Bull. Malays. Math. Sci. Soc. (2024) 47:19
https://doi.org/10.1007/s40840-023-01604-0

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MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

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Resonant-Superlinear and Resonant-Sublinear Dirichlet Problems

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Received: 30 September 2022 / Revised: 5 October 2023 / Accepted: 16 October 2023 /
Published online: 5 December 2023

Abstract

In this paper, we study elliptic equations in which the reaction (right hand side) exhibits an asymmetric behavior as $x \to \pm\infty$. More precisely, we assume that we have resonance as $x \to -\infty$, while as $x \to +\infty$ the equation is superlinear. Using variational tools combined with the theory of critical groups, we prove several multiplicity theorems for nonlinear, nonhomogeneous equations and for semilinear equations (driven by the Laplacian).

Keywords Asymmetric reaction · Regularity theory · Maximum principle · Resonance · Critical groups

Mathematics Subject Classification 35J20 · 35J60 · 58E05

Communicated by Rosihan M. Ali.

The work was supported by NNSF of China Grant No.12071413, NSF of Guangxi Grant No.2023GXNSFAA026085, the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N (N \geq 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following Dirichlet (p, q)-equation

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) = \hat{\lambda}_1(p)|u(z)|^{p-2}u(z) + f(z, u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 1 < q < p. \end{array} \right\}. \tag{1}$$

For $r \in (1, \infty)$, by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Equation (1) is driven by the sum of two such operators with distinct exponents (double phase problem with balanced growth). So, the differential operator in (1) is not homogeneous. In the reaction (right hand side) of (1), we have a resonant term $u \to \hat{\lambda}_1(p)|u|^{p-2}u$ with $\hat{\lambda}_1(p) > 0$ being the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and a Carathéodory perturbation $f(z, x)$ (that is, for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable and for a.e. $z \in \Omega$, $x \to f(z, x)$ is continuous) which exhibits asymmetric behavior as $x \to \pm\infty$. Our work here was motivated by that of Domingos da Silva-Ribeiro [8], who investigated the "resonant-superlinear" case for semilinear equations driven by the Dirichlet Laplacian. Similar problems were considered earlier by Cuesta–de Figueiredo–Srikanth [5] and Cuesta–De Coster [6]. Other versions of asymmetric equations can be found in the works of Recova–Rumbos [24] (semilinear equations), Motreanu–Motreanu–Papageorgiou [17] (nonlinear equations driven by the p-Laplacian) and Gasiński–Papageorgiou [12], Papageorgiou–Winkert [22] ($(p, 2)$-equations).

Here, in addition to the "resonant-superlinear" case (that is, the equation is resonant as $x \to -\infty$ and superlinear as $x \to +\infty$), we examine also the "resonant-sublinear" case which has not been considered in the literature. For both cases, we prove multiplicity results.

2 Mathematical Background

The main spaces in the analysis of problem (1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega=0}\}$.

On account of the Poincaré inequality, the norm of $W_0^{1,p}(\Omega)$ is given by

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\bar{\Omega}) : 0 \leq u(z) \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : 0 < u(z) \text{ for all } z \in \Omega, \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} < 0 \right\}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ and $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$.

For $r \in (1, \infty)$, let $A_r \colon W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ $(\frac{1}{r} + \frac{1}{r'} = 1)$ be the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,r}(\Omega).$$

If $r = 2$, then we write $A = A_2 \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. We set $V = A_p + A_q$: $W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$. This operator has the following properties (see Gasiński–Papageorgiou [10], Problem 2.192, p.279).

Proposition 1 *$V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (maps bounded sets to bounded ones), continuous, strictly monotone (thus, it is maximal monotone too) and of type $(S)_+$, that is,*

"if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$."

We will need some facts about the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$ and of $(-\Delta, H_0^1(\Omega))$. First we consider the nonlinear eigenvalue problem:

$$-\Delta_r u(z) = \widehat{\lambda}|u|^{r-2}u \text{ in } \Omega, u|_{\partial\Omega} = 0, 1 < r < \infty. \tag{2}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of (2), if the problem has a nontrivial solution $\widehat{u} \in W_0^{1,r}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. The set of eigenvalues is denoted by $\widehat{\sigma}(r)$. Acting on (2) with $\widehat{u}$, we see that $\widehat{\sigma}(r) \subseteq \mathbb{R}_+ = [0, \infty)$. In fact $\widehat{\sigma}(r)$ has a smallest element $\widehat{\lambda}_1(r)$ which has the following properties:

(a) $\widehat{\lambda}_1(r) > 0$.
(b) $\widehat{\lambda}_1(r)$ is isolated in $\widehat{\sigma}(r)$ (that is, we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(r), \widehat{\lambda}_1(r) + \varepsilon) \cap \widehat{\sigma}(r) = \emptyset$).
(c) $\widehat{\lambda}_1(r)$ is simple (that is, if $\widehat{u}, \widetilde{u}$ are two eigenfunctions corresponding to $\widehat{\lambda}_1(r)$, then $\widehat{u} = \vartheta\widetilde{u}$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$, that is the corresponding eigenspace is a one-dimensional vector space).

$$(d) \quad \widehat{\lambda}_1(r) = \inf\left[\frac{||Du||_r^r}{||u||_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right]. \tag{3}$$

The infimum in (3) is realized on the corresponding one dimensional eigenspace. Since in (3) u can be replaced by $|u|$, we see that the elements of this first eigenspace have fixed sign. In fact, $\widehat{\lambda}_1(r)$ is the only eigenvalue with eigenfunctions of fixed sign. All other eigenvalues have eigenfunctions which are nodal (sign-changing). By $\widehat{u}_1(r)$ we denote the positive, L^r-normalized (that is, $\|\widehat{u}_1(r)\|_r = 1$) eigenfunction. From Ladyzhenskaya–Uraltseva [14](p.286), we have that $\widehat{u}_1(r) \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [16] implies that $\widehat{u}_1(r) \in C_+ \setminus \{0\}$. In fact the nonlinear Hopf maximum principle (see Gasiński–Papageorgiou [9] and Pucci–Serrin [23]) implies that $\widehat{u}_1(r) \in \text{int}C_+$. The set $\widehat{\sigma}(r) \subseteq (0, \infty)$ is closed and so the second eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$ is defined by

$$\widehat{\lambda}_2(r) = \inf\left[\widehat{\lambda} \in \widehat{\sigma}(r) : \widehat{\lambda} > \widehat{\lambda}_1(r)\right].$$

Note that using the Lusternik–Schnirelmann minimax scheme (see Gasiński–Papageorgiou [9]), we can generate a whole sequence $\{\widetilde{\lambda}_k(r)\}_{k\in\mathbb{N}}$ of eigenvalues of $(-\Delta_r, W_0^{1,r}(\Omega))$, known as "variational eigenvalues", such that $\widetilde{\lambda}_k(r) \to +\infty$ as $k \in +\infty$. We have $\widetilde{\lambda}_1(r) = \widehat{\lambda}_1(r)$ and $\widetilde{\lambda}_2(r) = \widehat{\lambda}_2(r)$, but we do not know if the sequence of variational eigenvalues exhausts $\widehat{\sigma}(r)$. This is the case in the linear eigenvalue problem (that is, $r = 2$). So, we consider the following linear eigenvalue problem

$$-\Delta u = \widehat{\lambda} u \text{ in } \Omega, u|_{\partial\Omega} = 0. \tag{4}$$

The spectrum $\widehat{\sigma}(2)$ of (4) is a sequence $\{\widehat{\lambda}_k(2)\}_{k\in\mathbb{N}}$ of eigenvalues such that $\widehat{\lambda}_k(2) \to +\infty$ as $k \to \infty$ and the corresponding eigenspaces $E(\widehat{\lambda}_k(2)), k \in \mathbb{N}$ are all linear spaces and we have

$$H_0^1(\Omega) = \overline{\bigoplus_{k\in\mathbb{N}} E(\widehat{\lambda}_k(2))}.$$

Each eigenspace $E(\widehat{\lambda}_k(2))$ has the unique continuation property; that is, if $u \in E(\widehat{\lambda}_k(2))(k \in \mathbb{N})$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. Note that $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$.

In this case, all eigenvalues have variational characterizations. So, for $m \in \mathbb{N}$, let

$$\overline{H}_m = \bigoplus_{k=1}^{m} E(\widehat{\lambda}_k(2)). \text{ and } \widehat{H}_m = \overline{\bigoplus_{k\geq m} E(\widehat{\lambda}_k(2))}.$$

We have

$$\hat{\lambda}_1(2) = \inf\left[\frac{||Du||_2^2}{||u||_2^2} : u \in H_0^1(\Omega), u \neq 0\right] \tag{5}$$

and for $m \in \mathbb{N} \setminus \{1\}$ (that is, $m \geq 2$), we have

$$\begin{aligned}\hat{\lambda}_m(2) =& \sup\left[\frac{||Du||_2^2}{||u||_2^2} : u \in \overline{H}_m, u \neq 0\right] \\ =& \inf\left[\frac{||Du||_2^2}{||u||_2^2} : u \in \widehat{H}_m, u \neq 0\right].\end{aligned} \tag{6}$$

Note that (5) is a particular case of (3) (when $r = 2$) and the infimum is realized on $E(\widehat{\lambda}_1(2))$. In (6), both the supremum and the infimum are realized on $E(\widehat{\lambda}_m(2))$.

Using (5),(6) and the unique continuation property, we can have the following basic inequalities.

