm in $\mathbb{H}$.*

Corollary 2.1 *[11, 34] Assume that $g \in L^2_{loc}(\mathbb{R}; \mathbf{H})$, the triple $(v_{\tau,m}, F_{\tau,m}, M_{\tau,m}) \rightharpoonup (v_\tau, F_\tau, M_\tau)$ in $\mathbb{H}$, and let $\{(v_m(t), F_m(t), M_m(t))\}_{m\geq 1}$ be a sequence of weak solution for problem (1.2)–(1.5) such that $(v_m(\tau), F_m(\tau), M_m(\tau)) = (v_{\tau,m}, F_{\tau,m}, M_{\tau,m})$. For any $T \geq \tau$, $\tau \in \mathbb{R}$, if there exists a subsequence of $\{(v_m(t), F_m(t), M_m(t))\}_{m\geq 1}$ converging $(*-)$ weakly in the space $\big\{(v(t), F(t), M(t)) \in L^\infty(\tau, T; \mathbb{H}) \bigcap L^2(\tau, T; \mathbf{V} \times H^1(\Omega; \mathbb{R}^{2\times 2}) \times H^2(\Omega; \mathbb{R}^3)) : (v_t, F_t, M_t) \in L^2(\tau, T; \mathbf{V}' \times H^{-1}(\Omega) \times L^2(\Omega))\big\}$ to a certain function $(v(t), F(t), M(t))$. Then $(v(t), F(t), M(t))$ is a weak solution for problem (1.2)–(1.5) with $(v(x,\tau), F(x,\tau), M(x,\tau)) = (v_\tau, F_\tau, M_\tau)$.*

Next, we also present the following results in the section 3. Let X be a complete metric space with distance $d_X(\cdot,\cdot)$. We define the Hausdorff semidistance between A and B by

$$\mathrm{dist}(A, B) = \sup_{x\in A}\inf_{y\in B} d_X(x, y),\ \ A, B \subset X.$$

A two-parameter family of mappings $\{U(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$ is called a continuous process in X. If the two-parameter family of mappings $\{U(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$ from X to X satisfy

(i) $U(t,\tau) = U(t,r)U(r,\tau)$, for all $\tau \leq r \leq t$,
(ii) $U(\tau,\tau) = Id$, for all $\tau \in \mathbb{R}$,
(iii) $U(t,\tau)x_n \to U(t,\tau)x$, if $x_n \to x$ in X.

Let $\mathcal{D}$ be the family of nonempty sets parameterized with a real parameter $\hat{B} = \{B(t) : B(t) \neq \emptyset, t \in \mathbb{R}\}$ in X such that

$$\lim_{r\to-\infty} e^{\alpha_0 r}[B(r)] = 0,$$

where $[B(r)] = \sup\{\|u\|_X^2 : u \in B(r)\}, \alpha_0 > 0$. Similarly, let X_ℓ be a ℓ-trajectories space induced by X, and $\mathcal{D}_\ell$ be the family of nonempty sets parameterized with a real parameter $\hat{\mathcal{B}}_\ell = \{\mathcal{B}_\ell(t) : \mathcal{B}_\ell(t) \neq \emptyset, t \in \mathbb{R}\}$ in X_ℓ such that

$$\lim_{s\to-\infty} e^{\alpha_0 s}[\mathcal{B}_\ell(s)] = 0,$$

where $[\mathcal{B}_\ell(s)] = \sup\{\|u\|_{X_\ell}^2 : u \in \mathcal{B}_\ell(s)\}, \alpha_0 > 0$.

Additionally, for the sake of simplicity, we omit some basic definitions and some results in [1, 6, 15, 32] for the given non-autonomous dynamical systems (such as pullback absorbing sets, pullback attractors and fractal dimensions)

3 The Existence of Pullback Attractors

3.1 The Existence of Pullback Attractors in X_ℓ

In this section, we first make some priori estimates of solutions to establish the existence of pullback absorbing sets for problem (1.2)–(1.5).

Lemma 3.1 *Let (v, F, M) be a weak solution to problem (1.2)–(1.5), the basic energy $\mathcal{E}(t)$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \int_\Omega (\mu|\nabla v|^2 + \kappa|\nabla F|^2 + |W|^2)\mathrm{d}x = (g, v). \tag{3.1}$$

Proof See the proof of Lemma 2.2 in [28]. □

Lemma 3.2 *Assuming that* (A_1) *holds, then for any bounded subset* $\hat{B}(\tau) \in \mathcal{D}$ *and any* $\tau \in \mathbb{R}$*, there exists a time* $\tau_1 = \tau_1(\hat{B}(\tau)) \geq 0$ *such that for any weak solutions of problem (1.2)–(1.5) with initial data* $(v_\tau, F_\tau, M_\tau) \in B(\tau)$*, we have*

$$\|v(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 + \|\nabla M(t)\|_{L^2}^2 \leq R_1,$$

and

$$\int_0^\ell \left(\|v(t+\zeta)\|_{L^2}^2 + \|F(t+\zeta)\|_{L^2}^2 + \|\nabla M(t+\zeta)\|_{L^2}^2 \right) \mathrm{d}\zeta \leq R_2$$

for any $t - \tau \geq \tau_1$*, where* $R_1 := 1 + \frac{C_0}{\alpha_0} + (\frac{1}{2\nu\lambda_1} + \frac{1}{2\nu\lambda_1\alpha_0})R_g$; $R_2 := 1 + \frac{C_0\ell}{\alpha_0} + (\frac{\ell}{2\nu\lambda_1} + \frac{\ell}{2\nu\lambda_1\alpha_0})R_g$.

Proof Taking L^2-inner product in $L^2(\Omega)$ of $(2.2)_4$ with M, using Hölder's inequality and Young's inequality, we have

$$\begin{aligned} &(W, M) = -\|\nabla M\|_{L^2}^2 - \|M\|_{L^4}^4 + \|M\|_{L^2}^2, \\ &\|M\|_{L^2}^2 \leq \frac{1}{3}\|M\|_{L^4}^4 + \frac{3}{4}|\Omega|. \end{aligned} \tag{3.2}$$

On the other hand, using Hölder's inequality, Young's inequality and (3.2), we have

$$-(W, M) \leq \frac{1}{2}\|W\|_{L^2}^2 + \frac{1}{2}\|M\|_{L^2}^2 \leq \frac{1}{2}\|W\|_{L^2}^2 + \frac{1}{6}\|M\|_{L^4}^4 + \frac{3}{8}|\Omega|. \tag{3.3}$$

From (3.1)–(3.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \alpha_0\mathcal{E}(t) = \Psi(t), \tag{3.4}$$

where $\alpha_0 > 0$ is given later. Let

$$\begin{aligned} \Psi(t) :=& \frac{\alpha_0}{2}\|v\|_{L^2}^2 + \frac{\alpha_0}{2}\|F\|_{L^2}^2 + \frac{\alpha_0}{2}\|\nabla M\|_{L^2}^2 + \alpha_0\int_\Omega G(M)\mathrm{d}x - \mu\|\nabla v\|_{L^2}^2 - \kappa\|\nabla F\|_{L^2}^2 \\ &- \|W\|_{L^2}^2 + (-\|\nabla M\|_{L^2}^2 - \|M\|_{L^4}^4 + \|M\|_{L^2}^2 - (W, M)) + (g, v). \end{aligned}$$

Note that

$$\alpha_0\int_\Omega G(M)\mathrm{d}x = \alpha_0\int_\Omega \frac{1}{4}(|M|^2 - 1)^2\mathrm{d}x \leq \frac{\alpha_0}{2}\|M\|_{L^4}^4 + \frac{3\alpha_0|\Omega|}{4}. \tag{3.5}$$

Inserting (3.2), (3.3), (3.5) into (3.4), using Poincaré's inequality for v and F, we can obtain that

$$\Psi(t) \leq -(\frac{\mu}{2} - \frac{\alpha_0}{2\lambda_1})\|\nabla v\|_{L^2}^2 - (\kappa - \frac{\alpha_0}{2\lambda_1})\|\nabla F\|_{L^2}^2 - (1 - \frac{\alpha_0}{2})\|\nabla M\|_{L^2}^2 - (\frac{1}{2} - \frac{\alpha_0}{2})\|M\|_{L^4}^4 - \frac{1}{2}\|W\|_{L^2}^2 + \frac{1}{2\mu\lambda_1}\|g\|_{L^2}^2 + \frac{|\Omega|}{2}(\frac{9}{4} + \frac{3\alpha_0}{2}). \quad (3.6)$$

Taking $\alpha_0 = \min\{1, \mu\lambda_1, 2\kappa\lambda_1\}$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \alpha_0\mathcal{E}(t) \leq \frac{1}{2\mu\lambda_1}\|g\|_{L^2}^2 + C_0, \quad (3.7)$$

where $C_0 := \frac{|\Omega|}{2}(\frac{9}{4} + \frac{3\alpha_0}{2})$. Multiplying (3.7) by $e^{\alpha_0 t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^{\alpha_0 t}\mathcal{E}(t)] \leq C_0 e^{\alpha_0 t} + \frac{e^{\alpha_0 t}}{2\mu\lambda_1}\|g\|_{L^2}^2. \quad (3.8)$$

Now integrating (3.8) from τ to t, we obtain

$$\|v(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 + \|\nabla M(t)\|_{L^2}^2 \leq e^{-\alpha_0(t-\tau)}(\|v_\tau\|_{L^2}^2 + \|F_\tau\|_{L^2}^2 + \|\nabla M_\tau\|_{L^2}^2) + \frac{C_0}{\alpha_0} + \frac{1}{2\mu\lambda_1}\int_\tau^t e^{\alpha_0(s-t)}\|g(s)\|_{L^2}^2 \mathrm{d}s. \quad (3.9)$$

Note that

$$\begin{aligned}\int_\tau^t e^{\alpha_0(s-t)}\|g(s)\|_{L^2}^2 \mathrm{d}s &\leq e^{-\alpha_0 t}\sum_{n=0}^{\infty}\int_{t-(n+1)}^{t-n} e^{\alpha_0 s}\|g(s)\|_{L^2}^2 \mathrm{d}s \\ &\leq e^{-\alpha_0 t}\sum_{n=0}^{\infty} e^{\alpha_0(t-n)}\int_{t-(n+1)}^{t-n}\|g(s)\|_{L^2}^2 \mathrm{d}s \\ &\leq (1 + \frac{1}{\alpha_0})R_g,\end{aligned}$$

where $R_g := \sup_{t\in\mathbb{R}}\left(\int_{t-1}^t \|g(s)\|_{L^2}^2 \mathrm{d}s\right) < +\infty$.