Proposition 2 *(a) If $\vartheta \in L^{\infty}(\Omega)$ and $\vartheta(z) \geq \widehat{\lambda}_m(2)$ for a.e. $z \in \Omega$, $\vartheta \not\equiv \widehat{\lambda}_m(2)$, then there exists $c_1 > 0$ such that*

$$\|Du\|_2^2 - \int_{\Omega} \vartheta(z)|u|^2 dz \leq -c_1\|u\|^2 \quad \text{for all } u \in \overline{H}_m.$$

(b) If $\vartheta \in L^{\infty}(\Omega)$ and $\vartheta(z) \leq \widehat{\lambda}_m(2)$ for a.e. $z \in \Omega$, $\vartheta \not\equiv \widehat{\lambda}_m(2)$, then there exists $c_2 > 0$ such that

$$\|Du\|_2^2 - \int_{\Omega} \vartheta(z)|u|^2 dz \geq c_2\|u\|^2 \quad \text{for all } u \in \widehat{H}_m.$$

We will also consider a weighted version of (4). So, let $\eta \in L^{\infty}(\Omega) \setminus \{0\}$, $\eta(z) \geq 0$ for a.e. $z \in \Omega$ and consider the following linear eigenvalue problem

$$-\Delta u = \widetilde{\lambda}\eta(z)u \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{7}$$

The spectrum of this eigenvalue problem is a sequence $\{\widetilde{\lambda}_k(\eta, 2)\}_{k\in\mathbb{N}}$ of distinct eigenvalues such that $\widetilde{\lambda}_k(\eta, 2) \to \infty$ as $k \to \infty$. Again we have variational characterizations for all the eigenvalues using the Rayleigh quotient $\frac{\|Du\|_2^2}{\int_{\Omega}\eta(z)u^2 dz}$.

Proposition 3 *If $\eta, \widehat{\eta} \in L^{\infty}(\Omega)\setminus\{0\}$, $\eta(z) \leq \widehat{\eta}(z)$ for a.e. $z \in \Omega$, $\eta \neq \widehat{\eta}$, then $\widetilde{\lambda}_1(\widehat{\eta}, 2) < \widetilde{\lambda}_1(\eta, 2)$.*

Let X be a Banach and $\varphi \in C^1(X)$, $c \in \mathbb{R}$. We set

$$K_{\varphi} = \{u \in X : \varphi'(u) = 0\},$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

Also, if $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$, then by $H_k(Y_1, Y_2)$ we denote the $k \stackrel{th}{=}$-relative singular homology group with integer coefficients. Given $u \in K_{\varphi}$ isolated with $c = \varphi(u)$, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U\setminus\{u\}), \text{ for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_{\varphi} \cap \varphi^c \cap U = \{u\}$(isolating neighborhood). The excision property of singular homology implies that the above definition of critical groups is independent of the isolating neighborhood U.

We say that $\varphi \in C^1(X)$ satisfies the C-condition, if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence. Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $-\infty < \inf \varphi(K_{\varphi})$. Let $c < \inf \varphi(K_{\varphi})$. Then, the critical groups of $\varphi(\cdot)$ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c), \text{ for all } k \in \mathbb{N}_0.$$

Using the second deformation theorem (see [20], p.386), we see that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose K_φ is finite. We introduce the following series in $t \in \mathbb{R}$.

$$M(t,u) = \sum_{k\in\mathbb{N}_0} \text{rank } C_k(\varphi, u)t^k, \text{ for all } u \in K_\varphi,$$
$$P(t,\infty) = \sum_{k\in\mathbb{N}_0} \text{rank } C_k(\varphi, \infty)t^k.$$

These two series are related by the Morse identity

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t) \text{ for all } t \in \mathbb{R}. \tag{8}$$

Here, $Q(t) = \sum_{k\in\mathbb{N}_0} \alpha_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients α_k. For details, we refer to [20].

If $u : \Omega \to \mathbb{R}$ is measurable, then for all $z \in \Omega$ we define $u^\pm(z) = max\{\pm u(z), 0\}$. Evidently $z \to u^\pm(z)$ are both measurable and $u = u^+ - u^-$, $|u| = u^+ + u^-$. If $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. Finally if $1 < r < \infty$, then

$$r^* = \begin{cases} \frac{Nr}{N-r} \text{ if } r < N, \\ +\infty \text{ if } N \le r. \end{cases}$$

3 Resonant Superlinear Equations

The hypotheses on the data of (1) are the following:

H_0: If $q < N$, then $N \le \frac{pq}{p-q}$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.e. $z \in \Omega$ and

(i) $|f(z,x)| \le a(z)[1 + |x|^{r-1}]$ for a.e. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii) If $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \dfrac{F(z,x)}{x^p} = +\infty$ uniformly for a.e. $z \in \Omega$ and there exist $\mu \in \big((r-p)\max\{1, \frac{N}{p}\}, p^*\big)$ such that

$$0 < \beta_0 \le \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\mu} \text{ uniformly for } a.e.\ z \in \Omega;$$

(iii) $\lim_{x\to-\infty} \dfrac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.e. $z \in \Omega$;

(iv) there exist $\theta \in L^\infty(\Omega)$ and $\widehat{\theta}, \widehat{\eta} > 0$ such that

$$\widehat{\lambda}_1(q) \le \theta(z) \text{ for } a.e.\ z \in \Omega, \theta \not\equiv \widehat{\lambda}_1(q),$$

$$\theta(z) \leq \liminf_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} \leq \limsup_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} \leq \widehat{\theta} \text{ uniformly for } a.e.\ z \in \Omega,$$
$$e(z,x) = f(z,x)x - pF(z,x) \geq -\widehat{\eta} \text{ for } a.e.\ z \in \Omega,\ \text{ all } x \leq 0;$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for $a.e. z \in \Omega$, the function $x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, 0]$.
Remarks: Hypothesis $H_1(ii)$ implies that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for } a.e. z \in \Omega.$$

So, the perturbation $f(z,\cdot)$ is $(p-1)$ superlinear as $x \to +\infty$. However $f(z,\cdot)$ need not satisfy the Ambrosetti–Rabinowitz condition(see [2]), which is common in the literature when studying superlinear problems. Hypothesis $H_1(iii)$ implies that problem (1) is resonant with respect to the principal eigenvalue $\widehat{\lambda}_1(\rho)$ when $x \to -\infty$. So, the reaction of problem (1) exhibits an asymmetric behavior asymptotically as $x \to \pm\infty$.

The following two functions satisfy hypotheses H_1. For the sake of simplicity, we drop the z-dependence

$$f_1(x) = \begin{cases} ((\widehat{\theta}+1)|x|^{\tau-2} - |x|^{s-2})x & \text{if } x < -1 \\ \widehat{\theta}|x|^{q-2}x & \text{if } -1 \leq x \leq 1 \\ \widehat{\theta}x^{r-1} & \text{if } 1 < x \end{cases}$$

with $\widehat{\theta} > \widehat{\lambda}_1(q)$, $1 < \tau < s < p < r$,

$$f_2(x) = \begin{cases} ((\widehat{\theta}+1)|x|^{\tau-2} - |x|^{s-2})x & \text{if } x < -1 \\ \widehat{\theta}|x|^{q-2}x & \text{if } -1 \leq x \leq 1 \\ \widehat{\theta}(\ln x + 1)x^{p-1} & \text{if } 1 < x \end{cases}$$

with $\widehat{\theta} > \widehat{\lambda}_1(q)$, $1 < \tau < s < p$.

Note that $f_1(\cdot)$ satisfies the AR-condition but $f_2(\cdot)$ does not.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \frac{\widehat{\lambda}_1(p)}{p}\|u\|_p^p - \int_\Omega F(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Clearly, $\varphi \in C^1(W_0^{1,p}(\Omega))$. Also, let $\varphi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the "negative" truncation of $\varphi(\cdot)$ defined by

$$\varphi_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \frac{\widehat{\lambda}_1(p)}{p}\|u^-\|_p^p - \int_\Omega F(z,-u^-)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Proposition 4 *If hypotheses H_1 hold, then the functional $\varphi_-(\cdot)$ is coercive.*

Proof We argue by contradiction. So, suppose that $\varphi_-(\cdot)$ is not coercive. Then, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\|u_n\| \to \infty \ \text{ and } \varphi_-(u_n) \le c_3 \text{ for some } c_3 > 0, \ \text{all } n \in \mathbb{N}. \tag{9}$$

We have

$$\frac{1}{p}\|Du_n\|_p^p + \frac{1}{q}\|Du_n\|_q^q \le c_3 + \frac{\widehat{\lambda}_1(p)}{p}\|u_n^-\|_p^p + \int_\Omega F(z, -u_n^-)dz \text{ for all } n \in \mathbb{N}. \tag{10}$$

If $\{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, then from (10) and hypothesis $H_1(i)$ we see that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, which contradicts (9).

So, we may assume that $\|u_n^-\| \to \infty$. Let $v_n = \frac{u_n^-}{\|u_n^-\|}$, $n \in \mathbb{N}$. Then, $\|v_n\| = 1$, $v_n \ge 0$ for all $n \in \mathbb{N}$ and we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,p}(\Omega),\, v_n \to v \text{ in } L^p(\Omega),\, v \ge 0. \tag{11}$$

From (10), we have

$$\begin{aligned} &\frac{1}{p}\|Dv_n\|_p^p + \frac{1}{q\|u_n^-\|^{p-q}}\|Dv_n\|_q^q \le \frac{c_1}{\|u_n^-\|^p} \\ &+ \frac{\widehat{\lambda}_1(p)}{p}\|v_n^-\|_p^p - \int_\Omega \frac{F(z, -u_n^-)}{\|u_n^-\|^p}dz \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{12}$$

Hypotheses $H_1(i)$, (iii) imply that given $\varepsilon > 0$, we can find $c_4 = c_4(\varepsilon) > 0$ such that

$$|F(z, x)| \le c_4 + \varepsilon|p|^p \ \text{ for } a.e. z \in \Omega, \text{all } x \le 0. \tag{13}$$

From (13), we infer that

$$\left\{\frac{F(\cdot, -u_n^-(\cdot))}{\|u_n^-\|^p}\right\}_{n\in\mathbb{N}} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

The Dunford–Pettis theorem and hypothesis H_1(iii) imply that at least for a subsequence we have that

$$\frac{F(\cdot, -u_n^-(\cdot))}{\|u_n^-\|^p} \xrightarrow{w} 0 \text{ in } L^1(\Omega). \tag{14}$$

So, if in (12) we pass to the limit as $n \to \infty$ and use (11) and (14), we obtain

$$\|Dv\|_p^p \le \widehat{\lambda}_1(p)\|v\|_p^p,$$

$$\Rightarrow \|Dv\|_p^p = \widehat{\lambda}_1(p)\|v\|_p^p \ \text{ (see (3))},$$
$$\Rightarrow v = \vartheta \widehat{u}_1(p) \ \text{ for some } \vartheta \geq 0 \ \text{ (see (11))}.$$

If $\vartheta = 0$, then $v = 0$ and so we have

$$\|Dv_n\|_p^p \to 0 \Rightarrow v_n \to 0 \ \text{ in } W_0^{1,p}(\Omega),$$

which contradicts the fact that $\|v_n\| = 1$ for all $n \in \mathbb{N}$.

If $\vartheta > 0$, then $v \in \operatorname{int}C_+$ and so

$$v_n^-(z) \to +\infty \text{ for } a.e. z \in \Omega. \tag{15}$$

We have

$$\frac{d}{dx}\left[\frac{F(z,x)}{|x|^p}\right] = \frac{f(z,x)x - pF(z,x)}{|x|^p x}$$
$$\leq \frac{-\widehat{\eta}}{|x|^{p-2}x} \text{for a.e. } z \in \Omega, \ \text{all } x \leq 0 \ \text{ (see hypothesis } H_1(iv)),$$
$$\Rightarrow \frac{F(z,s)}{|s|^p} - \frac{F(z,x)}{|x|^p} \leq -\frac{\widehat{\eta}}{p}\left[\frac{1}{|x|^p} - \frac{1}{|s|^p}\right] \text{ for } a.e. \ z \in \Omega, \ \text{all } x < s < 0.$$

We pass to the limit as $x \to -\infty$ and use hypothesis $H_1(iii)$. We obtain

$$\frac{F(z,s)}{|s|^p} \leq \frac{\widehat{\eta}}{p}\frac{1}{|s|^p} \text{ for } a.e. \ z \in \Omega, \ \text{all } s < 0.$$
$$\Rightarrow -\widehat{\eta} \leq -pF(z,s) \ \text{ for } a.e. \ z \in \Omega, \ \text{all } s \leq 0. \tag{16}$$

From (8) we have

$$\frac{1}{q}\|Du_n^-\|_q^q - \int_\Omega F(z,-u_n^-)dz \leq c_1 \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \frac{1}{q}\widehat{\lambda}_1(q)\|u_n^-\|_q^q \leq c_5 \ \text{ for some } c_5 > 0, \ \text{all } n \in \mathbb{N} \ \text{ (see(16))}. \tag{17}$$

Fatou's lemma and (15) imply that

$$\|u_n^-\|_q \to +\infty, \ \text{which contradicts (17)}.$$

Therefore, we infer that

$$\{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}.$$

Then, from (10) it follows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}.$$

which contradicts (9). This proves that $\varphi_-(\cdot)$ is coercive. □

Using the above proposition, we can generate a negative solution for problem (1).

Proposition 5 *If hypotheses H_1 hold, then problem (1) admits a negative solution $v_0 \in -\,int\, C_+$ which is a local minimizer of the energy functional $\varphi(\cdot)$.*

Proof From Proposition 4, we know that $\varphi_-(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\varphi_-(\cdot)$ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass–Tonelli theorem, we can find $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_-(v_0) = \inf\left[\varphi_-(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{18}$$

On account of hypothesis $H_1(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{q}[\theta(z) - \varepsilon]|x|^q \leq F(z, x) \ \text{ for } a.e.\ z \in \Omega, \ \text{ all } |x| \leq \delta. \tag{19}$$

Recall that $\widehat{u}_1(q) \in \text{ int}C_+$. So, we can find $t \in (0, 1)$ small such that

$$t\widehat{u}_1(q)(z) \in [0, \delta] \ \text{ for all } z \in \overline{\Omega}. \tag{20}$$

Then, we have

$$\begin{aligned}\varphi_-(-t\hat{u}_1(q)) &\leq \frac{t^p}{p}\|D\hat{u}_1(q)\|_p^p + \frac{t^q}{q}\left[\int_\Omega (\widehat{\lambda}_1(q) - \theta(z))\hat{u}_1(q)^q dz + \epsilon\right] \\ &\text{(see (19), (20) and recall } \|\hat{u}_1(q)\|_q = 1).\end{aligned} \tag{21}$$

Since $\hat{u}_1(q) \in \text{ int}C_+$, using the hypothesis on $\theta(\cdot)$(see hypothesis $H_1(iv)$), we have

$$\int_\Omega [\theta(z) - \widehat{\lambda}_1(q)]\hat{u}_1(q)dz = \beta > 0.$$

Therefore, choosing $\varepsilon \in (0, \beta)$, from (21) we obtain

$$\varphi_-(-t\hat{u}_1(q)) \leq c_6 t^p - c_7 t^q \text{ for some } c_6, c_7 > 0 \text{ and } t \in (0, 1) \text{ small.}$$

Recall that $q < p$. So, choosing $t \in (0, 1)$ even smaller if necessary we have

$$\begin{aligned}&\varphi_-(-t\hat{u}_1(q)) < 0, \\ \Rightarrow &\varphi_-(v_0) < 0 = \varphi_-(0) \text{ (see (18))} \\ \Rightarrow &v_0 \neq 0.\end{aligned}$$

From (18), we have

$$\varphi'_-(v_0) = 0,$$

$$\Rightarrow\langle V(v_0), h\rangle = \widehat{\lambda}_1(p)\int_\Omega (v_0^-)^{p-1}hdz + \int_\Omega f(z, v_0^-)hdz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{22}$$

In (22), we use the test function $h = v_0^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}&\|Dv_0^+\|_p^p \le 0,\\ \Rightarrow &v_0 \le 0,\ v_0 \ne 0.\end{aligned} \tag{23}$$

From (22) and (23), it follows that

$$-\Delta_p v_0 - \Delta_q v_0 = \widehat{\lambda}_1(p)|v_0|^{p-2}v_0 + f(z, v_0) \text{ in } \Omega.$$

Theorem 7.1, p.286, of Ladyzhenskaya–Uraltseva [14] implies that $v_0 \in L^\infty(\Omega)$. Then, using the nonlinear regularity theory of Lieberman [16], we have $v_0 \in (-C_+)\backslash\{0\}$. Let $\rho = \|v_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$\begin{aligned}&\Delta_p(-v_0) + \Delta_q(-v_0) \le \hat{\xi}_\rho(-v_0)^{p-1},\\ \Rightarrow &v_0 \in \text{-int}C_+ \ \text{(see Pucci–Serrin [23]).}\end{aligned} \tag{24}$$

Note that

$$\varphi\Big|_{-C_+} = \varphi_-\Big|_{-C_+}.$$

So, from (24) it follows that v_0 is a local $C_0^1(\overline{\Omega})$-minimizer of $\varphi(\cdot)$. From Gasiński–Papageorgiou [11] it follows that v_0 is a local $W_0^{1,p}(\Omega)$-minimizer of $\varphi(\cdot)$. □

Using this constant sign solution $v_0 \in -\operatorname{int} C_+$ together with variational tools and critical groups, we will generate a second nontrivial smooth solution and have the first multiplicity theorem for problem (1). To this end, we need to strengthen hypothesis $H_1(iv)$ (the behavior of the perturbation $f(z, \cdot)$ near zero). The new hypotheses on the perturbation $f(z, x)$ are the following:

H_2: $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.e. $z \in \Omega$, hypotheses $H_2(i), (ii), (iii), (v)$ are the same as the corresponding hypotheses $H_1(i), (ii), (iii), (v)$ and the new condition is (iv) there exist $\hat{\theta} \in (\hat{\lambda}_2(q), \infty)\backslash\hat{\sigma}(q)$ and $\hat{\eta} > 0$ such that

$$\begin{aligned}&\lim_{x\to 0}\frac{f(z, x)}{|x|^{q-2}x} = \hat{\theta} \text{ uniformly for a.e.} z \in \Omega,\\ &e(z, x) = f(z, x)x - pF(z, x) \ge -\hat{\eta} \ \text{ for a.e.} z \in \Omega,\ \text{ all } x \le 0.\end{aligned}$$

The examples illustrating hypotheses H_1 (see functions $f_1(\cdot)$ and $f_2(\cdot)$) work also here, only now $\widehat{\theta} > \widehat{\lambda}_2(q), \widehat{\theta} \notin \widehat{\sigma}(q)$.

As we already mentioned earlier, our approach will combine variational arguments (the mountain pass theorem) with critical groups (Morse theory). To do this, we need to know that the energy functional $\varphi(\cdot)$ satisfies the compactness condition (the C-condition). This can be done using the initial (weaker) hypotheses H_1(since in H_2 we have modified only the behavior of $f(z,\cdot)$ near zero).

Proposition 6 *If hypotheses H_1 hold, then the energy functional $\varphi(\cdot)$ satisfies the C-condition.*

Proof Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi(u_n)| \leq c_8 \text{ for some } c_8 > 0, \text{ all } n \in \mathbb{N}, \tag{25}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W_0^{-1,p'}(\Omega). \tag{26}$$

From (25) we have

$$\|Du_n\|_p^p + \frac{p}{q}\|Du_n\|_q^q - \widehat{\lambda}_1(p)\|u_n\|_p^p - \int_\Omega pF(z,u_n)dz \leq pc_8 \text{ for all } n \in \mathbb{N}. \tag{27}$$

From (26), we have

$$|\langle\varphi'(u_n),h\rangle| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{28}$$

In (28), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$-\|Du_n^-\|_p^p - \|Du_n^-\|_q^q + \widehat{\lambda}_1(p)\|u_n^-\|_p^p + \int_\Omega f(z_1,-u_n^-)(-u_n^-)dz \leq \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{29}$$

Adding (28) and (29) and using hypothesis $H_1(iv)$ and the fact that $q < p$, we obtain

$$\|Du_n^+\|_p^p + \frac{p}{q}\|Du_n^+\|_q^q - \widehat{\lambda}_1(p)\|u_n^+\|_p^p - \int_\Omega pF(z,u_n^+)dz \leq c_9 \text{ for some } c_9 > 0, \text{ all } n \in \mathbb{N}. \tag{30}$$

In (28), we use the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$ and obtain

$$-\|Du_n^+\|_p^p - \|Du_n^+\|_q^q + \widehat{\lambda}_1(p)\|u_n^+\|_p^p + \int_\Omega f(z_1,u_n^+)u_n^+dz \leq \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{31}$$

We add (30) and(31) and use that $q < p$. We obtain

$$\int_\Omega [f(z,u_n^+)(u_n^+) - pF(z,u_n^+)]dz \leq c_{10} \text{ for some } c_{10} > 0, \text{ all } n \in \mathbb{N}. \tag{32}$$

Using (32), hypothesis $H_1(iii)$ and reasoning as in the "Claim" in the proof of Proposition 4 of Papageorgiou–Rădulescu–Zhang [21], we show that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{33}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned} &\varphi(u_n) = \varphi(u_n^+) + \varphi(-u_n^-) \\ \Rightarrow &\{\varphi(-u_n^-)\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ is bounded. (see (25), (33))} \end{aligned}$$

But then (33) and Proposition 4 imply that

$$\{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{34}$$

From (33) and (34), it follows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega). \tag{35}$$

In (28), we use the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (35). We obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0, \\ \Rightarrow &u_n \to u \text{ in } W_0^{1,p}(\Omega) \ \text{ (see Proposition (1))}, \\ \Rightarrow &\varphi(\cdot) \ \text{ satisfies the C-condition.} \end{aligned}$$

□

We assume that K_φ is finite or otherwise we already have an infinity of nontrivial solutions of (1) which by the nonlinear regularity theory are smooth(in $C_0^1(\overline{\Omega})$). Next we show the triviality of $C_1(\varphi, 0)$. To do this, we need hypothesis H_0 and also hypotheses H_2.