From (3.9), we conclude that for any $\hat{B}(\tau) \in \mathcal{D}$, there exists a time $\tau_0 = \tau_0(\hat{B}(\tau)) > 0$ such that for any $(v_\tau, F_\tau, M_\tau) \in B(\tau)$,

$$\|v(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 + \|\nabla M(t)\|_{L^2}^2 \leq 1 + \frac{C_0}{\alpha_0} + (\frac{1}{2\mu\lambda_1} + \frac{1}{2\mu\lambda_1\alpha_0})R_g \quad (3.10)$$

for any $t - \tau \geq \tau_0$.

Next, integrating (3.8) from $\tau + \zeta$ to $t + \zeta$ for any $\zeta \in (0, \ell)$, we derive that

$$\begin{aligned}
&\|v(t+\zeta)\|_{L^2}^2 + \|F(t+\zeta)\|_{L^2}^2 + \|\nabla M(t+\zeta)\|_{L^2}^2 \\
&\quad \le e^{-\alpha_0(t-\tau)}(\|v(\tau+\zeta)\|_{L^2}^2 + \|F(\tau+\zeta)\|_{L^2}^2 + \|\nabla M(\tau+\zeta)\|_{L^2}^2) \\
&\qquad + \frac{C_0}{\alpha_0} + \frac{1}{2\mu\lambda_1}\int_{\tau+\zeta}^{t+\zeta} e^{\alpha_0[s-(t+\zeta)]}\|g(s)\|_{L^2}^2 \mathrm{d}s. \qquad (3.11)
\end{aligned}$$

Combining (3.9) with (3.11), we derive that

$$\begin{aligned}
&e^{-\alpha_0(t-\tau)}(\|v(\tau+\zeta)\|_{L^2}^2 + \|F(\tau+\zeta)\|_{L^2}^2 + \|\nabla M(\tau+\zeta)\|_{L^2}^2) + \frac{C_0}{\alpha_0} \\
&\quad + \frac{1}{2\mu\lambda_1}\int_{\tau+\zeta}^{t+\zeta} e^{\alpha_0[s-(t+\zeta)]}\|g(s)\|_{L^2}^2 \mathrm{d}s \\
&\le e^{-\alpha_0(t-\tau)}\left[e^{-\alpha_1\zeta}(\|v_\tau\|_{L^2}^2 + \|F_\tau\|_{L^2}^2 + \|\nabla M_\tau\|_{L^2}^2) + \tilde{R}_g\right] + \tilde{R}_g, \qquad (3.12)
\end{aligned}$$

where $\tilde{R}_g = \frac{C_0}{\alpha_0} + (\frac{1}{2\mu\lambda_1} + \frac{1}{2\mu\lambda_1\alpha_0})R_g$.

Integrating (3.11) with respect to ζ over $(0, \ell)$ and using (3.12), we obtain that

$$\begin{aligned}
&\int_0^\ell \left(\|v(t+\zeta)\|_{L^2}^2 + \|F(t+\zeta)\|_{L^2}^2 + \|\nabla M(t+\zeta)\|_{L^2}^2\right) \mathrm{d}\zeta \\
&\le e^{-\alpha_1(t-\tau)}\left[\frac{1}{\alpha_1}(\|v_\tau\|_{L^2}^2 + \|F_\tau\|_{L^2}^2 + \|\nabla M_\tau\|_{L^2}^2) + \ell\tilde{R}_g\right] + \ell\tilde{R}_g. \qquad (3.13)
\end{aligned}$$

Therefore, for any $\hat{B}(\tau) \in \mathcal{D}$, there exists a time $\tau_1 = \tau_1(\hat{B}(\tau)) > \tau_0$ such that for any $(v_\tau, F_\tau, M_\tau) \in B(\tau)$, we have

$$\int_0^\ell \left(\|v(t+\zeta)\|_{L^2}^2 + \|F(t+\zeta)\|_{L^2}^2 + \|\nabla M(t+\zeta)\|_{L^2}^2\right) \mathrm{d}\zeta \le 1 + \ell\tilde{R}_g \qquad (3.14)$$

for any $t - \tau \ge \tau_1$. □

In what follows, we use the method of ℓ-trajectories to construct pullback attractors for system (1.2)–(1.5) in the phase space $\mathbb{H}$ (see, e.g., [1, 29, 30]).

Let $C_{\text{weak}}([\tau, \tau+\ell]; \mathbb{H})$ denote the space of weakly continuous functions from the interval $[\tau, \tau+\ell]$ to the Banach space $\mathbb{H}$ (see [36, 38]). Then we consider the solution $\mathbf{z}(t) = (v(t), F(t), M(t)) \in C_{\text{weak}}([\tau, \tau+\ell]; \mathbb{H})$ with the initial data $\mathbf{z}_\tau = (v_\tau, F_\tau, M_\tau) \in \mathbb{H}$. Let $\chi(s, \tau, \mathbf{z}_\tau) = \mathbf{z}|_{s\in[\tau,\tau+\ell]}$ denote ℓ-trajectory corresponding to the solution. Then we define the ℓ-trajectories space as follows:

$$X_\ell := \bigcup_{\mathbf{z}_\tau \in \mathbb{H}} \chi(s, \tau, \mathbf{z}_\tau),$$

where $\chi(s, \tau, \mathbf{z}_\tau)$ is a ℓ-trajectory associated with $\mathbf{z}_\tau$.

By Lemma 3.2, the ℓ-trajectories space X_ℓ is endowed with the topology of $L^2(\tau, \tau+\ell; \mathbb{H})$. Since $X_\ell \subset C_{\text{weak}}([\tau, \tau+\ell]; \mathbb{H})$, it makes sense to deal with the point values of trajectories. From Theorem 2.1, we can define a family of the continuous process $\{U(t,\tau)\}_{t\geq\tau}$ associated with problem (1.2)–(1.5) in the phase space $\mathbb{H}$ by

$$U(t,\tau)\mathbf{z}_\tau = \mathbf{z}(t) = \mathbf{z}(t,\tau;\mathbf{z}_\tau)$$

for all $t \geq \tau$, which is $(\mathbb{H},\mathbb{H})$-continuous, where $\mathbf{z}(t)$ is the solution of problem (1.2)–(1.5) with initial data $\mathbf{z}(x,\tau) = \mathbf{z}_\tau \in \mathbb{H}$. Considering the weak solution $\mathbf{z}(t)$ depends continuously on the initial data $\mathbf{z}_\tau$, we can infer that the process $\{U(t,\tau)\}_{t\geq\tau}$ associated with problem (1.2)–(1.5) in $\mathbb{H}$ is τ-continuous.

Based on the above results, we first define the mapping $b : \mathbb{H} \to X_\ell$ is given by

$$\{b(\mathbf{z}_\tau)\}(s) := \mathbf{z}(s,\tau;\mathbf{z}_\tau) = U(s,\tau)(\mathbf{z}_\tau) = \chi(s,\tau,\mathbf{z}_\tau)$$

for each $s \in [\tau, \tau+\ell]$ and any $\mathbf{z}_\tau = (v_\tau, F_\tau, M_\tau) \in \mathbb{H}$.
The second mapping $e_\theta : X_\ell \to \mathbb{H}$ is given by

$$e_\theta(\chi(s,\tau;\mathbf{z}_\tau)) := \mathbf{z}(\tau+\theta\ell,\tau;\mathbf{z}_\tau)$$

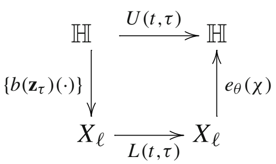
for any $\chi(s,\tau;\mathbf{z}_\tau) \in X_\ell$ and $\theta \in [0,1]$.

Then we can define a new process $\{L(t,\tau)\}_{t\geq\tau}$ acting on the ℓ-trajectories space induced by process $\{U(t,\tau)\}_{t\geq\tau}$ as

$$\begin{aligned} &L(t,\tau)\chi(s,\tau;\mathbf{z}_\tau) = \mathbf{z}(t+s-\tau,\tau;\mathbf{z}_\tau) \\ &= U(t+s-\tau,t)\mathbf{z}(t,\tau;\mathbf{z}_\tau) = \chi(t+s-\tau,t;\mathbf{z}(t,\tau,\mathbf{z}_\tau)), \quad s \in [\tau,\tau+\ell] \end{aligned}$$

for any $\mathbf{z}_\tau = (v_\tau, F_\tau, M_\tau) \in \mathbb{H}$.

The original phase space and the ℓ-trajectories space defined above satisfy the following commutative diagram (see [30] for more details):



Let

$$B_0 := \{(v,F,M) \in \mathbb{H} : \|v\|_{\mathbf{H}}^2 + \|F\|_{L^2}^2 + \|\nabla M\|_{L^2}^2 \leq R_1\},$$

then $\hat{B}_0 \in \mathcal{D}$. From Lemma 3.2, we can infer that any weak solutions of problem (1.2)–(1.5) with initial data $(v_\tau, F_\tau, M_\tau) \in B_0$, then there exists a time $\tau_0 = \tau_0(\hat{B}_0) \geq 0$

such that $(v(t), F(t), M(t)) \in B_0$ for any $t - \tau \geq \tau_0$. Thus, we get

$$U(t, \tau)B_0 \subset B_0$$

for any $t - \tau \geq \tau_0$.

Next, we can define for any $t \in \mathbb{R}$

$$B_1(t) := \overline{\bigcup_{\tau, s \in [t-\tau_0, t],\ \tau \leq s} \{U(s, \tau)(v_\tau, F_\tau, M_\tau) : \forall\, (v_\tau, F_\tau, M_\tau) \in B_0\}}^{\mathbb{H}},$$

and

$$\mathcal{B}_0^\ell(t) := \{\chi \in X_\ell : e_0(\chi) \in B_1(t)\}.$$

In particular, for $\tau = t$, $B_1(\tau) = \{(v_\tau, F_\tau, M_\tau) : \forall\, (v_\tau, F_\tau, M_\tau) \in B_0\}$, $\mathcal{B}_0^\ell(\tau) = \{\chi \in X_\ell : e_0(\chi) \in B_1(\tau)\}$.

From the proof of the bounded absorbing subset of Lemma 3.2, we derive that

$$U(t, \tau)B_1(\tau) \subset B_1(t),$$

and

$$L(t, \tau)\mathcal{B}_0^\ell(\tau) \subset \mathcal{B}_0^\ell(t)$$

for any $t \in \mathbb{R}$ with $\tau \leq t$, and $\hat{B}_1(t) \in \mathcal{D}$.

From Lemma 3.2, we immediately obtain that the following results.