Proposition 7 *If hypotheses H_0, H_2 hold, then $C_1(\varphi, 0) = 0$.*

Proof Let $\hat{\theta} \in (\hat{\lambda}_2(q), \infty)\backslash\hat{\sigma}(q)$ be as in hypothesis $H_2(iv)$. We consider the C^1 function $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{q}\|Du\|_q^q - \frac{\hat{\theta}}{q}\|u\|_q^q \text{ for all } u \in W_0^{1,p}(\Omega).$$

We introduce the homotopy

$$h_t(u) = t\varphi(u) + (1-t)\psi(u) \text{ for all } t \in [0,1],\ \text{all } u \in W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \text{ in } [0,1],\, u_n \to 0 \text{ in } W_0^{1,p}(\Omega),\, (h_{t_n})'(u_n) = 0 \ \text{ for all } \ n \in \mathbb{N}. \tag{36}$$

From the equality in (36), we have

$$\begin{aligned} &t_n\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle \\ =&\widehat{\lambda}_1(p)t_n\int_\Omega |u_n|^{p-2}u_n h dz + t_n\int_\Omega f(z,u_n)hdz + (1-t_n)\hat{\theta}\int_\Omega |u_n|^{q-2}u_n hdz \end{aligned} \tag{37}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

Let $\|\cdot\|_{1,q}$ denote the norm of $W_0^{1,q}(\Omega)(\|u\|_{1,q} = \|Du\|_q$ for all $u \in W_0^{1,q}(\Omega))$ and recall that $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$. We set $v_n = \frac{u_n}{\|u_n\|_{1,q}}$, $n \in \mathbb{N}$. Then, $\|v_n\|_{1,q} = 1$ and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,q}(\Omega) \text{ and } v_n \to v \text{ in } L^q(\Omega). \tag{38}$$

From (37) we have

$$\begin{aligned} &\|u_n\|_{1,q}^{p-q} t_n\langle A_p(v_n), h\rangle + \langle A_q(v_n), h\rangle \\ =&\|u_n\|_{1,q}^{p-q}\widehat{\lambda}_1(p)\int_\Omega |v_n|^{p-2}v_n h dz + t_n\int_\Omega \frac{f(z,u_n)}{\|u_n\|_{1,q}^{q-1}}hdz + (1-t_n)\hat{\theta}\int_\Omega |v_n|^{q-2}v_n hdz \\ &\qquad \text{for all } h \in W_0^{1,p}(\Omega),\ \text{all } n \in \mathbb{N}. \end{aligned} \tag{39}$$

Note that $\|u_n\|_{1,q} \to 0$ (see (36) and recall that $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$). In (39) we choose the test function $h = v_n - v \in W_0^{1,p}(\Omega)$ and exploit the monotonicity of $A_p(\cdot)$. We have

$$\begin{aligned} &\|u_n\|_{1,q}^{p-q} t_n\langle A_p(v), v_n - v\rangle + \langle A_q(v_n), v_n - v\rangle \\ \le&\|u_n\|_{1,q}^{p-q}\widehat{\lambda}_1(p)\int_\Omega |v_n|^{p-2}v_n(v_n - v)dz + t_n\int_\Omega \frac{f(z,u_n)}{\|u_n\|_{1,q}^{q-1}}(v_n - v)dz \\ &+ (1-t_n)\hat{\theta}\int_\Omega |v_n|^{q-2}v_n(v_n - v)dz. \end{aligned} \tag{40}$$

On account of hypothesis H_0, we have $p \le q^*$ and so $W_0^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ (by the Sobolev embedding theorem). Also, we have

$$\begin{aligned} & |\int_\Omega |v_n|^{p-2} v_n (v_n - v) dz| \\ \le & \int_\Omega |v_n|^{p-1} |v_n - v| dz \\ \le & \|v_n\|_p^{p-1} \|v_n - v\|_p \le c_{11} \text{ for some } c_{11} > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{41}$$

Let $\langle \cdot, \cdot \rangle_{1,q}$ denote the duality brackets for the pair $(W_0^{1,q}(\Omega), W^{-1,q'}(\Omega))$ and recall that $W^{-1,q'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ (see Gasiński–Papageorgiou [9], p.141). If in (40) we pass to the limit as $n \to \infty$ and use (38) and (41), we obtain

$$\begin{aligned} & \limsup_{n\to\infty} \langle A_q(v_n), v_n - v \rangle \\ = & \limsup_{n\to\infty} \langle A_q(v_n), v_n - v \rangle_{1,q} \le 0 \\ = & v_n \to v \text{ in } W_0^{1,q}(\Omega) \text{ and so } \|v\|_{1,q} = 1. \end{aligned} \tag{42}$$

In (39), we pass to the limit as $n \to \infty$ and use (42) and hypothesis $H_2(iv)$. We obtain

$$\begin{aligned} & \langle A_q(v), h \rangle_{1,q} = \int_\Omega \hat{\theta} |v|^{q-2} v h dz \text{ for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow & -\Delta_q v = \hat{\theta} |v|^{q-2} v \text{ in } \Omega, v|_{\partial\Omega} = 0 \text{ (since } W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega) \text{ densely)}. \end{aligned} \tag{43}$$

But by hypothesis $\hat{\theta} \notin \hat{\sigma}(p)$. So, from (43) we have $v = 0$, which contradicts (42). Therefore, (36) cannot happen and then the homotopy invariance property of critical groups (see Papageorgiou–Rădulescu–Repovš [20], p.509), we have

$$C_k(\varphi, 0) = C_k(\psi, 0) \text{ for all } k \in \mathbb{N}_0. \tag{44}$$

Since $\hat{\theta} > \hat{\lambda}_2(q), \hat{\theta} \notin \hat{\sigma}(p)$, from Theorem 1.1 of Dancer–Perera [7], we have

$$\begin{aligned} & C_1(\psi, 0) = 0, \\ \Rightarrow & C_1(\varphi, 0) = 0 \text{ (see (44))}. \end{aligned}$$

□

Now we have all the necessary tools to produce a second nontrivial solution for problem (1) and have the first multiplicity theorem.

Theorem 8 *If hypotheses H_0, H_2 hold, then problem (1) has at least two nontrivial solutions $v_0 \in -int\, C_+$, $u_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$.*

Proof From Proposition 5, we already have a solution $v_0 \in -\text{int}\, C_+$ which is a local minimizer of $\varphi(\cdot)$. Recall that without any loss of generality K_φ is assumed to be finite. Invoking Theorem 5.7.6, p.449, of Papageorgiou–Rădulescu–Repovš [20], we can find $\rho \in (0, 1)$ small such that

$$\varphi(v_0) < \inf \left[\varphi(u) : \|u - v_0\| = \rho \right] = m_\rho. \tag{45}$$

On account of hypothesis $H_2(ii) = H_1(ii)$, if $u \in \text{int}\, C_+$, then

$$\varphi(tu) \to -\infty \quad \text{as } t \in +\infty. \tag{46}$$

Moreover, from Proposition 6 we have that

$$\varphi(\cdot) \;\; \text{satisfies the C-condition.} \tag{47}$$

Then, (45), (46), (47) permit the use of the mountain pass theorem. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_0 \in K_\varphi, \varphi(v_0) < m_0 \le \varphi(u_0), \\ \Rightarrow &u_0 \in C_0^1(\overline{\Omega}) \;\; \text{(nonlinear regularity) is a solution of (1)}, u_0 \neq v_0. \end{aligned}$$

From Theorem 6.5.8, p.527, of [20], we have

$$C_1(\varphi_1, u_0) \neq 0. \tag{48}$$

Then, (48) and Proposition 7 imply $u_0 \neq 0$. □

When $q = 2$ (a $(p, 2)$-equation) and if strengthen the regularity of the perturbation $f(z, \cdot)$, we can generate a third nontrivial smooth solution.

The problem under consideration is now the following.

$$\left\{ \begin{array}{ll} -\Delta_p u(z) - \Delta u(z) = \hat{\lambda}_1(p)|u(z)|^{p-2}u(z) + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 2 < p. & \end{array} \right\}. \tag{49}$$

The new hypotheses on $f(z, x)$ are the following:

H_3: $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.e. $z \in \Omega$ $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z, x)| \le a(z)[1 + |x|^{r-2}]$ for a.e.$z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii), (iii) are the same as the corresponding hypotheses $H_1(ii), (iii)$;

(iv) there exists $m \ge 2$ such that

$$\begin{aligned} &f_x'(z, 0) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \;\; \text{for a.e.} z \in \Omega, \\ &f_x'(\cdot, 0) \not\equiv \hat{\lambda}_m(2), \; f_x'(\cdot, 0) \not\equiv \hat{\lambda}_{m+1}(2); \end{aligned}$$

(v) is the same with hypothesis $H_1(v)$.

The following function satisfies these hypotheses:

$$f(x) = \begin{cases} \theta x & \text{if } x < 1 \\ \theta x^{r-1} + (r-2)\theta \ln x & \text{if } 1 \leq x \end{cases}$$

with $\theta \in (\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2))$.

Now, the energy function $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ has the following form

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\hat{\lambda}_1(p)}{p}\|u\|_p^p - \int_\Omega F(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

In this case $\varphi \in C^2(W_0^{1,p}(\Omega))$(recall $p > 2$). The asymmetric behavior of the reaction as $x \to \pm\infty$ leads to the following result due to Papageorgiou–Winkert [22](Proposition 4.8).

Proposition 9 *If hypotheses H_3 hold, then $C_k(\varphi, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Using Morse theoretic tools (critical groups), we can generate a third nontrivial smooth solution and have the second multiplicity theorem.

Theorem 10 *If hypotheses H_0, H_3 hold, then problem (49) has at least three nontrivial solutions*

$$v_0 \in \textit{-int}\, C_+, u_0, w_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}.$$

Proof From Theorem 8, we already have two nontrivial solutions.

$$v_0 \in \text{ -int } C_+, u_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}.$$

Recall that v_0 is a local minimizer of $\varphi(\cdot)$(see Proposition 5). So, we have

$$C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{50}$$

Also, we know that

$$C_1(\varphi, u_0) \neq 0 \text{ (see (48))}.$$

Since $\varphi \in C^2(W_0^{1,p}(\Omega))$, from Claim 3 in the proof of Proposition 3.5 of Papageorgiou–Rădulescu [19], we have

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{51}$$

Consider the function $\hat{\psi} \in C^2(H_0^1(\Omega))$ defined by

$$\hat{\psi}(u) = \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega f_x'(z,0)u^2 dz \text{ for all } u \in H_0^1(\Omega).$$

On account of hypothesis $H_3(iv)$ and of the unique continuation property of the eigenspaces of $(-\Delta, H_0^1(\Omega))$, we have that $u = 0$ is a nondegenerate critical point of $\hat{\psi}(\cdot)$ with Morse index $d_m = \dim \overline{H}_m = \dim \bigoplus_{k=1}^m E(\widehat{\lambda}_k(2))$. Therefore,

$$C_k(\hat{\psi}, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see [20], Proposition 6.2.6, p.479).} \tag{52}$$

Let $\psi = \hat{\psi}|_{W_0^{1,p}(\Omega)}$(recall that $2 < p$). Then, Theorem 6.6.26, p.545, of [20] implies that

$$\begin{aligned} &C_k(\psi, 0) = C_k(\hat{\psi}, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow &C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \ \text{ (see (52))}. \end{aligned} \tag{53}$$

A homotopy invariance argument as in the proof of Proposition 7, shows that

$$\begin{aligned} &C_k(\varphi, 0) = C_k(\psi, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow &C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \end{aligned} \tag{54}$$

From Proposition 9, we know that

$$C_k(\varphi, \infty) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{55}$$

Suppose $K_\varphi = \{v_0, u_0, 0\}$. From (50),(51),(54),(55) and the Morse identity with $t = -1$(see (8)), we have

$$(-1)^0 + (-1)^1 + (-1)^{dm} = 0,$$

a contradiction. Hence, there exists $w_0 \in K_\varphi \subseteq C_0^1(\overline{\Omega})$ such that $w_0 \notin \{v_0, u_0, 0\}$. Therefore, w_0 is the third nontrivial smooth solution of (49). ☐

4 Semilinear Equations

In this section, we deal with the special case of semilinear equations driven by the Dirichlet Laplacian. In what follows $\hat{\lambda}_k = \hat{\lambda}_k(2)$ for all $k \in \mathbb{N}$ and $\hat{u}_1 = \hat{u}_1(2) \in \text{ int } C_+$.

The equation under consideration is the following

$$\left\{ \begin{array}{ll} -\Delta u(z) = \hat{\lambda}_1 u(z) + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{array} \right\}. \tag{56}$$

The hypotheses on the perturbation $f(z, x)$ are the following:

$H_4 : f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for $a.e.\, z \in \Omega,\ f(z, 0) = 0$ and

(i) $|f(z, x)| \le a(z)[1 + |x|^{r-1}]$ for $a.e.\ z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $2 < r < 2^*$;

(ii) if $F(z,x)=\int_0^x f(z,s)ds$, then $\lim_{x\to+\infty}\frac{F(z,x)}{x^2}=+\infty$ uniformly for $a.e.\ z\in\Omega$ and there exists $\mu\in((r-2)\frac{N}{2},2^*)$ such that

$$0<\beta_0\leq\liminf_{x\to+\infty}\frac{f(z,x)x-2F(z,x)}{x^{\mu}}\ \text{uniformly for a.e. } z\in\Omega;$$

(iii) $\lim_{x\to-\infty}\frac{f(z,x)}{x}=0$, $\lim_{x\to-\infty}f(z,x)=-\infty$ uniformly for a.e.$z\in\Omega$ and $\lim_{x\to-\infty}[f(z,x)x-2F(z,x)]=+\infty$ for a.e. $x\in\Omega$;

(iv) there exist $\theta\in L^{\infty}(\Omega)$ and $\hat{t},\hat{\eta}>0$ such that

$$\theta(z)\leq 0 \text{ for a.e. } z\in\Omega, \theta\not\equiv 0, \limsup_{x\to 0}\frac{f(z,x)}{x}\leq\theta(z) \text{ uniformly for a.e. } z\in\Omega,$$
$$\int_{\Omega}F(z,-\hat{t}\hat{u}_1)dz>0,\ e(z,x)=f(z,x)x-2F(z,x)\geq-\hat{\eta}\ \text{ for a.e. } z\in\Omega,\ \text{all } x\leq 0;$$

(v) for every $\rho>0$, there exists $\hat{\xi}_{\rho}>0$ such that for a.e. $z\in\Omega$, the function $x\to f(z,x)+\hat{\xi}_{\rho}x$ is nondecreasing on $[-\rho,\rho]$.

Remarks: The asymptotic conditions as $x\to\pm\infty$ (see $H_4(ii)$, (iii)) remain similar as before when we examined (p,q)-equations. Again we have a "resonant-superlinear" problem, similar to the one studied by Domingos da Silva-Ribeiro [8]. However, our conditions on the perturbation $f(z,x)$ are less restrictive. So, our multiplicity theorem (see Theorem 13) extends Theorem 1.2 and Corollary 1.1 of Domingos da Silva-Ribeiro [8].

The following function satisfies hypotheses H_4. As before for the sake of simplicity, we drop the z-dependence

$$f(x)=\begin{cases}-\ln|x|+\theta & \text{if } x<-1\\ cx-\sin x & \text{if } -1\leq x\leq 1\\ \theta x^{r-1} & \text{if } 1<x\end{cases}$$

with $\sin 1<c<1$ and $\theta=c-\sin 1>0$.

We introduce the C^1-functional $\zeta_{\pm}:H_0^1(\Omega)\to\mathbb{R}$ defined by

$$\zeta_+(u)=\frac{1}{2}\|Du\|_2^2-\frac{\hat{\lambda}_1}{2}\|u^+\|_2^2-\int_{\Omega}F(z,u^+)dz,$$
$$\zeta_-(u)=\frac{1}{2}\|Du\|_2^2-\frac{\hat{\lambda}_1}{2}\|u^-\|_2^2-\int_{\Omega}F(z,-u^-)dz,\ \text{for all } u\in H_0^1(\Omega).$$

Reasoning as in Proposition 4, we have

Proposition 11 *If hypotheses H_4 hold, then $\zeta_-(\cdot)$ is coercive.*

Next we determine what kind of critical point for $\zeta_{\pm}(\cdot)$ is the origin($u=0$).

Proposition 12 *If hypotheses H_4 hold, then $u=0$ is a local minimizer for the functionals $\zeta_{\pm}(\cdot)$.*

Proof On account of hypothesis $H_4(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z, x) \leq \frac{1}{2}[\theta(z) + \varepsilon]x^2 \text{ for a.e. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{57}$$

Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leq \delta$. We have

$$\begin{aligned} \zeta_-(u) &= \frac{1}{2}\|Du\|_2^2 - \frac{\hat{\lambda}_1}{2}\|u^-\|_2^2 - \int_\Omega F(z, -u^-)dz \\ &\geq \frac{1}{2}\|Du\|_2^2 - \frac{\hat{\lambda}_1}{2}\|u^-\|_2^2 - \frac{1}{2}\Big[\int_\Omega \theta(z)(u^-)^2 dz + \frac{\varepsilon}{2}\|u^-\|_2^2\Big] \\ &\geq \frac{1}{2}\Big[\|Du\|_2^2 - \int_\Omega [\hat{\lambda}_1 + \theta(z)](u^-)^2 dz - \frac{\varepsilon}{2\hat{\lambda}_1}\|Du^-\|_2^2\Big] \text{ (see (52)).} \end{aligned} \tag{58}$$

Note that

$$\hat{\lambda}_1 + \theta(z) \leq \hat{\lambda}_1 \text{ for a.e. } z \in \Omega, \hat{\lambda}_1 + \theta(\cdot) \not\equiv \hat{\lambda}_1.$$

So, from Proposition 2 we have

$$\|Du^-\|_2^2 - \int_\Omega [\hat{\lambda}_1 + \theta(z)](u^-)^2 dz \geq c_{12}\|Du^-\|_2^2 \text{ for some } c_{12} > 0.$$

Returning to (58) we have

$$\zeta_-(u) \geq \frac{1}{2}\Big[\|Du^+\|_2^2 + (c_{10} - \frac{\varepsilon}{\hat{\lambda}_1})\|Du^-\|_2^2\Big].$$

Choosing $\varepsilon \in (0, \hat{\lambda}_1 c_{10})$, we obtain

$$\zeta_-(u) \geq c_{13}\|u\|^2 \geq 0 = \zeta_-(0) \text{ for some } c_{13} > 0, \text{ all } u \in C_0^1(\overline{\Omega}), \|u\|_{C_0^1(\overline{\Omega})} \leq \delta.$$

This means that
$u = 0$ is a local $C_0^1(\overline{\Omega})$−minimizer of $\zeta_-(\cdot)$,
$\Rightarrow u = 0$ is a local $H_0^1(\overline{\Omega})$-minimizer of $\zeta_-(\cdot)$,(see Brezis-Nirenberg [3] and [11]).
Similarly we show that $u = 0$ is a local minimizer for $\zeta_+(\cdot)$ too. □

Now we can have our multiplicity theorem for problem (56).