Theorem 3.1 *Assuming that (A_1) holds, then for any $\hat{\mathcal{B}}_\ell(\tau) \in \mathcal{D}_\ell$ and any $\tau \in \mathbb{R}$, there exists a time $\tau_1 = \tau_1(\hat{\mathcal{B}}_\ell(\tau)) \geq 0$ such that for any weak solutions of problem (1.2)–(1.5) with ℓ-trajectory $\chi(s, \tau; \mathbf{z}_\tau) \in \mathcal{B}_\ell(\tau)$, we have*

$$\|v(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 + \|\nabla M(t)\|_{L^2}^2 \leq R_1,$$

and

$$\int_0^\ell \left(\|v(t+\zeta)\|_{L^2}^2 + \|F(t+\zeta)\|_{L^2}^2 + \|\nabla M(t+\zeta)\|_{L^2}^2\right) \mathrm{d}\zeta \leq R_2$$

for any $t - \tau \geq \tau_1$, where $\mathbf{z}_\tau := (v_\tau, F_\tau, M_\tau)$.

Next, we prove the existence of a compact pullback absorbing set in X_ℓ of the process $\{L(t, \tau)\}_{t \geq \tau}$.

Lemma 3.3 *Assuming that (A_1) holds, then for any $\tau \in \mathbb{R}$ and $\hat{\mathcal{B}}_0^\ell(\tau) \in \mathcal{D}_\ell$, there exist times $\tau_2 = \tau_2(\hat{\mathcal{B}}_0^\ell(\tau)) > 0$ such that for any weak solutions of problem (1.2)–(1.5)* *with ℓ-trajectory $\chi \in \mathcal{B}_0^\ell(\tau) \subset X_\ell$, we have*

$$\int_0^\ell \left(\|v(t+r)\|_{H^1}^2 + \|F(t+r)\|_{H^1}^2 + \|M(t+r)\|_{H^2}^2\right) \mathrm{d}r \le R_5$$

for any $t-\tau \ge \tau_2$, and

$$\int_0^\ell \left(\|v_t(t+r)\|_{\mathbf{V}'} + \|F_t(t+r)\|_{H^{-1}(\Omega)} + \|M_t(t+r)\|_{L^2}\right) \mathrm{d}r \le R_6$$

for any $t-\tau \ge \tau_2$, where R_5, R_6 are determined in the following proof.

Proof From (3.1), we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|v(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 + \|\nabla M(t)\|_{L^2}^2) + \mu\|\nabla v(t)\|_{L^2}^2 + 2\kappa\|\nabla F(t)\|_{L^2}^2 \\ +2\|W(t)\|_{L^2}^2 \le \frac{1}{\mu\lambda_1}\|g(t)\|_{L^2}^2. \tag{3.15}$$

Integrating (3.15) from $t-s$ to $t+\ell$, for any $t-\tau \ge \tau_1 + \frac{\ell}{2}$, $s \in (0, \frac{\ell}{2})$, and considering (3.9), (3.10), we find that

$$\begin{aligned}
&\|v(t+\ell)\|_{L^2}^2 + \|F(t+\ell)\|_{L^2}^2 + \|\nabla M(t+\ell)\|_{L^2}^2 \\
&\quad + \gamma_0 \int_{t-s}^{t+\ell} \left(\|\nabla v(r)\|_{L^2}^2 + \|\nabla F(r)\|_{L^2}^2 + \|W(r)\|_{L^2}^2\right) \mathrm{d}r \\
&\le \|v(t-s)\|_{L^2}^2 + \|F(t-s)\|_{L^2}^2 + \|\nabla M(t-s)\|_{L^2}^2 + \frac{1}{\mu\lambda_1}\int_{t-s}^{t+\ell} \|g(s)\|_{L^2}^2 \mathrm{d}s \\
&\le e^{-\alpha_1(t-s-\tau)}(\|v_\tau\|_{L^2}^2 + \|F_\tau\|_{L^2}^2 + \|\nabla M_\tau\|_{L^2}^2) + \frac{C_0}{\alpha_0} + C(\frac{1}{\mu\lambda_1} + \frac{1}{\alpha_0\mu\lambda_1})R_g \\
&\le 1 + \frac{C_0}{\alpha_0} + C(\frac{1}{\mu\lambda_1} + \frac{1}{\alpha_0\mu\lambda_1})R_g,
\end{aligned} \tag{3.16}$$

where $\gamma_0 = \min\{\mu, 2\kappa, 2\}$.
From (3.16), and taking $\tau_2 = \tau_1 + \frac{\ell}{2}$, we have

$$\int_{t-s}^{t+\ell} \left(\|\nabla v(r)\|_{L^2}^2 + \|\nabla F(r)\|_{L^2}^2 + \|W(r)\|_{L^2}^2\right) \mathrm{d}r \le \frac{1}{\gamma_0} + \frac{C_0}{\gamma_0\alpha_0} \\ + C\left(\frac{1}{\gamma_0\mu\lambda_1} + \frac{1}{\gamma_0\alpha_0\mu\lambda_1}\right) R_g \tag{3.17}$$

for any $t-\tau \ge \tau_2$.

Therefore, we conclude that

$$\int_0^\ell \left(\|\nabla v(t+r)\|_{L^2}^2 + \|\nabla F(t+r)\|_{L^2}^2 + \|W(t+r)\|_{L^2}^2 \right) \mathrm{d}r \le R_3 \tag{3.18}$$

for any $t-\tau \ge \tau_2$, where $R_3 := \frac{1}{\gamma_0} + \frac{C_0}{\gamma_0\alpha_0} + C(\frac{1}{\gamma_0\mu\lambda_1} + \frac{1}{\gamma_0\alpha_0\mu\lambda_1})R_g$.

From (3.2), (3.7), (3.10), we derive that

$$\|M\|_{H^1}^2 \le C(\|\nabla M\|_{L^2}^2 + \|M\|_{L^2}^2) \le C(\|\nabla M\|_{L^2}^2 + \|M\|_{L^4}^4 + 1) \le R_4, \tag{3.19}$$

where $R_4 := C + (\frac{C}{2\mu\lambda_1} + \frac{C}{2\mu\lambda_1\alpha_0})R_g$.

Applying (3.17), (3.19), the interpolation theorem and the Sobolev imbedding theorem (see, e.g., [40]), we have

$$\begin{aligned}
&\int_{t-s}^{t+\ell} \|M(r)\|_{H^2}^2 \mathrm{d}r \le C_1 \int_{t-s}^{t+\ell} (\|\Delta M(r)\|_{L^2}^2 + \|M(r)\|_{L^2}^2)\mathrm{d}r \\
&\le C_1 \int_{t-s}^{t+\ell} (\|\Delta M - (|M|^2-1)M)(r)\|_{L^2}^2)\mathrm{d}r \\
&\quad + C_1 \int_{t-s}^{t+\ell} (\|(|M|^2-1)M)(r)\|_{L^2}^2 + \|M(r)\|_{L^2}^2)\mathrm{d}r \\
&\le C_2 \int_{t-s}^{t+\ell} (\|W(r)\|_{L^2}^2 + \|M(r)\|_{L^6}^3 + \|M(r)\|_{L^2}^2)\mathrm{d}r \\
&\le C_3 \int_{t-s}^{t+\ell} (\|W(r)\|_{L^2}^2 + \|M(r)\|_{H^1}^3 + \|M(r)\|_{H^1}^2)\mathrm{d}r \\
&\le C_3 \left(R_3 + \frac{3\ell}{2} R_4^{3/2} + \frac{3\ell}{2} R_4 \right)
\end{aligned} \tag{3.20}$$

for any $t-\tau \ge \tau_2$.

So we conclude from (3.14), (3.18), (3.20) that

$$\int_0^\ell \left(\|v(t+r)\|_{H^1}^2 + \|F(t+r)\|_{H^1}^2 + \|M(t+r)\|_{H^2}^2 \right) \mathrm{d}r \le R_5 \tag{3.21}$$

for any $t-\tau \ge \tau_2$, where $R_5 := C(R_3 + R_4^{3/2} + R_4)$.

From (2.2), we have

$$\begin{cases} v_t = -\mu A v - B(v,v) + \mathbb{P}(\nabla\cdot(FF^T - \nabla^T M \nabla M)) + \mathbb{P}g, \\ F_t = \kappa\Delta F + \nabla v F - v\cdot\nabla F, \\ M_t = W - v\cdot\nabla M. \end{cases} \tag{3.22}$$

Taking $(\varphi, \psi) \in L^\infty(t, t+\ell; \mathbf{V} \times H_0^1(\Omega))$ such that $\|(\varphi, \psi)\|_{L^\infty(t,t+\ell;\mathbf{V}\times H_0^1(\Omega))} \le 1$, we estimate the time derivatives v_t, F_t as follows:

$$\begin{aligned}|\langle v_t, \varphi\rangle| &\le \mu \left|\int_\Omega \nabla v : \nabla\varphi \mathrm{d}x\right| + \left|\int_\Omega (v \otimes v) : \nabla\varphi \mathrm{d}x\right| \\ &\quad + \left|\int_\Omega (FF^T - \nabla^T M \nabla M) : \nabla\varphi \mathrm{d}x\right| + \|g\|_{L^2}\|\varphi\|_{L^2} \\ &\le \left(\mu\|\nabla v\|_{L^2} + \|v\|_{L^4}^2 + \|\nabla M\|_{L^4}^2 + \|F\|_{L^4}^2 + \frac{1}{\sqrt{\lambda_1}}\|g\|_{L^2}\right)\|\nabla\varphi\|_{L^2} \\ &\le C,\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}|\langle F_t, \psi\rangle| &= \left|\kappa \int_\Omega \nabla F : \nabla\psi \mathrm{d}x + \int_\Omega (v\cdot\nabla)F : \psi \mathrm{d}x - \int_\Omega \nabla v F : \psi \mathrm{d}x\right| \\ &\le \left[\kappa\|\nabla F\|_{L^2}\|\nabla\psi\|_{L^2} + (\|v\|_{L^4}\|\nabla F\|_{L^2} + \|\nabla v\|_{L^2}\|F\|_{L^4})\|\psi\|_{L^4}\right] \\ &\le C.\end{aligned} \tag{3.24}$$

Taking the supremum over all (φ, ψ) in (3.23) and (3.24), then we get

$$\int_0^\ell \|v_t(t+r)\|_{\mathbf{V}'}\mathrm{d}r + \int_0^\ell \|F_t(t+r)\|_{H^{-1}(\Omega)}\mathrm{d}r \le C. \tag{3.25}$$

The time derivative M_t is estimated from $(3.22)_3$ as follows:

$$\begin{aligned}\|M_t\|_{L^1(t,t+\ell;L^2(\Omega))} &\le \|(v\cdot\nabla)M\|_{L^1(t,t+\ell;L^2(\Omega))} + \|\Delta M\|_{L^1(t,t+\ell;L^2(\Omega))} \\ &\quad + \frac{1}{\gamma^2}\|(|M|^2-1)M\|_{L^1(t,t+\ell;L^2(\Omega))} \\ &\le \int_t^{t+\ell} \|v(r)\|_{L^4}\|\nabla M(r)\|_{L^4}\mathrm{d}r + \sqrt{\ell}\|\Delta M\|_{L^2(t,t+\ell;L^2(\Omega))} \\ &\quad + \frac{1}{\gamma^2}\int_t^{t+\ell} \|M(r)\|_{L^6}^3\mathrm{d}r + \frac{\sqrt{\ell}}{\gamma^2}\|M\|_{L^2(t,t+\ell;L^2(\Omega))} \le C.\end{aligned} \tag{3.26}$$

From (3.23)–(3.26), there is a constant $R_6 > 0$ such that

$$\int_0^\ell \left(\|v_t(t+r)\|_{\mathbf{V}'} + \|F_t(t+r)\|_{H^{-1}(\Omega)} + \|M_t(t+r)\|_{L^2}\right)\mathrm{d}r \le R_6 \tag{3.27}$$

for any $t - \tau \ge \tau_2$. □

Let

$$\mathfrak{X}_\ell := \Big\{\chi \in X_\ell : \chi \in L^2\left(\tau, \tau+\ell; \mathbf{V} \times H_0^1(\Omega) \times H^2(\Omega)\right); \\ \chi_t \in L^1\left(\tau, \tau+\ell; \mathbf{V}' \times H^{-1}(\Omega) \times L^2(\Omega)\right)\Big\} \tag{3.28}$$

endowed with the following norm

$$\|\chi\|_{\mathfrak{X}_\ell} := \left\{\int_\tau^{\tau+\ell} \|\chi\|^2_{\mathbf{V}\times H_0^1(\Omega)\times H^2(\Omega)} \mathrm{d}s + \left(\int_\tau^{\tau+\ell} \|\chi_t\|_{\mathbf{V}'\times H^{-1}(\Omega)\times L^2(\Omega)} \mathrm{d}s\right)^2\right\}^{\frac{1}{2}}$$

for any $\tau \in \mathbb{R}$.

Then we also define $\hat{\mathcal{B}}_1^\ell(t) := \{\mathcal{B}_1^\ell(t) : t \in \mathbb{R}\}$, where

$$\mathcal{B}_1^\ell(t) = \left\{\chi \in X_\ell : \|\chi\|_{\mathfrak{X}_\ell} \le R_7\right\},$$

where R_7 is a constant that depends on R_5 and R_6.

From Theorem 3.1 and Lemma 3.3, we conclude that $L(t,\tau)\mathcal{B}_0^\ell(\tau) \subset \mathcal{B}_0^\ell(t)$ for any $t \ge \tau$ and $L(t,\tau)\mathcal{B}_0^\ell(\tau) \subset \mathcal{B}_1^\ell(t)$ for any $t - \tau \ge \tau_2$. Furthermore, we have

Theorem 3.2 *Assuming that (A_1) holds, then*

$$\overline{L(t,\tau)\mathcal{B}_0^\ell(\tau)}^{L^2(\tau,\tau+\ell;\mathbb{H})} \subset \mathcal{B}_0^\ell(t)$$

for any $t \ge \tau$.

Proof Similar to the results proved in literature [1], we may just omit it. □

Lemma 3.4 *Assuming that (A_1) holds, then for any $\tau \in \mathbb{R}$, the mapping $L(t,\tau)$: $X_\ell \to X_\ell$ is Lipschitz continuous on $B_0^\ell(\tau)$ for any $t \ge \tau + \ell$.*

Proof For any fixed $\tau \in \mathbb{R}$ and any $\chi_1, \chi_2 \in \mathcal{B}_0^\ell(\tau)$, let $L(t,\tau)\chi_1 = (v_1(t), F_1(t), M_1(t))$, $L(t,\tau)\chi_2 = (v_2(t), F_2(t), M_2(t))$ for any fixed $t \ge \tau + \ell$ and denote by $(\bar{v}, \bar{F}, \bar{M}) = (v_1 - v_2, F_1 - F_2, M_1 - M_2)$. Since $e_0(\chi_1)$ and $e_0(\chi_2)$ are uniformly bounded in $\mathbb{H}$, we can infer from (2.2) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|^2_{L^2} + \mu\|\nabla\bar{v}\|^2_{L^2} = -\int_\Omega (\bar{v}\cdot\nabla v_1)\cdot\bar{v}\mathrm{d}x + \int_\Omega \mathrm{div}(F_1F_1^T - F_2F_2^T)\cdot\bar{v}\mathrm{d}x \\ - \int_\Omega \mathrm{div}(\nabla^T M_1\nabla M_1 - \nabla^T M_2\nabla M_2)\cdot\bar{v}\mathrm{d}x \\ := I_1 + I_2 + I_3, \tag{3.29}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{F}\|^2_{L^2} + \kappa\|\nabla\bar{F}\|^2_{L^2} = -\int_\Omega (\bar{v}\cdot\nabla F_1) : \bar{F}\mathrm{d}x + \int_\Omega (\nabla v_1F_1 - \nabla v_2F_2) : \bar{F}\mathrm{d}x \\ := I_4 + I_5, \tag{3.30}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{M}\|_{L^2}^2 + \|\nabla \bar{M}\|_{L^2}^2) + \|\nabla \bar{M}\|_{L^2}^2 + \|\Delta \bar{M}\|_{L^2}^2 + I_6 + I_7 = I_8 + I_9, \tag{3.31}$$

where

$$\begin{aligned}
I_6 &:= \frac{1}{\gamma^2}\int_\Omega ((|M_1|^2 - 1)M_1 - (|M_2|^2 - 1)M_2) \cdot \bar{M}\mathrm{d}x, \\
I_7 &:= -\frac{1}{\gamma^2}\int_\Omega ((|M_1|^2 - 1)M_1 - (|M_2|^2 - 1)M_2) \cdot \Delta\bar{M}\mathrm{d}x, \\
I_8 &:= -\int_\Omega (\bar{v} \cdot \nabla M_1) \cdot \bar{M}\mathrm{d}x, \ \ I_9 := \int_\Omega (v_1 \cdot \nabla M_1 - v_2 \cdot \nabla M_2) \cdot \Delta\bar{M}\mathrm{d}x.
\end{aligned}$$

For I_1, I_4, I_8, using Hölder's inequality, Young's inequality and Sobolev imbedding theorem, we have

$$I_1 \le \|\bar{v}\|_{L^4}^2 \|\nabla v_1\|_{L^2} \le C\|\bar{v}\|_{L^2}^2 \|\nabla v_1\|_{L^2}^2 + \frac{\mu}{8}\|\nabla \bar{v}\|_{L^2}^2, \tag{3.32}$$

$$\begin{aligned}
I_4 &\le \|\bar{v}\|_{L^4}\|\bar{F}\|_{L^4}\|\nabla F_1\|_{L^2} \le C\|\bar{v}\|_{L^2}\|\bar{F}\|_{L^2}\|\nabla F_1\|_{L^2}^2 + \frac{\sqrt{\mu\kappa}}{4}\|\nabla\bar{v}\|_{L^2}\|\nabla\bar{F}\|_{L^2} \\
&\le C(\|\bar{v}\|_{L^2}^2 + \|\bar{F}\|_{L^2}^2)\|\nabla F_1\|_{L^2}^2 + \frac{\mu}{8}\|\nabla\bar{v}\|_{L^2}^2 + \frac{\kappa}{8}\|\nabla\bar{F}\|_{L^2}^2,
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
I_8 &\le \|\bar{v}\|_{L^4}\|\bar{M}\|_{L^4}\|\nabla M_1\|_{L^2} \le C\|\bar{v}\|_{L^2}^{\frac{1}{2}}\|\nabla\bar{v}\|_{L^2}^{\frac{1}{2}}(\|\bar{M}\|_{L^2} + \|\bar{M}\|_{L^2}^{\frac{1}{2}}\|\nabla\bar{M}\|_{L^2}^{\frac{1}{2}})\|\nabla M_1\|_{L^2} \\
&\le C\left(\|\bar{v}\|_{L^2}^{\frac{2}{3}}\|\bar{M}\|_{L^2}^{\frac{4}{3}}\|\nabla M_1\|_{L^2}^{\frac{4}{3}} + \|\bar{v}\|_{L^2}\|\bar{M}\|_{L^2}\|\nabla M_1\|_{L^2}^2\right) + \frac{3\mu}{32}\|\nabla\bar{v}\|_{L^2}^2 \\
&\quad + \frac{\sqrt{\mu}}{8}\|\nabla\bar{v}\|_{L^2}\|\nabla\bar{M}\|_{L^2} \\
&\le C(\|\bar{v}\|_{L^2}^2 + \|\bar{M}\|_{L^2}^2)(1 + \|\nabla M_1\|_{L^2}^2) + \frac{\mu}{8}\|\nabla\bar{v}\|_{L^2}^2 + \frac{1}{8}\|\nabla\bar{M}\|_{L^2}^2.
\end{aligned} \tag{3.34}$$

Next, we estimate the terms $I_3 + I_9$ and $I_2 + I_5$. Using the divergence free condition on v and (2.1), we first obtain

$$\begin{aligned}
I_3 &= -\int_\Omega (\nabla^T M_1 \Delta M_1 - \nabla^T M_2 \Delta M_2) \cdot \bar{v}\mathrm{d}x \\
&= -\int_\Omega [(v_1 \cdot \nabla M_1) \cdot \Delta M_1 + (v_2 \cdot \nabla M_2) \cdot \Delta M_2 - (\nabla^T M_1 \Delta M_1) \cdot \\
&\quad\quad v_2 - (\nabla^T M_2 \Delta M_2) \cdot v_1]\mathrm{d}x.
\end{aligned} \tag{3.35}$$

From (3.31), (3.35), we find that

$$
\begin{aligned}
I_3 + I_9 &= -\int_\Omega (\nabla^T M_1 \Delta M_1 - \nabla^T M_2 \Delta M_2)\cdot \bar{v}\mathrm{d}x + \int_\Omega (v_1\cdot\nabla M_1 - v_2\cdot\nabla M_2)\cdot\Delta\bar{M}\mathrm{d}x \\
&= \int_\Omega [(\nabla^T\bar{M}\Delta\bar{M})\cdot v_2 - (\nabla^T\bar{M}\Delta M_2)\cdot\bar{v}]\mathrm{d}x. \qquad (3.36)
\end{aligned}
$$