Theorem 13 *If hypotheses H_4 hold, then problem (56) has at least three nontrivial solutions*

$$v_0, \hat{v} \in -int\, C_+, \hat{u} \in int\, C_+.$$

Proof From Proposition 11 we know that $\zeta_-(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\zeta_-(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $v_0 \in H_0^1(\Omega)$ such that

$$\zeta_-(v_0) = \inf\left[\zeta_-(u) : u \in H_0^1(\Omega)\right]. \tag{59}$$

Using hypothesis $H_4(iv)$ we have

$$\begin{aligned} &\zeta_-(-\hat{t}\hat{u}_1) = -\int_\Omega F(z, -\hat{t}\hat{u}_1)dz < 0, \\ \Rightarrow &\zeta_-(v_0) < 0 = \zeta_-(0)(\text{see}(59)), \\ \Rightarrow &v_0 \neq 0. \end{aligned}$$

From (59), we have

$$\begin{aligned} &\zeta_-'(v_0) = 0 \text{ in } H^{-1}(\Omega), \\ \Rightarrow &\langle \zeta_-'(v_0), h\rangle = 0 \ \text{ for all } \ h \in H_0^1(\Omega). \end{aligned} \tag{60}$$

In (60) we choose $h = v_0^+ \in H_0^1(\Omega)$. We obtain

$$\|Dv_0^+\|_2^2 = 0, \quad \Rightarrow \quad v_0 \leq 0, v_0 \neq 0.$$

We have

$$-\Delta v_0 = \hat{\lambda}_1 v_0 + f(z, v_0) \text{ in } \Omega, v_0|_{\partial\Omega} = 0.$$

Then, the classical regularity theory (see Gilbarg–Trudinger [13]) implies $v_0 \in (-C_+) \setminus \{0\}$. Let $\rho = \|v_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_4(v)$. We have

$$\begin{aligned} &-\Delta v_0 + \hat{\xi}_\rho v_0 = \hat{\lambda}_1 v_0 + f(z, v_0) + \hat{\xi}_\rho v_0 \geq 0 \\ \Rightarrow &\Delta(-v_0) \leq \hat{\xi}_\rho(-v_0), \\ \Rightarrow &v_0 \in -\text{int}\, C_+ \ \text{(by the Hopf maximum principle)}. \end{aligned}$$

We assume that K_ζ is finite. Otherwise we already have an infinity of negative smooth solutions and so we are done. From Proposition 12 we know that $u = 0$ is a local minimizer of $\zeta_-(\cdot)$. Using Theorem 5.7.6, p.449, of [20], we can find $\rho \in (0, 1)$ small such that

$$\zeta_-(v_0) < \zeta_-(0) < \inf\left[\zeta_-(u) : \|u\| = \rho\right] = m_-, \quad \rho < \|v_0\|. \tag{61}$$

Since $\zeta_-(\cdot)$ is coercive (see Proposition 11), we have that

$$\zeta_-(\cdot) \ \text{ satisfies the C-condition } \ \text{(see [20]), Proposition 5.1.15, p.369)}. \tag{62}$$

Then (61), (62) and the mountain pass theorem, imply that we can find $\hat{v} \in H_0^1(\Omega)$ such that

$$\begin{aligned}&\hat{v} \in K_{\zeta_-} \subseteq -C_+, m_- \leq \zeta_-(\hat{v}),\\ \Rightarrow &\hat{v} \in -\text{int } C_+ \ \text{ (by the Hopf maximum principle).}\end{aligned}$$

From Proposition 12 we know that $u = 0$ is also a local minimizer for $\zeta_+(\cdot)$. By the regularity theory $K_{\zeta_+} \subseteq C_+$ and again without any loss of generality, we assume that K_{ζ_+} is finite. So, as before we can find $\rho \in (0, 1)$ small such that

$$\zeta_+(0) = 0 < \inf\big[\zeta_+(u) : \|u\| = \rho\big] = m_+. \tag{63}$$

From Papageorgiou–Rădulescu–Zhang [21] (see the "Claim" in the proof of Proposition 4), we have that

$$\zeta_+(\cdot) \text{ satisfies the C-condition.} \tag{64}$$

Finally on account of hypothesis $H_4(ii)$, if $u \in \text{int } C_+$, then we have

$$\zeta_+(tu) \to -\infty \text{ as } t \to \infty. \tag{65}$$

Then, (63), (64), (65) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in H_0^1(\Omega)$ such that

$$\begin{aligned}&\hat{u} \in K_{\zeta_+} \subseteq C_+, \zeta_+(0) = 0 < m_+ \leq \zeta_+(\hat{u}),\\ \Rightarrow &\hat{u} \in \text{int } C_+ \ \text{ is a third solution of (56).}\end{aligned}$$

So, we have produced three nontrivial smooth solutions and provided sign information for all of them. □

Next for problem (56) we consider the case where the perturbation $f(z, \cdot)$ is sublinear "resonant-sublinear" equation). To the best of our knowledge, this case was not considered in the past.

The hypotheses on $f(z, x)$ are the following:

H_5 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for $a.e.\ z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for $a.e.\ z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $2 < r < 2^*$;

(ii) there exist $m \in \mathbb{N}$ and functions $\theta, \hat{\theta} \in L^\infty(\Omega)$ such that

$$\begin{aligned}&\hat{\lambda}_m - \hat{\lambda}_1 \leq \theta(z) \leq \hat{\theta}(z) \leq \hat{\lambda}_{m+1} - \hat{\lambda}_1 \ \text{ for a.e.} z \in \Omega,\\ &\theta \not\equiv \hat{\lambda}_m - \hat{\lambda}_1, \hat{\theta} \not\equiv \hat{\lambda}_{m+1} - \hat{\lambda}_1,\\ &\theta(z) \leq \liminf_{x \to +\infty} \frac{f(z, x)}{x} \leq \limsup_{x \to +\infty} \frac{f(z, x)}{x} \leq \hat{\theta}(z) \ \text{ uniformly for a.e.} z \in \Omega;\end{aligned}$$

(iii) $\lim_{x\to-\infty}\frac{f(z,x)}{x}=0$, $\lim_{x\to-\infty}f(z,x)=-\infty$ uniformly for a.e.$z\in\Omega$;

$$f(z,x)x-2F(z,x)\to+\infty \ \text{ for a.e.} z\in\Omega, \ \text{ as } x\to-\infty,$$
$$-\hat{\eta}\le f(z,x)x-2F(z,x) \ \text{ for a.e.} z\in\Omega \text{ with } \hat{\eta}>0;$$

(iv) there exists $l\in\mathbb{N}$ such that

$$f'_x(z,0)\in[\hat{\lambda}_l,\hat{\lambda}_{l+1}] \ \text{ for a.e.} z\in\Omega,$$
$$f'_x(\cdot,0)\not\equiv\hat{\lambda}_l,\ f'_x(\cdot,0)\not\equiv\hat{\lambda}_{l+1}.$$

(v) for every $\rho>0$, there exists $\hat{\xi}_\rho>0$ such that for a.e.$z\in\Omega$, the function $x\to f(z,x)+\hat{\xi}_\rho x$ is nondecreasing on $[-\rho,\rho]$.

In addition to the functionals $\zeta_\pm$, let $\zeta:H_0^1(\Omega)\to\mathbb{R}$ be the energy functional for problem (56) defined by

$$\zeta(u)=\frac{1}{2}\|Du\|_2^2-\frac{\hat{\lambda}_1}{2}\|u\|_2^2-\int_\Omega F(z,u)dz \ \text{ for all } u\in H_0^1(\Omega).$$

Note that $\zeta\in C^2(H_0^1(\Omega))$. Next we show that $\zeta(\cdot)$ satisfies the compactness condition (the C-condition).

Proposition 14 *If hypotheses H_5 hold, then the functional $\zeta(\cdot)$ satisfies the C-condition.*

Proof We consider a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq H_0^1(\Omega)$ such that

$$|\zeta(u_n)|\le c_{14} \ \text{ for some } \ c_{14}>0, \ \text{ all } n\in\mathbb{N}, \tag{66}$$
$$(1+\|u_n\|)\zeta'(u_n)\to 0 \text{ in } H^{-1}(\Omega) \text{ as } n\to\infty. \tag{67}$$

From (67), we have

$$|\langle A(u_n),h\rangle-\hat{\lambda}_1\int_\Omega u_n h dz-\int_\Omega f(z,u_n)hdz|\le\frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{68}$$
$$\text{for all } h\in H_0^1(\Omega), \text{ with } \varepsilon_n\to 0^+.$$

In (68), we use the test function $h=-u_n^-\in H_0^1(\Omega)$. We obtain

$$\|Du_n^-\|_2^2-\hat{\lambda}_1\|u_n^-\|_2^2-\int_\Omega f(z,-u_n^-)(-u_n^-)dz\le\varepsilon_n \ \text{ for all } n\in\mathbb{N}. \tag{69}$$

Suppose that $\|u_n^-\|\to\infty$ and let $v_n=\frac{v_n^-}{\|v_n^-\|}$, $n\in\mathbb{N}$. Then $\|v_n\|=1$, $v_n\ge 0$ for all $n\in\mathbb{N}$. So, we may assume that

$$v_n\xrightarrow{w} v \text{ in } H_0^1(\Omega),\ v_n\to v \text{ in } L^2(\Omega). \tag{70}$$

From (69) we have

$$\|Dv_n^-\|_2^2 - \hat{\lambda}_1\|v_n\|_2^2 - \int_\Omega \frac{f(z,-u_n^-)}{\|u_n^-\|} v_n dz \le \frac{\varepsilon_n}{\|u_n^-\|^2} \quad \text{for all } n \in \mathbb{N}. \tag{71}$$

Note that $\{\frac{f(\cdot,-u_n^-(\cdot))}{\|u_n^-\|}\}_{n\in\mathbb{N}} \subseteq L^2(\Omega)$ is bounded and so on account of hypothesis $H_5(iii)$, we have at least for a subsequence we have that

$$\frac{f(\cdot,-u_n^-(\cdot))}{\|u_n^-\|} \xrightarrow{w} 0 \text{ in } L^2(\Omega)$$

(see Aizicovici–Papageorgiou–Staicu [1], proof of Proposition 16).