Applying results in [34] and Hölder's inequality, Young's inequality, Sobolev imbedding theorem, we have

$$
\begin{aligned}
I_3 + I_9 &\le \|\nabla\bar{M}\|_{L^4}\|v_2\|_{L^4}\|\Delta\bar{M}\|_{L^2} + \|\nabla\bar{M}\|_{L^4}\|\bar{v}\|_{L^4}\|\Delta M_2\|_{L^2} \\
&\le C\|\nabla\bar{M}\|_{L^2}^{\frac12}\|\nabla^2\bar{M}\|_{L^2}^{\frac12}\|v_2\|_{L^2}^{\frac12}\|\nabla v_2\|_{L^2}^{\frac12}\|\Delta\bar{M}\|_{L^2} \\
&\quad + C\|\nabla\bar{M}\|_{L^2}^{\frac12}\|\nabla^2\bar{M}\|_{L^2}^{\frac12}\|\bar{v}\|_{L^2}^{\frac12}\|\nabla\bar{v}\|_{L^2}^{\frac12}\|\Delta M_2\|_{L^2} \\
&\le C\|\nabla\bar{M}\|_{L^2}(\|\bar{M}\|_{L^2} + \|\Delta\bar{M}\|_{L^2})\|v_2\|_{L^2}\|\nabla v_2\|_{L^2} + \frac18\|\Delta\bar{M}\|_{L^2}^2 \\
&\quad + C\|\nabla\bar{M}\|_{L^2}^{\frac12}(\|\bar{M}\|_{L^2}^{\frac12} + \|\Delta\bar{M}\|_{L^2}^{\frac12})\|\bar{v}\|_{L^2}^{\frac12}\|\nabla\bar{v}\|_{L^2}^{\frac12}\|\Delta M_2\|_{L^2} \\
&\le C(\|\bar{v}\|_{L^2}^2 + \|\bar{M}\|_{L^2}^2 + \|\nabla\bar{M}\|_{L^2}^2)(\|v_2\|_{L^2}\|\nabla v_2\|_{L^2} + \|v_2\|_{L^2}^2\|\nabla v_2\|_{L^2}^2 + \|\Delta M_2\|_{L^2}^2) \\
&\quad + \frac18(\mu\|\nabla\bar{v}\|_{L^2}^2 + 3\|\nabla\bar{M}\|_{L^2}^2 + 2\|\Delta\bar{M}\|_{L^2}^2). \qquad (3.37)
\end{aligned}
$$

Similarly, we have

$$
I_2 + I_5 \le C\|\bar{F}\|_{L^2}^2(\|F_1\|_{L^2}^2\|\nabla F_1\|_{L^2}^2 + \|\nabla v_1\|_{L^2}^2) + \frac18(\mu\|\nabla\bar{v}\|_{L^2}^2 + 2\kappa\|\nabla\bar{F}\|_{L^2}^2). \qquad (3.38)
$$

For I_6, I_7, using the following results

$$
\begin{aligned}
&(|M_1|^2 M_1 - |M_2|^2 M_2)\cdot(M_1 - M_2) \ge 0, \\
&\Big||M_1|^2 M_1 - |M_2|^2 M_2\Big| \le \frac32|\bar{M}|(|M_1|^2 + |M_2|^2),
\end{aligned}
$$

and Gagliardo–Nirenberg inequality (see, e.g., [40]), then we have

$$
I_6 = -\frac{1}{\gamma^2}\int_\Omega |\bar{M}|^2\mathrm{d}x + \frac{1}{\gamma^2}\int_\Omega (|M_1|^2 M_1 - |M_2|^2 M_2)\cdot\bar{M}\mathrm{d}x \ge -\frac{1}{\gamma^2}\int_\Omega |\bar{M}|^2\mathrm{d}x, \qquad (3.39)
$$

$$
\begin{aligned}
I_7 &= -\frac{1}{\gamma^2}\int_\Omega |\bar{M}|^2 \mathrm{d}x - \frac{1}{\gamma^2}\int_\Omega (|M_1|^2 M_1 - |M_2|^2 M_2)\cdot \Delta \bar{M} \mathrm{d}x \\
&\geq -\frac{1}{\gamma^2}\int_\Omega |\bar{M}|^2 \mathrm{d}x - \frac{3}{2\gamma^2}\int_\Omega (|M_1|^2 + |M_2|^2)|\bar{M}||\Delta \bar{M}| \mathrm{d}x \\
&\geq -C\left[\|\nabla \bar{M}\|_{L^2}^2 + \|\bar{M}\|_{L^4}^2(\|M_1\|_{L^8}^4 + \|M_2\|_{L^8}^4)\right] - \frac{1}{8}\|\Delta \bar{M}\|_{L^2}^2 \\
&\geq -C(\|\bar{M}\|_{L^2}^2 + \|\nabla \bar{M}\|_{L^2}^2)(1 + \|\nabla M_1\|_{L^2}^2\|M_1\|_{L^4}^2 + \|\nabla M_2\|_{L^2}^2\|M_2\|_{L^4}^2) - \frac{1}{8}\|\Delta \bar{M}\|_{L^2}^2.
\end{aligned}
\tag{3.40}
$$

Substituting the results of (3.32)–(3.40) into (3.29)–(3.31), respectively. Then we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{v}\|_{L^2}^2 + \|\bar{F}\|_{L^2}^2 + \|\bar{M}\|_{H^1}^2) + \mu\|\nabla \bar{v}\|_{L^2}^2 + \kappa\|\nabla \bar{F}\|_{L^2}^2 + \|\nabla \bar{M}\|_{L^2}^2 + \|\Delta \bar{M}\|_{L^2}^2 \\
&\leq C\rho(t)(\|\bar{v}\|_{L^2}^2 + \|\bar{F}\|_{L^2}^2 + \|\bar{M}\|_{H^1}^2),
\end{aligned}
\tag{3.41}
$$

where

$$
\begin{aligned}
\rho(t) = 1 &+ \|\nabla v_1\|_{L^2}^2 + \|\nabla F_1\|_{L^2}^2 + \|\nabla M_1\|_{L^2}^2 + \|v_2\|_{L^2}\|\nabla v_2\|_{L^2} + \|v_2\|_{L^2}^2\|\nabla v_2\|_{L^2}^2 \\
&+ \|\Delta M_2\|_{L^2}^2 + \|\nabla M_1\|_{L^2}^2\|M_1\|_{L^4}^2 + \|\nabla M_2\|_{L^2}^2\|M_2\|_{L^4}^2.
\end{aligned}
$$

Integrating (3.41) from $\tau + s$ to $t + s$, for any $s \in (0, \ell)$, we obtain

$$
\begin{aligned}
&\|\bar{v}(t+s)\|_{L^2}^2 + \|\bar{F}(t+s)\|_{L^2}^2 + \|\bar{M}(t+s)\|_{H^1}^2 \\
&\quad\leq \int_{\tau+s}^{t+s} \rho(r)(\|\bar{v}(r)\|_{L^2}^2 + \|\bar{F}(r)\|_{L^2}^2 + \|\bar{M}(r)\|_{H^1}^2)\mathrm{d}r \\
&\quad + \|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2.
\end{aligned}
\tag{3.42}
$$

From Lemmas 3.2, 3.3, using Gronwall's lemma, we obtain

$$
\begin{aligned}
&\|\bar{v}(t+s)\|_{L^2}^2 + \|\bar{F}(t+s)\|_{L^2}^2 + \|\bar{M}(t+s)\|_{H^1}^2 \\
&\leq (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2) exp\left(\int_{\tau+s}^{t+s}\rho(r)\mathrm{d}r\right) \\
&\leq (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2) exp\left(\int_{\tau}^{t+\ell}\rho(r)\mathrm{d}r\right) \\
&\leq N_\ell(t,\tau)(\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2),
\end{aligned}
\tag{3.43}
$$

where $N_\ell(t,\tau) = exp\left(\int_\tau^{t+\ell}\rho(r)\mathrm{d}r\right)$ is a finite number depending on $(v_{1,\tau}, F_{1,\tau}, M_{1,\tau})$ and $(v_{2,\tau}, F_{2,\tau}, M_{2,\tau})$.

Integrating (3.43) with respect to s over $(0, \ell)$, we derive that

$$\int_0^\ell (\|\bar{v}(t+s)\|_{L^2}^2 + \|\bar{F}(t+s)\|_{L^2}^2 + \|\bar{M}(t+s)\|_{H^1}^2)\mathrm{d}s$$
$$\leq N_\ell(t,\tau)\int_0^\ell (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2)\mathrm{d}s. \quad (3.44)$$

Therefore,

$$\|L(t,\tau)\chi_1 - L(t,\tau)\chi_2\|_{L^2(t,t+\ell;\mathbb{H})}^2 \leq N_\ell(t,\tau)\|\chi_1 - \chi_2\|_{L^2(\tau,\tau+\ell;\mathbb{H})}^2, \quad (3.45)$$

which implies the mapping $L(t,\tau) : X_\ell \to X_\ell$ is Lipschitz continuous on $\mathcal{B}_0^\ell(\tau)$ for all $t \geq \tau + \ell$. □

From Theorems 3.1, 3.2, Lemmas 3.3, 3.4 and Lemma 2.5 in [1], we can infer that $\hat{\mathcal{B}}_1^\ell(t) := \{\mathcal{B}_1^\ell(t) : t \in \mathbb{R}\}$ is a family of positive invariant, uniformly pullback absorbing compact subsets of X_ℓ, where

$$\mathcal{B}_1^\ell(t) = \left\{\chi \in X_\ell : \|\chi\|_{L^2(\tau,\tau+\ell;\mathbf{V}\times H_0^1(\Omega)\times H^2(\Omega))} + \|\chi_t\|_{L^2(\tau,\tau+\ell;\mathbf{V}'\times H^{-1}(\Omega)\times L^2(\Omega))} \leq R_7\right\}.$$

With a similar method to get the results of the autonomous case from Lemma 2.1 in [1], we can immediately obtain the following result.

Theorem 3.3 *Assuming that (A_1) holds, then the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) possesses a pullback attractor $\hat{\mathcal{A}}_\ell = \{\mathcal{A}_\ell(t) : t \in \mathbb{R}\}$ in X_ℓ and $e_1(\mathcal{A}_\ell(t-\ell)) \subset B_1(t)$ for any $t \in \mathbb{R}$, where*

$$e_1(\mathcal{A}_\ell(t-\ell)) = \{e_1(\chi) : \chi \in \mathcal{A}_\ell(t-\ell)\}$$

for any $t \in \mathbb{R}$.

Next, we still need to prove the smooth property of the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) to prove the pullback attractor with finite fractal dimension in X_ℓ.

Lemma 3.5 *Assuming that (A_1) holds, then there exists a constant $\kappa_1 > 0$ such that for any fixed $\tau \in \mathbb{R}$ and any $t \geq \tau + \ell$,*

$$\|L(t,\tau)\chi_1 - L(t,\tau)\chi_2\|_{\mathfrak{X}_\ell}^2$$
$$\leq (C_0(\tau,t,\ell) + \kappa_1 C_1(\tau,t,\ell))\int_0^\ell \|\chi_1(\tau+s) - \chi_2(\tau+s)\|_{\mathbb{H}}^2\mathrm{d}s,$$

where χ_1 and χ_2 are two ℓ-trajectories in $\mathcal{B}_0^\ell(\tau)$, κ_1, $C_0(\tau,t,\ell)$ and $C_1(\tau,t,\ell))$ are given in (3.52), (3.66), respectively.

Proof From (3.28) and Lemma 2.5 in [1], we can infer that $\mathfrak{X}_\ell \subset\subset X_\ell$. For any fixed $\tau \in \mathbb{R}$ and any $\chi_1, \chi_2 \in \mathcal{B}_0^\ell(\tau)$, let $L(t,\tau)\chi_1 = (v_1(t), F_1(t), M_1(t))$, $L(t,\tau)\chi_2 = (v_2(t), F_2(t), M_2(t))$ and $(\bar{v}, \bar{F}, \bar{M}) = (v_1 - v_2, F_1 - F_2, M_1 - M_2)$ for any $t \geq \tau + \ell$.

For any $t \geq \tau + \ell$, integrating (3.41) from $t - s$ to $t + \ell$ with $s \in [0, \frac{\ell}{2}]$, we obtain

$$\begin{aligned} &\|\bar{v}(t+\ell)\|_{L^2}^2 + \|\bar{F}(t+\ell)\|_{L^2}^2 + \|\bar{M}(t+\ell)\|_{H^1}^2 \\ &\quad + C_0 \int_{t-s}^{t+\ell} [\|\nabla \bar{v}\|_{L^2}^2 + \|\nabla \bar{F}\|_{L^2}^2 + \|\nabla \bar{M}\|_{L^2}^2 + \|\Delta \bar{M}\|_{L^2}^2](r)\mathrm{d}r \\ &\leq C \int_{t-s}^{t+\ell} \rho(r)[\|\bar{v}\|_{L^2}^2 + \|\bar{F}\|_{L^2}^2 + \|\bar{M}\|_{H^1}^2](r)\mathrm{d}r + \|\bar{v}(t-s)\|_{L^2}^2 + \|\bar{F}(t-s)\|_{L^2}^2 \\ &\quad + \|\bar{M}(t-s)\|_{H^1}^2, \end{aligned} \tag{3.46}$$

where $C_0 = \min\{\mu, \kappa, 1\} > 0$.

Similar to the proof of Lemma 3.4 in [1], using Gronwall's lemma to (3.46), we obtain

$$\begin{aligned} &\|\bar{v}(t+\ell)\|_{L^2}^2 + \|\bar{F}(t+\ell)\|_{L^2}^2 + \|\bar{M}(t+\ell)\|_{H^1}^2 \\ &\quad + C_0 \int_{t-s}^{t+\ell} [\|\nabla \bar{v}\|_{L^2}^2 + \|\nabla \bar{F}\|_{L^2}^2 + \|\nabla \bar{M}\|_{L^2}^2 + \|\Delta \bar{M}\|_{L^2}^2](r)\mathrm{d}r \\ &\leq K_\ell(t,\tau)(\|\bar{v}(t-s)\|_{L^2}^2 + \|\bar{F}(t-s)\|_{L^2}^2 + \|\bar{M}(t-s)\|_{H^1}^2) exp\left(\int_{t-s}^{t+\ell} \rho(r)\mathrm{d}r\right) \\ &\quad + \|\bar{v}(t-s)\|_{L^2}^2 \\ &\quad + \|\bar{F}(t-s)\|_{L^2}^2 + \|\bar{M}(t-s)\|_{H^1}^2, \end{aligned} \tag{3.47}$$

where $K_\ell(t,\tau) := \int_{\tau+\frac{\ell}{2}}^{t+\ell} \rho(r)\mathrm{d}r + 1$.

For any $t \geq \tau + \ell$, integrating (3.41) from $\tau + s$ to $t - s$ with $s \in [0, \frac{\ell}{2}]$, we have

$$\begin{aligned} &\|\bar{v}(t-s)\|_{L^2}^2 + \|\bar{F}(t-s)\|_{L^2}^2 + \|\bar{M}(t-s)\|_{H^1}^2 \\ &\leq \int_{\tau+s}^{t-s} \rho(r)(\|\bar{v}(r)\|_{L^2}^2 + \|\bar{F}(r)\|_{L^2}^2 + \|\bar{M}(r)\|_{H^1}^2)\mathrm{d}r + \|\bar{v}(\tau+s)\|_{L^2}^2 \\ &\quad + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2. \end{aligned} \tag{3.48}$$

Applying Gronwall's Lemma to (3.48), we get

$$\begin{aligned} &\|\bar{v}(t-s)\|_{L^2}^2 + \|\bar{F}(t-s)\|_{L^2}^2 + \|\bar{M}(t-s)\|_{H^1}^2 \\ &\leq (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2) exp\left(\int_{\tau+s}^{t-s} \rho(r)\mathrm{d}r\right) \\ &\leq (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2) exp\left(\int_{\tau}^{t-s} \rho(r)\mathrm{d}r\right). \end{aligned} \tag{3.49}$$

Considering (3.47) and (3.49), we have

$$
\begin{aligned}
&C_0\int_0^{\ell}[\|\nabla\bar{v}\|_{L^2}^2+\|\nabla\bar{F}\|_{L^2}^2+\|\nabla\bar{M}\|_{L^2}^2+\|\Delta\bar{M}\|_{L^2}^2](t+r)\mathrm{d}r\\
&\le K_\ell(t,\tau)(\|\bar{v}(\tau+s)\|_{L^2}^2+\|\bar{F}(\tau+s)\|_{L^2}^2+\|\bar{M}(\tau+s)\|_{H^1}^2)exp\left(\int_\tau^{t+\ell}\rho(r)\mathrm{d}r\right)\\
&\quad+(\|\bar{v}(\tau+s)\|_{L^2}^2+\|\bar{F}(\tau+s)\|_{L^2}^2+\|\bar{M}(\tau+s)\|_{H^1}^2)exp\left(\int_\tau^{t-s}\rho(r)\mathrm{d}r\right)\\
&\le 2K_\ell(t,\tau)N_\ell(t,\tau)(\|\bar{v}(\tau+s)\|_{L^2}^2+\|\bar{F}(\tau+s)\|_{L^2}^2+\|\bar{M}(\tau+s)\|_{H^1}^2),
\end{aligned}
\tag{3.50}
$$

where $N_\ell(t,\tau):=exp\left(\int_\tau^{t+\ell}\rho(r)\mathrm{d}r\right)$.

Integrating (3.50) with respect to s over $(0,\frac{\ell}{2})$, we get

$$
\begin{aligned}
&\int_0^{\ell}[\|\nabla\bar{v}\|_{L^2}^2+\|\nabla\bar{F}\|_{L^2}^2+\|\nabla\bar{M}\|_{L^2}^2+\|\Delta\bar{M}\|_{L^2}^2](t+r)\mathrm{d}r\\
&\le\frac{4K_\ell(t,\tau)N_\ell(t,\tau)}{C_0\ell}\int_0^{\frac{\ell}{2}}(\|\bar{v}(\tau+s)\|_{L^2}^2+\|\bar{F}(\tau+s)\|_{L^2}^2+\|\bar{M}(\tau+s)\|_{H^1}^2)\mathrm{d}s.
\end{aligned}
\tag{3.51}
$$

Noticing that $K_\ell(t,\tau)$, $N_\ell(t,\tau)$ are bounded for any fixed $t\in[\tau+\ell,+\infty)$, we can infer that

$$
\begin{aligned}
&\int_0^{\ell}[\|\nabla\bar{v}\|_{L^2}^2+\|\nabla\bar{F}\|_{L^2}^2+\|\nabla\bar{M}\|_{L^2}^2+\|\Delta\bar{M}\|_{L^2}^2](t+r)\mathrm{d}r\\
&\le C_0(\tau,t,\ell)\int_0^{\ell}(\|\bar{v}(\tau+s)\|_{L^2}^2+\|\bar{F}(\tau+s)\|_{L^2}^2+\|\bar{M}(\tau+s)\|_{H^1}^2)\mathrm{d}s.
\end{aligned}
\tag{3.52}
$$

Therefore,

$$
\|L(t,\tau)\chi_1-L(t,\tau)\chi_2\|_{L^2(t,t+\ell;\mathbf{V}\times H_0^1(\Omega)\times H^2(\Omega))}^2\le C_0(\tau,t,\ell)\|\chi_1-\chi_2\|_{L^2(\tau,\tau+\ell;\mathbb{H})}^2
\tag{3.53}
$$

for any $\chi_1,\chi_2\in\mathcal{B}_0^{\ell}(\tau)$ and any $t\ge\tau+\ell$.
Taking the difference of (2.2) solved by (v_1,F_1,M_1), (v_2,F_2,M_2), we have

$$
\begin{aligned}
&\bar{v}_t=-\mu A\bar{v}-B(\bar{v},v_1)-B(v_2,\bar{v})+\mathbb{P}(\nabla\cdot(F_1F_1^T-F_2F_2^T)\\
&\qquad-\nabla\cdot(\nabla^T M_1\nabla M_1-\nabla^T M_2\nabla M_2)),\\
&\bar{F}_t=\kappa\Delta\bar{F}-\bar{v}\cdot\nabla F_1-v_2\cdot\nabla\bar{F}+\nabla v_1F_1-\nabla v_2F_2,\\
&\bar{M}_t=\Delta\bar{M}-\bar{v}\cdot\nabla M_1-v_2\cdot\nabla\bar{M}+\frac{1}{\gamma^2}[(|M_1|^2-1)M_1-(|M_2|^2-1)M_2].
\end{aligned}
\tag{3.54}
$$