Therefore, if we pass to the limit as $n \to \infty$ in (71) and use (70) we obtain

$$\begin{aligned}
&\|Dv\|_2^2 \le \hat{\lambda}_1\|v\|_2^2,\\
\Rightarrow &\|Dv\|_2^2 = \hat{\lambda}_1\|v\|_2^2 \quad (\text{see}(5)),\\
\Rightarrow &v = \mu\hat{u}_1 \ \text{ with } \mu \ge 0 \quad (\text{recall } v \ge 0).
\end{aligned}$$

If $\mu = 0$, then $v = 0$ and so from (71) we have

$$\|Dv_n\|_2 \to 0, \quad \Rightarrow v_n \to 0 \text{ in } H_0^1(\Omega)$$

a contradiction to the fact that $\|v_n\| = 1$ for all $n \in \mathbb{N}$.

If $\mu > 0$, then $v = \mu\hat{u}_1 \in \text{ int } C_+$ and so we have

$$u_n^-(z) \to +\infty \ \text{ for a.e.} z \in \Omega. \tag{72}$$

From (66) we have

$$\begin{aligned}
&\|Du_n^+\|_2^2 + \|Du_n^-\|_2^2 - \|u_n^+\|_2^2 - \|u_n^-\|_2^2\\
&-\int_\Omega 2F(z,u_n^+)dz - \int_\Omega 2F(z,-u_n^-)dz \le 2c_{14} \text{ for all } n \in \mathbb{N}.
\end{aligned} \tag{73}$$

From (68) with $h = u_n^+ \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&-\|Du_n^+\|_2^2 + \hat{\lambda}_1\|u_n^+\|_2^2 + \int_\Omega f(z,u_n^+)u_n^+ dz \le \varepsilon_n,\\
\Rightarrow &-\|Du_n^+\|_2^2 + \hat{\lambda}_1\|u_n^+\|_2^2 + \int_\Omega 2F(z,u_n^+)dz \le c_{15}
\end{aligned} \tag{74}$$

for some $c_{15} > 0$, all $n \in \mathbb{N}$ (see hypothesis $H_5(iii)$).

Adding (73) and (74), we obtain

$$\|Du_n^-\|_2^2 - \hat{\lambda}_1 \|u_n^-\|_2^2 - \int_\Omega 2F(z, -u_n^-)dz \le c_{16} \text{ for some } c_{16} > 0, \text{ all } n \in \mathbb{N}. \tag{75}$$

In (68), we use the test function $h = -u_n^- \in H_0^1(\Omega)$ and obtain

$$-\|Du_n^-\|_2^2 + \hat{\lambda}_1 \|u_n^-\|_2^2 + \int_\Omega f(z, -u_n^-)(-u_n^-)dz \le \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{76}$$

We add (75) and (76) and have

$$\int_\Omega [f(z, -u_n^-)(-u_n^-) - 2F(z, -u_n^-)]dz \le c_{17} \text{ for some } c_{17} > 0, \text{ all } n \in \mathbb{N}.$$

Using hypothesis $H_5(iii)$,(72) and Fatou's lemma, we have a contradiction. This proves that

$$\{u_n^-\}_{n\in\mathbb{N}} \subseteq H_0^1(\Omega) \text{ is bounded.} \tag{77}$$

Now suppose that $\|v_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(\Omega),\ y_n \to y \text{ in } L^2(\Omega). \tag{78}$$

From (67) and (77), we have

$$\begin{aligned} &\langle A(u_n^+), h\rangle - \hat{\lambda}_1 \int_\Omega u_n^+ h dz - \int_\Omega f(z, u_n^+) h dz \le c_{18}\|h\| \\ &\qquad \text{for some } c_{18} > 0, \text{ all } h \in H_0^1(\Omega), \text{ all } n \in \mathbb{N}, \\ &\Rightarrow \langle A(y_n), h\rangle - \hat{\lambda}_1 \int_\Omega y_n h dz - \int_\Omega \frac{f(z, u_n^+)}{\|u_n^+\|} h dz \le \varepsilon_n' \text{ with } \varepsilon_n' \to 0^+ \text{ as } n \to \infty. \end{aligned} \tag{79}$$

In (79), we use the test function $h = y_n - y \in H_0^1(\Omega)$ and we note that $\{\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|}\}_{n\in\mathbb{N}} \subseteq L^2(\Omega)$ is bounded (see hypotheses $H_5(i)$, (ii)). So, if we pass to the limit as $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ &\Rightarrow \|Dy_n\|_2 \to \|Dy\|_2, \\ &\Rightarrow y_n \to y \text{ in } H_0^1(\Omega) \text{ and so } \|y\| = 1,\ y \ge 0 \\ &\qquad \text{(by the Kadec–Klee property of Hilbert spaces).} \end{aligned} \tag{80}$$

Recall that $\{\frac{f(\cdot,u_n^+(\cdot))}{\|u_n^+\|}\}_{n\in\mathbb{N}} \subseteq L^2(\Omega)$ is bounded. So, we may assume that

$$\begin{cases} \frac{f(\cdot,u_n^+(\cdot))}{\|u_n^+\|} \xrightarrow{w} \eta \quad \text{in } \mathrm{L}^2(\Omega), \\ \eta = \theta(\cdot)y, \quad \text{with } \theta(z) \leq \widetilde{\theta}(z) \ \text{ for a.e.} z \in \Omega. \end{cases} \tag{81}$$

(see hypothesis $H_5(ii)$ and see [1], proof of Proposition 16). So, if in (79) we pass to the limit as $n \to \infty$ and use (80) and (81), we obtain

$$\begin{aligned} &\langle A(y), h\rangle = \int_\Omega [\hat{\lambda}_1 + \widetilde{\theta}(z)]yhdz \text{ for all } h \in H_0^1(\Omega), \\ &\Rightarrow -A(y)(z) = [\hat{\lambda}_1 + \widetilde{\theta}(z)]y(z) \text{ in } \Omega,\ y|_{\partial\Omega} = 0. \end{aligned} \tag{82}$$

From (81) and hypothesis $H_5(ii)$, we have

$$\hat{\lambda}_m \leq \hat{\lambda}_1 + \widetilde{\theta}(z) \ \text{ for a.e.} z \in \Omega, \hat{\lambda}_1 + \widetilde{\theta}(\cdot) \not\equiv \hat{\lambda}_m.$$

Invoking Proposition 3 we have

$$\begin{aligned} &\widetilde{\lambda}_1(\hat{\lambda}_1 + \widetilde{\theta}(\cdot)) < \widetilde{\lambda}_1(\hat{\lambda}_m) \leq \widetilde{\lambda}_1(\hat{\lambda}_1) = 1, \\ &\Rightarrow y \text{ must be nodal, a contradiction (see (82), (80)).} \end{aligned}$$

This proves that

$$\begin{aligned} &\{u_n^+\}_{n\in\mathbb{N}} \subseteq H_0^1(\Omega) \ \text{ is bounded.} \\ &\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq H_0^1(\Omega) \ \text{ is bounded (see (77)).} \end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } H_0^1(\Omega), u_n \to u \text{ in } L^2(\Omega). \tag{83}$$

In (68) we choose the test function $h = u_n - u \in H_0^1(\Omega)$, pass to the limit as $n \to \infty$ and use (83). We obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\ &\Rightarrow \|Du_n\| \to \|Du\|_2, \\ &\Rightarrow u_n \to u \text{ in } H_0^1(\Omega) \ \text{ (Kadec–Klee property).} \end{aligned}$$

This proves that $\zeta(\cdot)$ satisfies the C-condition. □

Proposition 14 allows us to compute the critical groups of $\zeta(\cdot)$ at infinity. Recall that as before without any loss of generality, we assume that K_ζ is finite.

Proposition 15 *If hypotheses H_5 hold, then $C_k(\zeta,\infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof Let $\beta \in L^\infty(\Omega)$ such that $\beta(z) > 0$ for a.e.$z \in \Omega$ and $\vartheta_0 \in (\hat{\lambda}_m - \hat{\lambda}_1, \hat{\lambda}_{m+1} - \hat{\lambda}_1)$. We consider the homotopy $(t, u) \to \hat{h}_t(u)$ defined by

$$\hat{h}_t(u) = \frac{1}{2}\|Du\|_2^2 - \frac{\hat{\lambda}_1}{2}\|u\|_2^2 - \frac{(1-t)\theta_0}{2}\|u^+\|_2^2 - t\int_\Omega F(z,u)dz - (1-t)\int_\Omega \beta(z)udz$$
$$\text{for all } t \in [0,1], \text{ all } u \in H_0^1(\Omega).$$

Note that

$$\hat{h}_0(u) = \gamma(u) = \frac{1}{2}\|Du\|_2^2 - \frac{\hat{\lambda}_1}{2}\|u\|_2^2 - \frac{\theta_0}{2}\|u^+\|_2^2 - \int_\Omega \beta(z)udz,$$
$$\hat{h}_1(u) = \zeta(u) \text{ for all } u \in H_0^1(\Omega).$$

Suppose we can find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ and$\{u_n\}_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ such that

$$\hat{h}_{t_n}(u_n) \to -\infty \text{ and } (1+\|u_n\|)(\hat{h}_{t_n})'(u_n) \to 0 \text{ in } H^{-1}(\Omega). \tag{84}$$

From the second convergence in (84), we have

$$|\langle A(u_n), h\rangle - \hat{\lambda}_1\int_\Omega u_n hdz - (1-t)\theta_0\int_\Omega u_n^+ hdz - t_n\int_\Omega f(z,u_n)hdz$$
$$-(1-t_n)\int_\Omega \beta(z)hdz| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \text{ for all } h \in H_0^1(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{85}$$