Taking $(\varphi,\psi)\in L^\infty(t,t+\ell;\mathbf{V}\times H_0^1(\Omega))$ such that $\|(\varphi,\psi)\|_{L^\infty(t,t+\ell;\mathbf{V}\times H_0^1(\Omega))}\le 1$, we estimate the time derivatives $\bar{v}_t$, $\bar{F}_t$ as follows:

$$|\langle\bar{v}_t,\varphi\rangle| = \left|\mu\int_\Omega \nabla\bar{v}:\nabla\varphi\mathrm{d}x - \int_\Omega J_1:\nabla\varphi\mathrm{d}x - \int_\Omega J_2\cdot\varphi\mathrm{d}x + \int_\Omega J_3\cdot\varphi\mathrm{d}x\right|, \tag{3.55}$$

$$\left|\langle\bar{F}_t,\psi\rangle\right| = \left|\kappa\int_\Omega \nabla\bar{F}:\nabla\psi\mathrm{d}x + \int_\Omega J_4:\psi\mathrm{d}x - \int_\Omega J_5:\psi\mathrm{d}x\right|, \tag{3.56}$$

where

$$\begin{aligned}
&J_1 := \bar{v}\otimes v_1 + v_2\otimes\bar{v}, \quad J_2 := \nabla\cdot(F_1F_1^T - F_2F_2^T),\\
&J_3 := \nabla\cdot(\nabla^T M_1\nabla M_1 - \nabla^T M_2\nabla M_2),\\
&J_4 := \bar{v}\cdot\nabla F_1 - v_2\cdot\nabla\bar{F},\ J_5 := \nabla v_1 F_1 - \nabla v_2 F_2.
\end{aligned}$$

Let us estimate the terms on the right hand sides of (3.55), (3.55) one by one. From Lemma 3.3, using Hölder's, Young's, Sobolev's and Poincaré's inequalities, we derive that

$$\left|\mu\int_\Omega \nabla\bar{v}:\nabla\varphi\mathrm{d}x\right| \le \mu\|\bar{v}\|_{\mathbf{V}}\|\varphi\|_{\mathbf{V}} \le C\|\bar{v}\|_{\mathbf{V}}, \tag{3.57}$$

$$\left|\kappa\int_\Omega \nabla\bar{F}:\nabla\psi\mathrm{d}x\right| \le \kappa\|\bar{F}\|_{H^1}\|\psi\|_{H^1} \le C\|\bar{F}\|_{H^1}, \tag{3.58}$$

and

$$\left|\int_\Omega J_1:\nabla\varphi\mathrm{d}x\right| \le C\|\bar{v}\|_{\mathbf{V}}(\|v_1\|_{\mathbf{V}} + \|v_2\|_{\mathbf{V}}). \tag{3.59}$$

From (2.1), we conclude that

$$\begin{aligned}
\left|\int_\Omega J_2\cdot\varphi\mathrm{d}x\right| &\le \int_\Omega |\nabla\varphi||\bar{F}||F_1|\mathrm{d}x + \int_\Omega |\nabla\varphi||\bar{F}||F_2|\mathrm{d}x\\
&\le \|\nabla\varphi\|_{L^2}\|\bar{F}\|_{L^4}\|F_1\|_{L^4} + \|\nabla\varphi\|_{L^2}\|\bar{F}\|_{L^4}\|F_2\|_{L^4}\\
&\le C\|\bar{F}\|_{H^1}(\|F_1\|_{H^1} + \|F_2\|_{H^1}), \qquad (3.60)\\
\left|\int_\Omega J_3\cdot\varphi\mathrm{d}x\right| &\le \int_\Omega |\varphi||\nabla M_1||\Delta\bar{M}|\mathrm{d}x + \int_\Omega |\varphi||\nabla\bar{M}||\Delta M_2|\mathrm{d}x\\
&\le \|\varphi\|_{L^4}\|\nabla M_1\|_{L^4}\|\Delta\bar{M}\|_{L^2} + \|\varphi\|_{L^4}\|\nabla\bar{M}\|_{L^4}\|\Delta M_2\|_{L^2}\\
&\le C\|\nabla M_1\|_{L^2}^{\frac{1}{2}}\left(\|M_1\|_{L^2}^{\frac{1}{2}} + \|\Delta M_1\|_{L^2}^{\frac{1}{2}}\right)\|\Delta\bar{M}\|_{L^2}\\
&\quad + C\|\nabla\bar{M}\|_{L^2}^{\frac{1}{2}}\left(\|\bar{M}\|_{L^2}^{\frac{1}{2}} + \|\Delta\bar{M}\|_{L^2}^{\frac{1}{2}}\right)\|\Delta M_2\|_{L^2}, \qquad (3.61)
\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega J_4 : \psi \mathrm{d}x\right| &\le \int_\Omega |\bar{v}||\nabla F_1||\psi|\mathrm{d}x + \int_\Omega |v_2||\nabla \bar{F}||\psi|\mathrm{d}x \\ &\le \|\bar{v}\|_{L^4}\|\nabla F_1\|_{L^2}\|\psi\|_{L^4} + \|v_2\|_{L^4}\|\nabla \bar{F}\|_{L^2}\|\psi\|_{L^4} \\ &\le C(\|\bar{v}\|_{\mathbf{V}}\|F_1\|_{H^1} + \|\bar{F}\|_{H^1}\|v_2\|_{\mathbf{V}}), \qquad (3.62)\\ \left|\int_\Omega J_5 : \psi \mathrm{d}x\right| &\le \int_\Omega |\nabla v_1||\bar{F}||\psi|\mathrm{d}x + \int_\Omega |\nabla \bar{v}||F_2||\psi|\mathrm{d}x \\ &\le \|\nabla v_1\|_{L^2}\|\bar{F}\|_{L^4}\|\psi\|_{L^4} + \|\nabla \bar{v}\|_{L^2}\|F_2\|_{L^4}\|\psi\|_{L^4} \\ &\le C(\|v_1\|_{\mathbf{V}}\|\bar{F}\|_{H^1} + \|\bar{v}\|_{\mathbf{V}}\|F_2\|_{H^1}). \qquad (3.63)\end{aligned}$$

Plugging (3.57)–(3.63) into (3.55), (3.55) and taking the supremum over all (φ, ψ), we conclude that

$$\begin{aligned}\int_t^{t+\ell} \|\bar{v}_t\|_{\mathbf{V}'}\mathrm{d}r + \int_t^{t+\ell} \|\bar{F}_t\|_{H^{-1}(\Omega)}\mathrm{d}r \ &\le C(\|\bar{v}\|_{L^2(t,t+\ell;\mathbf{V})} + \|\bar{F}\|_{L^2(t,t+\ell;H^1)} \\ &\quad + \|\bar{M}\|_{L^2(t,t+\ell;H^2)}). \qquad (3.64)\end{aligned}$$

Similar to the above estimates, we can estimate the time derivative $\bar{M}_t$ as follows:

$$\begin{aligned}\int_t^{t+\ell} \|\bar{M}_t\|_{L^2}\mathrm{d}r &\le \|\Delta \bar{M}\|_{L^1(t,t+\ell;L^2(\Omega))} + \|(\bar{v}\cdot\nabla)M_1\|_{L^1(t,t+\ell;L^2)} \\ &\quad + \|(v_2\cdot\nabla)\bar{M}\|_{L^1(t,t+\ell;L^2)} \\ &\quad + \frac{1}{\gamma^2}\|\bar{M}\|_{L^1(t,t+\ell;L^2(\Omega))} \\ &\quad + \frac{3}{2\gamma^2}\int_t^{t+\ell}[\|\bar{M}\|_{L^4}(\|M_1\|_{L^8}^2 + \|M_2\|_{L^8}^2)](r)\mathrm{d}r \\ &\le C\|\bar{v}\|_{L^2(t,t+\ell;\mathbf{V})} + C\|\bar{M}\|_{L^2(t,t+\ell;H^2)}. \qquad (3.65)\end{aligned}$$

We obtain from (3.52), (3.64) and (3.65) that

$$\begin{aligned}&\left(\int_0^\ell (\|\bar{v}_t(t+r)\|_{\mathbf{V}'} + \|\bar{F}_t(t+r)\|_{H^{-1}(\Omega)} + \|\bar{M}_t(t+r)\|_{L^2})\mathrm{d}r\right)^2 \\ &\le \kappa_1 C_1(\tau,t,\ell)\int_0^\ell (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2)\mathrm{d}s. \qquad (3.66)\end{aligned}$$

Therefore,

$$\|L(t,\tau)\chi_1 - L(t,\tau)\chi_2\|_{\mathfrak{X}_\ell} \le \kappa_1 C_1(\tau,t,\ell)\|\chi_1 - \chi_2\|_{L^2(\tau,\tau+\ell;\mathbb{H})} \qquad (3.67)$$

for any $\chi_1, \chi_2 \in \mathcal{B}_0^\ell(\tau)$ and any $t \ge \tau + \ell$. □

Combining with Lemma 2.2 in [1], Theorem 3.3 and Lemma 3.5, we can obtain that the following result.

Theorem 3.4 *Assuming that (A_1) holds, then the fractal dimension of a pullback attractor $\hat{A}_\ell = \{\mathcal{A}_\ell(t) : t \in \mathbb{R}\}$ in X_ℓ of the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5) obtained in Theorem 3.3 is finite.*

Proof Since each section $\mathcal{A}_\ell(t)$ of pullback attractor $\hat{\mathcal{A}}_\ell = \{\mathcal{A}_\ell(t) : t \in \mathbb{R}\}$ is bounded in X_ℓ, there exist a positive constant R and some $\chi_0 \in \mathcal{A}_\ell(t)$ such that

$$\mathcal{A}_\ell(t) \subset B_{X_\ell}(\chi_0; R). \tag{3.68}$$

From Lemma 3.5, we can infer that there is a constant $C_2 = C_2(t, \ell) > 0$ such that

$$\|L(t,\tau)\chi_1 - L(t,\tau)\chi_2\|_{\mathfrak{X}_\ell} \leq C_2\|\chi_1 - \chi_2\|_{X_\ell}, \quad \text{forany} \quad \chi_1, \chi_2 \in \mathcal{A}_\ell(t),\ t \geq \tau + \ell. \tag{3.69}$$

On the other hand, from Theorem 3.3 and (3.69), we derive that

$$\mathcal{A}_\ell(t) = L(t,\tau)\mathcal{A}_\ell(\tau) \subset L(t,\tau)B_{X_\ell}(\chi_0; R) \subset B_{\mathfrak{X}_\ell}(L(t,\tau)\chi_0; R), \quad \text{forany} \quad t \geq \tau + \ell. \tag{3.70}$$

It follows from (3.69) and $\mathfrak{X}_\ell \subset\subset X_\ell$, then there exist a finite number N of balls with the same radius $\frac{R}{2}$ centered in $\xi_1, \xi_2, \cdots, \xi_N \in X_\ell$ such that

$$\mathcal{A}_\ell(t) \subset B_{\mathfrak{X}_\ell}(L(t,\tau)\chi_0; R) \subset \bigcup_{i=1}^{N} B_{X_\ell}(\xi_i; \frac{R}{2}), \quad \text{for any} \quad t \geq \tau + \ell. \tag{3.71}$$

Considering (3.69)–(3.71) and Theorem 3.3, we get

$$\begin{aligned}\mathcal{A}_\ell(t) &= L(t,\tau)\mathcal{A}_\ell(\tau) \subset \bigcup_{i=1}^{N} L(t,\tau)B_{X_\ell}(\xi_i; \frac{R}{2}) \subset \bigcup_{i=1}^{N} B_{\mathfrak{X}_\ell}(L(t,\tau)\xi_i; \frac{R}{2}),\\ &\text{for any} \quad t \geq \tau + \ell.\end{aligned} \tag{3.72}$$

Since each ball $B_{\mathfrak{X}_\ell}(L(t,\tau)\xi_i; \frac{R}{2})$ can be covered by N balls with the same radius $\frac{R}{2^2}$ centered in $\xi_{i1}, \xi_{i2}, \cdots, \xi_{iN} \in X_\ell$, we have

$$\mathcal{A}_\ell(t) \subset \bigcup_{j=1}^{N}\bigcup_{i=1}^{N} B_{X_\ell}(L(t,\tau)\xi_{ij}; \frac{R}{2^2}). \tag{3.73}$$

Repeatedly, we can obtain that

$$\mathcal{A}_\ell(t) \subset \bigcup_{j=1}^{N}\bigcup_{i=1}^{N^{n-1}} B_{X_\ell}(L(t,\tau)\xi_{ij}; \frac{R}{2^n}). \tag{3.74}$$

For any $\varepsilon > 0$, we can choose some positive integer n sufficiently large such that $\frac{R}{2^n} \le \varepsilon < \frac{R}{2^{n-1}}$; then we can infer from (3.74) that

$$d_F^{X_\ell}(\mathcal{A}_\ell(t)) = \lim_{\varepsilon\to 0^+} \sup \frac{\ln N_\varepsilon^{X_\ell}(\mathcal{A}_\ell(t))}{\ln(\frac{1}{\varepsilon})} \le \lim_{n\to\infty} \frac{\ln N_\varepsilon^{X_\ell}(\mathcal{A}_\ell(t))}{\ln(\frac{2^n}{R})} = \frac{\ln N}{\ln 2} < \infty. \tag{3.75}$$

This completes the proof of Theorem 3.4.

□

3.2 The Existence of Pullback Attractors in $\mathbb{H}$

In order to obtain that the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (1.2)–(1.5) has a pullback attractor with finite fractal dimension in the original phase space $\mathbb{H}$, we also need to prove the following results.

Lemma 3.6 *Assuming that* (A_1) *holds, then the mapping* e_1*:* $\mathcal{B}_0^\ell(\tau-\ell) \to B_1(\tau) = e_1(\mathcal{B}_0^\ell(\tau-\ell))$ *is Lipschitz continuous for any fixed* $\tau \in \mathbb{R}$*. That is, for any two* ℓ*-trajectories* $\chi_1, \chi_2 \in \mathcal{B}_0^\ell(\tau)$*, there exists a positive constant* C *dependent on* ℓ *such that*

$$\|e_1(\chi_1) - e_1(\chi_2)\|_{\mathbb{H}}^2 \le C\int_0^\ell \|\chi_1(\tau+s) - \chi_2(\tau+s)\|_{\mathbb{H}}^2 \mathrm{d}s. \tag{3.76}$$

Proof For any fixed $\tau \in \mathbb{R}$ and any $\chi_1, \chi_2 \in \mathcal{B}_0^\ell(\tau)$, let $L(t,\tau)\chi_1 = (v_1(t), F_1(t), M_1(t))$, $L(t,\tau)\chi_2 = (v_2(t), F_2(t), M_2(t))$ for any $t \ge \tau+\ell$. We use $(\bar{v}, \bar{F}, \bar{M}) = (v_1 - v_2, F_1 - F_2, M_1 - M_2)$ to denote the difference of two solutions for problem (1.2)–(1.5).

For any fixed $\tau \in \mathbb{R}$ and any $s \in (0,\ell)$, using Gronwall's lemma for (3.41), we get

$$\begin{aligned}
&\|\bar{v}(\tau+\ell)\|_{L^2}^2 + \|\bar{F}(\tau+\ell)\|_{L^2}^2 + \|\bar{M}(\tau+\ell)\|_{H^1}^2 \\
&\le (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2) exp\left(\int_{\tau+s}^{\tau+\ell} \rho(r)\mathrm{d}r\right) \\
&\le (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2) exp\left(\int_{\tau}^{\tau+\ell} \rho(r)\mathrm{d}r\right).
\end{aligned} \tag{3.77}$$

Integrating (3.77) with respect to s over $(0, \ell)$, we can conclude that

$$\begin{aligned}
&\|\bar{v}(\tau+\ell)\|_{L^2}^2 + \|\bar{F}(\tau+\ell)\|_{L^2}^2 + \|\bar{M}(\tau+\ell)\|_{H^1}^2 \\
&\leq \frac{1}{\ell} exp\left(\int_\tau^{\tau+\ell} \rho(r)\mathrm{d}r\right) \int_0^\ell (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2)\mathrm{d}s \\
&\leq C(\ell,\tau) \int_0^\ell (\|\bar{v}(\tau+s)\|_{L^2}^2 + \|\bar{F}(\tau+s)\|_{L^2}^2 + \|\bar{M}(\tau+s)\|_{H^1}^2)\mathrm{d}s, \qquad (3.78)
\end{aligned}$$

where $C(\ell,\tau) := \frac{1}{\ell} N_\ell(\tau)$, $N_\ell(\tau) := exp\left(\int_\tau^{\tau+\ell} \rho(r)\mathrm{d}r\right)$ is a finite number depending on $(v_{1,\tau}, F_{1,\tau}, M_{1,\tau})$ and $(v_{2,\tau}, F_{2,\tau}, M_{2,\tau})$.

Finally, we can infer from (3.78) that the mapping e_1: $\mathcal{B}_0^\ell(\tau-\ell) \to B_1(\tau)$ is Lipschitz continuous. The proof is complete. □

According to the above results, we now give the proof of Theorem 1.1 as follows:

Proof of Theorem 1.1. From Lemma 2.3 in [1], Theorem 3.4 and Lemma 3.6, we can infer that for any $t \in \mathbb{R}$, the sections $\mathcal{A}(t)$ of the pullback attractor $\hat{\mathcal{A}}$ are compact and their fractal dimensions are uniformly finite. From the invariance of $\hat{\mathcal{A}}_\ell$: $L(t-\ell, s-\ell)\mathcal{A}_\ell(s-\ell) = \mathcal{A}_\ell(t-\ell)$ for any $t \geq s$, we derive that

$$\begin{aligned}
&U(t,s)\mathcal{A}(s) = U(t,s)e_1(\mathcal{A}_\ell(s-\ell)) \\
&= e_1(L(t-\ell, s-\ell)\mathcal{A}_\ell(s-\ell)) = e_1(\mathcal{A}_\ell(t-\ell)) = \mathcal{A}(t) \qquad (3.79)
\end{aligned}$$

for any $t \geq s$.

For any bounded subset B of $\mathbb{H}$, from the definition of $B_1(t)$ and $\mathcal{B}_0^\ell(t)$, then there exists some time $\tilde{\tau} = \tilde{\tau}(B) > 0$ such that

$$U(t,\tau)B \subset B_1(t) = e_0(\mathcal{B}_0^\ell(t)) \qquad (3.80)$$

for any $t - \tau \geq \tilde{\tau}$.

Therefore, we only need to prove that

$$\lim_{\tau \to -\infty} \mathrm{dist}_{\mathbb{H}}(U(t,\tau)B_1(\tau), \mathcal{A}(t)) = 0.$$

Otherwise, there exist some sequence $\{(v_n, F_n, M_n)\}_{n\geq 1} \subset B_1(\tau_n)$, a positive constant ε_0 and some $\tilde{\tau}_{n_0} > 0$ such that for any $t - \tau_n \geq \tilde{\tau}_{n_0}$, we have

$$\mathrm{dist}_{\mathbb{H}}(U(t,\tau_n)(v_n, F_n, M_n), \mathcal{A}(t)) \geq \varepsilon_0. \qquad (3.81)$$

On the other hand, from the definition of B_1, we can infer that there exists a sequence $\{\chi_n\}_{n\geq 1} \subset \mathcal{B}_0^\ell(\tau_n)$ such that

$$(v_n, F_n, M_n) = e_0(\chi_n).$$

Since $\{\chi_n\}_{n\geq 1}$ is bounded in X_ℓ and $\hat{\mathcal{A}}_\ell$ is a pullback attractor in X_ℓ of the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (1.2)–(1.5), there exist a subsequence $\{\chi_{n_j}\}_{n_j\geq 1}$ of $\{\chi_n\}_{n\geq 1}$ and a subsequence $\{\tau_{n_j}\}_{n_j\geq 1}$ of $\{\tau_n\}_{n\geq 1}$ such that

$$L(t-\ell,\tau_{n_j})\chi_{n_j} \to \chi \in \mathcal{A}_\ell(t-\ell) \text{ in } X_\ell \text{as} j \to +\infty.$$

Using the continuity of e_1, we have

$$U(t,\tau_{n_j})(v_{n_j},F_{n_j},M_{n_j}) = e_1(L(t-\ell,\tau_{n_j})\chi_{n_j}) \to e_1(\chi) \in \mathcal{A}(t) \text{ in } \mathbb{H} \\ \text{as} j \to +\infty.$$

The contradiction to (3.81) completes the proof. □

Remark 3.1 For system (1.2)–(1.5), if we consider the external force term $\varepsilon g(x,t)$ depending on a small parameter $\varepsilon \in (0,1]$ as a small perturbation to the autonomous system, then we obtain a continuous process $U^\varepsilon(\cdot,\cdot)$ driven by the non-autonomous dynamical system. Since the upper semicontinuity implies some stability for the attractors of the systems under some perturbations (see [8, 16, 31, 32, 39] and references therein). It is also interesting and important to consider the relationship between the pullback attractors $\hat{\mathcal{A}}^\varepsilon = \{\mathcal{A}^\varepsilon(t) : t \in \mathbb{R}\}$ for the perturbed system (1.2)–(1.5) with $\varepsilon \in (0,1]$ and the global attractor $\mathcal{A}^0$ for the unperturbed system (1.2)–(1.5) with $\varepsilon = 0$ (see, e.g., [28]).

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Declarations

Conflict of interest This work does not have any conflicts of interest.

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