Assume that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and we may assume that

$$v_n \xrightarrow{w} v \text{ in } H_0^1(\Omega),\ v_n \to v \text{ in } L^2(\Omega). \tag{86}$$

From (85) we have

$$|\langle A(v_n), h\rangle - \hat{\lambda}_1\int_\Omega v_n hdz - (1-t)\theta_0\int_\Omega v_n^+ hdz - t_n\int_\Omega \frac{f(z,u_n)}{\|u_n\|}hdz$$
$$-(1-t_n)\int_\Omega \frac{\beta(z)}{\|u_n\|}hdz| \le \varepsilon_n' \text{ with } \varepsilon_n' \to 0 \text{ as } n \to \infty. \tag{87}$$

In (87) we use the test function $h = v_n - v \in H_0^1(\Omega)$, pass to the limit as $n \to \infty$ and use (86). Then,

$$\lim_{n\to\infty}\langle A(v_n), v_n - v\rangle = 0,$$
$$\Rightarrow \|Dv_n\|_2 \to \|Dv\|_2,$$
$$\Rightarrow v_n \to v \text{ in } H_0^1(\Omega) \text{ (Kadec–Klee property)}, \|v\| = 1. \tag{88}$$

Hypotheses $H_5(i)$, (ii), (iii) imply that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|} \xrightarrow{w} \theta^*(\cdot)v^+ \quad \text{in } \mathrm{L}^2(\Omega), \tag{89}$$

with $\theta^* \in L^\infty(\Omega)$ such that $\theta(z) \le \theta^*(z) \le \hat{\theta}(z)$ for a.e.$z \in \Omega$ (see [1]). Therefore, if in (87) we let $n \to \infty$ and use (88),(89), we obtain

$$\langle A(v_n), h\rangle = \hat{\lambda}_1 \int_\Omega vhdz + \int_\Omega [(1-t)\theta_0 + t\theta^*(z)]v^+ hdz \text{ for all } h \in H_0^1(\Omega). \tag{90}$$

Suppose $v^- \neq 0$ and in (90) choose the test function $h = -v^- \in H_0^1(\Omega)$. We have

$$\|Dv^-\|_2^2 = \hat{\lambda}_1 \|v^-\|_2^2, \quad \Rightarrow v = \mu\hat{u}_1 \text{ with } \mu < 0.$$

Then, $v(z) < 0$ for all $z \in \Omega$ and so

$$u_n(z) \to -\infty \text{ for a.e.} z \in \Omega.$$

Then, reasoning as in the proof of Proposition 4 (see the part of the proof after (15)), we reach a contradiction. This means that $v \ge 0$ and from (90) we have

$$-\Delta v(z) = [\hat{\lambda}_1 + \hat{\theta}_t(z)]v(z) \text{ in } \Omega, v|_{\partial\Omega} = 0, \tag{91}$$

with $\hat{\theta}_t(z) = (1-t)\theta_0 + t\theta^*(z)$, $\hat{\theta}_t \in L^\infty(\Omega)$, $0 \le t \le 1$. From the choice of θ_0 and (89) we see that

$$\left\{ \begin{array}{l} \hat{\lambda}_m \le \hat{\lambda}_1 + \hat{\theta}_t(z) \le \hat{\lambda}_{m+1} \text{ for a.e.} z \in \Omega, \\ \hat{\lambda}_m \not\equiv \hat{\lambda}_1 + \hat{\theta}_t(\cdot), \hat{\lambda}_{m+1} \not\equiv \hat{\lambda}_1 + \hat{\theta}_t(\cdot). \end{array} \right\}. \tag{92}$$

Using Proposition 3, we have

$$\widetilde{\lambda}_1(\hat{\lambda}_1 + \hat{\theta}_t) < \widetilde{\lambda}_1(\hat{\lambda}_m) \le \widetilde{\lambda}_1(\hat{\lambda}_1) = 1,$$

$\Rightarrow v$ must be nodal (see (91)), a contradiction.

Therefore $\{u_n\}_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ is bounded and this implies that $\{h_{t_n}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, contradicting (84). So, (84) cannot happen and then using Proposition 3.2 of Liang-Su [15] (see also Chang [4], Theorem 5.1.21, p.334), we have

$$\begin{aligned} &C_k(h_0, \infty) = C_k(h_1, \infty) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow &C_k(\gamma, \infty) = C_k(\zeta, \infty) \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{93}$$

Let $u \in K_\gamma$ We have

$$-\Delta u(z) = \hat{\lambda}_1 u(z) + \theta_0 u^+(z) + \beta(z) \text{ in } \Omega, u|_{\partial\Omega} = 0. \tag{94}$$

Suppose $u^- \neq 0$ and act on (94) with $-u^- \in H_0^1(\Omega)$. Then,

$$0 \leq \|Du^-\|_2^2 - \hat{\lambda}_1 \|u^-\|_2^2 - \int_\Omega \beta(z) u^- dz < 0$$

(recall $\beta(z) > 0$ for a.e.$z \in \Omega$), a contradiction. Hence, $u \geq 0$, $u \neq 0$(since $\beta \neq 0$). From (94), the regularity theory (see Gilbarg-Trudinger [13]) and the Hopf maximum principle we infer that $u \in \operatorname{int} C_+$ (note that since $\beta \neq 0$, then $u \neq 0$). Let $y \in \operatorname{int} C_+$. Using Picone's identity (see Motreanu–Motreanu–Papageorgiou [18], p.255), we have

$$\begin{aligned}
0 &\leq \|Dy\|_2^2 - \int_\Omega (Du, D(\frac{y^2}{u}))_{\mathbb{R}^N} dz \\
&= \|Dy\|_2^2 - \int_\Omega (-\Delta u)\frac{y^2}{u} dz \ \text{ (by Green's identity)} \\
&= \|Dy\|_2^2 - \int_\Omega [\hat{\lambda}_1 + \theta_0] y^2 dz - \int_\Omega \beta(z) \frac{y^2}{u} dz \ \text{ (see (94))} \\
&\leq \|Dy\|_2^2 - \int_\Omega [\hat{\lambda}_1 + \theta_0] y^2 dz
\end{aligned}$$

Let $y = \hat{u}_1 \in \operatorname{int} C_+$. We have

$$0 \leq -\theta_0 \int_\Omega \hat{u}_1^2 \, dz < 0 \quad \text{see (5)}$$

a contradiction. Therefore, $K_\gamma = \emptyset$ and the Proposition 6.2.28, p.491, in [20], implies that

$$\begin{aligned}
&C_k(\gamma, \infty) = 0 \ \text{ for all } k \in \mathbb{N}_0, \\
\Rightarrow &C_k(\zeta, \infty) = 0 \ \text{ for all } k \in \mathbb{N}_0 \ \text{ (see (93))} .
\end{aligned}$$

□

Remark: If $m = 2$ (see hypothesis $H_5(ii)$), then we can have an alternative proof that $K_\gamma = \emptyset$. We outline this alternative proof. Let $u \in K_\gamma$. We have

$$-\Delta u = \hat{\lambda}_1 u + \theta_0 u^+ + \beta(z) \text{ in } \Omega, \, u|_{\partial\Omega} = 0.$$

As before acting with $-u^- \in H_0^1(\Omega)$, we infer that $u \geq 0$, $u \neq 0$. On the other hand choosing θ_0 close to $\hat{\lambda}_2 - \hat{\lambda}_1$ and invoking the antimaximum principle (see Motreanu–Motreanu–Papageorgiou [18], p.263), we infer that $u \in -\operatorname{int} C_+$, a contradiction.

Note that hypothesis $H_5(iv)$ implies that $u = 0$ is a nondegenerate critical point of $\zeta(\cdot)$ with Morse index $d_l = \dim \overline{H}_l = \dim \bigoplus_{k=1}^l E(\widehat{\lambda}_k)$. Then using Proposition 6.2.6, p.479, of [20], we have:

Proposition 16 *If hypotheses H_5 hold, then $C_k(\zeta, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

We are ready for the multiplicity theorem of the "resonant-sublinear" case.

Theorem 17 *If hypotheses H_5 hold, then problem (56) has at least three nontrivial solutions $v_0 \in -int\, C_+, u_0, \hat{u} \in C_0^1(\overline{\Omega})$.*

Proof As before using the functional $\zeta_-(\cdot)$ which is coercive via the Weierstrass–Tonelli theorem, we produce $v_0 \in -\text{int}\, C_+$ a solution of (56) which is a local minimizer of $\zeta(\cdot)$. Hence,

$$C_k(\zeta, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{95}$$

Using v_0 and Proposition 14, as in the proof of Theorem 8, using the mountain pass theorem, we generate a second nontrivial solution $u_0 \in C_0^1(\overline{\Omega})$ (regularity theory). For this solution, we have

$$C_k(\zeta, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \text{ (see [20], p.529).} \tag{96}$$

From Proposition 15 and 16, we have

$$C_k(\zeta, \infty) = 0 \text{ for all } k \in \mathbb{N}_0, C_k(\zeta, 0) = \delta_{k,d_l}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{97}$$

Suppose $K_\zeta = \{v_0, u_0, 0\}$. Then from (95),(96),(97) and the Morse identity with $t = -1$, we have

$$\begin{aligned} &(-1)^0 + (-1)^1 + (-1)^{d_l} = 0, \\ \Rightarrow &(-1)^{d_l} = 0, \quad \text{a contradiction.} \end{aligned}$$

Therefore, there exists $\hat{u} \in K_\zeta, \hat{u} \notin \{v_0, u_0, 0\}$. Hence $\hat{u} \in C_0^1(\overline{\Omega})$(regularity theory) is the third nontrivial smooth solution of (56). ◻

Remark: It is interesting to know if the above result for the "resonant-sublinear" case remains valid if we consider (p, q)-equations.

Acknowledgements The authors wish to thank the referee for his/her remarks.

Funding This work was supported by Innovative Research Group Project of the National Natural Science Foundation of China (Grant No. 12071413).

Declarations

Conflict of interest We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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